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Research article

Fuzzy fixed point results for α-admissible ϕ-ψ-set-valued fuzzy contractions

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Abstract: Fixed point theory and fuzzy mathematics offer powerful tools for addressing real-world problems involving uncertainty and imprecision. In this paper, we introduced a generalized framework for $\alpha-$admissible fuzzy set-valued contractive mappings by extending the existing concepts of ϕ-set-valued and ϕ-ψ-set-valued contractions in the setting of b-metric spaces. We established several fixed point theorems under these generalized contractive conditions, thereby extending and enriching the existing theory. As an additional contribution, we also explored fixed point results under Kannan-type and Reich-type conditions within the same framework. Notably, several classical fixed point results were recovered as special cases of our findings, demonstrating the unifying nature of our approach. To illustrate the applicability and non-triviality of the results, a concrete example was provided.

Keywords: admissible mapping; fixed point; fuzzy fixed point; ϕ-ψ-set-valued contraction; b-metric space
Mathematics Subject Classification: 46S40, 54H25, 47H10, 37C25, 47H09

1. Introduction

Banach's contraction principle [1] is one of the most influential results in nonlinear analysis, ensuring the existence and uniqueness of fixed points for contraction mappings in complete metric spaces. Since its inception, numerous generalizations have been proposed to weaken the contraction condition or to extend the underlying space. Among these, Kannan's contraction [2], defined by

$$\rho(Tp, Tq) \leq \lambda[\rho(p, Tp) + \rho(q, Tq)], \quad \lambda \in \left[0, \tfrac{1}{2}\right),$$

introduced a fundamentally different mechanism independent of Banach's condition. Subsequent extensions, including those by Jleli and Samet [3] and Samet et al. [4], incorporated control functions and admissibility concepts, significantly broadening the applicability of fixed point techniques.

In parallel, Nadler [5] initiated the study of fixed points for set-valued mappings via the Hausdorff metric, opening a new direction for multivalued analysis. This line of research has since been extensively developed through various generalized contraction frameworks (see [6–8] and the references therein). These results have proven particularly effective in addressing problems arising in optimization, differential inclusions, and equilibrium theory.

A major structural generalization of metric spaces was introduced by Czerwik [9] through the notion of a b-metric space, where the classical triangle inequality is replaced by a relaxed inequality involving a coefficient $s \geq 1$. This framework captures a wide class of nonstandard distance structures and has stimulated extensive research on both single-valued and multivalued fixed point results (see [10–12]). However, many existing results in b-metric spaces still rely on contraction conditions that closely mirror their metric space counterparts.

The incorporation of fuzzy mathematics into fixed point theory represents another significant advancement. Since Zadeh's seminal introduction of fuzzy sets [13], fuzzy analysis has become an indispensable tool for modeling uncertainty and imprecision. In this context, fixed point theory has been extended to fuzzy settings by Weiss [14], Butnariu [15], and Heilpern [16], who established foundational fuzzy fixed point results for fuzzy set-valued mappings. Later works, including those of Kanwal et al. [17, 18], further demonstrated the effectiveness of fuzzy fixed point methods in both theoretical and applied problems.

More recently, admissibility-based techniques, particularly α-admissibility, have emerged as a powerful alternative to monotonicity assumptions. Originally introduced for single-valued mappings, α-admissibility has been extended to multivalued settings [19] and refined through triangular admissibility structures [20]. These approaches allow fixed point results to be obtained under weaker relational constraints between points and their images. Nevertheless, in much of the existing literature, α-admissibility is combined with relatively restrictive contraction conditions, often requiring strong continuity or compatibility assumptions.

At this stage, a noticeable gap arises in the literature. Although ϕ–ψ-contraction frameworks are known to provide flexible control over nonlinear contractive behavior, their integration with α-admissibility in the context of fuzzy set-valued mappings on b-metric spaces remains limited. In particular, existing works do not clearly address whether admissibility conditions can compensate for the absence of classical contraction constants, nor do they systematically examine how fuzzy set-valued dynamics interact with dual control functions. This gap is especially relevant for applications such as integral inclusions, dynamic programming equations, and fuzzy control systems, where non linearity, uncertainty, and nonstandard distance structures coexist.

The present paper addresses this gap by introducing and studying new classes of α-admissible ϕ-multivalued and α-admissible ϕ–ψ set-valued fuzzy contractions in b-metric spaces. The proposed framework genuinely weakens several assumptions used in closely related works: the contraction condition is governed by dual control functions rather than a single Lipschitz-type bound, no mixed monotonicity or continuity assumptions are imposed, and admissibility replaces stronger relational or ordering requirements. Moreover, the fuzzy b-metric setting further enhances the applicability by accommodating imprecision and non-standard distance structures, which frequently appear in real-world problems involving vagueness and uncertainty. Consequently, the obtained fixed point theorems strictly extend several known results in metric, b-metric, and fuzzy metric settings. Furthermore, Kannan-type and Reich-type variants are derived as corollaries, illustrating the unifying nature of the proposed approach.

The remainder of this paper is organized as follows. Section 2 introduces the necessary preliminaries and auxiliary results. Section 3 contains the main fixed point theorems along with illustrative examples that demonstrate the proper generalization of existing results. Concluding remarks and possible directions for future research are presented in the final section.

2. Preliminaries

Definition 2.1. *[9] Let Ω be a non-empty set and $\rho_b : \Omega \times \Omega \longrightarrow [0, \infty)$ be a function verifying for all $p, q, r \in \Omega$ (with $s \geq 1$):*
(1) $\rho_b(p, q) = 0$ if and only if $p = q$;
(2) $\rho_b(p, q) = \rho_b(q, p)$;
(3) $\rho_b(p, r) \leq s[\rho_b(p, q) + \rho_b(q, r)]$.
So ρ_b is referred to as a b-metric on Ω and the triplet (Ω, ρ_b, s) is named as a b-metric space (b-MS).

Example 2.2. *For $0 < i < 1$, take $l_i(\mathbf{R}) = \{\{p_n\} \subset \mathbf{R}/ \sum_{n=1}^{\infty}|p_n|^i < \infty\}$. The function $\rho_b : l_i(\mathbf{R})\times l_i(\mathbf{R}) \longrightarrow [0, \infty)$ defined by*

$$\rho_b(p, q) = \left(\sum_{n=1}^{\infty}|p_n - q_n|^i\right)^{\frac{1}{i}}$$

(where $p = \{p_n\}$ and $q = \{q_n\} \in l_i(\mathbf{R})$) is a b-MS along with $s = 2^{\frac{1}{i}} > 1$.

Remark 2.3. *Taking $s = 1$ in the structure of a b-MS, it will be turned into a metric space (MS). Hence, the structure of a b-MS is the generalized form of a MS.*

Let (Ω, ρ_b, s) be a b-MS and $\{p_n\}$ be a sequence in Ω.

(1) For $p \in \Omega$, $\{p_n\}$ is called convergent to an element $p \in \Omega$ if its limit exists. We can write it as $\lim_{n\to+\infty} \rho_b(p_n, p) = 0$.
(2) $\{p_n\}$ is called Cauchy, if for a number $\epsilon > 0$, there is an integer $n_\epsilon \in \mathbb{N}$ such that $\rho_b(p_n, p_m) < \epsilon$ for all $m, n > n_\epsilon$.
(3) The b-MS (Ω, ρ_b, s) is called complete if every Cauchy sequence within Ω converges in Ω.

Lemma 2.4. *[21]*

(1) Let $\{p_n\}$ and $\{q_n\}$ be two convergent sequences in a b-MS (Ω, ρ_b, s), which converge to p and q, respectively. Then, we have

$$\begin{aligned}\frac{1}{s^2}\rho_b(p, q) &\leq \lim_{n\to+\infty} \inf \rho_b(p_n, q_n)\\ &\leq \lim_{n\to+\infty} \sup \rho_b(p_n, q_n)\\ &\leq s^2\rho_b(p, q).\end{aligned}$$

(2) If, in addition, $p = q$, then $\lim_{n\to+\infty} \rho_b(p_n, q_n) = 0$. Also, for any $r \in \Omega$,

$$\begin{aligned}\frac{1}{s}\rho_b(p,r) &\leq \lim_{n\to+\infty}\inf \rho_b(p_n, r)\\ &\leq \lim_{n\to+\infty}\sup \rho_b(p_n, r)\\ &\leq s\,\rho_b(p,r).\end{aligned}$$

Lemma 2.5. *[22] Suppose $\{p_n\}$ is a sequence in a b-MS (Ω, ρ_b, s) so that*

$$\lim_{n\to+\infty} \rho_b(p_n, p_{n+1}) = 0.$$

If $\{p_n\}$ is not Cauchy, then there are $\epsilon > 0$ and two sequences of positive integers $\{n_k\}$ and $\{m_k\}$ such that

$$\begin{aligned}\epsilon &\leq \lim_{k\to+\infty}\inf \rho_b(p_{m_k}, p_{n_k}) \leq \lim_{k\to+\infty}\sup \rho_b(p_{m_k}, p_{n_k}) \leq s\,\epsilon,\\ \frac{\epsilon}{s} &\leq \lim_{k\to+\infty}\inf \rho_b(p_{m_k}, p_{n_{k+1}}) \leq \lim_{k\to+\infty}\sup \rho_b(p_{m_k}, p_{n_{k+1}}) \leq s^2\,\epsilon,\\ \frac{\epsilon}{s} &\leq \lim_{k\to+\infty}\inf \rho_b(p_{m_{k+1}}, p_{n_k}) \leq \lim_{k\to+\infty}\sup \rho_b(p_{m_{k+1}}, p_{n_k}) \leq s^2\,\epsilon,\\ \frac{\epsilon}{s^2} &\leq \lim_{k\to+\infty}\inf \rho_b(p_{m_{k+1}}, p_{n_{k+1}}) \leq \lim_{k\to+\infty}\sup \rho_b(p_{m_{k+1}}, p_{n_{k+1}}) \leq s^3\,\epsilon.\end{aligned}$$

Definition 2.6. *[3] Let Φ be the class of functions $\phi : (0, +\infty) \longrightarrow (1, +\infty)$ that satisfy the following conditions:*

(ϕ_1) *ϕ is non-decreasing,*

(ϕ_2) *for a sequence $\{p_n\} \subset (0, +\infty)$,*

$$\lim_{n\to+\infty} p_n = 0 \quad if\ and\ only\ if \quad \lim_{n\to+\infty} \phi(p_n) = 1,$$

(ϕ_3) *ϕ is continuous on $(0, +\infty)$,*

(ϕ_4) *there exist $i \in (0, 1)$ and $h \in (0, +\infty)$ such that*

$$\lim_{p\to 0^+} \frac{\phi(p) - 1}{p^i} = h.$$

Definition 2.7. *[23] Let Ψ be the set of functions $\psi : [1, +\infty) \longrightarrow [1, +\infty)$ that satisfy the following conditions:*

(ψ_1) *ψ is non-decreasing,*

(ψ_2) *for $p \in (1, +\infty)$,*

$$\lim_{n\to+\infty} \psi^n(p) = 1,$$

(ψ_3) *ψ is continuous on $[1, +\infty)$.*

Lemma 2.8. *[23] If $\psi \in \Psi$, then*

$$\psi(p) < p,\ for\ all\ p \in (1, +\infty)\ and\ \psi(1) = 1.$$

Zheng [23] introduced the concept of ϕ-ψ-contractions and presented the following findings.

Definition 2.9. *A self mapping $T : \Omega \longrightarrow \Omega$ is called a ϕ-ψ-contraction in a MS (Ω, ρ) if there are $\phi \in \Phi$ and $\psi \in \Psi$ such that*

$$\rho(Tp, Tq) > 0 \implies \phi[\rho(Tp, Tq)] \leq \psi(\phi[G(p, q)]), \ \forall \ p, q \in \Omega,$$

where

$$G(p, q) = \max\{\rho(p, q), \rho(p, Tp), \rho(q, Tq)\}.$$

Theorem 2.10. *Suppose $T : \Omega \longrightarrow \Omega$ is a ϕ-ψ-contraction on a complete MS (Ω, ρ). Then T has a unique fixed point.*

In the context of a b-MS, Rossafi et al. [29] in 2023 found some fixed point results for ϕ-ψ-contractions.

Definition 2.11. *A self mapping $T : \Omega \longrightarrow \Omega$ is called a ϕ-ψ-contraction in a b-MS (Ω, ρ_b, s) with $s > 1$, if there are $\phi \in \Phi$ and $\psi \in \Psi$ such that*

$$\rho_b(Tp, Tq) > 0 \implies \phi[s^3 \rho_b(Tp, Tq)] \leq \psi(\phi[G_b(p, q)]), \ \forall \ p, q \in \Omega,$$

where

$$G_b(p, q) = \max\left\{\rho_b(p, q), \rho_b(p, Tp), \rho_b(q, Tq), \frac{\rho_b(p, Tq) + \rho_b(Tp, q)}{2s^2}\right\}.$$

Theorem 2.12. *Assume $T : \Omega \longrightarrow \Omega$ is a ϕ-ψ-contraction on a complete b-MS (Ω, ρ_b, s). Then T has a unique fixed point.*

Definition 2.13. *[13] A fuzzy set (FS) A in Ω is a collection of ordered pairs, assuming that Ω is the family of objects represented generically by p.*

$$A = \{(p, \mu_A(p)) : p \in \Omega\},$$

where $\mu_A : \Omega \longrightarrow [0, 1]$ is referred to as the membership function (MF), and the membership value of p in A is $\mu_A(p)$. The family of all FSs in Ω is represented by I^Ω. $[A]_\beta$ and $[A]^*_\beta$, respectively, denote the β-level set and the strict β-level set of a fuzzy set A in Ω, and are defined by

$$[A]_\beta = \{p \in \Omega : \mu_A(p) \geq \beta\}, \text{ if } \beta \in (0, 1],$$

and

$$[A]^*_\beta = \{p \in \Omega : \mu_A(p) > \beta\}.$$

Example 2.14. *Let A be a FS in $\Omega = [-10, 10]$ and $\mu_A : \Omega \longrightarrow [0, 1]$ be a MF defined as:*

$$\mu_A(p) = \frac{|\sin p|}{|\sin p| + |\cos p|}, \quad p \in [-10, 10].$$

Clearly, A is the fuzzy set. Figure 1 shows the graphical representation of a fuzzy set A.

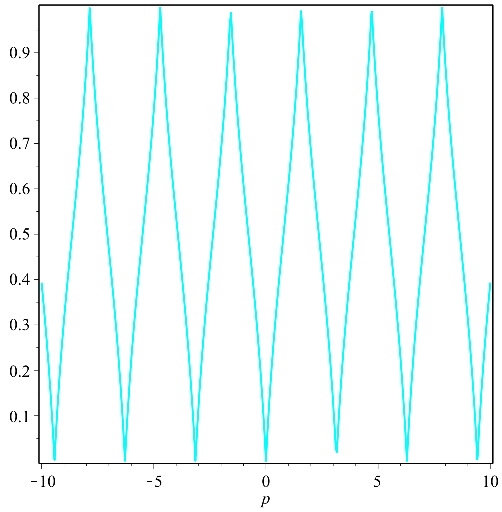

Figure 1. Graph of fuzzy set A.

Definition 2.15. *Let Θ be a MS and Ω be an arbitrary collection. A fuzzy mapping (FM) is defined as $\theta : \Omega \longrightarrow I^{\Theta}$. $\mu_{\theta(p)(q)}$ is the membership value of q in $\theta(p)$ if θ is a FM from Ω into I^{Θ} with $p \in \Omega$ and $q \in \Theta$. $[\theta(p)]_\beta$ represents the β-level collection of $\theta(p)$.*

Example 2.16. *Take $\Theta = \Omega = [0, 10]$. The membership value of FM $\theta : \Omega \longrightarrow I^{\Omega}$ is defined as:*

$$\mu_\theta(p)(q) = \frac{1}{e^{p+q}}, \quad p, q \ \in \Omega.$$

Observe that for every $p \in \Omega$, $\mu_\theta(p)(q) \in [0, 1]$. Figure 2 displays the graphical representation of the possible membership value of q in $\theta(p)$.

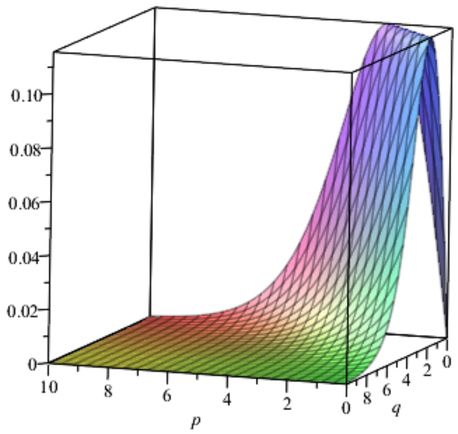

Figure 2. Graph of fuzzy mapping θ.

For a non-empty set Ω, let $\mathcal{F} = I^{\Omega}$ be the collection of all fuzzy sets. If (Ω, ρ_b, s) is a b-MS and $[A]_\beta, [B]_\beta \in \mathcal{F}$, then we define

$$CB(\Omega) = \left\{[A]_\beta \in \mathcal{F} : \ [A]_\beta \text{ is closed and bounded}\right\},$$

$$\begin{aligned}
\mathbb{K}(\Omega) &= \left\{[A]_\beta \in \mathcal{F} : \ [A]_\beta \text{ is compact}\right\}, \\
\rho_b(p,[A]_\beta) &= \inf\left\{\rho_b(p,q) : q \in [A]_\beta\right\}, \\
\delta([A]_\beta,[B]_\beta) &= \sup\left\{\rho_b(p,[B]_\beta) : p \in [A]_\beta\right\}, \\
H([A]_\beta,[B]_\beta) &= \max\left\{\delta([A]_\beta,[B]_\beta), \delta([B]_\beta,[A]_\beta)\right\},
\end{aligned}$$

where H is the Hausdorff metric.

Lemma 2.17. *[25] Suppose (Ω,ρ_b,s) is a b-MS and $p,q \in \Omega$. Then for all $[A]_\beta,[B]_\beta,[C]_\beta \in CB(\Omega)$, we have the following:*

(1) $\rho_b(p,[A]_\beta) \le \rho_b(p,a)$, for all $a \in [A]_\beta$,
(2) $\delta([A]_\beta,[B]_\beta) \le H([A]_\beta,[B]_\beta)$,
(3) $\rho_b(a,[B]_\beta) \le H([A]_\beta,[B]_\beta)$, for all $a \in [A]_\beta$,
(4) $H([A]_\beta,[A]_\beta) = 0$,
(5) $H([A]_\beta,[B]_\beta) = H([B]_\beta,[A]_\beta)$,
(6) $H([A]_\beta,[C]_\beta) \le s\left[H([A]_\beta,[B]_\beta) + H([B]_\beta,[C]_\beta)\right]$,
(7) $\rho_b(p,[A]_\beta) \le s\left[\rho_b(p,q) + \rho_b(q,[A]_\beta)\right]$.

Lemma 2.18. *[25] Let (Ω,ρ_b,s) be a b-MS and $p \in \Omega$. If $[A]_\beta \in CB(\Omega)$, then we have*

$$\rho_b(p,[A]_\beta) = 0 \quad \text{is equivalent to} \quad p \in \overline{[A]}_\beta = [A]_\beta,$$

where $\overline{[A]}_\beta$ is the closure of the set $[A]_\beta$.

3. Main results

In this section, we formulate new preliminary concepts in the fuzzy framework, motivated by the works [19, 26–28]. Subsequently, we introduce ϕ–ψ set-valued fuzzy contractive conditions for α-admissible mappings in b-metric spaces and derive several associated fixed point theorems. Our approach builds upon and extends the concepts developed by Rossafi et al. [29] and Taqbibt et al. [24], providing a broader and more flexible theoretical foundation for fuzzy fixed point analysis in generalized metric settings.

Definition 3.1. *Let $\theta : \Omega \longrightarrow I^\Omega$ be a fuzzy mapping and $\alpha : \Omega^2 \longrightarrow [0,+\infty)$ be a function for a non-empty set Ω. Then θ is said to be α-admissible, if for each $p \in \Omega$, $\beta \in (0,1]$, and $q \in [\theta(p)]_{\beta_{\theta(p)}}$, it holds that*

$$\alpha(p,q) \ge 1 \implies \alpha(q,r) \ge 1, \ \forall\, r \in [\theta(q)]_{\beta_{\theta(q)}}.$$

Definition 3.2. *Let (Ω,ρ_b,s) be a b-MS and $\alpha : \Omega^2 \longrightarrow [0,+\infty)$ be a function. Then (Ω,ρ_b,s) is called α-complete if and only if every Cauchy sequence $\{p_n\}$ converges in Ω, where $\alpha(p_n,p_{n+1}) \ge 1$ for all $n \in \mathbb{N}$.*

Definition 3.3. *Let $\theta : \Omega \longrightarrow I^\Omega$ be a fuzzy mapping for a b-MS (Ω,ρ_b,s) and $\alpha : \Omega^2 \longrightarrow [0,+\infty)$ be a function. Then θ is said to be an α-continuous fuzzy set-valued mapping on $(\mathbb{K}(\Omega),H)$, if for each* *sequence* $\{p_n\}$ *with* $\alpha(p_n, p_{n+1}) \geq 1$, *for every* $n \in \mathbb{N}$ *and* $\lim_{n\to+\infty} p_n = p \in \Omega$, *we have* $\lim_{n\to+\infty}[\theta(p_n)]_{\beta_{\theta(p_n)}} = [\theta(p)]_{\beta_{\theta(p)}}$ *so that*

$$\lim_{n\to+\infty} \rho_b(p_n, p) = 0 \text{ and } \alpha(p_n, p_{n+1}) \geq 1,\ \forall\, n \in \mathbb{N},$$

implies that

$$\lim_{n\to+\infty} H([\theta(p_n)]_{\beta_{\theta(p_n)}}, [\theta(p)]_{\beta_{\theta(p)}}) = 0,$$

where $\beta \in (0, 1]$.

Definition 3.4. *Let* $\theta : \Omega \longrightarrow I^{\Omega}$ *be a fuzzy mapping and* $\alpha : \Omega^2 \longrightarrow [0, +\infty)$ *be a function for a non-empty set* Ω. *Then* θ *is said to be triangular* α*-admissible if* θ *is* α*-admissible and verifies*

$$\alpha(p, q) \geq 1 \text{ and } \alpha(q, r) \geq 1 \implies \alpha(p, r) \geq 1,\ \forall\, r \in [\theta(q)]_{\beta_{\theta(q)}},\ \forall\, p, q \in \Omega,\ \text{and } \beta \in (0, 1].$$

Lemma 3.5. *Let* $\theta : \Omega \longrightarrow I^{\Omega}$ *be a triangular* α*-admissible fuzzy mapping. Let* $p_0 \in \Omega$, $\beta \in (0, 1]$, *and* $p_1 \in [\theta(p_0)]_{\beta_{\theta(p_0)}} \neq \emptyset$ *such that* $\alpha(p_0, p_1) \geq 1$, *and for a sequence* $\{p_n\}$, $p_{n+1} \in [\theta(p_n)]_{\beta_{\theta(p_n)}} \neq \emptyset$, *we have* $\alpha(p_n, p_{n+1}) \geq 1$, *for all* $n \in \mathbb{N}$.

Definition 3.6. *Let* $\theta : \Omega \longrightarrow I^{\Omega}$ *be a fuzzy mapping in a b-MS* (Ω, ρ_b, s) *with* $s > 1$. *Then* θ *is said to be an* α*-admissible* ϕ*-set-valued fuzzy contraction if there are* $\phi \in \Phi$, $L \geq 0$, $\beta \in (0, 1]$, *and* $i \in (0, 1)$ *such that for all* $p, q \in \Omega$ *with* $p \neq q$,

$$H([\theta(p)]_{\beta_{\theta(p)}}, [\theta(q)]_{\beta_{\theta(q)}}) > 0 \implies \phi[\alpha(p, q)s^3 H([\theta(p)]_{\beta_{\theta(p)}}, [\theta(q)]_{\beta_{\theta(q)}})] \leq \phi[E(p, q)]^i + LF(p, q), \quad (3.1)$$

where

$$E(p, q) = \max\left\{\rho_b(p, q), \rho_b(p, [\theta(p)]_{\beta_{\theta(p)}}), \rho_b(q, [\theta(q)]_{\beta_{\theta(q)}}), \frac{\rho_b(p, [\theta(q)]_{\beta_{\theta(q)}}) + \rho_b(q, [\theta(p)]_{\beta_{\theta(p)}})}{2s^2}\right\}$$

and

$$F(p, q) = \min\left\{\rho_b(p, [\theta(p)]_{\beta_{\theta(p)}}), \rho_b(q, [\theta(q)]_{\beta_{\theta(q)}}), \rho_b(p, [\theta(q)]_{\beta_{\theta(q)}}), \rho_b(q, [\theta(p)]_{\beta_{\theta(p)}})\right\}.$$

Remark 3.7. *Note that for* $p \neq q$, *we have* $\rho_b(p, q) > 0$, *hence* $E(p, q) > 0$ *and* $\phi(E(p, q))$ *is well-defined.*

Theorem 3.8. *Let* (Ω, ρ_b, s) *be a complete b-MS and* $\theta : \Omega \longrightarrow I^{\Omega}$ *be an* α*-admissible* ϕ*-set-valued fuzzy contraction mapping satisfying:*

(1) (Ω, ρ_b, s) *is an* α*-complete b-MS;*
(2) for $p_0 \in \Omega$ *and* $p_1 \in [\theta(p_0)]_{\beta_{\theta(p_0)}}$, *we have* $\alpha(p_0, p_1) \geq 1$, $\beta \in (0, 1]$;
(3) θ *is triangular* α*-admissible;*
(4) either
 (4a) θ *is an* α*-continuous set-valued fuzzy mapping, or*
 (4b) if $\{p_n\} \subset \Omega$ *so that* $\alpha(p_n, p_{n+1}) \geq 1$ *and* $\lim_{n\to+\infty} p_n = p \in \Omega$, *then we get* $\alpha(p_n, p) \geq 1$ *for all* $n \in \mathbb{N}$.

Then θ *has a unique fixed point.*

Proof. Assume $p_0 \in \Omega$ and $p_1 \in [\theta(p_0)]_{\beta_{\theta(p_0)}} \neq \emptyset$ such that $\alpha(p_0, p_1) \geq 1$, $\beta \in (0, 1]$. We consider a sequence $\{p_n\} \subset \Omega$ and define the iterative relation as

$$p_{n+1} \in [\theta(p_n)]_{\beta_{\theta(p_n)}} \neq \emptyset, \quad \forall\, n \in \mathbb{N}.$$

If we assume $n_0 \in \mathbb{N}$ with $\rho_b(p_{n_0}, [\theta(p_{n_0})]_{\beta_{\theta(p_{n_0})}}) = 0$, then it means that p_{n_0} is a member of $[\theta(p_{n_0})]_{\beta_{\theta(p_{n_0})}}$, and we can say that p_{n_0} is a fixed point of $[\theta(p_{n_0})]_{\beta_{\theta(p_{n_0})}}$. This completes the proof.

Now, we take $\rho_b(p_n, [\theta(p_n)]_{\beta_{\theta(p_n)}}) > 0$ for all $n \in \mathbb{N}$. Let $p_2 \in [\theta(p_1)]_{\beta_{\theta(p_1)}} \neq \emptyset$, and then the α-admissibility of θ implies that $\alpha(p_1, p_2) \geq 1$. By using Definition 3.1 and performing the same steps with α-admissibility of θ, we can obtain

$$\alpha(p_n, p_{n+1}) \geq 1, \quad for\ every\ \ n \in \mathbb{N} \cup \{0\}. \tag{3.2}$$

Now, from Eq (3.1), we can get

$$\begin{aligned} \phi[H([\theta(p_{n-1})]_{\beta_{\theta(p_{n-1})}}, [\theta(p_n)]_{\beta_{\theta(p_n)}})] &\leq \phi[s^3\, H([\theta(p_{n-1})]_{\beta_{\theta(p_{n-1})}}, [\theta(p_n)]_{\beta_{\theta(p_n)}})], \\ &\leq \phi[\alpha(p_{n-1}, p_n)\, s^3\, H([\theta(p_{n-1})]_{\beta_{\theta(p_{n-1})}}, [\theta(p_n)]_{\beta_{\theta(p_n)}})], \\ &\leq [\phi(E(p_{n-1}, p_n))]^i + L\, F(p_{n-1}, p_n), \quad \forall\, n \in \mathbb{N}, \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} E(p_{n-1}, p_n) &= \max\Big\{\rho_b(p_{n-1}, p_n), \rho_b(p_{n-1}, [\theta(p_{n-1})]_{\beta_{\theta(p_{n-1})}}), \rho_b(p_n, [\theta(p_n)]_{\beta_{\theta(p_n)}}), \\ &\quad \frac{\rho_b(p_{n-1}, [\theta(p_n)]_{\beta_{\theta(p_n)}}) + \rho_b(p_n, [\theta(p_{n-1})]_{\beta_{\theta(p_{n-1})}})}{2s^2}\Big\} \\ &= \max\left\{\rho_b(p_{n-1}, p_n), \rho_b(p_n, [\theta(p_n)]_{\beta_{\theta(p_n)}}), \frac{\rho_b(p_{n-1}, [\theta(p_n)]_{\beta_{\theta(p_n)}}) + 0}{2s^2}\right\}, \\ &\quad since\ \ p_n \ \in \ [\theta(p_{n-1})]_{\beta_{\theta(p_{n-1})}} \\ &= \max\left\{\rho_b(p_{n-1}, p_n), \rho_b(p_n, [\theta(p_n)]_{\beta_{\theta(p_n)}}), \frac{\rho_b(p_{n-1}, [\theta(p_n)]_{\beta_{\theta(p_n)}})}{2s^2}\right\}, \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} F(p_{n-1}, p_n) &= \min\Big\{\rho_b(p_{n-1}, [\theta(p_{n-1})]_{\beta_{\theta(p_{n-1})}}), \rho_b(p_n, [\theta(p_n)]_{\beta_{\theta(p_n)}}), \rho_b(p_{n-1}, [\theta(p_n)]_{\beta_{\theta(p_n)}}), \\ &\quad \rho_b(p_n, [\theta(p_{n-1})]_{\beta_{\theta(p_{n-1})}})\Big\} \\ &= \min\Big\{\rho_b(p_{n-1}, [\theta(p_{n-1})]_{\beta_{\theta(p_{n-1})}}), \rho_b(p_n, [\theta(p_n)]_{\beta_{\theta(p_n)}}), \rho_b(p_{n-1}, [\theta(p_n)]_{\beta_{\theta(p_n)}}), 0\Big\}, \\ &\quad since\ \ p_n \ \in \ [\theta(p_{n-1})]_{\beta_{\theta(p_{n-1})}} \\ &= 0. \end{aligned} \tag{3.5}$$

Now, from Eq (3.4), we have

$$\frac{\rho_b(p_{n-1}, [\theta(p_n)]_{\beta_{\theta(p_n)}})}{2s^2} \;\leq\; \frac{1}{2s^2}\Big\{s\Big[\rho_b(p_{n-1}, p_n) + \rho_b(p_n, [\theta(p_n)]_{\beta_{\theta(p_n)}})\Big]\Big\}, \ \ and\ \ using\ \ Lemma\ 2.17,$$

$$
\begin{aligned}
&= \frac{1}{2s}[\rho_b(p_{n-1}, p_n) + \rho_b(p_n, [\theta(p_n)]_{\beta_{\theta(p_n)}})] \\
&\leq \frac{1}{2}[\rho_b(p_{n-1}, p_n) + \rho_b(p_n, [\theta(p_n)]_{\beta_{\theta(p_n)}})] \\
&\leq \max\{\rho_b(p_{n-1}, p_n), \rho_b(p_n, [\theta(p_n)]_{\beta_{\theta(p_n)}})\}. \qquad (3.6)
\end{aligned}
$$

So, by using inequality (3.6), Eq (3.4) can be rewritten as:

$$E(p_{n-1}, p_n) = \max\{\rho_b(p_{n-1}, p_n), \rho_b(p_n, [\theta(p_n)]_{\beta_{\theta(p_n)}})\}.$$

If $E(p_{n-1}, p_n) = \rho_b(p_n, [\theta(p_n)]_{\beta_{\theta(p_n)}})$, since θ is compact, it follows that

$$\rho_b(p_n, p_{n+1}) \leq H([\theta(p_{n-1})]_{\beta_{\theta(p_{n-1})}}, [\theta(p_n)]_{\beta_{\theta(p_n)}}). \qquad (3.7)$$

We know that $p_{n+1} \in [\theta(p_n)]_{\beta_{\theta(p_n)}} \neq \emptyset$. By using Lemma 2.17, this implies that $\rho_b(p_n, [\theta(p_n)]_{\beta_{\theta(p_n)}}) \leq \rho_b(p_n, p_{n+1})$. By employing (3.3) and (3.7), we have

$$
\begin{aligned}
\phi(\rho_b(p_n, p_{n+1})) &\leq \phi(H([\theta(p_{n-1})]_{\beta_{\theta(p_{n-1})}}, [\theta(p_n)]_{\beta_{\theta(p_n)}})) \\
&\leq [\phi(E(p_{n-1}, p_n))]^i + L\, F(p_{n-1}, p_n) \\
&= [\phi(E(p_{n-1}, p_n))]^i + L \times 0, \quad \textit{by using } (3.5) \\
&\leq \phi(E(p_{n-1}, p_n)) \\
&= \phi(\rho_b(p_n, [\theta(p_n)]_{\beta_{\theta(p_n)}})) \\
&\leq \phi(\rho_b(p_n, p_{n+1})),
\end{aligned}
$$

which is a contradiction. Consequently, $E(p_{n-1}, p_n) = \rho_b(p_{n-1}, p_n)$. Therefore, by using inequality (3.3), we can process the following:

$$
\begin{aligned}
\phi(\rho_b(p_n, p_{n+1})) &\leq \phi(H([\theta(p_{n-1})]_{\beta_{\theta(p_{n-1})}}, [\theta(p_n)]_{\beta_{\theta(p_n)}})) \\
&\leq [\phi(E(p_{n-1}, p_n))]^i + L\, F(p_{n-1}, p_n) \\
&= [\phi(E(p_{n-1}, p_n))]^i + L \times 0, \quad \textit{by using } (3.5) \\
&= [\phi(\rho_b(p_{n-1}, p_n))]^i \\
&< \phi(\rho_b(p_{n-1}, p_n)). \qquad (3.8)
\end{aligned}
$$

From (3.8) and (ϕ_1), we have

$$\rho_b(p_n, p_{n+1}) < \rho_b(p_{n-1}, p_n).$$

Consequently, the sequence of non-negative real numbers $\{\rho_b(p_n, p_{n+1})\}_{n\in\mathbb{N}}$ is strictly decreasing. This implies the existence of a non-negative number γ such that

$$\lim_{n\to+\infty} \rho_b(p_n, p_{n+1}) = \gamma.$$

To prove $\gamma = 0$, we use a contradiction that $\gamma > 0$. Since the sequence $\{\rho_b(p_n, p_{n+1})\}_{n\in\mathbb{N}}$ is both positive and decreasing, we conclude that

$$\gamma \leq \rho_b(p_n, p_{n+1}) \implies \phi(\gamma) \leq \phi(\rho_b(p_n, p_{n+1})), \ \textit{for all } n \in \mathbb{N}.$$

By using (3.8) repeatedly, we have

$$\begin{aligned}\phi(\gamma) \leq \phi(\rho_b(p_n, p_{n+1})) &\leq [\phi(\rho_b(p_{n-1}, p_n))]^{i} \\ &\leq [\phi(\rho_b(p_{n-2}, p_{n-1}))]^{i^2} \\ &\vdots \\ &\leq [\phi(\rho_b(p_0, p_1))]^{i^n}. \end{aligned} \tag{3.9}$$

By using the function $\phi : (0, +\infty) \to (1, +\infty)$ and inequality (3.9), we get

$$1 < \phi(\gamma) \leq [\phi(\rho_b(p_0, p_1))]^{i^n}. \tag{3.10}$$

As $n \to +\infty$ in (3.10) with $i \in (0, 1)$, we can get

$$1 < \phi(\gamma) \leq 1.$$

This is a contradiction, so $\gamma = 0$. Hence,

$$\lim_{n\to+\infty} \rho_b(p_n, p_{n+1}) = 0. \tag{3.11}$$

We shall prove that the sequence $\{p_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Assume, by contradiction, that it fails to be Cauchy. Then, according to Lemma 2.5, there exists $\epsilon > 0$ such that there are two sequences of positive integers $\{m_k\}$ and $\{n_k\}$ for which

$$\begin{aligned}\epsilon &\leq \lim_{k\to+\infty} \inf \rho_b(p_{m_k}, p_{n_k}) \leq \lim_{k\to+\infty} \sup \rho_b(p_{m_k}, p_{n_k}) \leq s\, \epsilon, \\ \frac{\epsilon}{s} &\leq \lim_{k\to+\infty} \inf \rho_b(p_{m_k}, p_{n_{k+1}}) \leq \lim_{k\to+\infty} \sup \rho_b(p_{m_k}, p_{n_{k+1}}) \leq s^2\, \epsilon, \\ \frac{\epsilon}{s} &\leq \lim_{k\to+\infty} \inf \rho_b(p_{m_{k+1}}, p_{n_k}) \leq \lim_{k\to+\infty} \sup \rho_b(p_{m_{k+1}}, p_{n_k}) \leq s^2\, \epsilon, \\ \frac{\epsilon}{s^2} &\leq \lim_{k\to+\infty} \inf \rho_b(p_{m_{k+1}}, p_{n_{k+1}}) \leq \lim_{k\to+\infty} \sup \rho_b(p_{m_{k+1}}, p_{n_{k+1}}) \leq s^3\, \epsilon. \end{aligned}$$

Since θ is a triangular α-admissible fuzzy mapping, then by employing Lemma 3.5, we have

$$\alpha(p_{m_k}, p_{n_k}) \geq 1. \tag{3.12}$$

Now, from (3.12) and using the definition of contraction (3.1), we obtain

$$\begin{aligned}\phi[s^3 \rho_b(p_{m_{k+1}}, p_{n_{k+1}})] &\leq \phi[s^3\, H([\theta(p_{m_k})]_{\beta_{\theta(p_{m_k})}}, [\theta(p_{n_k})]_{\beta_{\theta(p_{n_k})}})] \\ &\leq \phi[\alpha(p_{m_k}, p_{n_k})\, s^3\, H([\theta(p_{m_k})]_{\beta_{\theta(p_{m_k})}}, [\theta(p_{n_k})]_{\beta_{\theta(p_{n_k})}})] \\ &\leq [\phi(E(p_{m_k}, p_{n_k}))]^{i} + L\, F(p_{m_k}, p_{n_k}, \end{aligned} \tag{3.13}$$

where

$$\begin{aligned} E(p_{m_k}, p_{n_k}) = \max\Big\{&\rho_b(p_{m_k}, p_{n_k}), \rho_b(p_{m_k}, [\theta(p_{m_k})]_{\beta_{\theta(p_{m_k})}}), \rho_b(p_{n_k}, [\theta(p_{n_k})]_{\beta_{\theta(p_{n_k})}}), \\ &\frac{\rho_b(p_{m_k}, [\theta(p_{n_k})]_{\beta_{\theta(p_{n_k})}}) + \rho_b(p_{n_k}, [\theta(p_{m_k})]_{\beta_{\theta(p_{m_k})}})}{2s^2}\Big\} \end{aligned}$$

$$
\begin{aligned}
&\leq \max\left\{\rho_b(p_{m_k}, p_{n_k}), \rho_b(p_{m_k}, p_{m_{k+1}}), \rho_b(p_{n_k}, p_{n_{k+1}}),\right. \\
&\quad \left.\frac{\rho_b(p_{m_k}, p_{n_{k+1}}) + \rho_b(p_{n_k}, p_{m_{k+1}})}{2s^2}\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
F(p_{m_k}, p_{n_k}) &= \min\Big\{\rho_b(p_{m_k}, [\theta(p_{m_k})]_{\beta_{\theta(p_{m_k})}}), \rho_b(p_{n_k}, [\theta(p_{n_k})]_{\beta_{\theta(p_{n_k})}}), \\
&\quad \rho_b(p_{m_k}, [\theta(p_{n_k})]_{\beta_{\theta(p_{n_k})}}), \rho_b(p_{n_k}, [\theta(p_{m_k})]_{\beta_{\theta(p_{m_k})}})\Big\} \\
&\leq \min\left\{\rho_b(p_{m_k}, p_{m_{k+1}}), \rho_b(p_{n_k}, p_{n_{k+1}}), \rho_b(p_{m_k}, p_{n_{k+1}}), \rho_b(p_{n_k}, p_{m_{k+1}})\right\}.
\end{aligned}
$$

By applying Lemma 2.5 and using (3.11) with $\lim_{k\to+\infty}$ on $E(p_{m_k}, p_{n_k})$ and $F(p_{m_k}, p_{n_k})$, we get

$$
\begin{aligned}
\lim_{k\to+\infty} E(p_{m_k}, p_{n_k}) &\leq \lim_{k\to+\infty} \max\Big\{\rho_b(p_{m_k}, p_{n_k}), \rho_b(p_{m_k}, p_{m_{k+1}}), \rho_b(p_{n_k}, p_{n_{k+1}}), \\
&\quad \frac{\rho_b(p_{m_k}, p_{n_{k+1}}) + \rho_b(p_{n_k}, p_{m_{k+1}})}{2s^2}\Big\} \\
&\leq \max\left\{s\epsilon, 0, 0, \frac{s^2\epsilon + s^2\epsilon}{2s^2}\right\} \\
&= \max\{s\epsilon, 0, 0, \epsilon\} \\
&= s\epsilon, \quad \textit{since } s \geq 1,
\end{aligned}
$$

and

$$
\begin{aligned}
\lim_{k\to+\infty} F(p_{m_k}, p_{n_k}) &\leq \lim_{k\to+\infty} \min\left\{\rho_b(p_{m_k}, p_{m_{k+1}}), \rho_b(p_{n_k}, p_{n_{k+1}}),\right. \\
&\quad \left.\rho_b(p_{m_k}, p_{n_{k+1}}), \rho_b(p_{n_k}, p_{m_{k+1}})\right\} \\
&\leq \min\{0, 0, s^2\epsilon, s^2\epsilon\} \\
&= 0.
\end{aligned}
$$

So, we have

$$
\begin{aligned}
\lim_{k\to+\infty} F(p_{m_k}, p_{n_k}) &= 0 \;\; \textit{and} \\
\lim_{k\to+\infty} E(p_{m_k}, p_{n_k}) &\leq s\epsilon.
\end{aligned}
\tag{3.14}
$$

By employing inequalities (3.14) and (3.13) and using the continuity of ϕ, we can get the following inequality by letting $k \to +\infty$:

$$
\begin{aligned}
\phi(s\epsilon) &= \phi\left(\frac{\epsilon}{s^2}\, s^3\right) \\
&\leq \phi\left(s^3 \lim_{k\to+\infty} \rho_b(p_{m_{k+1}}, p_{n_{k+1}})\right), \;\; \textit{by using Lemma } 2.5 \\
&\leq [\phi(\lim_{k\to+\infty} E(p_{m_k}, p_{n_k}))]^i + L \lim_{k\to+\infty} F(p_{m_k}, p_{n_k}), \;\; \textit{by } (3.13) \\
&= [\phi(s\epsilon)]^i + L \times 0, \;\; \textit{by } (3.14) \\
&\leq [\phi(s\epsilon)]^i < \phi(s\epsilon).
\end{aligned}
$$

But, by definition, ϕ is a monotonically increasing function, so

$$s\epsilon < s\epsilon.$$

This is a contradiction. Consequently, $\{p_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in Ω. The completeness of (Ω, ρ_b, s) ensures there is $\nu \in \Omega$ such that

$$\lim_{n\to+\infty} \rho_b(p_n, \nu) = 0. \tag{3.15}$$

Case 1: If θ is an α-continuous set-valued fuzzy mapping, then by condition **(4a)**, we conclude that

$$\lim_{n\to+\infty} H([\theta(p_n)]_{\beta_{\theta(p_n)}}, [\theta(\nu)]_{\beta_{\theta(\nu)}}) = 0. \tag{3.16}$$

Thus, by using Eqs (3.15) and (3.16), we get

$$\rho_b(\nu, [\theta(\nu)]_{\beta_{\theta(\nu)}}) = \lim_{n\to+\infty} \rho_b(p_{n+1}, [\theta(\nu)]_{\beta_{\theta(\nu)}}) \leq \lim_{n\to+\infty} H([\theta(p_n)]_{\beta_{\theta(p_n)}}, [\theta(\nu)]_{\beta_{\theta(\nu)}}) = 0.$$

Hence, $\nu \in [\theta(\nu)]_{\beta_{\theta(\nu)}}$ implies that θ has a fixed point.

Case 2: If θ is not an α-continuous set-valued fuzzy mapping, then, by condition **(4b)**, we show by contradiction that $\nu \in [\theta(\nu)]_{\beta_{\theta(\nu)}}$. Suppose that

$$\nu \notin [\theta(\nu)]_{\beta_{\theta(\nu)}}.$$

We know that $0 \leq \rho_b([\theta(p_n)]_{\beta_{\theta(p_n)}}, \nu) \leq \rho_b(p_{n+1}, \nu)$, hence

$$\lim_{n\to+\infty} \rho_b([\theta(p_n)]_{\beta_{\theta(p_n)}}, \nu) = 0.$$

As $\lim_{n\to+\infty}[\theta(p_n)]_{\beta_{\theta(p_n)}} = \nu$, by using Lemma 2.4, we can obtain

$$\begin{aligned}
\frac{1}{s^2}\rho_b(\nu, [\theta(\nu)]_{\beta_{\theta(\nu)}}) &\leq \lim_{n\to+\infty} \inf H([\theta(p_n)]_{\beta_{\theta(p_n)}}, [\theta(\nu)]_{\beta_{\theta(\nu)}}) \\
&\leq \lim_{n\to+\infty} \sup H([\theta(p_n)]_{\beta_{\theta(p_n)}}, [\theta(\nu)]_{\beta_{\theta(\nu)}}) \\
&\leq s^2 \rho_b(\nu, [\theta(\nu)]_{\beta_{\theta(\nu)}}).
\end{aligned} \tag{3.17}$$

Now, considering the definition of contraction (3.1), we have

$$\begin{aligned}
\phi[s^3 H([\theta(p_n)]_{\beta_{\theta(p_n)}}, [\theta(\nu)]_{\beta_{\theta(\nu)}})] &\leq \phi[\alpha(p_n, \nu)\, s^3 H([\theta(p_n)]_{\beta_{\theta(p_n)}}, [\theta(\nu)]_{\beta_{\theta(\nu)}})] \\
&\leq [\phi(E(p_n, \nu))]^i + L\, F(p_n, \nu), \ \forall\, n \in \mathbb{N},
\end{aligned}$$

where

$$E(p_n, \nu) = \max\left\{\rho_b(p_n, \nu), \rho_b(p_n, [\theta(p_n)]_{\beta_{\theta(p_n)}}), \rho_b(\nu, [\theta(\nu)]_{\beta_{\theta(\nu)}}), \frac{\rho_b(p_n, [\theta(\nu)]_{\beta_{\theta(\nu)}}) + \rho_b(\nu, [\theta(p_n)]_{\beta_{\theta(p_n)}})}{2s^2}\right\}$$

and

$$F(p_n, \nu) = \min\left\{\rho_b(p_n, [\theta(p_n)]_{\beta_{\theta(p_n)}}), \rho_b(\nu, [\theta(\nu)]_{\beta_{\theta(\nu)}}), \rho_b(p_n, [\theta(\nu)]_{\beta_{\theta(\nu)}}), \rho_b(\nu, [\theta(p_n)]_{\beta_{\theta(p_n)}})\right\}.$$

Now, by taking the $\lim_{n\to+\infty}$, we obtain

$$\begin{aligned}
\lim_{n\to+\infty}\sup E(p_n,\nu) &= \lim_{n\to+\infty}\sup\max\Big\{\rho_b(p_n,\nu),\rho_b(p_n,[\theta(p_n)]_{\beta_{\theta(p_n)}}),\rho_b(\nu,[\theta(\nu)]_{\beta_{\theta(\nu)}}),\\
&\quad \frac{\rho_b(p_n,[\theta(\nu)]_{\beta_{\theta(\nu)}})+\rho_b(\nu,[\theta(p_n)]_{\beta_{\theta(p_n)}})}{2s^2}\Big\}\\
&\le \lim_{n\to+\infty}\sup\max\Big\{\rho_b(p_n,\nu),\rho_b(p_n,p_{n+1}),\rho_b(\nu,[\theta(\nu)]_{\beta_{\theta(\nu)}}),\\
&\quad \frac{\rho_b(p_n,[\theta(\nu)]_{\beta_{\theta(\nu)}})+\rho_b(\nu,p_{n+1})}{2s^2}\Big\}\\
&\le \rho_b(\nu,[\theta(\nu)]_{\beta_{\theta(\nu)}}),\quad since\ \lim_{n\to+\infty}[\theta(p_n)]_{\beta_{\theta(p_n)}}=\nu,
\end{aligned}\tag{3.18}$$

and

$$\begin{aligned}
\lim_{n\to+\infty}\sup F(p_n,\nu) &= \lim_{n\to+\infty}\sup\min\{\rho_b(p_n,[\theta(p_n)]_{\beta_{\theta(p_n)}}),\rho_b(\nu,[\theta(\nu)]_{\beta_{\theta(\nu)}}),\rho_b(p_n,[\theta(\nu)]_{\beta_{\theta(\nu)}}),\\
&\quad \rho_b(\nu,[\theta(p_n)]_{\beta_{\theta(p_n)}})\}\\
&\le \lim_{n\to+\infty}\sup\min\Big\{\rho_b(p_n,p_{n+1}),\rho_b(\nu,[\theta(\nu)]_{\beta_{\theta(\nu)}}),\rho_b(p_n,[\theta(\nu)]_{\beta_{\theta(\nu)}}),\rho_b(\nu,p_{n+1})\Big\}\\
&= \min\{0,\rho_b(\nu,[\theta(\nu)]_{\beta_{\theta(\nu)}}),\rho_b(\nu,z),0\}\\
&= 0.
\end{aligned}\tag{3.19}$$

Therefore,

$$\begin{aligned}
\phi(s^3\,H([\theta(p_n)]_{\beta_{\theta(p_n)}},[\theta(\nu)]_{\beta_{\theta(\nu)}})) &\le \Big[\phi\Big(\max\Big\{\rho_b(p_n,\nu),\rho_b(p_n,[\theta(p_n)]_{\beta_{\theta(p_n)}}),\rho_b(\nu,[\theta(\nu)]_{\beta_{\theta(\nu)}}),\\
&\quad \frac{\rho_b(p_n,[\theta(\nu)]_{\beta_{\theta(\nu)}})+\rho_b(\nu,[\theta(p_n)]_{\beta_{\theta(p_n)}})}{2s^2}\Big\}\Big)\Big]^i\\
&\quad + L\ \min\Big\{\rho_b(p_n,[\theta(p_n)]_{\beta_{\theta(p_n)}}),\rho_b(\nu,[\theta(\nu)]_{\beta_{\theta(\nu)}}),\\
&\quad \rho_b(p_n,[\theta(\nu)]_{\beta_{\theta(\nu)}}),\rho_b(\nu,[\theta(p_n)]_{\beta_{\theta(p_n)}})\Big\}.
\end{aligned}\tag{3.20}$$

Taking the $\lim_{n\to+\infty}$ in (3.17) and (3.20), and by using (3.11) with (ϕ_3), we have

$$\begin{aligned}
\phi(s\,\rho_b(\nu,[\theta(\nu)]_{\beta_{\theta(\nu)}})) &= \phi\Big(s^3\times\frac{1}{s^2}\,\rho_b(\nu,[\theta(\nu)]_{\beta_{\theta(\nu)}})\Big)\\
&\le \phi[s^3\lim_{n\to+\infty}H([\theta(p_n)]_{\beta_{\theta(p_n)}},[\theta(\nu)]_{\beta_{\theta(\nu)}})],\quad by\ using\ (3.17)\\
&\le \lim_{n\to+\infty}\phi[s^3\,H([\theta(p_n)]_{\beta_{\theta(p_n)}},[\theta(\nu)]_{\beta_{\theta(\nu)}})]\\
&\le \lim_{n\to+\infty}\phi[\alpha(p_n,\nu)\,s^3\,H([\theta(p_n)]_{\beta_{\theta(p_n)}},[\theta(\nu)]_{\beta_{\theta(\nu)}})]\\
&\le \lim_{n\to+\infty}\Big(\phi[E(p_n,\nu)]^i+L\,F(p_n,\nu)\Big)\\
&\le \lim_{n\to+\infty}\phi[E(p_n,\nu)]^i+L\lim_{n\to+\infty}F(p_n,\nu)\\
&= \lim_{n\to+\infty}\phi[E(p_n,\nu)]^i+L\times 0,\quad by\ using\ Eq\ (3.19)\\
&\le [\phi(\rho_b(\nu,[\theta(\nu)]_{\beta_{\theta(\nu)}}))]^i,\quad by\ using\ Eq\ (3.18)\\
&< \phi(\rho_b(\nu,[\theta(\nu)]_{\beta_{\theta(\nu)}})).
\end{aligned}$$

Using (ϕ_1), we get

$$s\,\rho_b(\nu, [\theta(\nu)]_{\beta_{\theta(\nu)}}) < \rho_b(\nu, [\theta(\nu)]_{\beta_{\theta(\nu)}}).$$

This implies that

$$\rho_b(\nu, [\theta(\nu)]_{\beta_{\theta(\nu)}})(s-1) < 0,$$

hence $s < 1$. This is a contradiction to our supposition, so it follows that $\nu \in [\theta(\nu)]_{\beta_{\theta(\nu)}}$ and hence θ has a unique fixed point. □

We give an example to illustrate Theorem 3.8.

Example 3.9. *Consider the b-MS* $\Omega = [0, +\infty)$ *endowed with* $\rho_b(p, q) = |p - q|^2$ *(with* $s = 2$*) for all* $p, q \in \Omega$, $p \neq q$. *Let* $\theta : \Omega \longrightarrow I^{\Omega}$ *be a fuzzy mapping (see Figure 3) with membership value* μ_θ *defined as:*

$$\mu_\theta(p)(q) = \begin{cases} \frac{6}{7}\,, & if \quad 0 \le q \le \frac{p}{10}, \\ \frac{1}{5}\,, & if \quad \frac{p}{10} < q \le \frac{2p}{5}, \\ 0\,, & otherwise. \end{cases}$$

Take $\beta_{\theta(p)(q)} = \frac{1}{2}$. *Then the* β*-cut level of a fuzzy mapping* $\theta(p)(q)$ *is:*

$$[\theta(p)(q)]_{\beta_{\theta(p)(q)}} = [\theta(p)(q)]_{\frac{1}{2}} = \left[0, \frac{p}{10}\right].$$

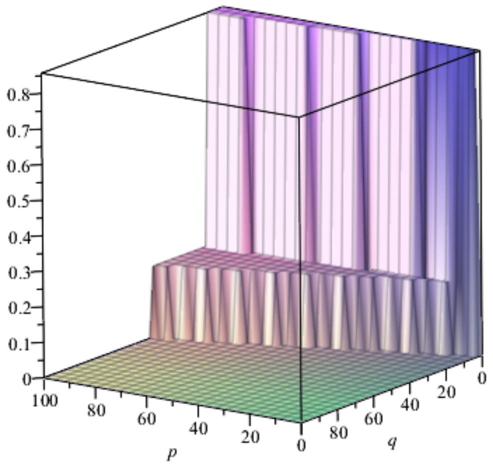

Figure 3. Graph of fuzzy mapping θ.

Now, we define the α*-admissible mapping* $\alpha : \Omega^2 \longrightarrow [0, +\infty]$ *as*

$$\alpha(p, q) = \begin{cases} 1\,, & if \quad p, q \in [0, 10], \\ \frac{1}{2}\,, & otherwise, \end{cases}$$

and take the function $\phi : (0, +\infty) \longrightarrow (1, +\infty)$ *as*

$$\phi(t) = \sqrt{t+1}.$$

Now, we will show that θ is a triangular α-admissible fuzzy mapping. Let $p \in \Omega$ and $q \in [\theta(p)]_{\beta_{\theta(p)}}$ such that $\alpha(p,q) \geq 1$. Then $p, q \in [0,10]$. Let $r \in [\theta(q)]_{\beta_{\theta(q)}}$, and then

$$r \in \left[0, \frac{q}{10}\right] = \left[0, \frac{p/10}{10}\right] = \left[0, \frac{p}{10^2}\right] \subset [0,1].$$

That is, $\alpha(q,r) \geq 1$. Hence, θ is an α-admissible fuzzy mapping. Moreover, let $p, q \in \Omega$ and $r \in [\theta(q)]_{\beta_{\theta(q)}}$ such that

$$\alpha(p,q) \geq 1 \ \text{ and } \ \alpha(q,r) \geq 1.$$

Since $p, q \in [0,10]$ and $r \in [0, \frac{q}{10}] \subset [0,1]$, this implies that $p, r \in [0,10]$, thus $\alpha(p,r) \geq 1$. So θ is a triangular α-admissible fuzzy mapping.

*For $p_0 = \frac{1}{10} \in \Omega$ and $p_1 \in [\theta(p_0)]_{\beta_{\theta(p_0)}} \neq \emptyset$, we have $\alpha(p_0, p_1) \geq 1$. In addition, for any sequence $\{p_n\} \subset \Omega$, $\lim_{n\to+\infty} p_n = p$, where $p \in \Omega$ and $\alpha(p_n, p_{n+1}) \geq 1$, $\forall\, n \in \mathbb{N}$, it holds that $p_n \in [0,10]$ for any $n \in \mathbb{N}$. Thus, $p_n, p \in [0,10]$, so $\alpha(p_n, p) \geq 1$, $\forall\, n \in \mathbb{N}$. Hence, condition **(4b)** holds.*

Subsequently, we will prove the assumptions of Theorem 3.8 are satisfied for every $p, q \in \Omega$ and $s = 2$.

***Case 1:** If $p, q \in [0,10]$, we have $\alpha(p,q) \geq 1$ and thus*

$$\begin{aligned}
\phi(\alpha(p,q)\ s^3\ H([\theta(p)]_{\beta_{\theta(p)}}, [\theta(q)]_{\beta_{\theta(q)}})) &= \sqrt{\alpha(p,q)\ s^3\ H([\theta(p)]_{\beta_{\theta(p)}}, [\theta(q)]_{\beta_{\theta(q)}}) + 1} \\
&= \sqrt{1.s^3 H([\theta(p)]_{\beta_{\theta(p)}}, [\theta(q)]_{\beta_{\theta(q)}}) + 1} \\
&= \sqrt{8H([\theta(p)]_{\beta_{\theta(p)}}, [\theta(q)]_{\beta_{\theta(q)}}) + 1}, \quad for\ s = 2.
\end{aligned}$$

Now, by using the definition of b-metric spaces, we can say that

$$\begin{aligned}
H([\theta(p)]_{\beta_{\theta(p)}}, [\theta(q)]_{\beta_{\theta(q)}}) &= \max\left\{\delta([\theta(p)]_{\beta_{\theta(p)}}, [\theta(q)]_{\beta_{\theta(q)}}), \delta([\theta(q)]_{\beta_{\theta(q)}}, [\theta(p)]_{\beta_{\theta(p)}})\right\} \\
&= \max\left\{\sup_{x\in[\theta(p)]_{\beta_{\theta(p)}}} \rho_b(x, [\theta(q)]_{\beta_{\theta(q)}}), \sup_{y\in[\theta(q)]_{\beta_{\theta(q)}}} \rho_b(y, [\theta(p)]_{\beta_{\theta(p)}})\right\} \\
&= \frac{(p-q)^2}{100}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\phi(\alpha(p,q)\ s^3\ H([\theta(p)]_{\beta_{\theta(p)}}, [\theta(q)]_{\beta_{\theta(q)}})) &= \sqrt{8\frac{(p-q)^2}{100} + 1} \\
&\leq \sqrt{(p-q)^2 + 1} \\
&= \sqrt{\rho_b(p,q) + 1} \\
&\leq \phi(E(p,q))^i + L\, F(p,q).
\end{aligned}$$

***Case 2:** If $p, q \in (10, +\infty)$, we have $\alpha(p,q) = \frac{1}{2}$ with $s = 2$ and*

$$\phi(\alpha(p,q)\ s^3\ H([\theta(p)]_{\beta_{\theta(p)}}, [\theta(q)]_{\beta_{\theta(q)}})) = \sqrt{\frac{1}{2} \times 2^3 H([\theta(p)]_{\beta_{\theta(p)}}, [\theta(q)]_{\beta_{\theta(q)}}) + 1}$$

$$
\begin{aligned}
&= \sqrt{4H([\theta(p)]_{\beta_{\theta(p)}}, [\theta(q)]_{\beta_{\theta(q)}}) + 1} \\
&= \sqrt{4\frac{(p-q)^2}{100} + 1} \\
&\leq \sqrt{(p-q)^2 + 1} \\
&= \sqrt{\rho_b(p,q) + 1} \\
&\leq \phi(E(p,q))^i + L\,F(p,q).
\end{aligned}
$$

Hence, θ satisfies all the assumptions of Theorem 3.8, so it must possess a unique fixed point.

Definition 3.10. *Let $\theta : \Omega \longrightarrow I^{\Omega}$ be a fuzzy mapping in a b-MS (Ω, ρ_b, s) with $s > 1$. Then θ is said to be an α-admissible ϕ-ψ-set-valued fuzzy contraction if there are $\phi \in \Phi$, $\psi \in \Psi$, $L \geq 0$, and $\beta \in (0, 1]$ such that for all $p, q \in \Omega$, $p \neq q$,*

$$
H([\theta(p)]_{\beta_{\theta(p)}}, [\theta(q)]_{\beta_{\theta(q)}}) > 0 \implies \phi[\alpha(p,q)s^3 H([\theta(p)]_{\beta_{\theta(p)}}, [\theta(q)]_{\beta_{\theta(q)}})] \leq \psi[\phi[E(p,q)]] + L\,F(p,q), \tag{3.21}
$$

where

$$
E(p,q) = \max\left\{\rho_b(p,q), \rho_b(p, [\theta(p)]_{\beta_{\theta(p)}}), \rho_b(q, [\theta(q)]_{\beta_{\theta(q)}}), \frac{\rho_b(p, [\theta(q)]_{\beta_{\theta(q)}}) + \rho_b(q, [\theta(p)]_{\beta_{\theta(p)}})}{2s^2}\right\}
$$

and

$$
F(p,q) = \min\left\{\rho_b(p, [\theta(p)]_{\beta_{\theta(p)}}), \rho_b(q, [\theta(q)]_{\beta_{\theta(q)}}), \rho_b(p, [\theta(q)]_{\beta_{\theta(q)}}), \rho_b(q, [\theta(p)]_{\beta_{\theta(p)}})\right\}.
$$

Theorem 3.11. *Let (Ω, ρ_b, s) be a complete b-MS and $\theta : \Omega \longrightarrow I^{\Omega}$ be an α-admissible ϕ-ψ-set-valued fuzzy contraction mapping satisfying:*

(1) (Ω, ρ_b, s) is an α-complete b-MS;
(2) for $p_0 \in \Omega$ and $p_1 \in [\theta(p_0)]_{\beta_{\theta(p_0)}}$, we have $\alpha(p_0, p_1) \geq 1$, $\beta \in (0, 1]$;
(3) θ is triangular α-admissible;
(4) either

(4a) θ is an α-continuous set-valued fuzzy mapping, or

(4b) if $\{p_n\} \subset \Omega$ so that $\alpha(p_n, p_{n+1}) \geq 1$ and $\lim_{n\to+\infty} p_n = p \in \Omega$, then we get $\alpha(p_n, p) \geq 1$ for all $n \in \mathbb{N}$.

Then θ has a unique fixed point.

Proof. Assume $p_0 \in \Omega$ and $p_1 \in [\theta(p_0)]_{\beta_{\theta(p_0)}} \neq \emptyset$ such that $\alpha(p_0, p_1) \geq 1$. We consider a sequence $\{p_n\} \subset \Omega$ defined by the iterative relation

$$
p_{n+1} \in [\theta(p_n)]_{\beta_{\theta(p_n)}} \neq \emptyset, \quad \forall\, n \in \mathbb{N}.
$$

If we assume $n_0 \in \mathbb{N}$ with $\rho_b(p_{n_0}, [\theta p_{n_0}]_{\beta(p_{n_0})}) = 0$, then it means that p_{n_0} is a member of $[\theta p_{n_0}]_{\beta(p_0)}$, and we can say that p_{n_0} is a fixed point of $[\theta(p_0)]_{\beta_{\theta(p_0)}} \neq \emptyset$, thus completing the proof.

Now, we take $\rho_b(p_n, [\theta(p_n)]_{\beta_{\theta(p_n)}}) > 0$, for all $n \in \mathbb{N}$. Let $p_2 \in [\theta(p_1)]_{\beta_{\theta(p_1)}} \neq \emptyset$, and then from the α-admissibility of θ, we have $\alpha(p_1, p_2) \geq 1$. By performing the same steps with the triangular α-admissibility of θ, we can obtain

$$
\alpha(p_n, p_{n+1}) \geq 1, \quad for\ every\ \ n \in \mathbb{N} \cup \{0\}. \tag{3.22}
$$

Now from Eq (3.21), we can get

$$\begin{aligned}\phi[H([\theta(p_{n-1})]_{\beta_{\theta(p_{n-1})}},[\theta(p_n)]_{\beta(p_n)})] &\leq \phi[s^3\ H([\theta(p_{n-1})]_{\beta_{\theta(p_{n-1})}},[\theta(p_n)]_{\beta(p_n)})]\\ &\leq \phi[\alpha(p_{n-1},p_n)\ s^3\ H([\theta(p_{n-1})]_{\beta_{\theta(p_{n-1})}},[\theta(p_n)]_{\beta(p_n)})]\\ &\leq \psi[\phi(E(p_{n-1},p_n))]+L\ F(p_{n-1},p_n),\ \ \forall\ n\in\mathbb{N},\end{aligned}\tag{3.23}$$

where

$$\begin{aligned}E(p_{n-1},p_n) &= \max\Big\{\rho_b(p_{n-1},p_n),\rho_b(p_{n-1},[\theta(p_{n-1})]_{\beta_{\theta(p_{n-1})}}),\rho_b(p_n,[\theta(p_n)]_{\beta_{\theta(p_n)}}),\\ &\quad \frac{\rho_b(p_{n-1},[\theta(p_n)]_{\beta_{\theta(p_n)}})+\rho_b(p_n,[\theta(p_{n-1})]_{\beta_{\theta(p_{n-1})}})}{2s^2}\Big\}\\ &= \max\left\{\rho_b(p_{n-1},p_n),\rho_b(p_n,[\theta(p_n)]_{\beta_{\theta(p_n)}}),\frac{\rho_b(p_{n-1},[\theta(p_n)]_{\beta_{\theta(p_n)}})+0}{2s^2}\right\},\\ &\quad \textit{because}\ \ p_n\ \in\ [\theta(p_{n-1})]_{\beta_{\theta(p_{n-1})}}\\ &= \max\left\{\rho_b(p_{n-1},p_n),\rho_b(p_n,[\theta(p_n)]_{\beta_{\theta(p_n)}}),\frac{\rho_b(p_{n-1},[\theta(p_n)]_{\beta_{\theta(p_n)}})}{2s^2}\right\}\end{aligned}\tag{3.24}$$

and

$$\begin{aligned}F(p_{n-1},p_n) &= \min\{\rho_b(p_{n-1},[\theta(p_{n-1})]_{\beta_{\theta(p_{n-1})}}),\rho_b(p_n,[\theta(p_n)]_{\beta_{\theta(p_n)}}),\rho_b(p_{n-1},[\theta(p_n)]_{\beta_{\theta(p_n)}}),\\ &\quad \rho_b(p_n,[\theta(p_{n-1})]_{\beta_{\theta(p_{n-1})}})\}\\ &= \min\{\rho_b(p_{n-1},[\theta(p_{n-1})]_{\beta_{\theta(p_{n-1})}}),\rho_b(p_n,[\theta(p_n)]_{\beta_{\theta(p_n)}}),\rho_b(p_{n-1},[\theta(p_n)]_{\beta_{\theta(p_n)}}),0\},\\ &\quad \textit{because}\ \ p_n\ \in\ [\theta(p_{n-1})]_{\beta_{\theta(p_{n-1})}}\\ &= 0.\end{aligned}\tag{3.25}$$

Now, from Eq (3.24), we have

$$\begin{aligned}\frac{\rho_b(p_{n-1},[\theta(p_n)]_{\beta_{\theta(p_n)}})}{2s^2} &\leq \frac{1}{2s^2}\Big\{s\Big[\rho_b(p_{n-1},p_n)+\rho_b(p_n,[\theta(p_n)]_{\beta_{\theta(p_n)}})\Big]\Big\},\ \ \textit{by using Lemma}\ 2.17\\ &= \frac{1}{2s}[\rho_b(p_{n-1},p_n)+\rho_b(p_n,[\theta(p_n)]_{\beta_{\theta(p_n)}})]\\ &\leq \frac{1}{2}[\rho_b(p_{n-1},p_n)+\rho_b(p_n,[\theta(p_n)]_{\beta_{\theta(p_n)}})]\\ &\leq \max\{\rho_b(p_{n-1},p_n),\rho_b(p_n,[\theta(p_n)]_{\beta_{\theta(p_n)}})\}.\end{aligned}\tag{3.26}$$

So, by using inequality (3.26), Eq (3.24) can be written as

$$E(p_{n-1},p_n)=\max\{\rho_b(p_{n-1},p_n),\rho_b(p_n,[\theta(p_n)]_{\beta_{\theta(p_n)}})\}.$$

If $E(p_{n-1},p_n)=\rho_b(p_n,[\theta(p_n)]_{\beta_{\theta(p_n)}})$, since θ is compact, it follows that

$$\rho_b(p_n,p_{n+1})\leq\ H([\theta(p_{n-1})]_{\beta_{\theta(p_{n-1})}},[\theta(p_n)]_{\beta_{\theta(p_n)}}).\tag{3.27}$$

We know that $p_{n+1}\in[\theta(p_n)]_{\beta_{\theta(p_n)}}\neq\emptyset$, which implies that $\rho_b(p_n,[\theta(p_n)]_{\beta_{\theta(p_n)}})\leq\rho_b(p_n,p_{n+1})$. By employing (3.23) and (3.27), we have

$$\phi(\rho_b(p_n,p_{n+1}))\ \leq\ \psi[\phi(H([\theta(p_{n-1})]_{\beta_{\theta(p_{n-1})}},[\theta(p_n)]_{\beta_{\theta(p_n)}}))]$$

$$
\begin{aligned}
&\leq \psi[\phi(E(p_{n-1}, p_n))] + L\,F(p_{n-1}, p_n) \\
&= \psi[\phi(E(p_{n-1}, p_n))] + L \times 0, \quad by \;\; using \;\; Eq\ (3.25) \\
&= \psi[\phi(E(p_{n-1}, p_n))] \\
&= \psi[\phi(\rho_b(p_n, [\theta(p_n)]_{\beta_{\theta(p_n)}}))] \\
&\leq \psi[\phi(\rho_b(p_n, p_{n+1}))] \\
&< \phi(\rho_b(p_n, p_{n+1})),
\end{aligned}
$$

which is a contradiction. Consequently,

$$E(p_{n-1}, p_n) = \rho_b(p_{n-1}, p_n).$$

Therefore, by using inequality (3.23), we can processes the following:

$$
\begin{aligned}
\phi(\rho_b(p_n, p_{n+1})) &\leq \psi[\phi(H([\theta(p_{n-1})]_{\beta_{\theta(p_{n-1})}}, [\theta(p_n)]_{\beta_{\theta(p_n)}}))] \\
&\leq \psi[\phi(E(p_{n-1}, p_n))] + L\,F(p_{n-1}, p_n) \\
&= \psi[\phi(E(p_{n-1}, p_n))] + L \times 0, \quad by \;\; using \;\; Eq\ (3.25) \\
&= \psi[\phi(E(p_{n-1}, p_n))] \\
&= \psi[\phi(\rho_b(p_{n-1}, p_n))] \\
&< \phi(\rho_b(p_{n-1}, p_n)).
\end{aligned} \tag{3.28}
$$

From (3.28) and (ϕ_1), we have

$$\rho_b(p_n, p_{n+1}) < \rho_b(p_{n-1}, p_n).$$

Consequently, the sequence of non-negative real numbers $\{\rho_b(p_n, p_{n+1})\}_{n\in\mathbb{N}}$ is strictly decreasing. This implies the existence of a non-negative number γ so that

$$\lim_{n\to+\infty} \rho_b(p_n, p_{n+1}) = \gamma.$$

To prove $\gamma = 0$, we use a contradiction that $\gamma > 0$. Since the sequence $\{\rho_b(p_n, p_{n+1})\}_{n\in\mathbb{N}}$ is both positive and decreasing, we conclude that $\gamma \leq \rho_b(p_n, p_{n+1})$, for all $n \in \mathbb{N}$. By using (3.28), we have

$$
\begin{aligned}
\phi(\gamma) \leq \phi(\rho_b(p_n, p_{n+1})) &\leq \psi[\phi(\rho_b(p_{n-1}, p_n))] \\
&\leq \psi^2[\phi(\rho_b(p_{n-2}, p_{n-1}))] \\
&\;\;\vdots \\
&\leq \psi^n(\phi(\rho_b(p_0, p_1))).
\end{aligned} \tag{3.29}
$$

By using the property of ϕ and inequality (3.29), we get

$$
\begin{aligned}
1 &< \phi(\gamma) \\
&\leq \phi(\rho_b(p_n, p_{n+1})) \\
&\leq \psi^n[\phi(\rho_b(p_0, p_1))].
\end{aligned} \tag{3.30}
$$

As $n \to +\infty$ in (3.30), we can get

$$1 < \phi(\gamma) \leq \lim_{n\to+\infty} \psi^n[\phi(\rho_b(p_0, p_1))] = 1.$$

This is a contradiction to our supposition, so $\gamma = 0$. Hence,

$$\lim_{n\to+\infty} \rho_b(p_n, p_{n+1}) = 0. \tag{3.31}$$

Our goal is now to show that the sequence $\{p_n\}_{n\in\mathbb{N}}$ is Cauchy. Suppose, on the contrary, that it is not. Then, according to Lemma 2.5, there exists $\epsilon > 0$ such that there are two sequences of positive integers $\{m_k\}$ and $\{n_k\}$ for which

$$\begin{aligned}
\epsilon &\leq \lim_{k\to+\infty} \inf \rho_b(p_{m_k}, p_{n_k}) \leq \lim_{k\to+\infty} \sup \rho_b(p_{m_k}, p_{n_k}) \leq s\,\epsilon,\\
\frac{\epsilon}{s} &\leq \lim_{k\to+\infty} \inf \rho_b(p_{m_k}, p_{n_{k+1}}) \leq \lim_{k\to+\infty} \sup \rho_b(p_{m_k}, p_{n_{k+1}}) \leq s^2\,\epsilon,\\
\frac{\epsilon}{s} &\leq \lim_{k\to+\infty} \inf \rho_b(p_{m_{k+1}}, p_{n_k}) \leq \lim_{k\to+\infty} \sup \rho_b(p_{m_{k+1}}, p_{n_k}) \leq s^2\,\epsilon,\\
\frac{\epsilon}{s^2} &\leq \lim_{k\to+\infty} \inf \rho_b(p_{m_{k+1}}, p_{n_{k+1}}) \leq \lim_{k\to+\infty} \sup \rho_b(p_{m_{k+1}}, p_{n_{k+1}}) \leq s^3\,\epsilon.
\end{aligned}$$

Since θ is triangular α-admissible, then by employing Lemma 3.5, we have

$$\alpha(p_{m_k}, p_{n_k}) \geq 1. \tag{3.32}$$

Also,

$$\begin{aligned}
E(p_{m_k}, p_{n_k}) &= \max\Big\{\rho_b(p_{m_k}, p_{n_k}), \rho_b(p_{m_k}, [\theta(p_{m_k})]_{\beta_{\theta(p_{m_k})}}), \rho_b(p_{n_k}, [\theta(p_{n_k})]_{\beta_{\theta(p_{n_k})}}),\\
&\quad \frac{\rho_b(p_{m_k}, [\theta(p_{n_k})]_{\beta_{\theta(p_{n_k})}}) + \rho_b(p_{n_k}, [\theta(p_{m_k})]_{\beta_{\theta(p_{m_k})}})}{2s^2}\Big\}\\
&\leq \max\Big\{\rho_b(p_{m_k}, p_{n_k}), \rho_b(p_{m_k}, p_{m_{k+1}}), \rho_b(p_{n_k}, p_{n_{k+1}}),\\
&\quad \frac{\rho_b(p_{m_k}, p_{n_{k+1}}) + \rho_b(p_{n_k}, p_{m_{k+1}})}{2s^2}\Big\}
\end{aligned}$$

and

$$\begin{aligned}
F(p_{m_k}, p_{n_k}) &= \min\Big\{\rho_b(p_{m_k}, [\theta(p_{m_k})]_{\beta_{\theta(p_{m_k})}}), \rho_b(p_{n_k}, [\theta(p_{n_k})]_{\beta_{\theta(p_{n_k})}}),\\
&\quad \rho_b(p_{m_k}, [\theta(p_{n_k})]_{\beta_{\theta(p_{n_k})}}), \rho_b(p_{n_k}, [\theta(p_{m_k})]_{\beta_{\theta(p_{m_k})}})\Big\}\\
&\leq \min\left\{\rho_b(p_{m_k}, p_{m_{k+1}}), \rho_b(p_{n_k}, p_{n_{k+1}}), \rho_b(p_{m_k}, p_{n_{k+1}}), \rho_b(p_{n_k}, p_{m_{k+1}})\right\}.
\end{aligned}$$

By applying Lemma 2.5 and using (3.31) with $\lim_{k\to+\infty}$, we get

$$\begin{aligned}
\lim_{k\to+\infty} E(p_{m_k}, p_{n_k}) &\leq \lim_{k\to+\infty} \max\Big\{\rho_b(p_{m_k}, p_{n_k}), \rho_b(p_{m_k}, p_{m_{k+1}}), \rho_b(p_{n_k}, p_{n_{k+1}}),\\
&\quad \frac{\rho_b(p_{m_k}, p_{n_{k+1}}) + \rho_b(p_{n_k}, p_{m_{k+1}})}{2s^2}\Big\}\\
&\leq \max\{s\epsilon, 0, 0, \frac{s^2\epsilon + s^2\epsilon}{2s^2}\} = s\epsilon
\end{aligned}$$

and

$$\lim_{k\to+\infty} F(p_{m_k}, p_{n_k}) \leq \lim_{k\to+\infty} \min\{\rho_b(p_{m_k}, p_{m_{k+1}}), \rho_b(p_{n_k}, p_{n_{k+1}}),$$

$$\begin{aligned}&\rho_b(p_{m_k},p_{n_{k+1}}),\rho_b(p_{n_k},p_{m_{k+1}})\}\\ \le\ & \min\{0,0,s^2\epsilon,s^2\epsilon\}\\ =\ & 0.\end{aligned}$$

So, we have

$$\begin{aligned}\lim_{k\to+\infty} F(p_{m_k},p_{n_k}) &= 0 \ \ \textit{and}\\ \lim_{k\to+\infty} E(p_{m_k},p_{n_k}) &\le s\epsilon. \end{aligned}\tag{3.33}$$

Now, from (3.32), we obtain

$$\begin{aligned}\phi[s^3\rho_b(p_{m_{k+1}},p_{n_{k+1}})] &\le \phi[s^3\ H([\theta(p_{m_k})]_{\beta_{\theta(p_{m_k})}},[\theta(p_{n_k})]_{\beta_{\theta(p_{n_k})}})]\\ &\le \phi[\alpha(p_{m_k},p_{n_k})\ s^3\ H([\theta(p_{m_k})]_{\beta_{\theta(p_{m_k})}},[\theta(p_{n_k})]_{\beta_{\theta(p_{n_k})}})]\\ &\le \psi[\phi(E(p_{m_k},p_{n_k}))]+L\ F(p_{m_k},p_{n_k}).\end{aligned}\tag{3.34}$$

By employing inequalities (3.33) and (3.34), and using the continuity of ϕ, we can get the following inequality by letting $k\to+\infty$:

$$\begin{aligned}\phi(s\epsilon) &= \phi(\frac{\epsilon}{s^2}\ s^3)\\ &\le \phi\left(s^3\ \lim_{k\to+\infty}\rho_b(p_{m_{k+1}},p_{n_{k+1}})\right)\ \ \textit{by using Lemma}\ 2.5\\ &\le \psi[\phi(\lim_{k\to+\infty}E(p_{m_k},p_{n_k}))]+L\ \lim_{k\to+\infty}F(p_{m_k},p_{n_k})\\ &= \psi[\phi(\lim_{k\to+\infty}E(p_{m_k},p_{n_k}))]+L\times 0\\ &= \psi[\phi(\lim_{k\to+\infty}E(p_{m_k},p_{n_k}))]\\ &\le \psi[\phi(s\epsilon)],\ \ \textit{by using Eq}\ (3.33).\end{aligned}$$

But, ϕ is a monotonically increasing function, so

$$s\epsilon < s\epsilon.$$

This is a contradiction. Consequently, $\{p_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in Ω. Using the completeness of (Ω,ρ_b,s), we take $\nu\in\Omega$ such that

$$\lim_{n\to+\infty}\rho_b(p_n,\nu)=0.\tag{3.35}$$

Case 1: If θ is an α-continuous set-valued fuzzy mapping, then by **(4a)**, we find that

$$\lim_{n\to+\infty}H([\theta(p_n)]_{\beta_{\theta(p_n)}},[\theta(\nu)]_{\beta_{\theta(\nu)}})=0.\tag{3.36}$$

Thus, by using Eqs (3.35) and (3.36), we get

$$\rho_b(\nu,[\theta(\nu)]_{\beta_{\theta(\nu)}})=\lim_{n\to+\infty}\rho_b(p_{n+1},[\theta(\nu)]_{\beta_{\theta(\nu)}})\le\lim_{n\to+\infty}H([\theta(p_n)]_{\beta_{\theta(p_n)}},[\theta(\nu)]_{\beta_{\theta(\nu)}})=0.$$

Hence, $\nu\in[\theta(\nu)]_{\beta_{\theta(\nu)}}$ implies that θ has a fixed point.

Case 2: If θ is not an α-continuous set-valued fuzzy mapping, then according to **(4b)**, we show that $\nu\in[\theta(\nu)]_{\beta_{\theta(\nu)}}$ by using a contradiction. Suppose that

$$\nu\notin[\theta(\nu)]_{\beta_{\theta(\nu)}}.$$

We know that $0 \leq \rho_b([\theta(p_n)]_{\beta_{\theta(p_n)}}, \nu) \leq \rho_b(p_{n+1}, \nu)$, hence

$$\lim_{n\to+\infty} \rho_b([\theta(p_n)]_{\beta_{\theta(p_n)}}, \nu) = 0.$$

As $\lim_{n\to+\infty}[\theta(p_n)]_{\beta_{\theta(p_n)}} = \nu$, by using Lemma 2.4, we obtain

$$\begin{aligned} \frac{1}{s^2}\rho_b(\nu, [\theta(\nu)]_{\beta_{\theta(\nu)}}) &\leq \lim_{n\to+\infty} \inf H([\theta(p_n)]_{\beta_{\theta(p_n)}}, [\theta(\nu)]_{\beta_{\theta(\nu)}}) \\ &\leq \lim_{n\to+\infty} \sup H([\theta(p_n)]_{\beta_{\theta(p_n)}}, [\theta(\nu)]_{\beta_{\theta(\nu)}}) \\ &\leq s^2\, \rho_b(\nu, [\theta(\nu)]_{\beta_{\theta(\nu)}}). \end{aligned} \tag{3.37}$$

Now, consider

$$\begin{aligned} \phi[s^3\, H([\theta(p_n)]_{\beta_{\theta(p_n)}}, [\theta(\nu)]_{\beta_{\theta(\nu)}})] &\leq \phi[\alpha(p_n, \nu)\, s^3\, H([\theta(p_n)]_{\beta_{\theta(p_n)}}, [\theta(\nu)]_{\beta_{\theta(\nu)}})] \\ &\leq \psi[\phi(E(p_n, \nu))] + L\, F(p_n, \nu),\ \forall\, n \in \mathbb{N}, \end{aligned}$$

where

$$E(p_n, \nu) = \max\left\{\rho_b(p_n, \nu), \rho_b(p_n, [\theta(p_n)]_{\beta_{\theta(p_n)}}), \rho_b(\nu, [\theta(\nu)]_{\beta_{\theta(\nu)}}), \frac{\rho_b(p_n, [\theta(\nu)]_{\beta_{\theta(\nu)}}) + \rho_b(\nu, [\theta(p_n)]_{\beta_{\theta(p_n)}})}{2s^2}\right\}$$

and

$$F(p_n, \nu) = \min\left\{\rho_b(p_n, [\theta(p_n)]_{\beta_{\theta(p_n)}}), \rho_b(\nu, [\theta(\nu)]_{\beta_{\theta(\nu)}}), \rho_b(p_n, [\theta(\nu)]_{\beta_{\theta(\nu)}}), \rho_b(\nu, [\theta(p_n)]_{\beta_{\theta(p_n)}})\right\}.$$

Now, by taking $n \to +\infty$, we obtain

$$\begin{aligned} \lim_{n\to+\infty} \sup E(p_n, \nu) &= \lim_{n\to+\infty} \sup \max\left\{\rho_b(p_n, \nu), \rho_b(p_n, [\theta(p_n)]_{\beta_{\theta(p_n)}}), \rho_b(\nu, [\theta(\nu)]_{\beta_{\theta(\nu)}}),\right. \\ &\qquad \left.\frac{\rho_b(p_n, [\theta(\nu)]_{\beta_{\theta(\nu)}}) + \rho_b(\nu, [\theta(p_n)]_{\beta_{\theta(p_n)}})}{2s^2}\right\} \\ &\leq \lim_{n\to+\infty} \sup \max\left\{\rho_b(p_n, \nu), \rho_b(p_n, p_{n+1}), \rho_b(\nu, [\theta(\nu)]_{\beta_{\theta(\nu)}}),\right. \\ &\qquad \left.\frac{\rho_b(p_n, [\theta(\nu)]_{\beta_{\theta(\nu)}}) + \rho_b(\nu, p_{n+1})}{2s^2}\right\} \\ &\leq \rho_b(\nu, [\theta(\nu)]_{\beta_{\theta(\nu)}}), \quad \textit{since} \quad \lim_{n\to+\infty}[\theta(p_n)]_{\beta_{\theta(p_n)}} = \nu, \end{aligned} \tag{3.38}$$

and

$$\begin{aligned} \lim_{n\to+\infty} \sup F(p_n, \nu) &= \lim_{n\to+\infty} \sup \min\{\rho_b(p_n, [\theta(p_n)]_{\beta_{\theta(p_n)}}), \rho_b(\nu, [\theta(\nu)]_{\beta_{\theta(\nu)}}), \rho_b(p_n, [\theta(\nu)]_{\beta_{\theta(\nu)}}), \\ &\qquad \rho_b(\nu, [\theta(p_n)]_{\beta_{\theta(p_n)}})\} \\ &\leq \lim_{n\to+\infty} \sup \min\left\{\rho_b(p_n, p_{n+1}), \rho_b(\nu, [\theta(\nu)]_{\beta_{\theta(\nu)}}), \rho_b(p_n, [\theta(\nu)]_{\beta_{\theta(\nu)}}), \rho_b(\nu, p_{n+1})\right\} \\ &= \min\{0, \rho_b(\nu, [\theta(\nu)]_{\beta_{\theta(\nu)}}), \rho_b(\nu, z), \rho_b(z, [\theta(\nu)]_{\beta_{\theta(\nu)}})\} \\ &= 0. \end{aligned} \tag{3.39}$$

Therefore,

$$\begin{aligned}\phi(s^3\, H([\theta(p_n)]_{\beta_{\theta(p_n)}}, [\theta(\nu)]_{\beta_{\theta(\nu)}})) \;\le\;& \phi\Big(\max\Big\{\rho_b(p_n,\nu), \rho_b(p_n,[\theta(p_n)]_{\beta_{\theta(p_n)}}), \rho_b(\nu,[\theta(\nu)]_{\beta_{\theta(\nu)}}),\\ & \frac{\rho_b(p_n,[\theta(\nu)]_{\beta_{\theta(\nu)}}) + \rho_b(\nu,[\theta(p_n)]_{\beta_{\theta(p_n)}})}{2s^2}\Big\}\Big)\\ & + L\,\min\Big\{\rho_b(p_n,[\theta(p_n)]_{\beta_{\theta(p_n)}}), \rho_b(\nu,[\theta(\nu)]_{\beta_{\theta(\nu)}}),\\ & \rho_b(p_n,[\theta(\nu)]_{\beta_{\theta(\nu)}}), \rho_b(\nu,[\theta(p_n)]_{\beta_{\theta(p_n)}})\Big\}. \qquad (3.40)\end{aligned}$$

Taking $n \to +\infty$ in (3.37) and (3.40), and by using (3.31) with the definitions of ϕ and ψ, we have

$$\begin{aligned}\phi(s\,\rho_b(\nu,[\theta(\nu)]_{\beta_{\theta(\nu)}})) &= \phi(s^3 \frac{1}{s^2}\, \rho_b(\nu,[\theta(\nu)]_{\beta_{\theta(\nu)}}))\\ &\le \phi[s^3 \lim_{n\to+\infty} H([\theta(p_n)]_{\beta_{\theta(p_n)}}, [\theta(\nu)]_{\beta_{\theta(\nu)}})]\\ &\le \lim_{n\to+\infty} \phi[s^3\, H([\theta(p_n)]_{\beta_{\theta(p_n)}}, [\theta(\nu)]_{\beta_{\theta(\nu)}})]\\ &\le \lim_{n\to+\infty} \phi[\alpha(p_n,\nu)\, s^3\, H([\theta(p_n)]_{\beta_{\theta(p_n)}}, [\theta(\nu)]_{\beta_{\theta(\nu)}})]\\ &\le \lim_{n\to+\infty} (\psi[\phi(E(p_n,\nu))] + L\, F(p_n,\nu))\\ &\le \lim_{n\to+\infty} \psi[\phi(E(p_n,\nu))] + L \lim_{n\to+\infty} F(p_n,\nu)\\ &= \lim_{n\to+\infty} \psi[\phi(E(p_n,\nu))] + L\times 0, \;\; by\;\; using\;\; Eq\, (3.39)\\ &\le \psi[\phi(\rho_b(\nu,[\theta(\nu)]_{\beta_{\theta(\nu)}}))], \;\; by\;\; using\;\; Eq\, (3.38)\\ &< \phi(\rho_b(\nu,[\theta(\nu)]_{\beta_{\theta(\nu)}})).\end{aligned}$$

Using (ϕ_1), we get

$$s\,\rho_b(\nu,[\theta(\nu)]_{\beta_{\theta(\nu)}}) < \rho_b(\nu,[\theta(\nu)]_{\beta_{\theta(\nu)}}).$$

This implies that

$$\rho_b(\nu,[\theta(\nu)]_{\beta_{\theta(\nu)}})(s-1) < 0,$$

hence

$$s < 1.$$

This is a contradiction to our supposition, so it follows that $\nu \in [\theta(\nu)]_{\beta_{\theta(\nu)}}$, and hence θ has a unique fixed point. □

The following fixed point theorems for α-admissible ϕ-ψ-set-valued Kannan-type and Reich-type fuzzy contractions are direct consequences of Theorem 3.11. These results not only extend but also strengthen the existing fixed point theorems for Kannan-type and Reich-type fuzzy contractions in the framework of b-metric spaces, thereby contributing new insights to the current literature.

Theorem 3.12. *Let (Ω, ρ_b, s) be a complete b-MS and $\theta : \Omega \longrightarrow I^\Omega$ be an α-admissible ϕ-ψ-set-valued fuzzy contraction mapping. Suppose that there are $\phi \in \Phi$, $\psi \in \Psi$, and $\beta \in (0,1]$ such that for all $p, q \in \Omega$ with $p \neq q$,*

$$H([\theta(p)]_{\beta_{\theta(p)}}, [\theta(q)]_{\beta_{\theta(q)}}) > 0 \implies \phi[s^3\, H([\theta(p)]_{\beta_{\theta(p)}}, [\theta(q)]_{\beta_{\theta(q)}})] \le \psi[\phi(M(p,q))], \qquad (3.41)$$

where

$$M(p,q)=\max\left\{\rho_b(p,q),\rho_b(p,[\theta(p)]_{\beta_{\theta(p)}}),\rho_b(q,[\theta(q)]_{\beta_{\theta(q)}})\right\}.$$

Then θ possesses a fixed point.

Proof. We have

$$\begin{aligned} M(p_n,\nu) &= \max\left\{\rho_b(p_n,\nu),\rho_b(p_n,[\theta(p_n)]_{\beta_{\theta(p_n)}}),\rho_b(\nu,[\theta(\nu)]_{\beta_{\theta(\nu)}})\right\} \\ &\le \max\left\{\rho_b(p_n,\nu),\rho_b(p_n,[\theta(p_n)]_{\beta_{\theta(p_n)}}),\rho_b(\nu,[\theta(\nu)]_{\beta_{\theta(\nu)}}),\frac{\rho_b(\nu,[\theta(p_n)]_{\beta_{\theta(p_n)}})+\rho_b(p_n,[\theta(\nu)]_{\beta_{\theta(\nu)}})}{2s^2}\right\} \\ &= E(p_n,\nu). \end{aligned}$$

Here, θ is an α-admissible ϕ-ψ-set-valued fuzzy contraction mapping, in which $\alpha(p,q)=1$ for all $p,q\in\Omega,\ p\neq q$. According to Theorem 3.11, we can find that θ possesses a fixed point. □

Definition 3.13. *Let $\theta:\Omega\longrightarrow I^{\Omega}$ be a fuzzy mapping in a b-MS (Ω,ρ_b,s) with $s>1$. Then θ is said to be an α-admissible ϕ-ψ-set-valued Kannan-type fuzzy contraction mapping if there are $\phi\in\Phi,\ \psi\in\Psi$, $L\geq 0$, and $\beta\in(0,1]$ such that for all $p,q\in\Omega$ with $p\neq q$,*

$$H([\theta(p)]_{\beta_{\theta(p)}},[\theta(q)]_{\beta_{\theta(q)}})>0$$

$$\Longrightarrow \phi[\alpha(p,q)s^3H([\theta(p)]_{\beta_{\theta(p)}},[\theta(q)]_{\beta_{\theta(q)}})]\le \psi\left[\phi\left(\frac{\rho_b(p,[\theta(p)]_{\beta_{\theta(p)}})+\rho_b(q,[\theta(q)]_{\beta_{\theta(q)}})}{2}\right)\right]. \tag{3.42}$$

Theorem 3.14. *Let (Ω,ρ_b,s) be a complete b-MS and $\theta:\Omega\longrightarrow I^{\Omega}$ be an α-admissible ϕ-ψ-set-valued Kannan-type fuzzy contraction mapping satisfying*

(1) (Ω,ρ_b,s) is an α-complete b-MS;
(2) for $p_0\in\Omega$ and $p_1\in[\theta(p_0)]_{\beta_{\theta(p_0)}}$, we have $\alpha(p_0,p_1)\geq 1,\ \ \beta\in(0,1]$;
(3) θ is triangular α-admissible;
(4) either
(4a) θ is an α-continuous set-valued fuzzy mapping, or
(4b) if $\{p_n\}\subset\Omega$ so that $\alpha(p_n,p_{n+1})\geq 1$ and $\lim_{n\to+\infty}p_n=p\in\Omega$, then we get $\alpha(p_n,p)\geq 1$ for all $n\in\mathbb{N}$.

Then θ has a unique fixed point.

Proof. The α-admissible ϕ-ψ-set-valued Kannan-type fuzzy contraction mapping implies that

$$\begin{aligned} \phi[\alpha(p,q)s^3H([\theta(p)]_{\beta_{\theta(p)}},[\theta(q)]_{\beta_{\theta(q)}})] &\le \psi\left[\phi\left(\frac{\rho_b(p,[\theta(p)]_{\beta_{\theta(p)}})+\rho_b(q,[\theta(q)]_{\beta_{\theta(q)}})}{2}\right)\right] \\ &\le \psi\left[\phi\left(\max\{\rho_b(p,[\theta(p)]_{\beta_{\theta(p)}}),\rho_b(q,[\theta(q)]_{\beta_{\theta(q)}})\}\right)\right] \\ &\le \psi\left[\phi(E(p,q))\right]. \end{aligned}$$

Therefore, θ is an α-admissible ϕ-ψ-set-valued Kannan-type fuzzy contraction mapping. According to Theorem 3.11, we find that θ has a unique fixed point. □

Definition 3.15. *Let $\theta : \Omega \longrightarrow I^{\Omega}$ be a fuzzy mapping in a b-MS (Ω, ρ_b, s) with $s > 1$. Then θ is said to be an α-admissible ϕ-ψ-set-valued Reich-type fuzzy contraction mapping if there are $\phi \in \Phi$, $\psi \in \Psi$, $L \geq 0$, and $\beta \in (0, 1]$ such that for all $p, q \in \Omega$ with $p \neq q$,*

$$H([\theta(p)]_{\beta_{\theta(p)}}, [\theta(q)]_{\beta_{\theta(q)}}) > 0$$

$$\implies \phi[\alpha(p,q)s^3 H([\theta(p)]_{\beta_{\theta(p)}}, [\theta(q)]_{\beta_{\theta(q)}})] \leq \psi\left[\phi\left(\frac{\rho_b(p,q) + \rho_b(p, [\theta(p)]_{\beta_{\theta(p)}}) + \rho_b(q, [\theta(q)]_{\beta_{\theta(q)}})}{3}\right)\right]. \tag{3.43}$$

Theorem 3.16. *Let (Ω, ρ_b, s) be a complete b-MS and $\theta : \Omega \longrightarrow I^{\Omega}$ be an α-admissible ϕ-ψ-set-valued Reich-type fuzzy contraction mapping satisfying:*

(1) (Ω, ρ_b, s) is an α-complete b-MS;
(2) for $p_0 \in \Omega$ and $p_1 \in [\theta(p_0)]_{\beta_{\theta(p_0)}} \implies \alpha(p_0, p_1) \geq 1$, $\beta \in (0, 1]$;
(3) θ is triangular α-admissible;
(4) either
(4a) θ is an α-continuous set-valued fuzzy mapping, or
(4b) if $\{p_n\} \subset \Omega$ so that $\alpha(p_n, p_{n+1}) \geq 1$ and $\lim_{n\to+\infty} p_n = p \in \Omega$, then we get $\alpha(p_n, p) \geq 1$ for all $n \in \mathbb{N}$.

Then θ has a unique fixed point.

Proof. From the definition of an α-admissible ϕ-ψ-set-valued Reich-type fuzzy contraction mapping, we have

$$\begin{aligned}
\phi[\alpha(p,q)s^3 H([\theta(p)]_{\beta_{\theta(p)}}, [\theta(q)]_{\beta_{\theta(q)}})] &\leq \psi\left[\phi\left(\frac{\rho_b(p,q) + \rho_b(p, [\theta(p)]_{\beta_{\theta(p)}}) + \rho_b(q, [\theta(q)]_{\beta_{\theta(q)}})}{3}\right)\right] \\
&\leq \psi\left[\phi\left(\max\{\rho_b(p,q), \rho_b(p, [\theta(p)]_{\beta_{\theta(p)}}), \rho_b(q, [\theta(q)]_{\beta_{\theta(q)}})\}\right)\right] \\
&\leq \psi\left[\phi(E(p,q))\right].
\end{aligned}$$

Hence, θ is an α-admissible ϕ-ψ-set-valued Reich-type fuzzy contraction mapping. Furthermore, with the help of Theorem 3.11, we prove that θ has a fixed point. □

On the basis of our findings, we can deduce many results of classical fixed point outcomes.

Corollary 3.17. *Let (ρ_b, Ω, s) be a complete b-MS and $\theta : \Omega \longrightarrow I^{\Omega}$ be a fuzzy mapping. Suppose that there are $i \in (0, 1)$, $\phi \in \Phi$, $\beta \in (0, 1]$, $\forall\ p, q \in \Omega$, $p \neq q$, such that*

$$H([\theta(p)]_{\beta_{\theta(p)}}, [\theta(q)]_{\beta_{\theta(q)}}) > 0 \implies \phi[s^3 H([\theta(p)]_{\beta_{\theta(p)}}, [\theta(q)]_{\beta_{\theta(q)}})] \leq [\phi(\rho_b(p,q))]^i.$$

Then θ has a fixed point.

Corollary 3.18. *Let (ρ_b, Ω, s) be a complete b-MS and $\theta : \Omega \longrightarrow I^{\Omega}$ be a fuzzy mapping. Suppose that there are $\phi \in \Phi$, $\psi \in \Psi$, and $\beta \in (0, 1]$ such that for all $p, q \in \Omega$ with $p \neq q$,*

$$H([\theta(p)]_{\beta_{\theta(p)}}, [\theta(q)]_{\beta_{\theta(q)}}) > 0 \implies \phi[s^3 H([\theta(p)]_{\beta_{\theta(p)}}, [\theta(q)]_{\beta_{\theta(q)}})] \leq \psi[\phi(\rho_b(p,q))].$$

Then θ has a fixed point.

Corollary 3.19. *Let (ρ_b, Ω, s) be a complete b-MS and $\theta : \Omega \longrightarrow I^{\Omega}$ be a fuzzy mapping. Suppose that there are $\phi \in \Phi$, $\psi \in \Psi$, and $\beta \in (0, 1]$ such that for all $p, q \in \Omega$ with $p \neq q$,*

$$H([\theta(p)]_{\beta_{\theta(p)}}, [\theta(q)]_{\beta_{\theta(q)}}) > 0$$

$$\Longrightarrow \phi[s^3 H([\theta(p)]_{\beta_{\theta(p)}}, [\theta(q)]_{\beta_{\theta(q)}})] \leq \psi\left[\phi\left(\frac{\rho_b(p, [\theta(p)]_{\beta_{\theta(p)}}) + \rho_b(q, [\theta(q)]_{\beta_{\theta(q)}})}{2}\right)\right].$$

Then θ has a fixed point.

Corollary 3.20. *Let (ρ_b, Ω, s) be a complete b-MS and $\theta : \Omega \longrightarrow I^{\Omega}$ be a fuzzy mapping. Suppose that there are $\phi \in \Phi$, $\psi \in \Psi$, and $\beta \in (0, 1]$ such that for all $p, q \in \Omega$ with $p \neq q$,*

$$H([\theta(p)]_{\beta_{\theta(p)}}, [\theta(q)]_{\beta_{\theta(q)}}) > 0$$

$$\Longrightarrow \phi[s^3 H([\theta(p)]_{\beta_{\theta(p)}}, [\theta(q)]_{\beta_{\theta(q)}})] \leq \psi\left[\phi\left(\frac{\rho_b(p, q) + \rho_b(p, [\theta(p)]_{\beta_{\theta(p)}}) + \rho_b(q, [\theta(q)]_{\beta_{\theta(q)}})}{3}\right)\right].$$

Then θ has a fixed point.

4. Conclusions

In this research, we have established a series of fuzzy fixed point theorems for α-admissible ϕ-ψ-set-valued fuzzy contractive mappings within the framework of b-metric spaces. Our results contribute to the generalization of classical fixed point theory by introducing a broader class of contractive conditions that incorporate both admissibility and dual control functions. In addition to the core results, we have formulated Kannan-type and Reich-type fuzzy fixed point theorems under the same α-admissible ϕ-ψ-set-valued fuzzy contraction structure. These extensions demonstrate the versatility and strength of the proposed framework. Furthermore, we explored the implications of our main theorems by deriving several known results from the literature as special cases, thereby confirming the generality of our approach. To support the theoretical developments, a carefully constructed nontrivial example was presented, illustrating the applicability and validity of the obtained results. Overall, this study offers a unified and extended perspective on fuzzy fixed point theory in b-metric spaces, laying the groundwork for further research in more generalized or application-driven settings. Furthermore, exploring the applicability of the proposed framework to concrete problems such as fuzzy integral inclusions, fractional differential equations, or dynamic programming models remains an appealing direction for future research.

Author contributions

All authors contributed equally and significantly in writing this article. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

References

1. S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, *Fundamenta Math.*, **3** (1922), 133–181.
2. R. Kannan, Some results on fixed points, *Bull. Calcutta Math. Soc.*, **60** (1968), 71–76.
3. M. Jleli, B. Samet, A new generalization of the Banach contraction principle, *J. Inequal. Appl.*, **38** (2014), 38. https://doi.org/10.1186/1029-242X-2014-38
4. B. Samet, C. Vetro, P. Vetro, Fixed point theorems for $\alpha - \psi$-contractive type mappings, *Nonlinear Anal. Theory Meth. Appl.*, **75** (2012), 2154–2165. https://doi.org/10.1016/j.na.2011.10.014
5. S. B. Nadler, Multi-valued contraction mappings, *Pacific J. Math.*, **30** (1969), 475–488.
6. A. Azam, S. Kanwal, Common fixed point results for multivalued mappings in Hausdorff intuitionistic fuzzy metric spaces, *Commun. Math. Appl.*, **9** (2018), 63–75. https://doi.org/10.26713/cma.v9i1.927
7. M. Berinde, V. Berinde, On a general class of multi-valued weakly Picard mappings, *J. Math. Anal. Appl.*, **326** (2007), 772–782. https://doi.org/10.1016/j.jmaa.2006.03.016
8. N. Mizoguchi, W. Takahashi, Fixed point theorems for multivalued mappings on complete metric spaces, *J. Math. Anal. Appl.*, **141** (1989), 177–188. https://doi.org/10.1016/0022-247X(89)90214-X
9. S. Czerwik, Contraction mapping in b-metric spaces, *Acta Math. Inform. Uni. Ostr.*, **1** (1993), 5–11.
10. H. Aydi, M. F. Bota, E. Karapinar, S. Mitrović, A fixed point theorem for set-valued quasi-contractions in b-metric space, *Fixed Point Theory Appl.*, **2012** (2012), 88. https://doi.org/10.1186/1687-1812-2012-88
11. A. Azam, S. Kanwal, Introduction to intuitionistic fuzzy b-metric spaces and fixed point results, *Thai J. Math.*, **20** (2022), 141–163.
12. A. Azam, S. Kanwal, A. F. Shami, On Coincidence theorem in intuitionistic fuzzy b-metric spaces with application, *J. Function Spaces*, **2022** (2022), 5616824. https://doi.org/10.1155/2022/5616824
13. L. A. Zadeh, Fuzzy sets, *Inform. Control*, **8** (1965), 338–353.
14. M. D. Weiss, Fixed points, separation, and induced topologies for fuzzy sets, *J. Math. Anal. Appl.*, **50** (1975), 142–150.

15. D. Butnariu, Fixed points for fuzzy mappings, *Fuzzy Sets Syst.*, **7** (1982), 191–207. https://doi.org/10.1016/0165-0114(82)90049-5
16. S. Heilpern, Fuzzy mappings and fixed point theorem, *J. Math. Anal. Appl.*, **83** (1981), 566–569. https://doi.org/10.1016/0022-247X(81)90141-4
17. S. Kanwal, S. M. Shagari, H. Aydi, A. Mukheimer, T. Abdeljawad, Common fixed-point results of fuzzy mappings and applications on stochastic Volterra integral equations, *J. Inequal. Appl.*, **2022** (2022), 110. https://doi.org/10.1186/s13660-022-02849-2
18. S. Kanwal, A. Ali, A. A. Mazrooie, G. Santos-Garcia, Existence of fuzzy fixed points of set-valued fuzzy mappings in metric and fuzzy metric spaces, *AIMS Math.*, **8** (2023), 10095–10112. https://doi.org/10.3934/math.2023511
19. B. Mohammadi, S. Rezapour, N. Shahzad, Some results on fixed points of α-ψ Ćirić generalized multifunctions, *Fixed Point Theory Appl.*, **2013** (2013), 24. https://doi.org/10.1186/1687-1812-2013-24
20. D. K. Patel, Fixed points of multivalued contractions via generalized class of simulation functions, *Bolet. Soc. Paranaense Mat.*, **38** (2018), 161–179.
21. A. Aghajani, M. Abbas, J. Roshan, Common fixed point of generalized weak contractive mappings in partially ordered b-metric spaces, *Math. Slovaca*, **64** (2014), 941–960. https://doi.org/10.2478/s12175-014-0250-6
22. R. J. Rezaei, V. Parvaneh, K. Zoran, Common fixed point theorems for weakly isotone increasing mappings in ordered b-metric spaces, *J. Nonlinear Sci. Appl.*, **7** (2014), 229–245.
23. D. W. Zheng, Z. Y. Cai, P. Wang, New fixed point theorems for $\theta-\phi$-contraction in complete metric spaces, *J. Nonlinear Sci. Appl.*, **10** (2017), 2662–2670. https://doi.org/10.22436/jnsa.010.05.32
24. A. Taqbibt, M. Savatović, M. Elomari, S. Melliani, Some fixed point theorems for α-admissible θ-ϕ-multivalued contraction mappings in b-Metric spaces, *Appl. Anal. Discrete Math.*, **19** (2025), 150–175. https://doi.org/10.2298/AADM231005024T
25. S. Czerwick, K. Dlutek, S. L. Singh, Round-off stability of iteration procedures for set-valued operators in b-metric spaces, *J. Natural Phys. Sci.*, **11** (2011), 87–94.
26. N. Hussain, M. A. Kutbi, P. Salimi, Fixed point theory in α-complete metric spaces with applications, *Abstract Appl. Anal.*, **2014** (2014), 1–11. https://doi.org/10.1155/2014/280817
27. M. A. Kutbi, W. Sintunavarat, On new fixed point results for (α, ψ, ζ)-contractive multivalued mappings on α-complete metric spaces and their consequences, *Fixed Point Theory Appl.*, **2015** (2015), 2. https://doi.org/10.1186/1687-1812-2015-2
28. D. K. Patel, Fixed points of multivalued contractions via generalized class of simulation functions, *Bolet. Soc. Paranaense Mat.*, **38** (2018), 161–179.
29. M. Rossafi, A. Kari, C. Park, R. J. Lee, New fixed point theorems for θ-ϕ-contraction on b-metric spaces, *J. Math. Comput. Sci.*, **29** (2023), 12–27. http://doi.org/10.22436/jmcs.029.01.02

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