Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(4): 9736–9759.
DOI:10.3934/math.2026403
Received: 28 December 2025
Revised: 25 March 2026
Accepted: 01 April 2026
Published: 13 April 2026

Research article

Direct and inverse problems for a nonlocal heat equation under generalized-periodic boundary conditions

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Abstract: This paper studies direct and inverse source problems for a nonlocal heat equation with an involution in the spatial variable under generalized periodic-type boundary conditions. These conditions couple the traces of the solution and its derivatives at the endpoints of the interval and lead to a non-self-adjoint spatial operator, for which the classical separation-of-variables approach may fail due to the lack of the Riesz basis property of root functions. To overcome this difficulty, we develop a functional decomposition method that represents the solution as a specially chosen sum of two auxiliary components. This decomposition reduces the original nonlocal problem to the sequential solution of two classical heat equations with standard self-adjoint boundary conditions (a Dirichlet problem for the odd component and a Robin problem for the auxiliary even component). Using this reduction, we prove existence, uniqueness, and stability of strong Sobolev solutions for the direct initial-boundary value problem for all admissible real values of the model parameters, without imposing any arithmetic restrictions on the associated spectral ratio. We also investigate the inverse problem of recovering a stationary (space-dependent) source from a final-time observation and establish well-posedness and stability estimates for the reconstructed source and the corresponding solution.

Keywords: nonlocal heat equations; involution of the spatial variable; generalized boundary conditions of periodic type; inverse source recovery problem; Riesz basis; Fourier method
Mathematics Subject Classification: 35K05, 35K20, 35R10, 35R30

1. Introduction

Differential equations with argument transformation, and in particular equations with involution, form an important class of nonlocal mathematical models. The interest in them is due to the fact that they describe processes that violate the classical principle of locality, where the state of the system at a given point depends on its state at a symmetric or shifted point in space.

In the present paper, in the domain $\Omega = \{(x,t) : -1 < x < 1,\ 0 < t < T\}$, we consider a nonlocal heat equation:

$$u_t(x,t) - u_{xx}(x,t) + \varepsilon u_{xx}(-x,t) = f(x,t), \tag{1.1}$$

where ε is a real parameter, $|\varepsilon| < 1$. Equations of this type arise, for example, in the mathematical modeling of the heat propagation process in a closed metal wire wrapped around a permeable insulating material [1]. The term $\varepsilon u_{xx}(-x,t)$ describes the heat flux through the insulation, creating a nonlocal coupling between diametrically opposite points x and $-x$.

The parameter ε obeys condition $|\varepsilon| < 1$, so that term $\varepsilon u_{xx}(-x,t)$ in Eq (1.1) is, in a sense, subordinate to the principal part $u_t(x,t) - u_{xx}(x,t)$ of the equation. In the case where $|\varepsilon| \geq 1$, the study of the corresponding initial-boundary value problems must be conducted using a completely different methodology. The problems we investigate are model problems, and the coefficients of the equation are dimensionless, meaning they do not carry a direct physical meaning.

We consider Eq (1.1) with the classical condition

$$u(x,0) = \varphi(x), \qquad x \in [-1,1] \tag{1.2}$$

and the following boundary conditions of generalized-periodic type:

$$\begin{cases} U_1(u) \equiv u_x(-1,t) + a u_x(1,t) + b u(-1,t) = 0, \\ U_2(u) \equiv u(-1,t) - u(1,t) = 0, \end{cases} \qquad t \in [0,T]. \tag{1.3}$$

Here, $a, b \in \mathbb{R}$, $a \neq 1$.

The condition $U_2(u) = 0$ expresses the continuity of temperature at the connection point of the wire ends. The condition $U_1(u) = 0$ represents a generalized balance of fluxes. The introduction of the new term $bu(-1,t)$ (for $b \neq 0$) has a clear physical meaning: It models convective heat exchange or the presence of a local heat source/sink directly at the connection point (joint), whose intensity is proportional to the current temperature.

The study of boundary value problems for differential equations with nonlocal conditions is a relevant direction in modern mathematical physics. Fundamental contribution to the development of the theory of nonlocal problems for parabolic equations was made in the classical works of Ionkin [2, 3]. These works first considered problems of the Samarskii–Ionkin type. It was shown that the corresponding differential operators are, as a rule, non-self-adjoint, and the system of their eigenfunctions is not complete, which requires the introduction of associated functions. The spectral properties of such problems have been studied by many authors (see, for example, the works of Ilyin [4], his students [5], and followers). The solvability conditions in Sobolev spaces were studied by Ashyralyev and Sarsenbi [6] and Beilin [7].

In recent decades, interest in differential equations containing the involution $x \to -x$ has increased. In this paper, we consider one of the simplest cases of the involution $x \to -x$, as it is precisely this transformation that commutes with the second-order differentiation operator. This allows us to employ symmetry methods in our study. The general theory of such equations is presented in the monograph [1]. The spectral properties of differential operators with involution differ significantly from classical Sturm–Liouville operators. The study of their spectral structure is the subject of the works [8–12]. Equations with involution also find application in hyperbolic problems [13].

A separate important class of problems is inverse source recovery problems. Fundamental results are systematized in the monographs [14, 15]. The problem of recovering the source $f(x)$ with respect to the final overdetermination $u(x, T) = \psi(x)$ is classical [17, 18]. Stability issues were studied in [16]. The specifics of inverse problems for equations with nonlocal conditions were considered in [19, 20].

However, as shown in our recent work [21], the applicability of the classical spectral method to problem (1.1) with conditions (1.3) (for $b = 0$) encounters a problem of basis property: A Riesz basis exists if and only if the parameter $r = \sqrt{(1-\varepsilon)/(1+\varepsilon)}$ is rational. The introduction of the parameter $b \neq 0$ makes the spectral problem even more complicated (transcendental).

The goal of the present work is to develop the *method of functional decomposition*, which allows one to circumvent the problem of Riesz basis property and to construct a well-posed theory of solvability for the direct and inverse problems for conditions (1.3) for any real values of the parameters ε, $|\varepsilon| < 1$, a, and b, without resorting to the study of the arithmetic properties of the number r.

The overall structure of the work is as follows. The paper consists of seven sections, including the Introduction (Section 1) and the Conclusion (Section 7). Section 2 demonstrates the impossibility of directly applying the method of separation of variables to solve the problem, as the system of root functions of the resulting spectral problem may not form an unconditional basis (depending on whether the parameter r is rational). In this regard, Section 3 provides a justification for the method of functional decomposition, the application of which reduces the original problem to the sequential solution of two problems with self-adjoint boundary conditions (with respect to the spatial variable). In Section 4, this method of functional decomposition is employed to prove the well-posedness of the direct problem under study. Section 5 applies the same method to investigate the inverse problem of recovering an external source. Section 6 is devoted to demonstrating the application of the method of functional decomposition through a specific example: constructing a solution for a particular direct initial-boundary value problem. Finally, Section 7 provides a brief summary of the results obtained throughout the paper.

2. Limitation of possibilities of the spectral method

An attempt to solve problem (1.1)–(1.3) using the Fourier method leads to a spectral problem for the operator

$$\mathcal{L}y \equiv -y'' + \varepsilon y''(-x) = \lambda y(x), \tag{2.1}$$

with boundary conditions

$$\begin{cases} U_1(y) \equiv y'(-1) + ay'(1) + by(-1) = 0, \\ U_2(y) \equiv y(-1) - y(1) = 0. \end{cases} \tag{2.2}$$

For $b \neq 0$, the characteristic equation for the eigenvalues λ splits into two conditions:

- Odd series: $\sin(\mu_2) = 0$, where $\mu_2 = \sqrt{\lambda/(1+\varepsilon)}$.
- Even series: $\tan(\mu_1) = \frac{b}{a-1}\frac{1}{\mu_1}$, where $\mu_1 = \sqrt{\lambda/(1-\varepsilon)}$.

The multiplicity of the spectrum arises when λ from both series coincide, which leads to the equation

$$\tan(rn\pi) = \frac{b}{(a-1)rn\pi}.$$

For such multiple eigenvalues, associated functions arise. The questions of the basis property of the system of root vectors with an infinite number of associated functions are currently unstudied in this case.

In the paper [22], a spectral problem of the form (2.2) was considered in the case $b = 0$. It was proved that if $r = \sqrt{(1-\varepsilon)/(1+\varepsilon)}$ is irrational, then the system of its eigenfunctions is complete and minimal in $L_p(-1,1)$ for any $p > 1$ but does not form a basis. In the case of a rational value of r, the way of supplying this system with associated functions is specified to make all the root functions a basis in $L_p(-1,1)$.

In the case when the number of multiple eigenvalues is finite, the problem of small denominators arises, also requiring additional investigation. Thus, the analysis of the basis property of the system of root functions of problems (2.1) and (2.2) is an open problem for both rational r and irrational r. This makes the classical Fourier method currently inapplicable for solving problems (1.1)–(1.3) in the general case.

3. The method of functional decomposition

To overcome these difficulties, we use the decomposition of the solution into a sum of two functions:

$$u(x,t) = v(x,t) + \alpha(x)s(x,t), \quad \text{where } \alpha(x) = kx + 1, \quad k = \frac{1+a}{1-a}. \tag{3.1}$$

Here, $s(x,t)$ is an odd part of the solution

$$s(x,t) = \frac{u(x,t) - u(-x,t)}{2}, \tag{3.2}$$

and $v(x,t)$ is a new auxiliary function.

Lemma 1. *The initial-boundary value problems* (1.1)–(1.3) *are equivalent to the sequential solution of two classical problems:*

- **Problem** D_s (Independent): The classical Dirichlet problem for an odd component:

$$\begin{cases} s_t - (1+\varepsilon)s_{xx} = f_s(x,t), \\ s(x,0) = \varphi_s(x), \\ s(-1,t) = 0, \quad s(1,t) = 0. \end{cases} \tag{3.3}$$

- **Problem** R_v (Sequential): The problem with homogeneous Robin boundary conditions for the function $v(x,t)$:

$$\begin{cases} v_t - (1-\varepsilon)v_{xx} = f_v(x,t), \\ v(x,0) = \varphi_v(x), \\ (1-a)v_x(-1,t) + bv(-1,t) = 0, \\ (a-1)v_x(1,t) + bv(1,t) = 0, \end{cases} \tag{3.4}$$

  where $f_s(x,t), \varphi_s(x)$ are odd parts of the initial data, and $f_v(x,t)$ and $\varphi_v(x)$ are known functions defined from $f(x,t)$, $\varphi(x)$ and the solution of problem D_s.

The following notations are used here:

$$\begin{cases} f_s(x,t) = \frac{1}{2}\left(f(x,t) - f(-x,t)\right), \\ f_v(x,t) = f(x,t) - \alpha(x) f_s(x,t) + 2k(1-\varepsilon)s_x(x,t) - 2\varepsilon k x s_{xx}(x,t), \end{cases} \tag{3.5}$$

$$\begin{cases} \varphi_s(x) = \frac{1}{2}(\varphi(x) - \varphi(-x)), \\ \varphi_v(x) = \varphi(x) - \alpha(x)\varphi_s(x). \end{cases} \tag{3.6}$$

Proof. Let us first show that although the function $v(x,t)$ is not the even part of the solution $u(x,t)$, it is an even function.

From definition (3.1), we express $v(x,t)$:

$$v(x,t) = u(x,t) - (kx+1)s(x,t).$$

We check the parity of the function $v(x,t)$.

$$v(-x,t) = u(-x,t) - (-kx+1)s(-x,t).$$

Using the identity $u(-x,t) = u(x,t) - 2s(x,t)$ and the odd property of $s(-x,t) = -s(x,t)$, we get

$$v(-x,t) = (u(x,t) - 2s(x,t)) - (-kx+1)(-s(x,t)) = u(x,t) - (kx+1)s(x,t) = v(x,t).$$

Thus, the function $v(x,t)$ is even with respect to the variable x.

Let us find boundary conditions for the introduced new functions. From the periodicity condition $U_2(u) = 0$, it follows

$$[v(-1,t) + \alpha(-1)s(-1,t)] - [v(1,t) + \alpha(1)s(1,t)] = 0.$$

Due to the even property $v(-1,t) = v(1,t)$ proven above, this condition is reduced to $s(-1,t)\alpha(-1) - s(1,t)\alpha(1) = 0$. Since $s(x,t)$ is the odd part of $u(x,t)$, then

$$s(1,t) = -s(-1,t) = 0. \tag{3.7}$$

Now, from the condition $U_1(u) = 0$ by direct check, we easily get that $v(x,t)$ satisfies the boundary conditions from (3.4).

Taking into account that $s(x,t)$ is the odd part of $u(x,t)$, by direct check, it is easy to verify that $s_t = (1+\varepsilon)s_{xx} + f_s$.

The formulas for f_v and φ_v are obtained by directly substituting $u = v + \alpha s$ into the original equation and the initial condition.

The lemma is proven. □

4. Solution of the direct problem

Theorem 1. *Let* $|\varepsilon| < 1$, $a \neq 1$, $b \in \mathbb{R}$. *Let* $f \in L_2(\Omega)$ *and* $\varphi \in W_2^2(-1,1)$ *and satisfy the boundary conditions* (1.3) *for* $t = 0$. *Then there exists a unique strong generalized solution* $u(x,t) \in W_2^{2,1}(\Omega)$ *of problems* (1.1)–(1.3). *The sustainability estimate is true:*

$$\|u\|_{W_2^{2,1}(\Omega)} \leq C\left(\|f\|_{L_2(\Omega)} + \|\varphi\|_{W_2^2(-1,1)}\right). \tag{4.1}$$

Remark 1. *This statement is true for any real value $r = \sqrt{(1-\varepsilon)/(1+\varepsilon)}$, in contrast to the results of the work [21], where the well-posedness of problems* (1.1)–(1.3) *for $b = 0$ was considered only in the case of a rational parameter r.*

Proof. The proof is based on Lemma 1 on the reduction of problems (1.1)–(1.3) to the sequential solution of two problems: $\mathbf{D_s}$ and $\mathbf{R_v}$. Although these problems are problems with self-adjoint boundary conditions (which are sufficiently well studied), in our case, there are some details concerning the oddness or evenness of their solutions, respectively, as well as the issue of the inclusion of derivatives of the solution of problem D_s in the right-hand side of the equation for problem R_v. Therefore (despite the apparent simplicity), we provide a detailed proof here.

We first consider problem D_s.

Lemma 2. *Let $|\varepsilon| < 1$. Let $f_s \in L_2(\Omega)$ and $\varphi_s \in W_2^1(-1, 1)$ and satisfy the boundary conditions*

$$\varphi_s(-1) = 0, \ \ \varphi_s(1) = 0. \tag{4.2}$$

Then, there exists a unique strong generalized solution $s(x, t) \in W_2^{2,1}(\Omega)$ of the problem D_s. The stability estimate is true:

$$\|s\|_{W_2^{2,1}(\Omega)} \le C\left(\|f_s\|_{L_2(\Omega)} + \|\varphi_s\|_{W_2^1(-1,1)}\right). \tag{4.3}$$

To prove the lemma, following the classical method, we look for a solution in the form of a product $s(x, t) = X(x)T(t)$. The separation of variables leads to the Sturm–Liouville spectral problem for the spatial part:

$$\begin{cases} -X''(x) = \mu X(x), & x \in (-1, 1), \\ X(-1) = 0, & X(1) = 0. \end{cases}$$

Substitute the general solution of the equation

$$X(x) = C_1 \cos(\sqrt{\mu}x) + C_2 \sin(\sqrt{\mu}x)$$

into the boundary conditions. We obtain the system

$$\begin{cases} 2C_1 \cos(\sqrt{\mu}) = 0, \\ 2C_2 \sin(\sqrt{\mu}) = 0. \end{cases}$$

Since we are looking for an odd component of the solution $s(x, t)$, we are only interested in odd eigenfunctions, that is, the case $C_1 = 0, C_2 \ne 0$. This leads to a condition for eigenvalues: $\sin(\sqrt{\mu}) = 0$. Hence, $\sqrt{\mu_k} = k\pi, \ k = 1, 2, \ldots$.

Thus, we obtain a set of eigenvalues and normalized eigenfunctions:

$$\mu_k = (k\pi)^2, \quad X_k(x) = \sin(k\pi x), \quad k \in \mathbb{N},$$

$$\|X_k\|^2 = \int_{-1}^{1} \sin^2(k\pi x)dx = 1.$$

The system of functions $\{X_k(x)\}_{k=1}^{\infty} = \{\sin(k\pi x)\}_{k=1}^{\infty}$ forms an orthonormal basis in the space of odd functions from $L_2(-1, 1)$.

We look for the solution of non-homogeneous problem D_s (3.3) in the form of a series with respect to the eigenfunctions

$$s(x,t) = \sum_{k=1}^{\infty} s_k(t)\sin(k\pi x). \tag{4.4}$$

We also expand the right-hand side of the equation $f_s(x,t)$ and the initial function $\varphi_s(x)$ into a Fourier series:

$$f_s(x,t) = \sum_{k=1}^{\infty} f_{s,k}(t)\sin(k\pi x), \quad \text{where } f_{s,k}(t) = \int_{-1}^{1} f_s(x,t)\sin(k\pi x)dx,$$

$$\varphi_s(x) = \sum_{k=1}^{\infty} \varphi_{s,k}\sin(k\pi x), \quad \text{where } \varphi_{s,k} = \int_{-1}^{1} \varphi_s(x)\sin(k\pi x)dx.$$

Substitute series (4.4) in the differential equation from (3.3). Taking into account that $X_k''(x) = -(k\pi)^2 X_k(x)$, we get

$$\sum_{k=1}^{\infty}\left[\frac{ds_k(t)}{dt} + (1+\varepsilon)(k\pi)^2 s_k(t)\right]\sin(k\pi x) = \sum_{k=1}^{\infty} f_{s,k}(t)\sin(k\pi x).$$

Due to the orthogonality of the basis, we can equate the coefficients for the same harmonics. Denote the eigenvalues of the original operator (taking into account the thermal diffusivity coefficient) as

$$\lambda_k^{(s)} = (1+\varepsilon)(k\pi)^2.$$

We obtain a countable system of ordinary differential equations for $s_k(t)$:

$$\frac{ds_k(t)}{dt} + \lambda_k^{(s)} s_k(t) = f_{s,k}(t), \quad t > 0. \tag{4.5}$$

The initial condition for $s_k(t)$ is obtained from the expansion $\varphi_s(x)$:

$$s_k(0) = \varphi_{s,k}. \tag{4.6}$$

Thus, the sought Fourier coefficients are solutions of the Cauchy problems (4.5) and (4.6), can be constructed in explicit form, and have the form

$$s_k(t) = \varphi_{s,k}e^{-\lambda_k^{(s)}t} + \int_0^t e^{-\lambda_k^{(s)}(t-\tau)} f_{s,k}(\tau)d\tau. \tag{4.7}$$

Substituting the found coefficients (4.7) into series (4.4), we obtain a formal solution to problem D_s:

$$s(x,t) = \sum_{k=1}^{\infty}\left[\varphi_{s,k}e^{-(1+\varepsilon)(k\pi)^2 t} + \int_0^t e^{-(1+\varepsilon)(k\pi)^2(t-\tau)} f_{s,k}(\tau)d\tau\right]\sin(k\pi x). \tag{4.8}$$

To complete the study of problem D_s, it is necessary to justify the convergence of the obtained formal solution. Since $|\varepsilon| < 1$, $1+\varepsilon > 0$. The exponential factors $e^{-\lambda_k^{(s)}t}$ ensure the rapid convergence of the series and its derivatives for $t > 0$, which guarantees that the solution belongs to the class C^{∞} inside the domain for smooth f_s.

For the solution to belong to the Sobolev space $W_2^{2,1}(\Omega)$, it is sufficient that $\varphi_s \in W_2^1(-1,1)$ and $f_s \in L_2(\Omega)$.

Indeed, the norm in the Sobolev space $W_2^{2,1}(\Omega)$ is defined as the sum of the L_2-norms of the function itself and its derivatives (the first with respect to t and the second with respect to x):

$$\|s\|^2_{W_2^{2,1}} = \|s_t\|^2_{L_2(\Omega)} + \|s_{xx}\|^2_{L_2(\Omega)} + \|s\|^2_{L_2(\Omega)}.$$

Using Parseval's identity for the orthonormal system $\{\sin(k\pi x)\}_{k=1}^{\infty}$, we can rewrite these norms in terms of the Fourier coefficients $s_k(t)$.

Since $\frac{d^2}{dx^2}\sin(k\pi x) = -(k\pi)^2\sin(k\pi x)$, the action of the derivative ∂_{xx} in the coefficient space is equivalent to multiplication by $-(k\pi)^2$. Consequently, the condition $s \in W_2^{2,1}(\Omega)$ is equivalent to the convergence of the series

$$\sum_{k=1}^{\infty}\int_0^T \left(|\dot{s}_k(t)|^2 + k^4|s_k(t)|^2\right)dt < \infty. \tag{4.9}$$

The coefficients $s_k(t)$ satisfy to the differential equation

$$\dot{s}_k(t) + \lambda_k^{(s)} s_k(t) = f_{s,k}(t),$$

where $\lambda_k^{(s)} = (1+\varepsilon)(k\pi)^2 \sim k^2$. Squaring both sides of this equation yields:

$$(\dot{s}_k)^2 + 2\lambda_k^{(s)} s_k\dot{s}_k + (\lambda_k^{(s)})^2 s_k^2 = f_{s,k}^2.$$

Integrate the obtained equality with respect to time from 0 to T:

$$\int_0^T \dot{s}_k^2(t)dt + \int_0^T 2\lambda_k^{(s)} s_k(t)\dot{s}_k(t)dt + \int_0^T (\lambda_k^{(s)})^2 s_k^2(t)dt = \int_0^T f_{s,k}^2(t)dt.$$

For the middle term in this equality, we have

$$\int_0^T 2\lambda_k^{(s)} s_k\dot{s}_k dt = \lambda_k^{(s)}\int_0^T \frac{d}{dt}(s_k^2)dt = \lambda_k^{(s)}[s_k^2(T) - s_k^2(0)].$$

Substitute this back into the integral identity:

$$\int_0^T \dot{s}_k^2 dt + \lambda_k^{(s)} s_k^2(T) + \int_0^T (\lambda_k^{(s)})^2 s_k^2 dt = \int_0^T f_{s,k}^2 dt + \lambda_k^{(s)} s_k^2(0).$$

Since $\lambda_k^{(s)} > 0$ and $s_k^2(T) \geq 0$, we can omit the term $\lambda_k^{(s)} s_k^2(T)$ on the left side, strengthening the inequality (moving from an equality to an inequality):

$$\int_0^T \dot{s}_k^2(t)dt + \int_0^T (\lambda_k^{(s)})^2 s_k^2(t)dt \leq \int_0^T |f_{s,k}(t)|^2 dt + \lambda_k^{(s)}|s_k(0)|^2.$$

Sum the resulting inequality over all k from 1 to ∞:

$$\sum_{k=1}^{\infty}\left(\int_0^T |\dot{s}_k|^2 dt + \int_0^T (\lambda_k^{(s)})^2|s_k|^2 dt\right) \leq \sum_{k=1}^{\infty}\int_0^T |f_{s,k}|^2 dt + \sum_{k=1}^{\infty}\lambda_k^{(s)}|\varphi_{s,k}|^2.$$

Combining all of the above, we obtain the final a priori estimate (4.3). Lemma 2 is proven.

Lemma 3. *Let the conditions of Lemma 2 be satisfied. Additionally, let $f \in L_2(\Omega)$ and $\varphi \in W_2^2(-1,1)$. Then the functions $f_v(x,t)$ and $\varphi_v(x)$, defined by formulas* (3.5) *and* (3.6)*, satisfy the conditions*

$$f_v \in L_2(\Omega), \quad \varphi_v \in W_2^2(-1,1).$$

The following estimates hold:

$$\|f_v\|_{L_2(\Omega)} \le C_1 \left(\|f\|_{L_2(\Omega)} + \|\varphi\|_{W_2^1(-1,1)}\right), \tag{4.10}$$

$$\|\varphi_v\|_{W_2^2(-1,1)} \le C_2 \|\varphi\|_{W_2^2(-1,1)}. \tag{4.11}$$

Proof. Let us first estimate the initial function $\varphi_v(x)$ given by formula (3.6). Since $\varphi(x) \in W_2^2(-1,1)$, its odd part $\varphi_s(x)$ also belongs to $W_2^2(-1,1)$ and $\|\varphi_s\| \le \|\varphi\|$. The product of a smooth function $\alpha(x)$ and a function from a Sobolev space preserves the smoothness class. Consequently, $\varphi_v \in W_2^2(-1,1)$ and the estimate (4.11) hold.

To estimate the right-hand side $f_v(x,t)$, we apply the triangle inequality for the $L_2(\Omega)$ norm to expression (3.5):

$$\|f_v\| \le \|f\| + \max_x |\alpha(x)| \|f_s\| + |2k(1-\varepsilon)| \|s_x\| + |2\varepsilon k| \max_x |x| \|s_{xx}\|.$$

Since $x \in [-1,1]$, $\max|x| = 1$ and $\max|\alpha(x)| \le |k| + 1$. The inequality for the odd part $\|f_s\|_{L_2} \le \|f\|_{L_2}$ is also evident.

To estimate the norms of the derivatives $\|s_x\|$ and $\|s_{xx}\|$, we use the result of Lemma 2. Since $s \in W_2^{2,1}(\Omega)$, then by the definition of the norm of this space

$$\|s\|^2_{W_2^{2,1}} = \|s\|^2_{L_2} + \|s_t\|^2_{L_2} + \|s_{xx}\|^2_{L_2}.$$

It follows from this that $\|s_{xx}\|_{L_2} \le \|s\|_{W_2^{2,1}}$.

Additionally, from interpolation inequalities for Sobolev spaces, it follows that the norm of the first derivative is dominated by the norm of the second: $\|s_x\|_{L_2} \le C\|s\|_{W_2^{2,1}}$.

Using the stability estimate (4.3) from Lemma 2, we obtain

$$\|s\|_{W_2^{2,1}} \le C(\|f_s\| + \|\varphi_s\|_{W_2^1}) \le C(\|f\| + \|\varphi\|_{W_2^1}).$$

Substituting this into the inequality for $\|f_v\|$, we get

$$\|f_v\| \le \|f\| + C_3\|f\| + C_4(\|f\| + \|\varphi\|_{W_2^1}).$$

Grouping the terms, we arrive at the final estimate (4.10). The lemma is proven. □

Let us pass to the consideration of problem R_v for the even component.

Lemma 4. *Let the conditions of Lemma 3 be satisfied. Furthermore, let $|\varepsilon| < 1$, $a \neq 1$, $b \in \mathbb{R}$, and let the data–matching condition for the function $v(x,t)$ be fulfilled:*

$$(a-1)\varphi_v'(1) + b\varphi_v(1) = 0. \tag{4.12}$$

Then, there exists a unique strong generalized solution $v(x,t) \in W_2^{2,1}(\Omega)$ to problem R_v. The following estimate holds:

$$\|v\|_{W_2^{2,1}(\Omega)} \le C\left(\|f_v\|_{L_2(\Omega)} + \|\varphi_v\|_{W_2^2(-1,1)}\right). \tag{4.13}$$

Proof. Problem R_v is a parabolic problem with homogeneous Robin-type boundary conditions. Applying the Fourier method leads to an auxiliary spectral problem on the interval $(-1,1)$ for even functions $Y(x)$:

$$\begin{cases} -Y''(x) = \nu Y(x), \quad x \in (-1,1), \\ Y(-x) = Y(x), \\ (a-1)Y'(1) + bY(1) = 0. \end{cases} \tag{4.14}$$

Note that the condition at the left boundary $(1-a)Y'(-1) + bY(-1) = 0$ is satisfied automatically for even functions. The general even solution to the equation from (4.14) takes the form $Y(x) = \cos(\sqrt{\nu}x)$. Substituting this into the boundary condition at $x = 1$ yields a transcendental equation for the eigenvalues $\mu = \sqrt{\nu}$:

$$\tan\mu = \frac{b}{a-1}\frac{1}{\mu}.$$

Let $\{\mu_n\}_{n=0}^{\infty}$ be the positive roots of this equation. As follows from Rouché's theorem, they possess the following asymptotics for large values of n:

$$\mu_n = n\pi + O\left(\frac{1}{n}\right).$$

The eigenfunctions $Y_n(x) = \cos(\mu_n x)$ form an orthogonal basis in the subspace of even functions in $L_2(-1,1)$. We denote the squared norm as $\|Y_n\|^2$.We seek a solution in the form of a series:

$$v(x,t) = \sum_{n=0}^{\infty} v_n(t)Y_n(x).$$

We also expand the right-hand side $f_v(x,t)$ and the initial condition $\varphi_v(x)$:

$$f_v(x,t) = \sum_{n=0}^{\infty} f_{v,n}(t)Y_n(x), \quad \varphi_v(x) = \sum_{n=0}^{\infty} \varphi_{v,n}Y_n(x).$$

Substituting these series into problem (4.14), we obtain the Cauchy problem for the coefficients $v_n(t)$:

$$\frac{dv_n}{dt} + \lambda_n^{(v)} v_n(t) = f_{v,n}(t), \quad v_n(0) = \varphi_{v,n}, \tag{4.15}$$

where $\lambda_n^{(v)} = (1-\varepsilon)\mu_n^2$.

The solution of this Cauchy problem has the form

$$v_n(t) = \varphi_{v,n}e^{-\lambda_n^{(v)}t} + \int_0^t e^{-\lambda_n^{(v)}(t-\tau)} f_{v,n}(\tau)d\tau.$$

Thus, we have constructed the formal solution to problem (4.14).

To complete the proof of the lemma, we must justify the convergence of all the series involved. Since $|\varepsilon| < 1$, $\lambda_n^{(v)} > 0$ ($\lambda_n^{(v)} \sim n^2$ at $n \to \infty$).

The estimate of the solution's norm in $W_2^{2,1}(\Omega)$ via Fourier coefficients is analogous to the estimate in Lemma 2. The square of the norm $\|v\|^2_{W_2^{2,1}}$ is equivalent to the convergence of the series:

$$\sum_{n=0}^{\infty} \int_0^T \left(|\dot{v}_n|^2 + (\lambda_n^{(v)})^2 |v_n|^2\right) dt.$$

Using the energy inequality for the ordinary differential equation (4.15) (see the proof of Lemma 2), we obtain

$$\|v\|^2_{W_2^{2,1}} \le C\left(\sum_{n=0}^{\infty} \int_0^T |f_{v,n}(t)|^2 dt + \sum_{n=0}^{\infty} \lambda_n^{(v)} |\varphi_{v,n}|^2\right).$$

The first term on the right is $\|f_v\|^2_{L_2(\Omega)}$, which is finite according to the conditions of Lemma 3.

The second term represents the square of the norm of the initial function $\|\varphi_v\|^2_{W_2^1}$.

It should be noted here that to satisfy the boundary condition (4.12) (ensuring the series converges to a function that satisfies the derivative condition), a higher degree of smoothness $\varphi_v \in W_2^2$ is required, which is also guaranteed by Lemma 3 (φ_v inherits the smoothness of φ).

Thus, the series converges, the function $v(x, t)$ belongs to the required class, and the estimate (4.13) holds. The lemma is proven. □

Lemma 5. *If the original initial function $\varphi(x) \in W_2^2(-1, 1)$ satisfies the boundary conditions* (1.3)*, then:*

1) The function $\varphi_s(x)$ satisfies the Dirichlet conditions $\varphi_s(1) = \varphi_s(-1) = 0$.
2) The function $\varphi_v(x)$ satisfies the matching conditions of problem R_v:

$$(a - 1)\varphi_v'(1) + b\varphi_v(1) = 0.$$

Proof. From the boundary condition $U_2(\varphi) = \varphi(-1) - \varphi(1) = 0$, it follows that $\varphi(-1) = \varphi(1)$. Then

$$\varphi_s(1) = \frac{\varphi(1) - \varphi(-1)}{2} = 0.$$

Due to the oddness, $\varphi_s(-1) = -\varphi_s(1) = 0$.

We check the condition for φ_v. Through standard transformations, it is easy to verify that

$$(a - 1)\varphi_v'(1) + b\varphi_v(1) = a\varphi'(1) + \varphi'(-1) + b\varphi(1) = U_1(\varphi) = 0.$$

The lemma is proven. □

Let us now proceed to the completion of the proof of Theorem 1.

According to Lemma 1, the solution of the original problem is represented as $u(x, t) = v(x, t) + \alpha(x)s(x, t)$. Since the function $\alpha(x) = kx + 1$ is infinitely differentiable and bounded on the interval $[-1, 1]$, then multiplying by it is a bounded operator in the space $W_2^{2,1}(\Omega)$. Consequently,

$$\|u\|_{W_2^{2,1}(\Omega)} \le \|v\|_{W_2^{2,1}(\Omega)} + \|\alpha \cdot s\|_{W_2^{2,1}(\Omega)} \le \|v\|_{W_2^{2,1}(\Omega)} + C_\alpha \|s\|_{W_2^{2,1}(\Omega)}.$$

Let us use the estimates for the solutions of the auxiliary problems obtained in Lemmas 2 and 4:

$$\|u\|_{W_2^{2,1}(\Omega)} \le C_v \left(\|f_v\|_{L_2(\Omega)} + \|\varphi_v\|_{W_2^2(-1,1)}\right) + C_s \left(\|f_s\|_{L_2(\Omega)} + \|\varphi_s\|_{W_2^1(-1,1)}\right).$$

Now, we apply Lemma 3, which expresses the norms of the auxiliary functions through the initial data f and φ:

$$\|f_v\|_{L_2(\Omega)} \le C_1 \left(\|f\|_{L_2(\Omega)} + \|\varphi\|_{W_2^1(-1,1)}\right), \quad \|\varphi_v\|_{W_2^2(-1,1)} \le C_2\|\varphi\|_{W_2^2(-1,1)}.$$

Also, it clearly follows from the definition of the odd part that

$$\|f_s\|_{L_2(\Omega)} \le \|f\|_{L_2(\Omega)}, \quad \|\varphi_s\|_{W_2^1(-1,1)} \le \|\varphi\|_{W_2^1(-1,1)}.$$

Combining all the inequalities and taking into account the embedding $W_2^2(-1,1) \subset W_2^1(-1,1)$, we arrive at the final a priori estimate:

$$\|u\|_{W_2^{2,1}(\Omega)} \le C \left(\|f\|_{L_2(\Omega)} + \|\varphi\|_{W_2^2(-1,1)}\right).$$

The existence of the solution follows from the constructive writing of the functions $s(x,t)$ and $v(x,t)$. Furthermore, Lemma 5 guarantees that the matching conditions for the initial functions φ_s and φ_v are satisfied, which is a necessary condition for the solutions to belong to the class $W_2^{2,1}(\Omega)$. The uniqueness of the solution follows from the linearity of the problem and the obtained stability estimate. Theorem 1 is proved. □

5. Investigation of the inverse problem

In many heat-transfer experiments, the distributed source term is not known a priori and has to be identified from additional measurements. For instance, the term $f(x)$ may represent spatially distributed heat generation/absorption caused by imperfect insulation, contact resistance at the joint, or external heating, while the temperature evolution $u(x,t)$ can be measured. A standard way to reconstruct such an unknown source is to use an additional final observation $u(x,T) = \psi(x)$ (final overdetermination), which is typical in inverse source problems for parabolic equations. Therefore, together with the direct problem, it is natural to study the inverse problem of recovering a stationary source $f = f(x)$ and to establish existence, uniqueness and stability of the reconstruction. Moreover, the functional decomposition method developed in Sections 3 and 4 applies to the inverse setting as well, reducing the original nonlocal inverse problem to two classical inverse problems with Dirichlet and Robin boundary conditions, respectively.

In this section, we consider the problem of identifying a stationary (depending only on the spatial variable) external source $f(x)$.

In the domain $\Omega = \{(x,t) : -1 < x < 1,\ 0 < t < T\}$, it is required to find the source function $f(x)$ in the nonlocal equation

$$u_t(x,t) - u_{xx}(x,t) + \varepsilon u_{xx}(-x,t) = f(x) \tag{5.1}$$

and a solution $u(x,t)$ satisfying the initial condition

$$u(x,0) = \varphi(x), \qquad x \in [-1,1], \tag{5.2}$$

the final overdetermination condition

$$u(x,T) = \psi(x), \qquad x \in [-1,1], \tag{5.3}$$

and the boundary conditions

$$\begin{cases} U_1(u) \equiv u_x(-1,t) + au_x(1,t) + bu(-1,t) = 0, \\ U_2(u) \equiv u(-1,t) - u(1,t) = 0, \end{cases} \qquad t \in [0,T]. \tag{5.4}$$

Here, $\varphi(x)$ and $\psi(x)$ are given sufficiently smooth functions consistent with the boundary conditions; ε, a, b are real parameters, $|\varepsilon| < 1, a \neq 1$.

The solution is sought in the class $u \in W_2^{2,1}(\Omega)$, $f \in L_2(-1,1)$.

By analogy with the method for solving the direct problem, we represent the solution of the inverse problems (5.1)–(5.4) as a sum of two functions:

$$u(x,t) = v(x,t) + \alpha(x)s(x,t), \quad \text{where } \alpha(x) = kx + 1, \quad k = \frac{1+a}{1-a}. \tag{5.5}$$

Here, $s(x,t)$ is the odd part of the solution

$$s(x,t) = \frac{u(x,t) - u(-x,t)}{2}, \tag{5.6}$$

and $v(x,t)$ is a new auxiliary function. We also represent the sought source as a sum of even and odd components: $f(x) = f_c(x) + f_s(x)$.

Lemma 6. *The inverse problems* (5.1)–(5.4) *are equivalent to the sequential solution of two inverse problems with classical boundary conditions:*

- **Problem** D_s^{-1} (Independent): The inverse problem with Dirichlet boundary conditions for the odd component. Find a pair of odd functions $\{s(x,t), f_s(x)\}$,

  $$s(-x,t) = -s(x,t), \quad f_s(-x) = -f_s(x),$$

  which are the solution to the inverse problem:

  $$\begin{cases} s_t(x,t) - (1+\varepsilon)s_{xx}(x,t) = f_s(x), \\ s(x,0) = \varphi_s(x), \quad s(x,T) = \psi_s(x), \\ s(-1,t) = 0, \quad s(1,t) = 0, \end{cases} \tag{5.7}$$

  where $\varphi_s(x), \psi_s(x)$ are the odd parts of the initial data

  $$\varphi_s(x) = \frac{\varphi(x) - \varphi(-x)}{2}, \quad \psi_s(x) = \frac{\psi(x) - \psi(-x)}{2}. \tag{5.8}$$

- **Problem** R_v^{-1} (Sequential): The inverse problem with homogeneous Robin conditions. Find a pair of functions $\{v(x,t), f_c(x)\}$, where $v(x,t)$ is an even function and $f_c(x)$ is the even part of the sought source, which are the solution to the inverse problem:

  $$\begin{cases} v_t - (1-\varepsilon)v_{xx} = f_c(x) + H(x,t), \\ v(x,0) = \varphi_v(x), \quad v(x,T) = \psi_v(x), \\ (1-a)v_x(-1,t) + bv(-1,t) = 0, \\ (a-1)v_x(1,t) + bv(1,t) = 0, \end{cases} \tag{5.9}$$

where the known functions are defined by the formulas

$$\begin{cases} \varphi_v(x) = \varphi(x) - \alpha(x)\varphi_s(x), \\ \psi_v(x) = \psi(x) - \alpha(x)\psi_s(x), \\ H(x,t) = -kxf_s(x) + 2k(1-\varepsilon)s_x(x,t) - 2\varepsilon kxs_{xx}(x,t). \end{cases} \tag{5.10}$$

Proof. Let us first prove the necessity. Let the pair $\{u(x,t), f(x)\}$ be a solution to the original problem. We define $s(x,t)$ and $f_s(x)$ as the odd parts of u and f. Substitution into Eq (5.1) and boundary conditions (5.4) (analogously to the proof of Lemma 1) shows that they satisfy problem D_s^{-1}.

We define $v(x,t) = u(x,t) - \alpha(x)s(x,t)$ and $f_c(x) = f(x) - f_s(x)$.

According to the proof of Lemma 1, the function $v(x,t)$ satisfies the homogeneous boundary conditions of problem R_v^{-1}.

The fulfillment of the differential equation, the initial condition, and the final condition from (5.9) is verified by direct substitution.

Let us prove the sufficiency. Suppose the solutions to the auxiliary problems have been found. We construct $u = v + \alpha s$ and $f = f_c + f_s$. By virtue of the invertibility of all the algebraic transformations performed, the pair $\{u, f\}$ satisfies the equation and all conditions of the original problem.

The lemma is proved. □

Let us consider these auxiliary inverse problems sequentially.

5.1. Problem D_s^{-1}: Recovery of the odd component of the source

For the odd part $s(x,t)$, problem (5.7) is formulated independently. We expand the sought function $f_s(x)$ and the solution $s(x,t)$ into the Fourier series using the orthonormal eigenfunctions $\{X_k(x)\}_{k=1}^{\infty} = \{\sin(k\pi x)\}_{k=1}^{\infty}$, which form an orthonormal basis in the space of odd functions in $L_2(-1,1)$:

$$s(x,t) = \sum_{k=1}^{\infty} s_k(t)\sin(k\pi x), \quad f_s(x) = \sum_{k=1}^{\infty} f_{s,k}\sin(k\pi x). \tag{5.11}$$

The coefficients $f_{s,k}$ are unknown constants. From the equation, we obtain the Cauchy problem for $s_k(t)$:

$$\dot{s}_k(t) + \lambda_k^{(s)} s_k(t) = f_{s,k}, \quad s_k(0) = \varphi_{s,k}.$$

The solution to this problem is given by formula (4.7).

From this, taking into account that $f_{s,k}$ are constants, we find

$$s_k(t) = \varphi_{s,k}e^{-\lambda_k^{(s)}t} + f_{s,k}\frac{1 - e^{-\lambda_k^{(s)}t}}{\lambda_k^{(s)}}. \tag{5.12}$$

Therefore, for $t = T$, we obtain the following relation:

$$s_k(T) = \varphi_{s,k}e^{-\lambda_k^{(s)}T} + f_{s,k}\frac{1 - e^{-\lambda_k^{(s)}T}}{\lambda_k^{(s)}}.$$

Using the condition $s_k(T) = \psi_{s,k}$, we uniquely determine coefficients of the sought source:

$$f_{s,k} = \frac{\lambda_k^{(s)}}{1 - e^{-\lambda_k^{(s)}T}} \left(\psi_{s,k} - \varphi_{s,k} e^{-\lambda_k^{(s)}T}\right). \tag{5.13}$$

Substituting the values found from (5.12) and (5.13) into series (5.11), we get a formal solution to the inverse problem D_s^{-1}. To complete the investigation, it is necessary to justify the convergence of the obtained series.

Lemma 7. *Let* $|\varepsilon| < 1$, $\varphi_s(x), \psi_s(x) \in W_2^2(-1, 1)$ *and satisfy the Dirichlet conditions*

$$\varphi_s(-1) = \varphi_s(1) = 0, \quad \psi_s(-1) = \psi_s(1) = 0.$$

Then, there exists a unique solution $\{s(x, t), f_s(x)\}$ of problem D_s^{-1} such that $s(x, t) \in W_2^{2,1}(\Omega)$, $f_s(x) \in L_2(-1, 1)$. The following estimate holds:

$$\|s\|_{W_2^{2,1}(\Omega)} + \|f_s\|_{L_2(-1,1)} \leq C \left(\|\varphi_s\|_{W_2^2(-1,1)} + \|\psi_s\|_{W_2^2(-1,1)}\right).$$

Proof. From formula (5.13) and the asymptotics $\lambda_k^{(s)} \sim k^2$, it follows that $|f_{s,k}| \sim k^2|\psi_{s,k}|$. The convergence of the series $\sum |f_{s,k}|^2$ (i.e, $f_s \in L_2$) is equivalent to the convergence of $\sum k^4|\psi_{s,k}|^2$, which holds since $\psi_s \in W_2^2$.

The estimate for s follows from the direct problem (Lemma 2). □

5.2. Problem R_v^{-1}: Recovery of the even part of the source

Once problem D_s^{-1} is solved, the functions $s(x, t)$ and $f_s(x)$ are considered known. The second part of the inverse problem is reduced to finding the even component of the source $f_c(x)$ and the function $v(x, t)$ as the solution to the inverse problem R_v^{-1} (5.9), where the known functions $H(x, t)$, $\varphi_v(x)$, and $\psi_v(x)$ are determined through the original data and the solution of the auxiliary inverse problem D_s^{-1} according to formulas (5.10).

Since $s \in W_2^{2,1}(\Omega)$ and $f_s \in L_2(-1, 1)$, $H \in L_2(\Omega)$.

We seek a solution to problem R_v^{-1} as a series with respect to the eigenfunctions $\{Y_n(x)\}_{n=0}^{\infty}$ of the spectral problem (4.14) (the even eigenfunctions with homogeneous Robin conditions introduced in Lemma 4):

$$v(x, t) = \sum_{n=0}^{\infty} v_n(t) Y_n(x), \quad f_c(x) = \sum_{n=0}^{\infty} f_{c,n} Y_n(x). \tag{5.14}$$

We also expand into a series the known function

$$H(x, t) = \sum_{n=0}^{\infty} H_n(t) Y_n(x)$$

and the initial and final functions

$$\varphi_v(x) = \sum_{n=0}^{\infty} \varphi_{v,n} Y_n(x), \quad \psi_v(x) = \sum_{n=0}^{\infty} \psi_{v,n} Y_n(x).$$

Projecting the equation from (5.9) onto the basis Y_n, we obtain the Cauchy problem

$$\frac{dv_n}{dt} + \lambda_n^{(v)} v_n(t) = f_{c,n} + H_n(t), \quad v_n(0) = \varphi_{v,n} \tag{5.15}$$

for the coefficients $v_n(t)$, where $\lambda_n^{(v)} = (1-\varepsilon)\mu_n^2$.

The solution of this problem is written out as

$$v_n(t) = \varphi_{v,n} e^{-\lambda_n^{(v)} t} + f_{c,n} \frac{1 - e^{-\lambda_n^{(v)} T}}{\lambda_n^{(v)}} + \int_0^t e^{-\lambda_n^{(v)}(t-\tau)} H_n(\tau) d\tau. \tag{5.16}$$

To determine the unknown constants $f_{c,n}$, we use the final overdetermination condition from (5.9), which in terms of coefficients means $v_n(T) = \psi_{v,n}$.

Substituting $t = T$ into (5.16), we obtain the equation

$$\psi_{v,n} = \varphi_{v,n} e^{-\lambda_n^{(v)} T} + f_{c,n} \frac{1 - e^{-\lambda_n^{(v)} T}}{\lambda_n^{(v)}} + \mathcal{H}_n(T),$$

where

$$\mathcal{H}_n(T) = \int_0^T e^{-\lambda_n^{(v)}(T-\tau)} H_n(\tau) d\tau. \tag{5.17}$$

From this, we express the sought coefficients $f_{c,n}$:

$$f_{c,n} = \frac{\lambda_n^{(v)}}{1 - e^{-\lambda_n^{(v)} T}} \left[\psi_{v,n} - \varphi_{v,n} e^{-\lambda_n^{(v)} T} - \mathcal{H}_n(T) \right]. \tag{5.18}$$

Since $\lambda_n^{(v)} > 0$ (for sufficiently large n and assuming the absence of resonance for small n, i.e., $\lambda_n^{(v)} \neq 0$), the coefficients are uniquely determined.

Lemma 8. *Let $|\varepsilon| < 1$, $a \neq 1$, $b \in \mathbb{R}$. Let the given functions $\varphi(x), \psi(x) \in W_2^2(-1,1)$ and satisfy the original conditions of problem (5.4). Also let the function $H(x,t)$, defined in (5.10), belong to $L_2(\Omega)$ (which is guaranteed by the solvability of problem D_s^{-1}). Then there exists a unique strong generalized solution $\{v(x,t), f_c(x)\}$ to problem R_v^{-1} such that $v \in W_2^{2,1}(\Omega)$ and $f_c \in L_2(-1,1)$. The stability estimate holds:*

$$\|v\|_{W_2^{2,1}(\Omega)} + \|f_c\|_{L_2(-1,1)} \leq C \left(\|\varphi\|_{W_2^2(-1,1)} + \|\psi\|_{W_2^2(-1,1)} \right). \tag{5.19}$$

Proof. The functions $\varphi_v(x)$ and $\psi_v(x)$ are determined by formulas (5.10). Since according to the condition $\varphi, \psi \in W_2^2(-1,1)$, their odd parts φ_s, ψ_s also belong to $W_2^2(-1,1)$. The function $\alpha(x) = kx + 1$ is infinitely differentiable. Consequently, the linear combinations $\varphi_v, \psi_v \in W_2^2(-1,1)$ and their norms are estimated by the norms of the original functions.

According to Lemma 5, if φ and ψ satisfy the original boundary conditions (5.4), then the functions φ_v and ψ_v satisfy the matching conditions for problem R_v^{-1}.

The Fourier coefficients of the source are determined by the previously obtained formula (5.18). As $n \to \infty$, the eigenvalues have the asymptotic behavior $\lambda_n^{(v)} \sim n^2$, while the denominator of the fraction tends to 1.

Let us estimate the integral term $\mathcal{H}_n(T)$. We present the function $H(x,t)$, defined in (5.10), as a sum of two summands: $H(x,t) = H^{(1)}(x) + H^{(2)}(x,t)$, where

$$H^{(1)}(x) = -kxf_s(x),$$

$$H^{(2)}(x,t) = 2k(1-\varepsilon)s_x(x,t) - 2\varepsilon kxs_{xx}(x,t).$$

Let us separately consider the contribution of each summand to the integral term of formula (5.17).

(A) Contribution of the stationary part $H^{(1)}(x)$. Let us denote the Fourier coefficients of the function $H^{(1)}(x)$ by $h_n^{(1)}$. Since they do not depend on time, the integral is calculated explicitly:

$$\int_0^T e^{-\lambda_n^{(v)}(T-\tau)} h_n^{(1)} d\tau = h_n^{(1)} \left[\frac{e^{-\lambda_n^{(v)}(T-\tau)}}{\lambda_n^{(v)}} \right]_{t=0}^{t=T} = h_n^{(1)} \frac{1 - e^{-\lambda_n^{(v)}T}}{\lambda_n^{(v)}}.$$

Substituting this into formula (5.18), we see that the factor $\frac{\lambda_n^{(v)}}{1-e^{-\lambda_n^{(v)}T}}$ completely cancels out with the result of the integration. Therefore, the contribution of $H^{(1)}$ is

$$-\frac{\lambda_n^{(v)}}{1 - e^{-\lambda_n^{(v)}T}} \cdot \left(h_n^{(1)} \frac{1 - e^{-\lambda_n^{(v)}T}}{\lambda_n^{(v)}} \right) = -h_n^{(1)}.$$

Since $f_s \in L_2(-1,1)$ (by Lemma 7), $H^{(1)}(x) \in L_2(-1,1)$, too. Consequently, the series of the squares of these coefficients converges: $\sum |h_n^{(1)}|^2 < \infty$.

(B) The contribution of the non-stationary part $H^{(2)}(x,t)$. The function $H^{(2)}$ consists of derivatives of the solution $s(x,t)$. It is well known that the solution of a parabolic equation with a source in L_2 possesses a smoothing property for $t > 0$. For any $\delta > 0$, the function $s(x,t)$ is infinitely differentiable with respect to t in the domain $t \in [\delta, T]$.

We split the integral into two parts: $\int_0^T = \int_0^\delta + \int_\delta^T$. The integral over the first interval $[0,\delta]$ yields a contribution that is exponentially small for large n due to the factor $e^{-\lambda_n^{(v)}(T-\delta)}$.

On the interval $[\delta, T]$, the function $H_n^{(2)}(\tau)$ is smooth in time. Applying integration by parts, we obtain

$$J_n = \int_\delta^T e^{-\lambda_n(T-\tau)} H_n^{(2)}(\tau)\, d\tau = \frac{1}{\lambda_n} e^{-\lambda_n T} \int_\delta^T \left(e^{\lambda_n \tau}\right)' H_n^{(2)}(\tau)\, d\tau$$
$$= \frac{1}{\lambda_n} H_n^{(2)}(T) - \frac{1}{\lambda_n} e^{-\lambda_n(T-\delta)} H_n^{(2)}(\delta) - \frac{1}{\lambda_n} \int_\delta^T e^{-\lambda_n(T-\tau)} \left(H_n^{(2)}\right)'(\tau)\, d\tau.$$

From the analysis of this formula, we obtain the asymptotics as $n \to \infty$:

$$J_n = \frac{1}{\lambda_n^{(v)}} H_n^{(2)}(T) + O\left(\frac{1}{\lambda_n^{(v)}} e^{-\lambda_n^{(v)}(T-\delta)} \right) + O\left(\frac{1}{\left(\lambda_n^{(v)}\right)^2} \right).$$

After multiplication by the external factor λ_n, this term behaves like $H_n^{(2)}(T)$, meaning it is determined by the value of the function at the final moment in time. Since $\psi_s \in W_2^2(-1,1)$, $H^{(2)}(x,T) \in L_2(-1,1)$.

(C) Final estimate. Collecting all the summands in (5.18), we obtain the asymptotics as $n \to \infty$ ($\lambda_n^{(v)} \sim n^2$):

$$f_{c,n} \approx \lambda_n^{(v)} \psi_{v,n} - h_n^{(1)} - H_n^{(2)}(T).$$

The series $\sum |\lambda_n \psi_{v,n}|^2$ converges because $\psi_v \in W_2^2(-1,1)$ (which is equivalent to $\sum n^4 |\psi_{v,n}|^2 < \infty$). The series for the components of H converge because $H^{(1)}, H^{(2)}(T) \in L_2(-1,1)$. Thus, $f_c \in L_2(-1,1)$, and its norm is estimated by the norms of the boundary data ψ, φ.

Once the source $f_c \in L_2(-1,1)$ is found, the problem reduces to the direct problem R_v with a known right-hand side $\tilde{f}(x,t) = f_c(x) + H(x,t)$.

Since $f_c(x) \in L_2(-1,1)$ and $H(x,t) \in L_2(\Omega)$, $\tilde{f}(x,t) \in L_2(\Omega)$. The initial function $\varphi_v \in W_2^2(-1,1)$ and satisfies the matching condition. Then, according to Lemma 4 (on the well-posedness of the direct problem R_v), there exists the unique solution $v(x,t) \in W_2^{2,1}(\Omega)$.

Summing the estimates for $f_c(x)$ and $v(x,t)$, we obtain the required inequality (5.19). Lemma is proved. □

5.3. The main result for the inverse problem

Theorem 2. *Let the conditions of Theorem 1 be satisfied. Let additionally:*

1) The initial function $\varphi(x) \in W_2^2(-1,1)$ and satisfies the matching conditions (5.4) for $t = 0$.
2) The final function $\psi(x) \in W_2^2(-1,1)$ and satisfies the matching conditions (5.4) for $t = T$.

Then, there exists a unique solution $\{u(x,t), f(x)\}$ to the inverse problems (5.1)–(5.4) such that $u \in W_2^{2,1}(\Omega)$ and $f \in L_2(-1,1)$. The solution is stable with respect to small perturbations of the input data, and the following estimate holds:

$$\|f\|_{L_2(-1,1)} + \|u\|_{W_2^{2,1}(\Omega)} \leq C \left(\|\varphi\|_{W_2^2(-1,1)} + \|\psi\|_{W_2^2(-1,1)} \right). \tag{5.20}$$

Remark 2. *This statement is valid for any real value $r = \sqrt{(1-\varepsilon)/(1+\varepsilon)}$, in contrast to the results of the work [21], where the well-posedness of the inverse problems* (5.1)–(5.4) *for $b = 0$ is considered only for the case of a rational parameter r.*

Proof. The solution is constructed as the sum $u = v + \alpha s$ and $f = f_c + f_s$. Estimates for the components s, f_s have been obtained in Lemma 7 and for v, f_c in Lemma 8. By virtue of the linearity of the expansions and the boundedness of the function $\alpha(x)$ and its derivatives, the total estimate (5.20) follows from the sum of the estimates of the auxiliary problems. Uniqueness follows from the uniqueness of the expansion into even/odd parts and the uniqueness of the solutions to the auxiliary problems.

Theorem 2 is proved. □

6. Examples (direct and inverse problems)

To demonstrate the effectiveness of the proposed functional expansion method, let us consider one direct and one inverse initial-boundary value problem in the case of an irrational value of the parameter r, when the classical method of separation of variables (in a basis of sines and cosines) is inapplicable.

Let the parameters have the following values: $\varepsilon = 1/2$ (then $r = 1/\sqrt{3}$ is irrational), $a = 2$ and $b = 1$. In the domain $\Omega = (-1, 1) \times (0, T)$, consider the problem

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) + \frac{1}{2}u_{xx}(-x,t) = f(x,t), \\ u(x,0) = \varphi(x), \quad x \in [-1,1], \\ u_x(-1,t) + 2u_x(1,t) + u(-1,t) = 0, \quad u(-1,t) = u(1,t), \quad t \in [0,T]. \end{cases} \tag{6.1}$$

Set an exact solution of the problem as

$$u_{exact}(x,t) = e^{-t}\left(\pi x^2 + \sin(\pi x)\right). \tag{6.2}$$

It corresponds to the input data

$$f(x,t) = e^{-t}\left[-\pi(x^2+1) + \left(\frac{3}{2}\pi^2 - 1\right)\sin(\pi x)\right],$$

$$\varphi(x) = \pi x^2 + \sin(\pi x)$$

and satisfies the boundary conditions of problem (6.1).

Let us demonstrate the solution of this problem by the functional expansion method.

Step 1. Problem D_s for the odd component. Single out the odd parts of the data:

$$\varphi_s(x) = \sin(\pi x), \quad f_s(x,t) = e^{-t}\left(\frac{3}{2}\pi^2 - 1\right)\sin(\pi x).$$

Solve the first initial-boundary value problem with Dirichlet boundary conditions:

$$\begin{cases} s_t(x,t) - \frac{3}{2}s_{xx}(x,t) = f_s(x,t), \\ s(x,0) = \sin(\pi x), \quad x \in [-1,1], \\ s(-1,t) = 0, \quad s(1,t) = 0, \quad t \in [0,T]. \end{cases} \tag{6.3}$$

Since $\sin(\pi x)$ is an eigenfunction of the operator arising from the Fourier method, we seek the solution in the form $s(x,t) = A(t)\sin(\pi x)$. Substituting this into (6.3) yields

$$A'(t) + \frac{3}{2}\pi^2 A(t) = e^{-t}\left(\frac{3}{2}\pi^2 - 1\right), \quad A(0) = 1.$$

The solution to this Cauchy problem is $A(t) = e^{-t}$. Thus, the odd part is found:

$$s(x,t) = e^{-t}\sin(\pi x).$$

Step 2. Problem for the auxiliary function $v(x,t)$. For our specific case, we have $k = (1+a)/(1-a) = -3$, $\alpha(x) = 1 - 3x$.

We form the data for problem R_v (Lemma 1):

$$\varphi_v(x) = \varphi(x) - \alpha(x)\varphi_s(x) = \pi x^2 + 3x\sin(\pi x).$$

We calculate the right-hand side $f_v(x,t)$:

$$f_v(x,t) = f(x,t) - \alpha(x) f_s(x,t) + 2k(1-\varepsilon)s_x(x,t) - 2\varepsilon k x s_{xx}(x,t)$$
$$= e^{-t}\left(-\pi x^2 - \pi - 3\pi\cos(\pi x) + \left(\frac{3}{2}\pi^2 x - 3x\right)\sin(\pi x)\right).$$

It corresponds to the exact solution

$$v_{exact} = e^{-t}(\pi x^2 + 3x\sin(\pi x)). \tag{6.4}$$

This function is the exact solution to the problem with the Robin boundary conditions:

$$\begin{cases} v_t(x,t) - \frac{1}{2}v_{xx}(x,t) = f_v(x,t), \\ v(x,0) = \pi x^2 + 3x\sin(\pi x), \quad x \in [-1,1], \\ v_x(-1,t) - v(-1,t) = 0, \quad v_x(1,t) + v(1,t) = 0, \quad t \in [0,T]. \end{cases} \tag{6.5}$$

Since the input data for problem (6.5) are of a special type (a combination of polynomials and trigonometric functions), its solution can be constructed by determining the coefficients of an expression in the form of a finite combination of elementary functions:

$$v(x,t) = Ax^2 + Bx + C + (Dx+E)\sin(\pi x) + (Fx+G)\cos(\pi x).$$

By substituting this expression into problem (6.5) and equating the coefficients of the linearly independent summands, we find the solution (6.4).

Step 3. Final step. Adding the found components together, we obtain the full solution to problem (6.1):

$$u(x,t) = v(x,t) + (1-3x)s(x,t) = e^{-t}(\pi x^2 + \sin(\pi x)).$$

The obtained solution completely coincides with the exact one.

Let us now consider a demonstration example of solving the inverse problem. Let $\varepsilon = \frac{1}{2}$ and choose $a = -1$, $b = 1$. Consider the inverse problem

$$u_t - u_{xx} + \frac{1}{2}u_{xx}(-x) = f(x), \qquad (x,t) \in (-1,1)\times(0,T),$$

with boundary conditions

$$u_x(-1,t) - u_x(1,t) + u(-1,t) = 0, \qquad u(-1,t) = u(1,t),$$

initial data $u(x,0) = \varphi(x)$, and final overdetermination $u(x,T) = \psi(x)$.

Set the exact stationary source $f(x) = \sin(\pi x)$ and look for the solution in the form $u(x,t) = A(t)\sin(\pi x)$. Since u is odd and $\sin(\pm\pi) = 0$, both boundary conditions are satisfied. Substitution into the equation gives

$$A'(t) + \lambda A(t) = 1, \qquad \lambda = (1+\varepsilon)\pi^2 = \frac{3}{2}\pi^2.$$

Taking $A(0) = 0$ (i.e., $\varphi(x) \equiv 0$), we obtain

$$A(t) = \frac{1-e^{-\lambda t}}{\lambda}, \qquad u(x,t) = \frac{1-e^{-\lambda t}}{\lambda}\sin(\pi x),$$

and hence,

$$\psi(x) = u(x,T) = \frac{1-e^{-\lambda T}}{\lambda}\sin(\pi x).$$

To recover f, we apply the formula for the odd component coefficients:

$$f_{s,1} = \frac{\lambda}{1-e^{-\lambda T}}\left(\psi_{s,1} - \varphi_{s,1}e^{-\lambda T}\right).$$

Here, $\varphi_{s,1} = 0$ and $\psi_{s,1} = (1-e^{-\lambda T})/\lambda$, thus $f_{s,1} = 1$ and

$$f(x) = f_s(x) = \sin(\pi x).$$

Therefore, the proposed reconstruction yields the exact stationary source.

These examples confirm the effectiveness of the proposed functional decomposition method.

7. Conclusions

This paper has developed a functional decomposition approach for the analysis of direct and inverse source problems for a nonlocal heat equation with spatial involution under generalized periodic-type boundary conditions. The main analytical difficulty of such problems is that the corresponding spatial operator is generally non-self-adjoint, and its system of root functions may fail to form a Riesz basis. As a consequence, the classical Fourier method and the standard separation-of-variables technique can become inapplicable or lead to additional arithmetic restrictions on the parameters.

The key idea proposed here is to represent the unknown solution as a specially chosen sum of two auxiliary components. This functional decomposition is constructed so that the odd component satisfies a classical heat equation with Dirichlet boundary conditions, while the remaining auxiliary component is an even function that satisfies a classical heat equation with homogeneous Robin boundary conditions. In this way, the original nonlocal boundary value problem is reduced to the sequential solution of two well-studied self-adjoint problems in the spatial variable. An important feature of the method is that it avoids any direct spectral analysis of the non-self-adjoint operator associated with the initial nonlocal formulation.

On this basis, we established a complete well-posedness theory for the direct initial-boundary value problem in Sobolev spaces. In particular, for admissible real parameters and square-integrable right-hand sides, we proved existence and uniqueness of a strong generalized solution and obtained an a priori stability estimate in the anisotropic Sobolev space $W_2^{2,1}(\Omega)$. Notably, these results hold for all real values of the model parameters (including those corresponding to irrational values of the associated spectral ratio), thus removing the arithmetic restrictions that typically arise in the classical spectral approach.

We also considered an inverse source recovery problem in which the source term depends only on the spatial variable and is reconstructed from a final-time observation. Using the same decomposition framework, the inverse problem was reduced to two classical inverse problems: a Dirichlet inverse problem for the odd component and a Robin inverse problem for the even component, with explicit reconstruction formulas for the Fourier coefficients of the unknown source. Under natural smoothness and compatibility assumptions on the initial and final data, we proved existence, uniqueness, and stability of the recovered source in L_2 together with the corresponding solution in $W_2^{2,1}(\Omega)$. The obtained estimates show continuous dependence of the reconstructed source and the solution on the input data, which is essential for both qualitative analysis and the design of stable numerical procedures.

Finally, an example was presented to illustrate the constructive character of the method and to demonstrate that the decomposition strategy remains effective in a parameter regime where the classical separation-of-variables approach is obstructed by basis issues. The developed reduction and the proved a priori estimates provide a robust analytical foundation for further studies of more general models. Possible directions include inverse problems with time-dependent sources, identification of parameters in the boundary conditions, extensions to multi-dimensional settings, and the development of efficient numerical algorithms based on the proposed decomposition scheme.

Author contributions

Makhmud Sadybekov: Methodology, investigation, writing–review and editing, supervision, project administration, funding acquisition; Gulnar Dildabek: Methodology, validation, investigation, writing–review and editing. All authors contributed equally to the writing of this paper. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have used Artificial Intelligence (AI) tools in the creation of this article at the stage of final formatting of articles in the LaTeX template.

Acknowledgments

This research was carried out under the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan, the projects (No. AP23485279 and No. BR24993094).

Conflict of interest

No conflict of interest exists regarding this research.

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