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Research article

Incorporating complex N-fuzzy set with classical semigroups and application in real life industrial systems

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Abstract: In recent years, complex fuzzy sets and N-fuzzy sets have gained substantial attention within the research community, leading to their extensive utilization in different algebraic systems including, groups, rings, and modules. This research work presents a pioneering mathematical framework, the complex N-fuzzy set, which combines the complex fuzziness of parameters with the negative fuzziness of data. Complex N-fuzzy sets represent a recent generalization of complex fuzzy sets and N-fuzzy sets in which the membership degree of each element is allowed to take values in a complex negative domain of the form $[-1,0]+[-1,0]i$. The real part represents a negative, counter-supportive or inhibitive degree of membership, while the imaginary part models an independent direction of orthogonal uncertainty, conflicting evidence, or dual hesitation. Based on this new structure, we introduce and examine the notions of complex N-fuzzy characteristic functions, level (α,β)-cut, and the product of two complex N-fuzzy sets. Also, we present some fundamental operations of complex N-fuzzy sets and certain examples of them. In addition, this paper presents the novel mathematical framework of complex N-fuzzy semigroups, an algebraic structure that arises by superimposing complex N-fuzzy sets on crisp semigroup theory. Here, we present the notions of complex N-fuzzy sub-semigroups, complex N-fuzzy left (right) ideals, and complex N-fuzzy ideals of semigroups. We study certain theorems and their corresponding proofs to underpin these foundational concepts. Furthermore, we investigate some characterizations of these concepts using level (α,β)-cut, and the product of two complex N-fuzzy sets. Lastly, we use complex N-fuzzy semigroup structure in real life industrial systems.

Keywords: semigroups; subsemigroups; ideals; complex N-fuzzy sets; complex N-fuzzy subsemigroups; complex N-fuzzy left (right) ideals; complex N-fuzzy ideals
Mathematics Subject Classification: 03B05, 03B52, 03G25, 06F35

1. Introduction

Semigroup theory occupies a central position in algebra as one of the most basic algebraic frameworks controlled only by the associativity axiom. Despite this minimal structural requirement, semigroups exhibit remarkably rich internal behavior, particularly through the study of congruences, sub-semigroups, ideals, and homomorphic images. Their structural flexibility makes them indispensable not only in pure algebra but also in applied domains such as formal languages, automata theory, and theoretical computer science. The investigation of substructures plays a crucial role in understanding decomposition theory, stability properties, and algebraic invariants within semigroups.

In recent decades, classical semigroup theory has been substantially extended through fuzzy and generalized membership frameworks, allowing the incorporation of graded information and uncertainty into algebraic systems. These developments have deepened the structural analysis of semigroups and broadened their applicability to models involving imprecision and partial truth. However, the continuous evolution of fuzzy paradigms suggests the need for more expressive algebraic models capable of capturing richer informational behavior beyond classical and complex-valued settings.

Fuzzy mathematics is an emerging domain where the classical notions are extended to fuzzy settings. The idea of fuzzy frameworks was adopted by Zadeh [1] as a broad statement of ordinary (crisp) set theory to handle vagueness and unpredictability. The groundbreaking paper "Fuzzy Sets" by Zadeh laid the groundwork for the theory of fuzzy sets and sparked major studies in fuzzy mathematics. Fuzzy set theory provides a way to represent and reason with ambiguous and imprecise data. The notion of a membership function is central to fuzzy structure, unlike a crisp set where an element is either part or not part of a set, a fuzzy set allows an element to have degrees of membership between 0 and 1. A membership function assigns a degree of membership to every element that possesses the characteristics of the set. Fuzzy sets have become a fundamental tool in several fields, including pattern recognition, medicine, engineering design, and decision-making; see [2–4].

Fuzzy set theory has limitations in handling imprecise, inconsistent, and incomplete periodic data. To resolve this matter, Ramot et al. [5] implemented complex fuzzy sets as an extension of the fuzzy set theory. In complex fuzzy sets, the range is extended from a closed interval $[0,1]$ to a unit disk in a complex plane. The membership function of a complex fuzzy set, denoted as $\mu_s(x)$, is defined on the universal U. In other words, for all $x \in U$, $\mu_s(x)$ takes a complex value within the disk with a radius of one in the complex plane, where all values of $\mu_s(x)$ lie inside a circle with a radius of one, and $\mu_s(x) = r_s(x)e^{i\omega_s(x)}$, where $i = \sqrt{-1}$. The term $r_s(x)$ represents the amplitude term, and $\omega_s(x)$ represents the phase term; both are real-valued with $r_s(x) \in [0,1]$. A complex fuzzy set is represented as $\{(x, \mu_s(x)) | x \in U\}$. Tamir et al. [6] introduced complex fuzziness sets in another way and discussed several results and properties of this concept. The conception of an N-fuzzy set is one of the broader formulations of a fuzzy set, as a fuzzy set is unable to cover the negative supportive grade or negative opinion of human beings. Thus, Jun et al. [7] initiated the idea of an N-fuzzy set to cover the negative opinions of human beings by modifying the codomain of the associated membership function of a fuzzy set [0, 1] to the codomain of the associated membership function N-fuzzy set $[-1, 0]$.

Rosenfeld [8] was the pioneer in introducing fuzzy set theory to the realm of groups, leading to the development of structured fuzzy groups. Kuroki [9–11] further delved into the interpretation of fuzzy semigroups, examining concepts like bi-ideals and ideals. Also, Dib and Galhum [12] discussed and extended this exploration to fuzzy semigroups by introducing fuzzy ideals and bi-ideals. Moreover, in the context of fuzzy semigroups, Budimirović et al. [13] laid the foundation for fuzzy identities and their applications. Jun and Song [14], and Wang and Liu [15] presented generalized fuzzy interior ideals and fuzzy regular subsemigroups, respectively. In the study of ordered semigroups, fuzzy bi-ideals [16], fuzzy prime ideals, and fuzzy radicals [17] are fundamental concepts. Kehayopulu and Tsingelis [18], as well as Xie and Tang [19], formulated the notions of regular and intra-regular ordered semigroups. Moreover, several features of intra-regular semigroups by their generalized fuzzy ideals were presented by Khan et al. [20]. Based on the combination of numerous expansions of fuzziness structures and the theory of semigroups, the concept of neutrosophic N-interior ideals in semigroups was studied by Porselvi et al. [21]. Also, Abuhijleh et al. [22] adopted and studied complex fuzzy groups based on Rosenfeld's strategy. Jun et al. [23] investigated the context of subsemigroups and ideals of semigroups in view of roughness single valued neutrosophic structures. Rehman et al. [24] studied the idea of bipolar complex fuzzy semigroups. They implemented the notions of crisp subsemigroups and ideals in view of bipolar complex fuzzy sets and investigated some characterizations of these concepts.

The idea of complex fuzzy semigroups is a concept that combines the conception of fuzzy sets and classical semigroups in complex algebraic structures. Majumder and Das [25] presented complex fuzzy sub-semigroups, bi-ideals, and ideals in semigroups. Also, they presented complex fuzzy interior ideals and complex fuzzy characteristic interior ideals in semigroups and obtained some important characterizations. Mahmood et al. [26] applied the idea of bipolar complex fuzzy sets to Γ-semigroups, and they implemented bipolar complex fuzzy sub-semigroups and certain types of bipolar complex fuzzy ideals. In the connection between complex neutrosophic structures and semigroups, Gulistan et al. [27] explored the idea of subsemigroups based on complex neutrosophic frameworks and defined the Cartesian product of complex neutrosophic sub-semigroups in semigroups. Also, they proposed the concepts of right, left, and interior ideals in view of complex neutrosophic structures. Al-Masarwah et al. [28] studied commutative semigroups and groups based on cubic multi-polar frameworks. Habib et al. [29] studied (interior) ideals in ordered semigroups with a view to the possibility multi-fuzzy soft models.

1.1. Research gap

Although fuzzy sets and their extensions, such as bipolar (intuitionistic) fuzzy sets and neutrosophic sets, have been extensively examined in algebraic structures, the idea of complex N-fuzzy sets has received little attention in the context of semigroups. Most existing works focus on the characterization of fuzzy subsets, fuzzy (subsemigroups) ideals, and related structures, leaving the role of complex N-fuzzy membership functions largely unexamined. While complex N-fuzzy sets have found applications in fields like uncertainty modeling and decision-making, their algebraic significance, especially in analyzing semigroups, subsemigroups, and ideals, remains underdeveloped. This lack of systematic investigation demonstrates the need to establish a solid theoretical foundation for complex N-fuzzy sets in classical semigroups and to clarify their relationship with other fuzziness extensions.

1.2. Motivation

The motivation for introducing complex N- fuzzy sets, whose membership values lie in $[-1,0]+i[-1,0]$ rather than in the classical complex fuzzy domain $[0,1]+i[0,1]$, lies in the following conceptual and structural considerations:

1) Dual representation of influence: Complex fuzzy sets with $[0,1]+i[0,1]$ show uncertainty in magnitude and phase but remain positively oriented. Conversely, complex N- fuzzy sets with $[-1,0]+i[-1,0]$ obtain the degree of rejection (negative magnitude) and the direction of counter-effect (negative phase), giving a richer algebraic interpretation, especially in systems involving antagonistic operations.
2) Modeling negative information explicitly: Classical fuzzy and complex fuzzy sets model degrees of membership (positive support). However, in many applied and algebraic systems (especially in decision systems, AI, and conflict modeling), we encounter negative influence, contradiction, anti-membership, or inhibitory behavior. The notation $[-1,0]+i[-1,0]$ allows direct modeling of opposition, rejection, or inhibitory strength, which cannot be naturally represented in $[0,1]+i[0,1]$.
3) Algebraic compatibility with subtractive structures: In algebraic structures, such as semigroups, Lie algebras, and BCK (BCI)-algebras, operations often involve subtraction like or cancellation behavior. The negative interval provides better compatibility with these structural features, especially when defining anti-subalgebras, negative homomorphisms, and stability under inverse type operations.

In semigroups, the impetus for studying the notion of complex N-fuzzy sets arises from the need to develop more comprehensive tools for representing uncertainty and dual behavior in algebraic models. While fuzzy subsets capture the degree of belonging of objects, they have limited ability to represent contradictory or opposing features that naturally arise in many processes. The introduction of complex N-fuzzy sets provides an avenue to formally describe such dual behaviors within semigroups, thereby providing the algebraic framework. This not only extends the theoretical reach of fuzzy algebra but also opens new opportunities for applications in some fields, such as information systems, computational models, and automata theory, where both complex positive and complex negative contributions of elements must be simultaneously considered. By introducing complex N-fuzzy sets, scholars can gain deeper insights into the structure of classical semigroups and establish connections with broader classes of fuzziness extensions.

To clarify the novelty of our framework, Table 1 compares our framework with previous expansions of fuzziness sets.

Table1. Comparison of membership domains in different fuzziness frameworks.

Framework	Codomain of the associated membership function(s)	Complex	Handles negativity	Dual uncertainty
Fuzzy set	$[0,1]$	No	No	No
Intuitionistic fuzzy set	$[0,1]^2$	No	No	Moderate
Bipolar fuzzy set	$[-1,0] \times [0,1]$	No	Yes	Yes
N-fuzzy set	$[-1,0]$	No	Yes	No uncertainty
Complex fuzzy set	$[0,1] + [0,1]i$	Yes	No	Yes (comes from both positive real and positive imaginary parts)
Complex N-fuzzy set	$[-1,0] + [-1,0]i$	Yes	Yes	Strong (comes from both negative real and negative imaginary parts)

Figure 1 depicts the research framework (relationship between proposed and existing frameworks) of this study.

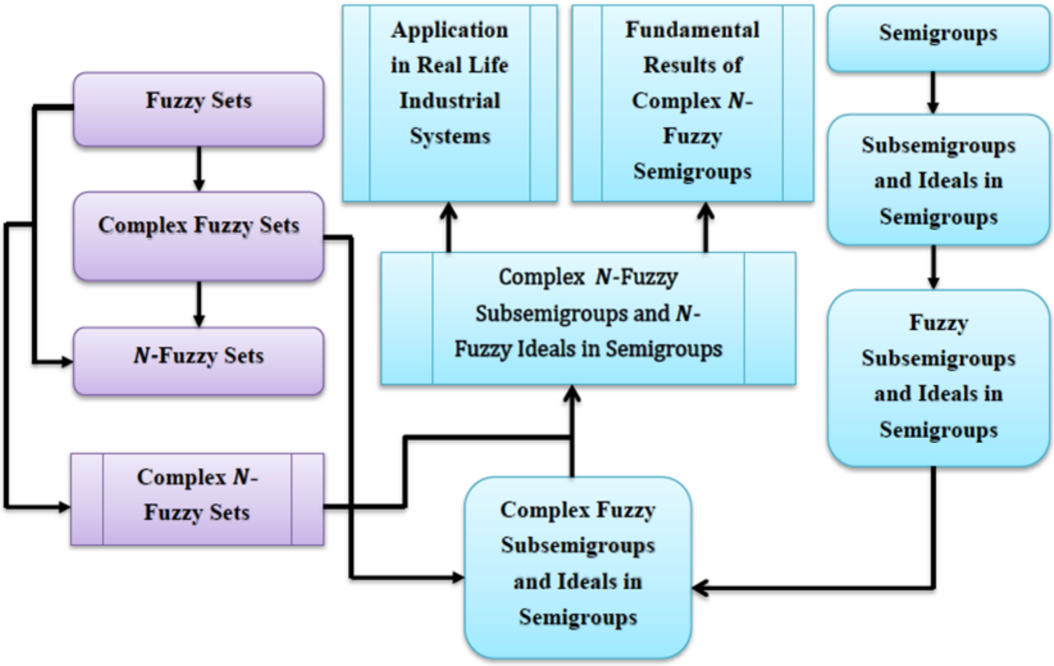

Figure 1. The relationship between proposed and existing models.

1.3. Main contribution

The objectives of this research study are listed as follows:

1) Demonstrates how to propose the concepts of a complex N-fuzzy set, complex N-fuzzy

characteristic function and level (α,β)-cut sets, and study some operations on them.

2) A novel mathematical model that integrates N- fuzzy set and complex fuzzy set concepts into classical semigroups.
3) Applies the proposed structure to a classical semigroup theory, showcasing its effectiveness and improvements over existing fuzzy semigroup methods. Here, we demonstrate how to define the concepts of complex N-fuzzy subsemigroups, complex N-fuzzy left ideals, complex N-fuzzy right ideals, and complex N-fuzzy ideals of classical semigroups, and we discuss their related properties.
4) Provides some characterizations of complex N-fuzzy subsemigroups, complex N-fuzzy left and right ideals, and complex N-fuzzy ideals of semigroups using level (α,β)-cut sets and the product of two complex N-fuzzy sets. Also, the idea of complex N-fuzzy semigroups is used in real life industrial systems.

1.4. Blueprint

This paper is structured into six sections. The main definitions related to this work, such as semigroups, subsemigroups, left ideals, right ideals in semigroups, fuzzy sets, N-fuzzy sets, with some results and properties are illustrated in Section 2. Section 3 illustrates the notions of complex N-fuzzy sets, complex N-fuzzy characteristic functions, the level (α,β)-cut and the product of two complex N-fuzzy sets, and investigates several properties and results of semigroups in terms of these concepts. Some operations of complex N-fuzzy sets and certain examples of them are presented. Also, the idea of a complex N-fuzzy subsemigroup is introduced and some characterizations of this concept are investigated using the level (α,β)-cut and the product of two complex N-fuzzy sets. Section 4 illustrates the notions of complex N-fuzzy left (right) ideals and complex N-fuzzy ideals of semigroups. Some properties and results of these notions are studied. Also, some characterizations of these notions are investigated using the level (α,β)-cut and the product of two complex N-fuzzy sets. In Section 5, we discuss the notion of complex N-fuzzy semigroups in real-life industrial systems. In Section 6, we provide a comparative analysis of the current study with classical and N-fuzzy models. This article is concluded in Section 7 with a research conclusion and certain recommendations for future work.

2. Preliminaries

Here, we cover the main definitions and preliminary results to develop our idea of complex N-fuzzy semigroups. In this regard, we review the notions of semigroups, subsemigroups and ideals of semigroups. Thereafter, we recall the main notions related to (complex) fuzziness sets, N-fuzzy sets, fuzzy subsemigroups, fuzzy (left and right) ideals of semigroups, and with some results and properties related to these notions. Throughout the article, S denotes a semigroup, unless otherwise specified.

Definition 2.1. [30,31] A semigroup is a system consisting of a set S and an associative binary operation in S.

Definition 2.2. [31,32] A nonempty $K \subseteq S$ is called a subsemigroup of S if it is closed under the operation.

Definition 2.3. [28,29] A non-empty $L \subseteq S$ is called:

i. A left ideal of S if $SL \subseteq L$;
ii. A right ideal of S if $LS \subseteq L$;
iii. An ideal of S if $SL \subseteq L$ and $LS \subseteq L$.

Definition 2.4. [1] Let S be a nonempty set. A mathematical shape

$$A = \{(w, S_A(w)) : w \in S\}$$

is a fuzzy set over S, where $S_A(w) \in [0,1]$.

Definition 2.5. [9] A fuzzy set A over a semigroup S is a fuzzy subsemigroup of S if $\forall\, z, y \in S$,

$$S_A(zy) \geq min\{S_A(z),\ S_A(y)\}.$$

Definition 2.6. [10] Suppose a fuzzy set A over a semigroup S. Then, A is said to be a fuzzy left (right) ideal of S if $\forall\, z, y \in S$,

$$S_A(zy) \geq S_A(y)\big(S_A(zy) \geq S_A(z)\big).$$

If a nonempty set A is both a fuzzy left ideal and a fuzzy right ideal, then it is a fuzzy ideal of S.

Definition 2.7. [6] A complex fuzzy set is in the shape of $C = \big\{\big(z, H_C(z)\big) : z \in S\big\} = \{z, H_{RC}(z) + iH_{IC}(z)\} : z \in S\}$ on a universal set S, where $H_C(z)$ is a complex membership degree that gives the values in a unit square of a complex plane, where $H_{RC}(z), H_{IC}(z) \in [0, 1]$ and $i = \sqrt{-1}$.

Definition 2.8. [33] An N-fuzzy set (negative fuzzy set) in a nonempty set S is a function $N_C : \mathrm{S} \rightarrow [-1, 0]$.

Jun et al. [7] used the term negative-valued function and N-function for a negative fuzzy set and N-fuzzy set.

3. Complex N-fuzzy subsemigroups in semigroups

Here, the idea of complex N-fuzzy set is presented and supported by an example. Then, some operations of complex N-fuzzy sets are defined. Also, the complex N-fuzzy characteristic function, complex N-fuzzy product of two complex N- fuzzy sets, and the level (α, β)-cut are introduced. In addition, the complex N-fuzzy subsemigroup is presented and its related theorems are studied using the level (α, β)-cut and the product of two complex N-fuzzy sets.

Definition 3.1. Let S be a semigroup. A mathematical shape

$$C = \big\{\big(w, N_C(w)\big) : w \in S\big\} = \{w, N_{RC}(w) + iN_{IC}(w)\} : w \in S\}$$

is called a complex N-fuzzy set over a nonempty S, where $N_{RC}(w), N_{IC}(w) \in [-1, 0]$.

Definition 3.2. For any two complex N- fuzzy sets

$$C = \big\{\big(w, N_C(w)\big) : w \in S\big\} \text{ and } \quad V = \big\{\big(w, N_V(w)\big) : w \in S\big\}.$$

We have

1) $C \subseteq V$ if $N_C(w) \geq N_V(w)$. That is,

$$N_{RC}(w) \geq N_{RV}(w) \text{ and } N_{IC}(w) \geq N_{IV}(w).$$

2) $C^c = \{(w, N_{C^c}(w)) : w \in S\} = \left\{\left(w, \left(-1 - N_{RC}(w)\right) + i\left(-1 - N_{IC}(w)\right)\right) : w \in S\right\}$.

3) $C \cup V = \{(w, min\{N_C(w), N_V(w)\}) : w \in S\}$
$= \{(w, min\,\{N_{RC}(w), N_{RV}(w)\} + i\, min\,\{N_{IC}(w), N_{IV}(w)\}) : w \in S\}$,

4) $C \cap V = \{(w, max\,\{N_C(w), N_V(w)\}) : w \in S\}$
$= \{(w, max\,\{N_{RC}(w), N_{RV}(w)\} + i\, max\,\{N_{IC}(w), N_{IV}(w)\}) : w \in S\}$.

In the above definition, C^c is the complement of C, and $C \cup V$ and $C \cap V$ are called the union and intersection of C and V, respectively.

Example 3.1. Take a semigroup $S = \{e, u, v, t\}$ with the following multiplication in Table 2:

Table 2. Cayley table of $S = \{e, u, v, t\}$.

$\cdot$	e	u	v	t
e	e	e	e	e
u	e	e	e	e
v	e	e	e	u
t	e	e	u	v

Let

$$C(w) = \begin{cases} -0.3 - 0.4i, & \text{if } w = e \\ -0.1 - 0.3i, & \text{if } w = u \\ -0.3 - 0.2i, & \text{if } w = v \\ -0.4 - 0.2, & \text{if } w = t \end{cases}$$

and

$$V(w) = \begin{cases} -0.3 - 0.3i, & \text{if } w = e \\ -0.2 - 0.5i, & \text{if } w = u \\ -0.4 - 0.8i, & \text{if } w = v \\ -0.3 - 0.1, & \text{if } w = t \end{cases}$$

be two complex N-fuzzy sets of S. Then,

$$C^c(w) = \begin{cases} -0.7 - 0.6i, & \text{if } w = e \\ -0.9 - 0.7i, & \text{if } w = u \\ -0.7 - 0.8i, & \text{if } w = v\text{'} \\ -0.6 - 0.8i, & \text{if } w = t \end{cases}$$

$$(C \cup V)(w) = \begin{cases} -0.3 - 0.4i, & \text{if } w = e \\ -0.2 - 0.5i, & \text{if } w = u \\ -0.4 - 0.8i, & \text{if } w = v\text{'} \\ -0.4 - 0.2i, & \text{if } w = t \end{cases}$$

and

$$(C \cap V)(w) = \begin{cases} -0.2 - 0.3i, & \text{if } w = e, \\ -0.1 - 0.3i, & \text{if } w = u, \\ -0.3 - 0.2i, & \text{if } w = v, \\ -0.3 - 0.1i, & \text{if } w = t. \end{cases}$$

Definition 3.3. The complex N-fuzzy characteristic function of $\mathcal{B} \subseteq S$, is presented by $C^{\mathcal{B}} = \{(w, N_{C^{\mathcal{B}}}(w)): w \in S\}$, where

$$N_{C^{\mathcal{B}}}(w) = \begin{cases} -1 - 1i, & \text{if } w \in \mathcal{B}, \\ 0 + 0i, & \text{otherwise.} \end{cases}$$

Remark 3.4. *We observe that* S *can be taken as a complex* N*-fuzzy set of itself and write* $N_{C^{\mathcal{B}}}(w) = N_S(w)$.

Definition 3.5. Suppose $C = \{(w, N_C(w)): w \in S\}$ is a complex N-fuzzy set over S. Then, for each $\alpha, \beta \in [-1, 0]$. The set

$$\aleph(C, (\alpha, \beta)) = \{w \in S: N_{RC}(w) \leq \alpha \text{ and } N_{IC}(w) \leq \beta\}$$

is known as a level (α, β)-cut of S.

Definition 3.6. Let $C = \{(w, N_C(w)): w \in S\}$ be a complex N-fuzzy set over S. Then, C is said to be a complex N-fuzzy subsemigroup of S if $\forall\, w, z \in S$,

$$N_C(wz) \leq \max\{N_C(w), N_C(z)\}.$$

That is,

$$N_{RC}(wz) \leq \max\{N_{RC}(w), N_{RC}(z)\} \text{ and } N_{IC}(wz) \leq \max\{N_{IC}(w), N_{IC}(z)\}.$$

Example 3.2. Let $S = \{e, u, v, t, s\}$ be a semigroup given by the following Table 3:

Table 3. Cayley Table of $S = \{e, u, v, t, s\}$.

$\cdot$	e	u	v	t	s
e	e	e	e	e	e
u	e	e	e	e	e
v	e	e	v	t	s
t	e	e	v	t	s
s	e	e	v	t	s

Define a complex N-fuzzy set $C = \{(w, N_C(w)): w \in S\}$ over S as follows:

$$C(w) = \begin{cases} -0.23 - 0.25i, & \text{if } w = e, \\ -0.33 - 0.36i, & \text{if } w = u, \\ -0.60 - 0.30i, & \text{if } w = v, t, s. \end{cases}$$

Then, for $e, u \in S$, we have

$$N_{RC}(eu) = N_{RC}(e) = -0.23,$$

$$max\{N_{RC}(e), N_{RC}(u)\} = max\{-0.23, -0.33\} = -0.23,$$

$$\Rightarrow N_{RC}(eu) = -0.23 \leq max\{N_{RC}(e), N_{RC}(u)\} = -0.23,$$

and

$$N_{IC}(eu) = N_{IC}(e) = -0.25,$$

$$max\{N_{IC}(e), N_{IC}(u)\} = max\{-0.25, -0.36\} = -0.25,$$

$$\Rightarrow N_{RC}(eu) = -0.25 \leq max\{N_{RC}(e), N_{RC}(u)\} = -0.25.$$

The other elements of S are the same. Hence, by Definition 3.6, C is a complex N-fuzzy subsemigroup of S.

Definition 3.7. Let $C = \{(w, N_C(w)) : w \in S\}$ and $V = \{(w, N_V(w)) : w \in S\}$ be two complex N-fuzzy sets over S. Then, the product of C and V is presented by:

$$C \odot V = \left\{\left(w, N_{C\odot V}(w)\right) : w \in S\right\} = \{(w, (N_C . N_V)(w)) : w \in S\}$$

$$=\{(w, (N_{RC} . N_{RV})(w) + i(N_{IC} . N_{IV})(w)) : w \in S\},$$

where

$$(N_{RC} \cdot N_{RV})(w) = \begin{cases} \inf\limits_{w=zy}\{\max\{N_{RC}(z), N_{RV}(y)\}\}, & \text{if } \ w = zy \text{ for some } z, y \in S, \\ 0, & \text{if } \quad \text{otherwise}, \end{cases}$$

and

$$(N_{IC} \cdot N_{IV})(w) = \begin{cases} \inf\limits_{w=zy}\{\max\{N_{IC}(z), N_{IV}(y)\}\}, & \text{if } \ w = zy \text{ for some } z, y \in S, \\ 0, & \text{if } \quad \text{otherwise}. \end{cases}$$

Remark 3.8. *Clearly, the operation "$\odot$" is an associative on the set of all complex N-fuzzy sets over a semigroup S, that is, $(C \odot V) \odot D = C \odot (V \odot D)$ for all C, D, and V.*

Theorem 3.9. *Suppose that $C = \{(w, N_C(w)) : w \in S\}$ is a complex N-fuzzy set over S. Then, C is a complex N-fuzzy subsemigroup of S if and only if $C \odot C \subseteq C$.*

Proof. Suppose that C is a complex N-fuzzy subsemigroup over S and $w \in S$. If

$$(N_{RC} \cdot N_{RC})(w) = 0 \geq N_{RC}(w) \text{ and } (N_{IC} \cdot N_{IC})(w) = 0 \geq N_{IC}(w),$$

then clearly, $C \odot C \subseteq C$. Otherwise, there are $z, y \in S$ such that $w = zy$. Then,

$$(N_{RC} \cdot N_{RC})(w) = \inf_{w=zy}\{max\{N_{RC}(z), N_{RC}(y)\}\} \geq \inf_{w=zy}\{N_{RC}(zy)\} = N_{RC}(w)$$

and

$$(N_{IC}\cdot N_{IC})(w)=\inf_{w=zy}\{max\,(N_{IC}(z),N_{IC}(y))\}\geq \inf_{w=zy}\{N_{IC}(zy)\}=N_{IC}(w).$$

Thus,

$$(N_{RC}\cdot N_{RC})(w)\geq N_{RC}(w) \text{ and } (N_{IC}\cdot N_{IC})(w)\geq N_{IC}(w).$$

This implies $(N_C\cdot N_C)(w)\geq N_C(w)$. Hence, $C\odot C\subseteq C$.

Conversely, let C be a complex N-fuzzy set over S such that $C\odot C\subseteq C$, and let $w,z,y\in S$ such that $w=zy$. Then,

$$N_C(zy)=N_C(w)=N_{RC}(w)+iN_{IC}(w).$$

Now, take

$$N_{RC}(w)\ \leq (N_{RC}\cdot N_{RC})(w)=\underset{w=zy}{inf}\{max\{N_{RC}(z),N_{RC}(y)\}\}\leq max\{N_{RC}(z),N_{RC}(y)\},$$

and

$$N_{IC}(w)\leq (N_{IC}\cdot N_{IC})(w)=\underset{w=zy}{inf}\{max\{N_{IC}(z),N_{IC}(y)\}\}\ \leq max\{N_{IC}(z),N_{IC}(y)\}.$$

Therefore, $N_C(zy)\leq max\{N_C(z),N_C(y)\}$. Hence, C is a complex N-fuzzy subsemigroup over S.

Theorem 3.10. *Suppose* $C=\{(w,N_C(w)):w\in S\}$ *is a complex* N*-fuzzy set over* S. *Then, for each* $\alpha,\beta\in[-1,0]$, *the nonempty set* $\aleph(C,(\alpha,\beta))$ *is a subsemigroup of* S *if and only if* C *is a complex* N*-fuzzy subsemigroup over* S.

Proof. Suppose that $\aleph(C,(\alpha,\beta))$ is a subsemigroup over S, then let $w,y\in S$, and let $\alpha=max\{N_{RC}(w),N_{RC}(y)\}$ and $\beta=max\{N_{IC}(w),N_{IC}(y)\}$. Evidently,

$$N_{RC}(w)\leq max\{N_{RC}(w),N_{RC}(y)\}=\alpha,$$

$$N_{RC}(y)\leq max\{N_{RC}(w),N_{RC}(y)\}=\alpha,$$

and

$$N_{IC}(w)\leq max\{N_{IC}(w),N_{IC}(y)\}=\beta,$$

$$N_{IC}(y)\leq max\{N_{IC}(w),N_{IC}(y)\}=\beta.$$

This implies that $w,y\in\aleph(C,(\alpha,\beta))$. Since $\aleph(C,(\alpha,\beta))$ is a subsemigroup over S, then $wy\in\aleph(C,(\alpha,\beta))$. Thus,

$$N_{RC}(wy)\leq\alpha=max\{N_{RC}(w),N_{RC}(y)\}$$

and

$$N_{IC}(wy) \leq \beta = max\{N_{IC}(w), N_{IC}(y)\}.$$

Thus, C is a complex N-fuzzy subsemigroup over S.

Conversely, let $C = \{(w, N_C(w)) : w \in S\}$ be a complex N-fuzzy subsemigroup over S and $w, y \in S$ such that $w, y \in \aleph(C, (\alpha, \beta))$ $\forall$ $\alpha, \beta \in [-1, 0]$. Since $N_{RC}(w) \leq \alpha$, $N_{IC}(y) \leq \alpha$, $N_{IC}(w) \leq \beta$, and $N_{IC}(y) \leq \beta$, then

$$N_{RC}(wy) \leq max\{N_{RC}(w), N_{RC}(y)\} \leq \alpha$$

and

$$N_{IC}(wy) \leq max\{N_{IC}(w), N_{IC}(y)\} \leq \beta.$$

Thus, $wy \in \aleph(C, (\alpha, \beta))$, and so $\aleph(C, (\alpha, \beta))$ is a subsemigroup of S.

Theorem 3.11. *Suppose that $C^{\mathcal{B}} = \{\left(w, N_{C^{\mathcal{B}}}(w)\right) : w \in S\}$ is a complex N-fuzzy characteristic function over S. Then, $C^{\mathcal{B}}$ is a complex N-fuzzy subsemigroup over S if and only if $\mathcal{B}$ is a subsemigroup of S.*

Proof. Assume that $\mathcal{B}$ is a subsemigroup of S and let $w, y \in \mathcal{B}$. Then,

$$N_{C^{\mathcal{B}}}(w) = -1 - 1i = N_{C^{\mathcal{B}}}(y).$$

Since $wy \in \mathcal{B}$, then

$$N_{C^{\mathcal{B}}}(wy) = -1 - 1i = max\,\{-1, -1\} + i\,max\,\{-1, -1\} = max\left\{N_{C^{\mathcal{B}}}(w), N_{C^{\mathcal{B}}}(y)\right\}.$$

Next, if $w \notin \mathcal{B}$ or $y \notin \mathcal{B}$, then

$$N_{C^{\mathcal{B}}}(w) = 0 + 0i \text{ or } N_{C^{\mathcal{B}}}(y) = 0 + 0i.$$

Thus,

$$N_{C^{\mathcal{B}}}(wy) \leq 0 + 0i = max\left\{N_{C^{\mathcal{B}}}(w), N_{C^{\mathcal{B}}}(y)\right\}.$$

Hence, $C^{\mathcal{B}} = \left\{(w, N_{C^{\mathcal{B}}}(w)) : w \in S\right\}$ is a complex N-fuzzy subsemigroup over S.

Conversely, let $C^{\mathcal{B}} = \left\{(w, N_{C^{\mathcal{B}}}(w)) : w \in S\right\}$ be a complex N-fuzzy subsemigroup over S and $w \in S$ such that $w \in \mathcal{B}$. Thus, we get $N_{C^{\mathcal{B}}}(w) = -1 - 1i$. This implies that $w \in \aleph(C, (-1, -1))$. Now, let $y \in S$ such that $y \in \aleph(C, (-1, -1))$. This shows that $N_{RC^{\mathcal{B}}}(w) \leq -1, N_{IC^{\mathcal{B}}}(w) \leq -1$, and so $y \in \mathcal{B}$. Hence, $\mathcal{B} = \aleph(C, (-1, -1))$. By Theorem 3.10, we obtained that $\mathcal{B}$ is a subsemigroup of S.

Lemma 3.12. *Let*

$$C^{\mathcal{B}} = \left\{(w, N_{RC^{\mathcal{B}}}(w) + iN_{IC^{\mathcal{B}}}(w)) : w \in S\right\} \textit{ and}$$

$$C^{\mathcal{R}} = \{(w, N_{RC^{\mathcal{R}}}(w) + iN_{IC^{\mathcal{R}}}(w)) : w \in S\}$$

be two characteristic complex N-fuzzy sets *over* S. *Then,*

1) $C^{\mathcal{B}} \cap C^{\mathcal{R}} = C^{\mathcal{B} \cap \mathcal{R}}$;

2) $C^{\mathcal{B}} \odot C^{\mathcal{R}} = C^{\mathcal{B}\mathcal{R}}$.

Proof. 1) Let $w \in S$. If $w \in \mathcal{B} \cap \mathcal{R}$, then,

$$\begin{aligned}(N_{C^{\mathcal{B}}} \cap N_{C^{\mathcal{R}}})(w) &= max\{N_{C^{\mathcal{B}}}(w), N_{C^{\mathcal{R}}}(w)\} \\ &= max\{N_{RC^{\mathcal{B}}}(w), N_{RC^{\mathcal{R}}}(w)\} + i\, max\{N_{IC^{\mathcal{B}}}(w), N_{IC^{\mathcal{R}}}(w)\} \\ &= -1 - 1i = N_{C^{\mathcal{B}\cap\mathcal{R}}}(w).\end{aligned}$$

Hence, $C^{\mathcal{B}} \cap C^{\mathcal{R}} = C^{\mathcal{B} \cap \mathcal{R}}$. If $w \notin \mathcal{B} \cap \mathcal{R}$, then $w \notin \mathcal{B}$ or $w \notin \mathcal{R}$. Thus,

$$\begin{aligned}(N_{C^{\mathcal{B}}} \cap N_{C^{\mathcal{R}}})(w) &= max\{N_{C^{\mathcal{B}}}(w), N_{C^{\mathcal{R}}}(w)\} \\ &= max\{N_{RC^{\mathcal{B}}}(w), N_{RC^{\mathcal{R}}}(w)\} + i\, max\{N_{IC^{\mathcal{B}}}(w), N_{IC^{\mathcal{R}}}(w)\} \\ &= 0 + 0i = N_{C^{\mathcal{B}\cap\mathcal{R}}}(w).\end{aligned}$$

Hence, $C^{\mathcal{B}} \cap C^{\mathcal{R}} = C^{\mathcal{B} \cap \mathcal{R}}$.

2) Let $w \in S$. Then, we have two cases:

Case (1). If $w \notin \mathcal{B}\mathcal{R}$, then it is clear that

$$(N_{RC^{\mathcal{B}}} \cdot N_{RC^{\mathcal{R}}})(w) = 0 \text{ and } (N_{IC^{\mathcal{B}}} \cdot N_{IC^{\mathcal{R}}})(w) = 0i.$$

This implies that

$$(N_{C^{\mathcal{B}}} \cdot N_{C^{\mathcal{R}}})(w) = 0 + 0i = (N_{C^{\mathcal{B}\mathcal{R}}})(w).$$

Hence, $N_{C^{\mathcal{B}}} \odot N_{C^{\mathcal{R}}} = N_{C^{\mathcal{B}\mathcal{R}}}$.

Case (2). If $w \in \mathcal{B}\mathcal{R}$, then $w = br$ for some $b \in \mathcal{B}$ and $r \in \mathcal{R}$.

$$\begin{aligned}(N_{RC^{\mathcal{B}}} \cdot N_{RC^{\mathcal{R}}})(w) &= \inf_{w=br}\{max\{N_{RC^{\mathcal{B}}}(w), N_{RC^{\mathcal{R}}}(w)\}\} \\ &\leq max\{N_{RC^{\mathcal{B}}}(b), N_{RC^{\mathcal{R}}}(r)\} \\ &= -1 = N_{RC^{\mathcal{B}\mathcal{R}}}(w)\end{aligned}$$

and

$$\begin{aligned}(N_{\mathrm{I}C^{\mathcal{B}}} \cdot N_{IC^{\mathcal{R}}})(w) &= \inf_{w=br}\{max\{N_{IC^{\mathcal{B}}}(w), N_{IC^{\mathcal{R}}}(w)\}\} \\ &\leq max\{N_{IC^{\mathcal{B}}}(w), N_{IC^{\mathcal{R}}}(w)\} \\ &= -1i = N_{IC^{\mathcal{B}\mathcal{R}}}(w).\end{aligned}$$

Hence, $\left(N_{C^{\mathcal{B}}} \cdot N_{C^{\mathcal{R}}}\right)(w) = \left(N_{RC^{\mathcal{B}}} \cdot N_{RC^{\mathcal{R}}}\right)(w) + i\left(N_{\mathrm{I}C^{\mathcal{B}}} \cdot N_{IC^{\mathcal{R}}}\right)(w) = -1 - 1i = N_{C^{\mathcal{B}\mathcal{R}}}(w)$. Thus, $N_{C^{\mathcal{B}}} \odot N_{C^{\mathcal{R}}} = N_{C^{\mathcal{B}\mathcal{R}}}$.

Theorem 3.13. *Suppose that* $C = \{(w, N_C(w)) : w \in S\}$ *and* $D = \{(w, N_D(w)) : w \in S\}$ *are two complex* N*-fuzzy subsemigroups over* S*, then* $C \cap D$ *is a complex* N*-fuzzy subsemigroup over* S.

Proof. For any $w, z \in S$, we have

$$(N_C \cap N_D)(wz) = max\{N_{RC}(wz), N_{RD}(wz)\} + i\, max\{N_{IC}(wz), N_{ID}(wz)\}.$$

Now, take

$$\begin{aligned} max\,\{N_{RC}(wz), N_{ID}(wz)\} &\leq max\{max\{N_{RC}(w), N_{RC}(z)\}, max\{N_{RD}(w), N_{RD}(z)\}\} \\ &= max\{max\{N_{RC}(w), N_{RD}(w)\}, max\{N_{RC}(z), N_{RD}(z)\}\} \\ &= max\{(N_{RC} \cap N_{RD})(w), (N_{RC} \cap N_{RD})(z)\}, \end{aligned}$$

and

$$\begin{aligned} max\{N_{IC}(wz), N_{ID}(wz)\} &\leq max\{N_{IC}(w), N_{IC}(z)\}, max\{N_{ID}(w), N_{ID}(z)\} \\ &= max\{max\{N_{IC}(w), N_{ID}(w)\}, max\{N_{IC}(z), N_{ID}(z)\}\} \\ &= max\{(N_{IC} \cap N_{ID})(w), (N_{IC} \cap N_{ID})(z)\}. \end{aligned}$$

Thus, $C \cap D$ is a N-fuzzy subsemigroup over S.

4. Complex N-fuzzy ideals in semigroups

In the current section, the notions of complex N-fuzzy left (right) ideals and complex N-fuzzy ideals of semigroups are illustrated. Some properties and results of these notions are studied. Also, some characterizations of these notions are investigated using the (α, β)-cut and the product of two complex N-fuzzy sets.

Definition 4.1. Let $C = \{(w, N_C(w)) : w \in S\}$ be a complex N-fuzzy set over S. Then, C is a complex N-fuzzy left ideal of S if $\forall\, w, z \in S$,

$$N_C(wz) \leq N_C(z) \Longrightarrow N_{RC}(wz) \leq N_{RC}(z) \text{ and } N_{IC}(wz) \leq N_{IC}(z).$$

Definition 4.2. Let $C = \{(w, N_C(w)) : w \in S\}$ be a complex N- fuzzy set over S. Then, C is a complex N-fuzzy right ideal of S if $\forall\, w, z \in S$,

$$N_C(wz) \leq N_C(w) \Longrightarrow N_{RC}(wz) \leq N_{RC}(w) \text{ and } N_{IC}(wz) \leq N_{IC}(w).$$

Definition 4.3. Let $C = \{(w, N_C(w)) : w \in S\}$ be a complex N-fuzzy set over S, then C is a complex N-fuzzy ideal if it is both a complex N-fuzzy left ideal and a complex N-fuzzy right ideal.

Example 4.1. Consider the semigroup S of Example 3.2 and a complex N-fuzzy set $C = \{(w, N_C(w)) : w \in S\}$ over S, where

$$C(w) = \begin{cases} -0.60 - 0.30i, & \text{if } w = e, \\ -0.33 - 0.36i, & \text{if } w = u, \\ -0.23 - 0.25i, & \text{if } w = v, t, s. \end{cases}$$

Then, C is a complex N-fuzzy left ideal and a complex N-fuzzy right ideal over S. Therefore, C is a complex N-fuzzy ideal over S.

Remark 4.4. *Every complex N-fuzzy left ideal, complex N-fuzzy right ideal, and complex N-fuzzy ideal over S is a complex N-fuzzy subsemigroup, but the converse is not true.*

Example 4.2. The complex N-fuzzy subsemigroup $C = \{(w, N_C(w)): w \in S\}$ over S in Example 3.2 is not a complex N-fuzzy left ideal, since

$$N_{RC}(ue) = N_{RC}(e) = -0.2 \text{ and } N_{RC}(u) = -0.33.$$

Thus,

$$N_{RC}(ue) \nleq N_{RC}(u) \Longrightarrow N_C(ue) \nleq N_C(u).$$

Also, it is not a complex N-fuzzy right ideal since

$$N_{RC}(eu) = N_{RC}(e) = -0.23 \text{ and } N_{RC}(u) = -0.33.$$

Thus,

$$N_{RC}(eu) \nleq N_{RC}(u) \Longrightarrow N_C(eu) \nleq N_C(u).$$

Hence, C is also not a complex N-fuzzy ideal over S.

The following two theorems explain that the complex N-fuzzy set $C = \{(w, N_C(w)): w \in S\}$ of S is a complex N-fuzzy left (right) ideal over S if and only if $D \odot C \subseteq C$ $(C \odot D \subseteq C)$ for any complex N-fuzzy set D over S.

Theorem 4.5. *Suppose that $C = \{(w, N_C(w)): w \in S\}$ is a complex N-fuzzy set over S. Then, C is a complex N-fuzzy left ideal over S if and only if $D \odot C \subseteq C$ for any complex N-fuzzy set D over S.*

Proof. Assume that $C = \{(w, N_C(w)): w \in S\}$ is a complex N-fuzzy left ideal over S and let $w \in S$. If $(N_{RD} \cdot N_{RC})(w) = 0 \geq N_{RC}(w)$ and $(N_{ID} \cdot N_{IC})(w) = 0 \geq N_{RC}(w)$, then $D \odot C \subseteq C$. Otherwise, there are elements $z, y \in S$ such that $w = zy$. Then,

$$(N_{RD} \cdot N_{RC})(w) = \inf_{w=zy}\{max\{N_{RD}(z), N_{RC}(y)\}\} = \inf_{w=zy}\{max\{-1, N_{RC}(y)\}\}$$

$$= \inf_{w=zy}\{N_{RC}(y)\} \geq \inf_{w=zy}\{N_{RC}(zy)\} = N_{RC}(w),$$

and

$$(N_{ID} \cdot N_{IC})(w) = \inf_{w=zy}\left\{max\left(N_{ID}(z), N_{IC}(y)\right)\right\} = \inf_{w=zy}\{max\{-1, N_{IC}(y)\}\}$$

$$= \inf_{w=zy}\{N_{IC}(y)\} \geq \inf_{w=zy}\{N_{IC}(zy)\} = N_{IC}(w).$$

Thus,

$$(N_{RD} \cdot N_{RC})(w) \geq N_{RC}(w) \text{ and } (N_{ID} \cdot N_{IC})(w) \geq N_{IC}(w).$$

Hence, $(N_D \cdot N_C)(w) \geq N_C(w)$, and so $D \odot C \subseteq C$.

Conversely, let $C = \{(w, N_C(w)) : w \in S\}$ be a complex N-fuzzy set such that $D \odot C \subseteq C$, and let $w, z, y \in S$ such that $w = zy$. Then,

$$N_C(zy) = N_C(w) = N_{RC}(w) + iN_{IC}(w).$$

Now, take

$$\begin{aligned} N_{RC}(w) \leq (N_{RD} \cdot N_{RC})(w) &= \inf_{w=zy} \{max\{N_{RD}(z), N_{RC}(y)\}\} \\ &= \inf_{w=zy} \{max\{-1, N_{RC}(y)\}\} \leq max\{-1, N_{RC}(y)\} = N_{RC}(y) \\ &\Longrightarrow N_{RC}(zy) \leq N_{RC}(y) \end{aligned}$$

and

$$\begin{aligned} N_{IC}(w) \leq (N_{ID} \cdot N_{IC})(w) &= \inf_{w=zy} \{max\{N_{ID}(z), N_{IC}(y)\}\} \\ &= \inf_{w=zy} \{max\{-1, N_{IC}(y)\}\} \leq max\{-1, N_{IC}(y)\} = N_{IC}(y) \\ &\Longrightarrow N_{IC}(zy) \leq N_{IC}(y). \end{aligned}$$

This implies that C is a complex N-fuzzy left ideal of S.

Theorem 4.6. *Suppose that* $C = \{(w, N_C(w)) : w \in S\}$ *is a complex* N*-fuzzy set over* S*. Then,* C *is a complex* N*-fuzzy right ideal over* S *if and only if* $C \odot D \subseteq C$ *for any complex* N*-fuzzy set* D *over* S.

Proof. The proof here is likewise the proof of Theorem 4.5.

Theorem 4.7. *Suppose that* $C = \{(w, N_C(w)) : w \in S\}$ *is a complex* N*-fuzzy set over* S*. Then,* C *is a complex* N*-fuzzy ideal* S *if and only if* $D \odot C \subseteq C$ *and* $C \odot D \subseteq C$ *for any complex* N*-fuzzy set* D *over* S.

Proof. The proof follows from Theorems 4.5 and 4.6.

Theorem 4.8. *Suppose* $C = \{(w, N_C(w)) : w \in S\}$ *is a complex* N*-fuzzy set over* S*. Then, for any* $\alpha, \beta \in [-1, 0]$, $\aleph(C, (\alpha, \beta)) \neq \phi$ *is a left ideal of* S *if and only if* C *is a complex* N*-fuzzy left ideal over* S.

Proof. Assume that $\aleph(C, (\alpha, \beta)) \neq \phi$ for $\alpha, \beta \in [-1, 0]$. Let C be a complex N-fuzzy left ideal of S, and let $w, y \in \aleph(C, (\alpha, \beta))$. Thus,

$$N_{RC}(wy) \leq N_{RC}(y) \leq \alpha, \; N_{IC}(wy) \leq N_{IC}(y) \leq \beta.$$

This implies that $wy \in \aleph(C, (\alpha, \beta))$. Therefore, $\aleph(C, (\alpha, \beta))$ is a left ideal of S.

Conversely, assume that $\aleph(C,(\alpha,\beta))$ is a left ideal over S. If $w,y \in S$ such that

$$N_{RC}(wy) > N_{RC}(y) \text{ and } N_{IC}(wy) > N_{IC}(y),$$

then there exist $t_\alpha, t_\beta \in [-1,0)$ such that

$$N_{RC}(wy) > t_\alpha \geq N_{RC}(y) \text{ and } N_{IC}(wy) > t_\beta \geq N_{IC}(y).$$

Thus,$y \in \aleph\left(C,\left(t_\alpha,t_\beta\right)\right)$, but $wy \notin \aleph\left(C,\left(t_\alpha,t_\beta\right)\right)$, a contradiction. Hence, $N_{RC}(wy) \leq N_{RC}(y)$ and $N_{IC}(wy) \leq N_{IC}(y)$. Therefore, C is a complex N-fuzzy left ideal over S.

Theorem 4.9. *Suppose* $C = \{(w, N_C(w)) : w \in S\}$ *is a complex* N*-fuzzy set over* S*. Then, for any* $\alpha,\beta \in [-1,0]$, $\aleph(C,(\alpha,\beta)) \neq \phi$ *is a right ideal of* S *if and only if* C *is a complex* N*-fuzzy right ideal over* S.

Proof. The proof here is likewise the proof of Theorem 4.8.

Theorem 4.10. *Suppose* $C = \{(w, N_C(w)) : w \in S\}$ *is a complex* N*-fuzzy set over* S*. Then, for any* $\alpha,\beta \in [-1,0]$, $\aleph(C,(\alpha,\beta)) \neq \phi$ *is an ideal of* S *if and only if* C *is a complex* N*-fuzzy ideal over* S.

Proof. The proof follows from Theorems 4.8 and 4.9.

Theorem 4.11. *Suppose that* $C^{\mathcal{B}} = \{(w, N_{C^{\mathcal{B}}}(w)) : w \in S\}$ *is a complex* N*-fuzzy characteristic function over* S*. Then,* $C^{\mathcal{B}}$ *is a complex* N*-fuzzy left ideal over* S *if and only if* $\mathcal{B}$ *is a left ideal of* S.

Proof. The proof is likewise the proof of Theorem 3.11.

Theorem 4.12. *Suppose that* $C^{\mathcal{B}} = \{(w, N_{C^{\mathcal{B}}}(w)) : w \in S\}$ *is a complex* N*-fuzzy characteristic function over* S*. Then,* $C^{\mathcal{B}}$ *is a complex* N*-fuzzy right ideal over* S *if and only if* $\mathcal{B}$ *is a right ideal of* S.

Proof. The proof is likewise the proof of Theorem 3.11.

Theorem 4.13. *Suppose that* $C^{\mathcal{B}} = \{(w, N_{C^{\mathcal{B}}}(w)) : w \in S\}$ *is a complex* N*-fuzzy characteristic function over* S*. Then,* $C^{\mathcal{B}}$*is a complex* N*-fuzzy ideal over* S *if and only if* $\mathcal{B}$ *is a two-sided ideals of* S.

Proof. The proof follows from Theorems 4.11 and 4.12.

Theorem 4.14. *If* $C = \{(w, N_C(w)) : w \in S\}$ *and* $D = \{(y, N_D(y)) : y \in S\}$ *are two complex* N*-fuzzy sets over* S*, then* $C \cap D$ *is a complex* N*-fuzzy left ideal over* S.

Proof. Let C and D be two complex N-fuzzy left ideals of S. For any $w,z \in S$, we have

$$(N_C \cap N_D)(wz) = max\,\{N_{RC}(wz), N_{RD}(wz)\} + imax\,\{N_{IC}(wz), N_{ID}(wz)\}.$$

Now,

$$max\{N_{RC}(wz), N_{RD}(wz)\} \leq max\{N_{RC}(z), N_{RD}(z)\} = (N_{RC} \cap N_{RD})(z).$$

Also,

$$max\{N_{IC}(wz), N_{ID}(wz)\} \leq max\{N_{IC}(z), N_{ID}(z)\} = (N_{IC} \cap N_{ID})(z).$$

Hence, $(N_C \cap N_D)(wz) \leq (N_C \cap N_D)(z)$, and so $C \cap D$ is a complex N-fuzzy left ideal of S.

Theorem 4.15. *If* $C = \{(w, N_C(w)): w \in S\}$ *and* $D = \{(y, N_D(y)): y \in S\}$ *are two complex* N*-fuzzy sets over* S*, then* $C \cap D$ *is a complex* N*-fuzzy right ideal over* S.

Proof. Let C and D be two complex N-fuzzy right ideals of S. For any $w, z \in S$, we have

$$(N_C \cap N_D)(wz) = max\{N_{RC}(wz), N_{RD}(wz)\} + imax\{N_{IC}(wz), N_{ID}(wz)\}.$$

Now,

$$max\{N_{RC}(wz), N_{RD}(wz)\} \leq max\{N_{RC}(w), N_{RD}(w)\} = (N_{RC} \cap N_{RD})(w).$$

Also,

$$max\{N_{IC}(wz), N_{ID}(wz)\} \leq max\{N_{IC}(w), N_{ID}(w)\} = (N_{IC} \cap N_{ID})(w).$$

Hence, $(N_C \cap N_D)(wz) \leq (N_C \cap N_D)(z)$, and so $C \cap D$ is a complex N-fuzzy right ideal of S.

Theorem 4.16. *If* $C = \{(w, N_C(w)): w \in S\}$ *and* $D = \{(y, N_D(y)): y \in S\}$ *are two complex* N*-fuzzy sets over* S*, then* $C \cap D$ *is a complex* N*-fuzzy ideal over* S.

Proof. The proof is true using Theorems 4.14 and 4.15.

Theorem 4.17. *Suppose* $C = \{(w, N_C(w)): w \in S\}$ *is a complex* N*-fuzzy right ideal over* S*. Then,* $C \cup (S \odot C)$ *is a complex* N*-fuzzy ideal over* S.

Proof. As S is a complex N-fuzzy left ideal,

$$S\odot \Big(C \cup \big((S \odot C)\big)\Big) = (S \odot C) \cup (S \odot S \odot C) \subseteq (S \odot C) \cup (S \odot C) = S \odot C \subseteq C \cup (S \odot C).$$

This proves that $C \cup (S \odot C)$ is a complex N-fuzzy left ideal over S. Now,

$$\Big(C \cup \big((S\odot C)\big)\Big) \odot S = (C \odot S) \cup (S \odot C \odot S) \subseteq C \cup (S \odot C).$$

This proves that $C \cup (S \odot C)$ is a complex N-fuzzy right ideal over S. Thus, $C \cup (S \odot C)$ is a complex N-fuzzy ideal over S.

Corollary 4.18. *Suppose* $C = \{(w, N_C(w)): w \in S\}$ *is a complex* N*-fuzzy left ideal over* S*. Then,* $C \cup (C \odot S)$ *is a complex* N*-fuzzy ideal over* S.

5. Application

To illustrate the applicability of the proposed complex N-fuzzy sets, we consider the following application as a simple industrial decision system in which negative preference values represent undesirable operational states. The following example in this section demonstrates how complex

N-fuzzy sets can model such situations within a semigroup structure.

We create a fundamental algorithm to demonstrate how the complex N-fuzzy semigroup may be applied to industrial systems.

In applied mathematics, there are many industrial systems, such as manufacturing pipelines, electric-power substations, chemical processing chains, and water-treatment lines. These systems operate as a sequence of stages (mechanical filter → chemical neutralization → ultraviolet sterilization, etc.). Every stage converts an input and passes it to the subsequent one, establishing a natural semigroup structure through the operation of stage composition. In practice, each stage may present observable degradation (measurable defects, hidden underperformance, residue, energy loss) and latent degradation (microstructural damage, hidden chemical interactions, sensor blind-spots). Classical fuzziness frameworks cannot quantify both types of negative influence at once. Complex N-fuzzy sets offer a mathematically robust framework to model both measurable degradation (real part) and latent degradation (imaginary part) within a unified structure.

The following algorithm describes step by step the procedure of the proposed model.

Algorithm 1. Complex N-fuzzy semigroup algorithm for real-life industrial systems.

```
Step 1: Define the industrial process semigroup.
    ➢ Identify the main stages $S = \{d_1, d_2, \dots, d_n\}$.
    ➢ Define the semigroup operation that models stage composition: $d_i * d_j$ denotes the
      operation: "apply stage $d_i$ then stage $d_j$".
    ➢ Verify associativity: $(d_i * d_j) * d_k = d_i * (d_j * d_k)$.
Step 2: Construct the complex $N$-fuzzy set mapping for stages and assign fuzzy values to each stage.
    ➢ For each stage $d_i$, define
          $N_C(d_i) = N_{RC}(d_i) + iN_{IC}(d_i)$.
    ➢ Assign fuzzy values of the form $[-1, 0] + i[-1,0]$ to each stage.
Step 3: Define a complex $N$-fuzzy propagation under semigroup composition.
    ➢ Take $\beta = 0.5$ as equal influence (simple baseline).
    ➢ Compute stepwise propagation.
Step 4: Give a real world interpretation.
    ➢ Calculate the final complex $N$-fuzzy score, $N_C(\text{final})$.
    ➢ Calculate the readability index (magnitude score), $R = |N_C(\text{final})|$.
Step 4: Choose thresholds $\varepsilon_1$ and $\varepsilon_2$ such that $0 \le \varepsilon_1 \le \varepsilon_2 \le 1$.
Step 5: Make decisions.
    ➢ Decision = $\begin{cases} \text{Accept}, & R \le \varepsilon_1 \\ \text{Rework}, & \varepsilon_1 < R \le \varepsilon_2 \\ \text{Reject}, & R > \varepsilon_2 \end{cases}$
Step 6: End and investigate the significance of the application.
```

The following flowchart (Figure 2) describes the algorithm for a complex N-fuzzy industrial evaluation.

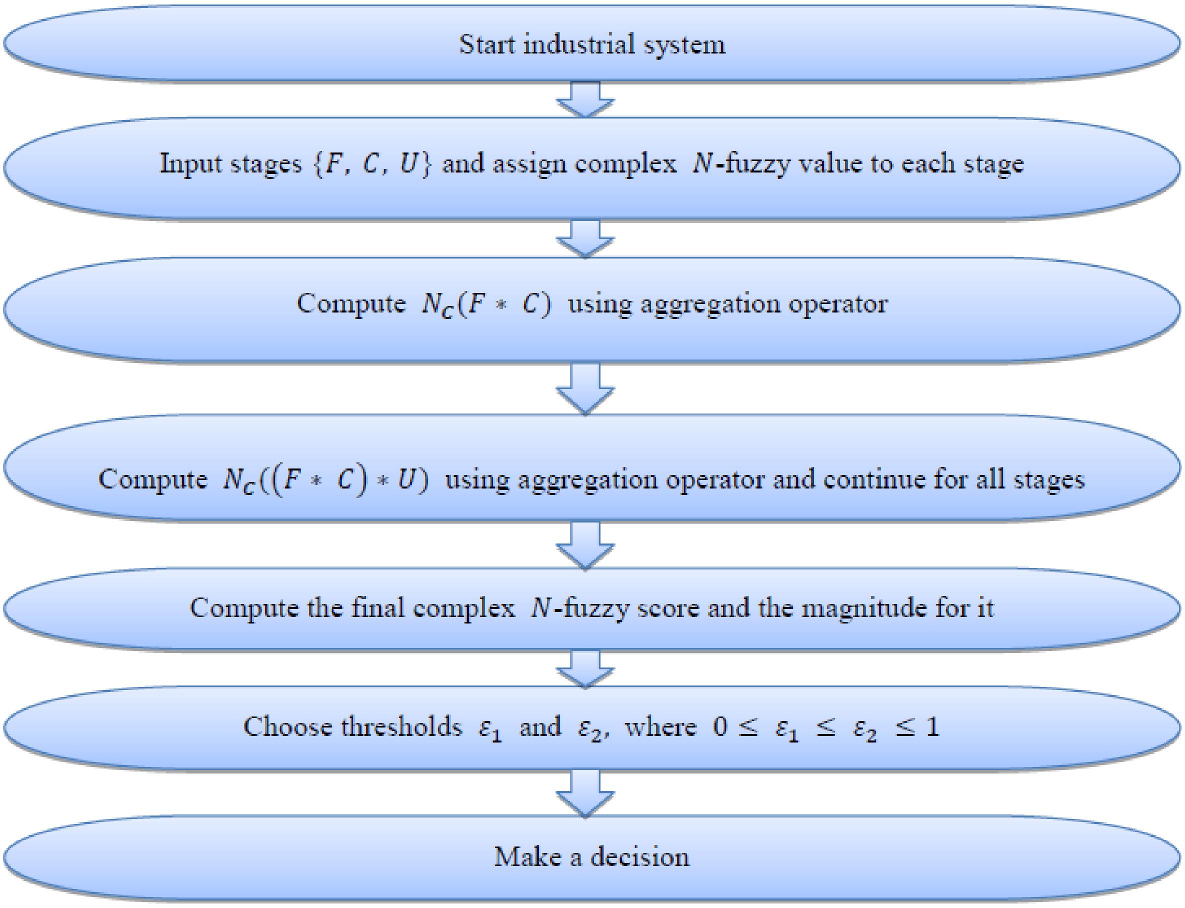

Figure 2. Algorithm flowchart for complex N-fuzzy industrial evaluation.

Step 1. Define the semigroup (system of stages and operation):

Consider $S = \{F =$ mechanical filtering, $C =$ chemical neutralization, $U =$ ultraviolet sterilization$\}$ as a set of sequential stages of an industrial water-treatment process. We impose the natural order $F < C < U$. The composition of stages defines a semigroup $(S, *)$, where $x * y$ denotes the operation: "apply stage x then stage y". Each stage may introduce measurable and hidden negative effects. Now, define $x * y = max\{x, y\}$ with respect to the order $F < C < U$. This is a well-defined binary operation.

Explicitly: The operation $*: S \times S \to S$ is defined in Table 4.

Table 4. Cayley table of $S = \{F, C, U\}$.

$*$	F	C	U
F	F	C	U
C	C	C	U
U	U	U	U

It is clear that S is closed under the operation "$*$", i.e., $x * y = max\{x, y\} \in S$ for all $x, y \in S$. Also, the operation "$*$" is associative, since "$*$" is a maximum operation,

$$(x * y) * z = \max\{max\{x, y\}, z\},$$

$$x * (y * z) = max\ \{x, max\{y, z\}\},$$

for all $x, y, z \in S$. However, maximum on a totally ordered set is associative:

$$max\ \{max\{x, y\}, z\} = max\{x, y, z\} = max\ \{x, max\{y, z\}\}.$$

That is, $(x * y) * z = x * (y * z)$ for all $x, y, z \in S$. Thus, $(S, *)$ is a semigroup.

Here, we can check with F, C, and U:

1) $(F * C) * U$:

$$F * C = C, \qquad C * U = U.$$

2) $F * (C * U)$:

$$C * U = U, \qquad F * U = U.$$

This gives U, and so associativity holds. Similarly, any other combination, e.g., $(F * F) * C$, $(U * F) * C$, etc., also satisfies associativity.

Step 2. Define the complex N-fuzzy set mapping for the following stages.

For each stage in S, assign the negative influence in the following form:

$$N_C(w) = N_{RC}(w) + iN_{IC}(w),$$

where $N_{RC}(w), N_{IC}(w) \in [-1, 0]$ for all $w \in S$.

For each stage in S, take

$$N_C(F) = -0.30 - 0.10i,$$

$$N_C(C) = -0.40 - 0.20i,$$

$$N_C(U) = -0.20 - 0.30i.$$

Here, the values $N_C(w)$ for any $w \in S$ are interpretable. That is, the real part, $N_{RC}(w)$ for any $w \in S$, measures observable degradation (chemical residue, particle leakage, UV aging). The imaginary part, $N_{IC}(w)$ for any $w \in S$, measures hidden degradation (chemical by-products, sensor-undetectable faults, thermal instability). In this case, the value $N_{RC}(C) = -0.40$ indicates stronger negative effects from the chemical neutralization stage and mechanical filtering.

Step 3. Complex N-fuzzy propagation under semigroup composition:

Define a complex N-fuzzy propagation as follows: $N_C(w * y) = \beta N_C(w) + i(1 - \beta)N_C(y)$. That is,

$$N_C(w) = N_{RC}(w) + iN_{IC}(w) \quad \text{and}\ N_C(y) = N_{RC}(y) + iN_{IC}(y),$$

where $\beta \in [0,1]$, and it reflects the dominant influence in the combined action of stages w and y. The common choices for β:

- $\beta = 0.5$: equal influence (simple baseline).
- $\beta > 0.5$: earlier stage dominates.
- β: chosen by empirical calibration (expert judgment or from data).

Now, if we take $\beta = 0.5$ for equal contributions and consider the complex N-fuzzy set for the stages $F, C,$ and U, then:

- Mechanical filtering followed by chemical neutralization:

$$N_C(F * C) = 0.5(-0.30 - 0.10i) + 0.5(-0.40 - 0.20i) = -0.35 - 0.15i.$$

- Result followed by ultraviolet sterilization:

$$N_C((F * C) * U) = 0.5(-0.35 - 0.15i) + 0.5(-0.20 - 0.30i) = -0.275 - 0.225i.$$

Step 4. Real world interpretation:

The final result $N_C((F * C) * U) = -0.275 - 0.225i$ consists of:

- $N_{RC}((F * C) * U) = -0.275$: quantifiable degradation (residues, partial treatment inefficiency, chemical traces).
- $N_{IC}((F * C) * U) = -0.225$: latent degradation that is not directly measurable (chemical by product, hidden energy loss, UV bulb instability).
- The final complex N-fuzzy score: $N_C((F * C) * U) = -0.275 - 0.225i$ represents the total degradation introduced across all stages, encompassing both visible and hidden impacts.
- The readability index (a magnitude measure):

$$|N_C((F * C) * U)| = \sqrt{(-0.275)^2 + (-0.225)^2} \approx 0.356.$$

Step 5. Result: If we choose thresholds ε_1 and ε_2, where $0 \leq \varepsilon_1 \leq \varepsilon_2 \leq 1$, then our decision depends on the readability index (a magnitude measure) R:

1) Accept if $R \leq \varepsilon_1$.
2) Rework if $\varepsilon_1 < R \leq \varepsilon_2$.
3) Reject if $R > \varepsilon_2$.

Using thresholds $\varepsilon_1 = 0.40$ (safe and very small degradation) and $\varepsilon_2 = 0.60$ (moderate to high degradation and unacceptable). Then, $|N_C((F * C) * U)| \approx 0.356 < 0.40$, so the decision is accepted. This means the industrial system is stable, safe, clean, and does not require reprocessing.

Step 6. Significance of the application:

The suggested complex N-fuzzy semigroup model provides:

1) Quantification of hidden or latent faults, which classical fuzziness systems fail to identify.
2) An optimization foundation, such as reordering stages to minimize degradation or selecting parameters.
3) A consolidated structure for modeling multi-stage degradation processes.
4) Dynamic propagation of negative effects across sequential operations.

6. Comparative analysis with classical and N-fuzzy models

Classical fuzzy sets, presented by Zadeh (1965), assign each element w a membership value $S_A(w) \in [0,1]$, representing the degree of truth or belonging. In industrial systems, classical fuzzy models are typically used to describe risk levels, failure likelihood, efficiency degradation, and performance quality indices. Propagation across stages is commonly modeled using: $S_A(w \circ z) = min\{S_A(w), S_A(z)\}$ or $S_A(w \circ z) = max\{S_A(x), S_A(y)\}$, where these operators provide a scalar aggregation of uncertainty. N-fuzzy sets extend the membership range to $S_A(w) \in [-1,0]$, allowing modeling of the structural damage index, inhibitory influence, and degradation intensity. However, the information remains one-dimensional. Both classical and negative fuzzy systems represent uncertainty using a single scalar parameter.

Consider the multi-stage industrial system $S = \{F$ = mechanical filtering, C = chemical neutralization, U = ultraviolet sterilization$\}$ in the previous application. In real industrial environments, degradation has two independent components:

- Structural degradation includes mechanical wear, corrosion, and material fatigue.
- Dynamic instability includes chemical reaction instability, power fluctuation, vibration, and oscillations.

A scalar fuzzy value cannot separate these two phenomena. For example, if we take $S_A(C) = 0.6$, there is no indication of whether the risk is caused by mechanical failure or oscillatory instability.

In the proposed complex N-fuzzy framework, each stage is assigned as $N_C(d_i) = N_{RC}(d_i) + iN_{IC}(d_i)$, where $N_{RC}(d_i)$, $N_{IC}(d_i) \in [-1, 0]$, where $N_{RC}(d_i)$ and $N_{IC}(d_i)$ are a structural degradation index and a dynamic instability index, respectively. Thus, uncertainty becomes two-dimensional. In this case, we have a complex N-fuzzy propagation $N_C(w * y) = \beta N_C(w) + i(1-\beta)N_C(y)$. This introduces a tunable weight parameter β, directional influence, and cross-coupling between structural and dynamic effects. Table 5 describes the mathematical comparison with classical fuzzy models.

Table 5. Mathematical comparison with classical fuzzy models.

Feature	Classical fuzzy set	N-Fuzzy set	Complex N –Fuzzy semigroup
Membership codomain	$[0, 1]$	$[-1, 0]$	$[-1, 0] + i\,[-1, 0]$
Dimensionality	1D	1D	2D
Structural degradation modeling	√	√	√
Dynamic instability modeling	×	×	√
Propagation flexibility	max/min only	max only	Weighted complex operator
Directional influence	No	No	Yes
Algebraic semigroup structure	Optional	Optional	Explicitly defined

If we use the industrial values for F, C, and U in the previous application section,

$$N_C(F) = -0.30 - 0.10i,$$

$$N_C(C) = -0.40 - 0.20i,$$

$$N_C(U) = -0.20 - 0.30i.$$

A classical negative fuzzy model would only compare: – 0.30, – 0.40, and – 0.20 and conclude that stage C is the worst. However, the complex model reveals that C has the highest structural degradation and U has the highest dynamic instability. Thus, the maintenance strategy differs as follows:

- Complex model: prioritize structural stabilization at C and dynamic stabilization at U.
- Classical model: prioritize C only.

This produces more refined engineering decisions**.** Finally, the proposed work constitutes a strict extension of negative fuzzy systems and a structural enhancement over classical fuzzy frameworks by introducing:

1) Complex-valued N-fuzzy membership.
2) Associative internal composition.
3) Directional propagation.
4) Algebraic modeling of industrial stage interaction.

Hence, the contribution is not incremental but structural.

7. Conclusions and future works

The concept of a semigroup has played a pivotal role in various scholarly endeavors and has found applications across diverse fields. Building upon this foundation, numerous scholars extended and modified the notion of a semigroup to introduce innovative concepts like fuzzy semigroups, bipolar fuzzy semigroups, and more. In recent years, researchers expanded the scope of fuzzy algebraic structures, applying them to real-world problems in different scientific domains.

This study delved into the algebraic structure of semigroups, employing the complex N-fuzzy set to address existing research gaps. Complex N-fuzzy sets, complex N-fuzzy characteristic functions, the level (α,β)-cut, and the product of two complex N-fuzzy sets were defined. Also, some operations of complex N-fuzzy sets and certain examples of them were presented. In addition, the notions of complex N-fuzzy subsemigroups, complex N-fuzzy left ideals, complex N-fuzzy right ideals, and complex N-fuzzy ideals over semigroups were presented. Theorems and their corresponding proofs were studied to underpin these foundational concepts. Also, some characterizations of these notions were investigated using the level (α,β)-cut and the product of two complex N-fuzzy sets. Finally, we discussed the notion of complex N-fuzzy semigroup structure in real life industrial systems.

The incorporation between classical semigroups and complex N-fuzzy sets not only contributes to the theoretical foundations of semigroups but also provides a bridge between abstract algebraic structures, such as gamma semigroups, ordered semigroups, hemirings, semirings, Lie algebras, hoops, lattice implication algebras, etc., and their potential applications in various scientific domains. In other words, the incorporation of a negative version of complex structures with classical semigroups opens up new avenues for exploring semigroup theory and offers a fresh perspective that can be further extended to address real-world issues.

Author contributions

Anas Al-Al-Masarwah: Conceptualization, methodology, supervision, modified and verified the result; Moa'th Algarager: Conceptualization, investigation, writing–original draft, writing–review and editing; Hasan Almutairi: Methodology, investigation, writing–review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare that they have no competing interests.

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