AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 11(4): 9633–9654.
DOI: 10.3934/math.2026399
Received: 17 January 2026
Revised: 16 March 2026
Accepted: 02 April 2026
Published: 10 April 2026

Research article

Convergence analysis of Mann-type hybrid inertial Yosida approximation iterative schemes for split variational inclusions

Doaa Filali[1], **Mohammad Dilshad**[2,*], **Mohammad Akram**[3,*], **Md. Nasiruzzaman**[2] **and Esmail Alshaban**[2]

[1] Department of Mathematical Science, College of Sciences, Princess Nourah Bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia

[2] Department of Mathematics, Faculty of Science, University of Tabuk, Tabuk-71491, Saudi Arabia

[3] Department of Mathematics, Faculty of Science, Islamic University of Madinah, Madinah-107, Saudi Arabia

* **Correspondence:** Email: mdilshaad@gmail.com, akramkhan_20@rediffmail.com.

Abstract: This manuscript introduces two Mann-type hybrid inertial Yosida approximation iterative schemes for exploring a split variational inclusion problem and a fixed point of a nonexpansive mapping. Unlike existing methods, our schemes initiate the process by computing a Mann-type iteration that incorporates both an inertial extrapolation and a fixed-point iteration. The Yosida approximation operators associated with the corresponding monotone mappings are employed. We establish strong convergence theorems for the proposed schemes under suitable assumptions without estimating the norm of a bounded linear operator. Numerical examples are presented to validate the theoretical results, and a comparison of the proposed iterative schemes with existing methods is provided. Finally, an application of our schemes for solving the split common fixed point problem (SCFPP) is also discussed.

Keywords: split variational inclusion; fixed point; Inertial; algorithm; Yosida approximation; strong convergence
Mathematics Subject Classification: 47H05, 47H06, 49J40

1. Introduction

The fixed point problem (FPP) of a nonexpansive mapping $T : \mathbb{V} \to \mathbb{V}$ is defined as:

$$\text{Find } z \in \mathbb{V} \text{ such that } T(z) = z.$$

The set of fixed points of T is denoted by $\mathrm{Fix}(T)$. Notice that $z \in \mathrm{Fix}(T)$ can equivalently be expressed as a zero of the operator $I - T$, where I denotes the identity operator on Hilbert space $\mathbb{V}$. This reformulation offers a natural and coherent framework for studying various nonlinear problems, with applications in optimization, finance, and analysis, see, [10, 13, 20].

A widely used scheme for estimating fixed points of nonlinear mappings in Hilbert or Banach spaces is the Mann iterative scheme [22], given by

$$z_{r+1} = (1 - \zeta_r)z_r + \zeta_r T z_r, \tag{1.1}$$

where $\{\zeta_r\} \subset (0, 1)$ is a control sequence. This iteration process guarantees the weak convergence, meanwhile it is noted that the convergence of $\{z_r\}$ is slow toward the fixed point of nonexpansive mappings T.

To improve the rate of convergence, various modifications of the classical Mann iteration have been proposed. One of such extensions introduces an additional control parameter θ_r, resulting in the efficiency of the iterative process

$$z_{r+1} = (1 - \zeta_r - \theta_r)z_r + \zeta_r T z_r. \tag{1.2}$$

This modified scheme generalizes the classical Mann iteration (recovered when $\theta_r = 0$) and accelerates the rate of convergence under suitable assumptions.

One of the most notable and widely used problems studied in [15] is referred to as a split feasibility problem (SFP), which is the extension of the convex feasibility problem. A broad range of inverse problems including phase retrieval, reconstruction of medical images, signal and data recovery, computed tomography, and treatment of radiation therapy can be formulated within the SFP framework. For comprehensive discussions and additional applications, see, [12, 14, 16] and the references therein.

Byrne [12] investigated the CQ method together with several iterative schemes and their convergence properties for approximating solutions of the SFP. In parallel, various forms of feasible sets have been explored within the SFP framework. Building on these developments, Moudafi [24] instigated the split monotone variational inclusion problem (SMVIP), formulated as below:

$$\text{Find } x^* \in \mathbb{V}_1 \text{ such that } 0 \in G_1(x^*) + F_1(x^*) \text{ and } 0 \in G_2(Mx^*) + F_2(Mx^*), \tag{1.3}$$

where $M : \mathbb{V}_1 \to \mathbb{V}_2$ is a bounded linear operator; $F_1 : \mathbb{V}_1 \to 2^{\mathbb{V}_1}$, $F_2 : \mathbb{V}_2 \to 2^{\mathbb{V}_2}$ are monotone mappings; and $G_1 : \mathbb{V}_1 \to \mathbb{V}_1$, $G_2 : \mathbb{V}_2 \to \mathbb{V}_2$ are two single-valued mappings. To examine SMVIP, he proposed the following scheme: For arbitrary $z_0 \in \mathbb{V}_1$ and $\mu > 0$, compute

$$z_{r+1} = W[z_r + \sigma M^*(X - I)Mz_r], \tag{1.4}$$

where $\sigma \in (0, \frac{1}{\alpha})$ and α is the spectral radius of M^*M, $W = R_\mu^{F_1}(I - \mu G_1)$, $X = R_\mu^{F_2}(I - \mu G_2)$, $R_\mu^{F_1} = (I + \mu F_1)^{-1}$ and $R_\mu^{F_2} = (I + \mu F_1)^{-1}$ are resolvents of F_1 and F_2, respectively. The split variational inclusion problem (SVIP) is a special case of SMVIP, which can be obtained by considering $G_1 = G_2 = 0$ in (1.3):

$$\text{Find } x^* \in \mathbb{V}_1 \text{ such that } 0 \in F_1(x^*) \text{ and } 0 \in F_2(Mx^*).$$

Byrne et al. [11] conducted a detailed study of the SVIP, analyzed it through the following method:

$$z_{r+1} = R_{\mu}^{F_1}[z_r + \sigma M^*(R_{\mu}^{F_2} - I)Mz_r], \tag{1.5}$$

and proved that the weak limit of $\{z_r\}$ approximates the solution of the SVIP. Subsequently, the common solution to the SVIP and FPP for a mapping $T : \mathbb{V}_1 \to \mathbb{V}_1$ was reported by Kazmi and Rizvi [19], using the following iterative approach:

$$\begin{cases} u_r = R_{\mu}^{F_1}[z_r + \sigma M^*(R_{\mu}^{F_2} - I)Mz_r], \\ z_{r+1} = \varsigma_r f(z_r) + (1 - \varsigma_r)Tu_r, \end{cases} \tag{1.6}$$

where f is a contraction mapping, T is nonexpansive, $\mu > 0$, $\sigma \in (0, \frac{1}{\|M\|^2})$, and the sequence $\varsigma_r \in (0, 1)$ satisfies $\lim\limits_{r\to\infty} \varsigma_r = 0$, $\sum\limits_{r=1}^{\infty} \varsigma_r = \infty$, and $\sum\limits_{r=1}^{\infty} |\varsigma_r - \varsigma_{r-1}| < \infty$.

More recently, Akram et al. [1] presented a modification of scheme (1.6) and estimated a common solution of the SVIP and FPP, using the following method:

$$\begin{cases} u_r = z_r - \sigma[(I - R_{\mu_1}^{F_1})z_r + M^*(I - R_{\mu_2}^{F_2})Mz_r], \\ x_{r+1} = \varsigma_r f(z_r) + (1 - \varsigma_r)T(u_r), \end{cases} \tag{1.7}$$

where $\sigma = \frac{1}{1+\|M\|^2}$.

The study of solutions to split problems and FPP, along with their applications in Banach and Hilbert spaces, has attracted considerable interest. Numerous authors have contributed to this area for more details; see [6, 31, 33].

Notice that all the aforementioned techniques required the step size based on the computationally challenging operator norm $\|M\|$. This limitation was addressed by López et al. [21] by designing the new scheme with a step size independent of the matrix norm:

$$z_{r+1} = P_C[I - \sigma_r M^*(I - P_Q)M]z_r, \ \forall n \geq 1, \tag{1.8}$$

where $\sigma_r = \frac{\gamma_r f(z_r)}{\|\nabla f(z_r)\|^2}$ with $f(x) = \frac{1}{2}\|(I - P_Q Mx\|^2, \nabla f(x) = M^*(I - P_Q)Mx,\ r \geq 0,\ 0 < \gamma_r < 4, \inf \gamma_r(4 - \gamma_r) > 0$, and P_C and P_Q are the orthogonal projections on closed and convex subsets C and Q, respectively. Moudafi [25] addressed the SFP without requiring prior computation of the operator norm. Some iterative schemes that investigated the SVIP without the need for estimating the norm of the bounded linear operator can be seen in [18, 28, 34].

To enhance convergence, Polyak [30] proposed the scheme involving the inertial term for smooth convex optimization problems. Its efficiency has motivated the integration of inertial terms in many algorithms. Alvarez and Attouch [4] developed an inertial iterative method to deal with the null point problem. Their method incorporates arbitrary initial points z_0, z_1, and parameter $\epsilon_r \in [0, 1)$ is

$$z_{r+1} = R_{\mu_r}^{F_1}[z_r + \epsilon_r(z_r - z_{r-1})],\ n \geq 1, \tag{1.9}$$

where $R_{\mu_r}^{F_1}$ is the resolvent of F_1, and $\mu_r > 0$. To accelerate the convergence, inertial step has been adopted and applied by numerous authors to solve nonlinear problems; see [26, 27, 35].

It is worth emphasizing that set-valued monotone mappings can be transformed into single-valued monotone operators through the Yosida approximation, which serves as an effective tool for the analysis of variational inclusions and related problems in both linear and nonlinear spaces. The Yosida approximation operator for a monotone mapping F_1 is characterized as $J_\mu^{F_1} = \frac{1}{\mu}(I - R_\mu^{F_1})$, where $R_\mu^{F_1}$ denotes the resolvent of F_1.

Following the research work discussed above, Dilshad et al. [17] investigated an inertial Yosida approximation method to examine the SVIP and FPP, in which the computation of step-size was free from the estimation of $\|M\|$. They proved the weak as well as strong convergence theorems of the suggested methods. The iterative scheme can be summarized as follows:

$$\begin{aligned} u_r &= z_r + \epsilon_n(z_r - z_{r-1}), \\ v_r &= u_r - \sigma_r[J_{\mu_1}^{F_1}(u_r) + M^* J_{\mu_2}^{F_2}(Mu_r)], \\ z_{r+1} &= (1 - \varsigma_r - \theta_r)v_r + \varsigma_r T(v_r), \end{aligned}$$

where T is nonexpansive mapping; $J_{\mu_1}^{F_1}$ and $J_{\mu_1}^{F_1}$ are Yosida approximation operators of F_1 and F_2, respectively; $\{\varsigma_r\}$, $\{\theta_r\}$ are control parameters and $\epsilon_n(z_r - z_{r-1})$ is the inertial term. The step size $\sigma_r = \frac{\|J_{\mu_1}^{F_1}(u_r)\|^2 + \|J_{\mu_2}^{F_2}(Mu_r)\|^2}{\|J_{\mu_1}^{F_1}(u_r) + M^* J_{\mu_2}^{F_2}(Mu_r)\|^2}$, will be further discussed in Section 3.

The Yosida approximation operators have been extensively employed as a fundamental tool in the analysis of nonlinear problems, particularly in the study of variational inclusions. They provide an effective framework for handling multivalued operators and facilitating the convergence analysis of iterative schemes. Consequently, these operators have been widely utilized in the investigation of variational inclusions [5, 7, 8], systems of variational inclusions [2, 3], as well as split variational inclusions [17, 18].

Motivated by the above mentioned discussion, we propose Mann-type hybrid inertial iterative schemes for the SVIP and FPP. Unlike the traditional methods, our schemes are designed in such a way that the Mann iteration is computed jointly with the inertial extrapolation and fixed point iteration in the starting of the iteration process. We use the Yosida approximation operators of monotone mappings F_1 and F_2 such that the estimation of norm of a bounded linear is not required. Our schemes are simple, easily implementable, and significantly reduce the number of iterations and CPU time in comparison to some related works.

The organization of the paper is as follows: In Section 2, we present some pivotal tools and basic definitions. In Section 3, Mann-type hybrid inertial Yosida approximation schemes are derived, and strong convergence theorems are proved. In Section 4, we verify the effectiveness of the considered schemes by comparing them with some related works with the help of some numerical examples. Section 5 describes the application of the proposed schemes to solve the split common fixed point problem (SCFPP) for firmly nonexpansive mappings.

2. Preliminaries

Let $\mathbb{V}$ denote a real Hilbert space endowed with the inner product $\langle \cdot, \cdot \rangle$ and its induced norm $\| \cdot \|$. Strong and weak convergence will be written as $\to$ and $\rightharpoonup$, respectively.

Definition 2.1. *For all $\varrho, \varsigma \in \mathbb{V}$, an operator $F : \mathbb{V} \to \mathbb{V}$ is called*

(i) contraction, if

$$\|F\varrho - F\varsigma\| \le \kappa \|\varrho - \varsigma\|, \qquad \kappa \in [0, 1);$$

(ii) firmly nonexpansive, if

$$\|F\varrho - F\varsigma\|^2 \leq \langle \varrho - \varsigma\, F\varrho - F\varsigma\rangle;$$

(iii) τ-inverse strongly monotone, if $\exists\tau > 0$ such that

$$\langle F\varrho - F\varsigma\, \varrho - \varsigma\rangle \geq \tau\|F\varrho - F\varsigma\|^2.$$

If $\tau = 1$, then F is nonexpansive.

Remark 2.1. *For all $\varrho, \varsigma, \varepsilon \in \mathbb{V}, \alpha_1, \alpha_2, \alpha_3 \in [0, 1]$ with $\alpha_1 + \alpha_2 + \alpha_3 = 1$, the following characteristic equalities and inequality hold:*

(i)

$$\|\alpha_1\varrho + \alpha_2\varsigma + \alpha_3\varepsilon\|^2 = \alpha_1|\varrho\|^2 + \alpha_2\|\varsigma\|^2 + \alpha_3\|\varepsilon\|^2 - \alpha_1\alpha_2\|\varrho - \varsigma\|^2 \\ - \alpha_2\alpha_3\|\varsigma - \varepsilon\|^2 - \alpha_3\alpha_1\|\varrho - \varepsilon\|^2.$$

(ii)

$$\|\varsigma \pm \varepsilon\|^2 = \|\varsigma\|^2 \pm 2\langle\varepsilon,\, \varepsilon\rangle + \|\varepsilon\|^2.$$

(iii)

$$\|\varsigma + \varepsilon\|^2 \leq \|\varsigma\|^2 + 2\langle\varepsilon,\, \varsigma + \varepsilon\rangle.$$

Definition 2.2. *[19] Let $\varsigma \in \mathbb{V}$ and the projection $P_{\mathbb{D}}\varsigma$ of ς onto $\mathbb{D} \subset \mathbb{V}$ is expressed as*

$$\|\varsigma - P_{\mathbb{D}}\varsigma\| \leq \|\varsigma - \varepsilon\|,\ \forall\varepsilon \in \mathbb{D}.$$

The following key relations hold for $P_{\mathbb{D}}\varsigma$:

$$\|P_{\mathbb{D}}\varsigma - P_{\mathbb{D}}\varepsilon\|^2 \leq \langle\varsigma - \varepsilon, P_{\mathbb{D}}\varsigma - P_{\mathbb{D}}\varepsilon\rangle, \forall\varsigma, \varepsilon \in \mathbb{V}$$

$$\text{and} \quad P_{\mathbb{D}}\varsigma = \varrho \Leftrightarrow \langle\varsigma - \varrho, \varepsilon - \varrho\rangle \geq 0,\ \varepsilon \in \mathbb{D}.$$

Definition 2.3. *[9] A mapping $F : \mathbb{V} \to 2^{\mathbb{V}}$ is said to be monotone if $\langle\varsigma - \varrho,\ x - y\rangle \geq 0, \forall\varsigma \in F(x)$ and $\varrho \in F(y)$.*

Lemma 2.1. *[29] Let $\mathbb{D}(\neq \emptyset) \subset \mathbb{V}$ and $\{u_r\}$ be a bounded sequence in $\mathbb{V}$ such that*

(i) $\lim_{r\to\infty} \|u_r - p\|$ exists for every $p \in \mathbb{D}$,
(ii) $\omega_w(u_r) \subset \mathbb{D}$.

Then, there exists $p \in \mathbb{D}$ such that $u_r \rightharpoonup p$ as $r \to \infty$.

Lemma 2.2. *[32] If $\{w_r\}$ is a nonnegative real sequence satisfying*

$$w_{r+1} \leq (1 - \varsigma_r)w_r + \varphi_r,$$

where $\{\varsigma_r\} \in (0, 1)$ and $\{\varphi_r\} \in (-\infty, +\infty)$ are sequences such that *(i)* $\sum_{r=1}^{\infty} \varsigma_r = \infty$,

(ii) $\limsup_{r\to\infty} \frac{\varphi_r}{\varsigma_r} \leq 0$ *or* $\limsup_{r\to\infty} |\varphi_r| < \infty$,

then $\lim_{r\to\infty} w_r = 0$.

Lemma 2.3. *[23] Let* $\{\Gamma_r\} \in (-\infty, +\infty)$ *be a sequence that does not decrease at infinity in the sense that there exists a subsequence* $\{\Gamma_{r_k}\}$ *of* $\{\Gamma_r\}$ *such that* $\Gamma_{r_k} < \Gamma_{r_k+1}$ *for all* $k \geq 0$*. Also, consider the sequence of integers* $\{g(r)\}_{r\geq r_0}$ *defined by*

$$g(r) = \max\{k \leq r : \Gamma_r \leq \Gamma_{r+1}\}.$$

Then, $\{g(r)\}_{r\geq r_0}$ *is a nondecreasing sequence verifying* $\lim_{r\to\infty} g(r) = \infty$*, and for all* $r \geq r_0$,

$$\max\{\Gamma_{g(r)}, \Gamma_r\} \leq \Gamma_{g(r)+1}.$$

3. Main results

We denote by $\mathbb{S}_1$ and $\mathbb{S}_2$ the solution set of the FPP and SVIP, respectively, such that $\mathbb{S}_1 \cap \mathbb{S}_2 \neq \emptyset$. We suppose the following assumptions.

Assumption 3.1. ($\mathbf{A_1}$) *Suppose* $J_{\mu_1}^{F_1}$ *and* $J_{\mu_2}^{F_2}$ *are Yosida approximation operators of monotone mappings* $F_1 : \mathbb{V}_1 \rightrightarrows \mathbb{V}_1$ *and* $F_2 : \mathbb{V}_2 \rightrightarrows \mathbb{V}_2$*, respectively, and* $T : \mathbb{V}_1 \to \mathbb{V}_1$ *is nonexpasnive mapping;*
($\mathbf{A_2}$) $M : \mathbb{V}_1 \to \mathbb{V}_2$ *is a bounded linear operator;*
($\mathbf{A_3}$) $\{\varsigma_r\}, \{\theta_r\} \subseteq (0, 1)$ *so that* $\lim_{r\to\infty} \theta_r = 0$ *and* $\sum_{r=1}^{\infty} \theta_r = \infty$, $\inf \varsigma_r(1 - \varsigma_r - \theta_r) > 0$*;*
($\mathbf{A_4}$) $\{\delta_r\}$ *is a positive sequence for which* $\sum_{r=1}^{\infty} \delta_r < \infty$ *holds and* $\lim_{r\to\infty} \frac{\delta_r}{\theta_r} = 0$.

Algorithm 3.1. *Hybrid inertial Yosida approximation iterative scheme-I*

Step 0: *Choose* $\epsilon \in [0, 1)$, $\mu_1 > 0, \mu_2 > 0$ *and* $\mu = \min\{\mu_1, \mu_2\}$ *such that* $\mu > \frac{1}{2}$.

Step 1: *For arbitrary* z_0 *and* z_1 *and* $r \geq 1$*, choose* $0 < \epsilon_r < \bar{\epsilon}_r$*, where*

$$\bar{\epsilon}_r = \begin{cases} \min\{\frac{\delta_r}{\|z_r - z_{r-1}\|}, \quad \epsilon\}, & \text{if } z_r \neq z_{r-1}, \\ \epsilon, & \text{otherwise.} \end{cases} \tag{3.1}$$

Compute

$$u_r = (1 - \varsigma_r - \theta_r)z_r + \varsigma_r T z_r + \epsilon_r(z_r - z_{r-1}), \tag{3.2}$$

$$z_{r+1} = u_r - \sigma_r[J_{\mu_1}^{F_1}(u_r) + M^* J_{\mu_2}^{F_2}(Mu_r)], \tag{3.3}$$

where

$$\sigma_r = \begin{cases} \frac{\|J_{\mu_1}^{F_1}(u_r)\|^2 + \|J_{\mu_2}^{F_2}(Mu_r)\|^2}{\|J_{\mu_1}^{F_1}(u_r) + M^* J_{\mu_2}^{F_2}(Mu_r)\|^2}, & \text{if } \|J_{\mu_1}^{F_1}(u_r) + M^* J_{\mu_2}^{F_2}(Mu_r)\| \neq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{3.4}$$

Stopping criteria: *Stop if* $z_{r+1} = z_r = u_r$*; otherwise, go to Step 1.*

Remark 3.1. *From* (3.1)*, we have* $\epsilon_r\|z_r - z_{r-1}\| \le \delta_r$*. With the assumption on* δ_r*, we obtain* $\lim_{r\to\infty} \epsilon_r\|z_r - z_{r-1}\| = 0$*. Since* $\lim_{r\to\infty} \frac{\delta_r}{\theta_r} \to 0$ *as* $r \to \infty$*, we have* $\frac{\epsilon_r\|z_r - z_{r-1}\|}{\theta_r} = 0$.

Lemma 3.1. *[17] If* $\lim_{r\to\infty} \frac{(\|J_{\mu_1}^{F_1}(u_r)\|^2+\|J_{\mu_2}^{F_2}(Mu_r)\|^2)^2}{\|J_{\mu_1}^{F_1}(u_r)+M^*J_{\mu_2}^{F_2}(Mu_r)\|^2} = 0$*, then* $\lim_{r\to\infty} \|J_{\mu_1}^{F_1}(u_r)\| = \lim_{r\to\infty} \|J_{\mu_2}^{F_2}(Mu_r)\| = 0$.

Lemma 3.2. *[17] The assertions given below for the operators* $R_{\mu_1}^{F_1}$ *and* $J_{\mu_1}^{F_1}$ *of the monotone mapping* F_1 *are analogous.*

(i) $p \in \mathbb{V}_1$ *is the solution of* $(F_1)^{-1}(0)$,
(ii) $R_{\mu_1}^{F_1}(p) = p$,
(iii) $J_{\mu_1}^{F_1}(p) = 0$.

Lemma 3.3. *[9] Let* $T : \mathbb{V} \to \mathbb{V}$ *be an operator. If* T *is nonexpansive, then* $I - T$ *is demiclosed at zero. Moreover, if* T *is firmly noexpansive, then* $I - T$ *is also firmly nonexpansive.*

Remark 3.2. *If* F_1 *is maximal monotone, then* $R_{\mu_1}^{F_1}$ *and* $[I - R_{\mu_1}^{F_1}]$ *are firmly nonexpansive [9, Corollary 23.10]. Hence, by [5, Lemma 1(v)], we conclude that the operator* $J_{\mu_1}^{F_1} = \frac{1}{\mu}[I - R_{\mu_1}^{F_1}]$ *is* μ_1*-inverse strongly monotone.*

Remark 3.3. *Let* $z_{r+1} = z_r = u_r$*. Then, if* $\|J_{\mu_1}^{F_1}(u_r) + M^*J_{\mu_2}^{F_2}(Mu_r)\| = 0$*, it implies that*

$$0 = \|J_{\mu_1}^{F_1}(z_r) + M^*J_{\mu_2}^{F_2}(Mz_r)\|^2 \ge 2\|J_{\mu_1}^{F_1}(z_r)\|^2 + 2\|M^*J_{\mu_2}^{F_2}(Mz_r)\|^2 \ge 0.$$

Hence, we get $\|J_{\mu_1}^{F_1}(z_r)\| = 0$ *and* $\|M^*J_{\mu_2}^{F_2}(Mz_r)\| = 0$*. Since* M *is bounded, we have* $\|J_{\mu_2}^{F_2}(Mz_r)\| = 0$*, hence,* $z_r \in \mathbb{S}_1$*. If* $\|J_{\mu_1}^{F_1}(z_r) + M^*J_{\mu_2}^{F_2}(Mz_r)\| \ne 0$*, then recalling* (3.3)*, we acquire*

$$\frac{\|J_{\mu_1}^{F_1}(z_r)\|^2 + \|J_{\mu_2}^{F_2}(Mz_r)\|^2}{\|J_{\mu_1}^{F_1}(z_r) + M^*J_{\mu_2}^{F_2}(Mz_r)\|^2}[J_{\mu_1}^{F_1}(z_r) + M^*J_{\mu_2}^{F_2}(Mz_r)] = 0.$$

Now implementing the norm on both sides, we get

$$\frac{\|J_{\mu_1}^{F_1}(z_r)\|^2 + \|J_{\mu_2}^{F_2}(Mz_r)\|^2}{\|J_{\mu_1}^{F_1}(z_r) + M^*J_{\mu_2}^{F_2}(Mz_r)\|} = 0.$$

Lemma 3.1 yields, $\|J_{\mu_1}^{F_1}(z_r)\| = \|J_{\mu_2}^{F_2}(Mz_r)\| = 0$*, and Lemma 2.1 implies that* $z_r \in \mathbb{S}_1$*. Furthermore, from* (3.2)*,* $z_{r+1} = (1 - \varsigma_r - \theta_r)z_r + \varsigma_r Tz_r$ *implies that* $\{z_r\}$ *converges strongly to some fixed point of* T *in* $\mathbb{S}_2$.

Theorem 3.1. *If Assumptions* (A_1)*–*(A_4) *hold and* $\{z_r\}$ *is produced by Algorithm 3.1, then the sequence* $\{z_r\}$ *converges to* $p \in \mathbb{S}_1 \cap \mathbb{S}_2$ *such that* $p = P_{\mathbb{S}_1\cap\mathbb{S}_2}(0)$.

Proof. Let $p = P_{\mathbb{S}_1\cap\mathbb{S}_2}(0)$. Now, using Remark 2.1 (*iii*) and (3.3), we obtain

$$\begin{aligned} \|z_{r+1} - p\|^2 &= \|u_r - \sigma_r[J_{\mu_1}^{F_1}(u_r) + M^*J_{\mu_2}^{F_2}(Mu_r)] - p\|^2 \\ &\le \|u_r - p\|^2 + \sigma_r^2\|J_{\mu_1}^{F_1}(u_r) + M^*J_{\mu_2}^{F_2}(Mu_r)\|^2 \\ &\quad - 2\sigma_r\Big\langle J_{\mu_1}^{F_1}(u_r) + M^*J_{\mu_2}^{F_2}(Mu_r), \quad u_r - p\Big\rangle. \end{aligned} \tag{3.5}$$

For $p \in \mathbb{S}_1 \cap \mathbb{S}_2$, Lemma 3.2 yields $J_{\mu_1}^{F_1}(p) = 0$ and $J_{\mu_2}^{F_2}(Mp) = 0$. Utilizing the μ_1-inverse strongly monotone property of $J_{\mu_1}^{F_1}$ (Remark 3.2), we can write

$$\begin{aligned}
&\Big\langle J_{\mu_1}^{F_1}(u_r) + M^* J_{\mu_2}^{F_2}(Mu_r), \quad u_r - p\Big\rangle \\
&= \Big\langle J_{\mu_1}^{F_1}(u_r), \quad u_r - p\rangle + \langle M^* J_{\mu_2}^{F_2}(Mu_r), \quad u_r - p\Big\rangle \\
&= \Big\langle J_{\mu_1}^{F_1}(u_r) - J_{\mu_1}^{F_1}(p), \quad u_r - p\rangle + \langle M^*(J_{\mu_2}^{F_2}(Mu_r) - J_{\mu_2}^{F_2}(Mp)), \quad u_r - p\Big\rangle \\
&= \Big\langle J_{\mu_1}^{F_1}(u_r) - J_{\mu_1}^{F_1}(p), \quad u_r - p\Big\rangle + \Big\langle J_{\mu_2}^{F_2}(Mu_r) - J_{\mu_2}^{F_2}(Mp), \quad M(u_r) - M(p)\Big\rangle \\
&\geq \mu_1 \|J_{\mu_1}^{F_1}(u_r) - J_{\mu_1}^{F_1}(p)\|^2 + \mu_2 \|J_{\mu_2}^{F_2}(Mu_r) - J_{\mu_2}^{F_2}(Mp)\|^2 \\
&\geq \min\{\mu_1, \mu_2\}\{\|J_{\mu_1}^{F_1}(u_r)\|^2 + \|J_{\mu_2}^{F_2}(Mu_r)\|^2\} \\
&\geq \mu\Big\{\|J_{\mu_1}^{F_1}(u_r)\|^2 + \|J_{\mu_2}^{F_2}(Mu_r)\|^2\Big\}, \qquad (3.6)
\end{aligned}$$

and

$$\begin{aligned}
\sigma_r^2 \|J_{\mu_1}^{F_1}(u_r) \;&+\; M^* J_{\mu_2}^{F_2}(Mu_r)\|^2 - 2\sigma_r \Big\langle J_{\mu_1}^{F_1}(u_r) + M^* J_{\mu_2}^{F_2}(Mu_r), \quad u_r - p\Big\rangle \\
&\leq \frac{\Big(\|J_{\mu_1}^{F_1}(u_r)\|^2 + \|J_{\mu_2}^{F_2}(Mu_r)\|^2\Big)^2}{\|J_{\mu_1}^{F_1}(u_r) + M^* J_{\mu_2}^{F_2}(Mu_r)\|^2} - 2\mu \frac{\Big(\|J_{\mu_1}^{F_1}(u_r)\|^2 + \|J_{\mu_2}^{F_2}(Mu_r)\|^2\Big)^2}{\|J_{\mu_1}^{F_1}(u_r) + M^* J_{\mu_2}^{F_2}(Mu_r)\|^2} \\
&= (1 - 2\mu) \frac{\Big(\|J_{\mu_1}^{F_1}(u_r)\|^2 + \|J_{\mu_2}^{F_2}(Mu_r)\|^2\Big)^2}{\|J_{\mu_1}^{F_1}(u_r) + M^* J_{\mu_2}^{F_2}(Mu_r)\|^2}. \qquad (3.7)
\end{aligned}$$

From (3.5)–(3.7), we achieve

$$\|z_{r+1} - p\|^2 \leq \|u_r - p\|^2 + (1 - 2\mu) \frac{\Big(\|J_{\mu_1}^{F_1}(u_r)\|^2 + \|J_{\mu_2}^{F_2}(Mu_r)\|^2\Big)^2}{\|J_{\mu_1}^{F_1}(u_r) + M^* J_{\mu_2}^{F_2}(Mu_r)\|^2}. \qquad (3.8)$$

Since $\mu > \frac{1}{2}$, we can have

$$\|z_{r+1} - p\| \leq \|u_r - p\|. \qquad (3.9)$$

Next, we show that $\{u_r\}$ is bounded. Since $\frac{\epsilon_r}{\theta_r}\|z_r - z_{r-1}\| \to 0$ as $r \to \infty$, it implies there exists a constant K_1 such that $\frac{\epsilon_r}{\theta_r}\|z_r - z_{r-1}\| \leq K_1$. From (3.2), we have

$$\begin{aligned}
\|u_r - p\| &= \|(1 - \varsigma_r - \theta_r)z_r + \varsigma_r T(z_r) + \epsilon_r(z_r - z_{r-1}) - p\| \\
&= \|(1 - \varsigma_r - \theta_r)(z_r - p) + \varsigma_r(T(z_r) - p) + \theta_r(-p) + \epsilon_r(z_r - z_{r-1})\| \\
&\leq (1 - \varsigma_r - \theta_r)\|z_r - p\| + \varsigma_r\|T(z_r) - p\| + \theta_r\|p\| + \epsilon_r\|z_r - z_{r-1}\| \\
&\leq (1 - \theta_r)\|z_r - p\| + \theta_r\left[\|p\| + \frac{\epsilon_r}{\theta_r}\|z_r - z_{r-1}\|\right] \\
&\leq (1 - \theta_r)\|z_r - p\| + \theta_r\left[\|p\| + K_1\right].
\end{aligned}$$

Applying mathematical induction and using (3.9), we find that

$$\|u_r - p\| \leq max\Big\{\|z_0 - p\|, \|p\| + K_1\Big\},$$

that is, $\{u_r\}$ is bounded, and therefore $\{z_r\}$ is also bounded. Now, let $x_r = (1-\varsigma_r-\theta_r)z_r + \varsigma_r T(z_r)$, then by using Remark 2.1 (ii), we estimate

$$\begin{aligned}
\|u_r - p\|^2 &= \|x_r + \epsilon_r(z_r - z_{r-1}) - p\|^2 \\
&\le \|x_r - p\|^2 + 2\Big\langle \epsilon_r(z_r - z_{r-1}),\, u_r - p, \Big\rangle \\
&\le \|x_r - p\|^2 + 2\epsilon_r\|z_r - z_{r-1}\|\|u_r - p\| \\
&\le \|x_r - p\|^2 + 2K_2\epsilon_r\|z_r - z_{r-1}\|. \qquad (3.10)
\end{aligned}$$

Also,

$$\begin{aligned}
\|x_r - p\|^2 &= \|(1-\varsigma_r-\theta_r)z_r + \varsigma_r T(z_r) - p\|^2 \\
&= \|(1-\varsigma_r-\theta_r)(z_r - p) + \varsigma_r(T(z_r) - p) + \theta_r(-p)\|^2 \\
&\le (1-\varsigma_r-\theta_r)\|z_r - p\|^2 + \varsigma_r\|T(z_r) - p\|^2 + \theta_r\|p\|^2 \\
&\quad - \varsigma_r(1-\varsigma_r-\theta_r)\|z_r - T(z_r)\|^2 \\
&= (1-\theta_r)\|z_r - p\|^2 + \theta_r\|p\|^2 - \varsigma_r(1-\varsigma_r-\theta_r)\|z_r - T(z_r)\|^2. \qquad (3.11)
\end{aligned}$$

From (3.10) and (3.11), we have

$$\|u_r - p\|^2 \le (1-\theta_r)\|z_r - p\|^2 + \theta_r\|p\|^2 - \varsigma_r(1-\varsigma_r-\theta_r)\|z_r - T(z_r)\|^2 + 2K_2\epsilon_r\|z_r - z_{r-1}\|. \qquad (3.12)$$

Using (3.8) and (3.12), we obtain

$$\begin{aligned}
\|z_{r+1} - p\|^2 \;\le\; & (1-\theta_r)\|z_r - p\|^2 + \theta_r\|p\|^2 - \varsigma_r(1-\varsigma_r-\theta_r)\|z_r - T(z_r)\|^2 \\
& + 2K_2\epsilon_r\|z_r - z_{r-1}\| + (1-2\mu)\frac{(\|J_{\mu_1}^{F_1}(u_r)\|^2 + \|J_{\mu_2}^{F_2}(Mu_r)\|^2)^2}{\|J_{\mu_1}^{F_1}(u_r) + M^* J_{\mu_2}^{F_2}(Mu_r)\|^2}. \qquad (3.13)
\end{aligned}$$

There are possible two cases.

Case I. If the sequence $\{\|z_r - p\|\}$ is monotonically decreasing, $\exists N \in \mathbb{N}$ such that

$$\|z_{r+1} - p\| \le \|z_r - p\|, \forall r > N.$$

As the sequence $\{\|z_r - p\|\}$ is nonnegative and is bounded below, $\{\|z_r - p\|\}$ converges. Hence, from (3.13), it follows that

$$\begin{aligned}
&\varsigma_r(1-\varsigma_r-\theta_r)\|z_r - T(z_r)\|^2 - (1-2\mu)\frac{(\|J_{\mu_1}^{F_1}(u_r)\|^2 + \|J_{\mu_2}^{F_2}(Mu_r)\|^2)^2}{\|J_{\mu_1}^{F_1}(u_r) + M^* J_{\mu_2}^{F_2}(Mu_r)\|^2} \\
&\le \theta_r\left(\|p\|^2 - \|z_r - p\|^2\right) + 2K_2\epsilon_r\|z_r - z_{r-1}\|.
\end{aligned}$$

Since $\mu > 1/2, \inf \varsigma_r(1-\varsigma_r-\theta_r) > 0$ and using $\theta_r \to 0$ and $\epsilon_r\|z_r - z_{r-1}\| \to 0$ as $r \to \infty$, we get

$$\lim_{r\to\infty}\|z_r - T(z_r)\| = 0, \qquad (3.14)$$

and

$$\lim_{r\to\infty}\frac{\left(\|J_{\mu_1}^{F_1}(u_r)\|^2 + \|J_{\mu_2}^{F_2}(Mu_r)\|^2\right)^2}{\|J_{\mu_1}^{F_1}(u_r) + M^* J_{\mu_2}^{F_2}(Mu_r)\|^2} = 0. \qquad (3.15)$$

Applying Lemma 3.1 to (3.15), we get

$$\lim_{r\to\infty} \|J_{\mu_1}^{F_1}(u_r)\| = 0, \quad \lim_{r\to\infty} \|J_{\mu_2}^{F_2}(Mu_r)\| = 0. \tag{3.16}$$

Hence, from (3.3), we get

$$\lim_{r\to\infty} \|z_{r+1} - u_r\| = 0. \tag{3.17}$$

Since $\{z_r\}$ is bounded, there exists a subsequence $\{z_{r_k}\}$ of $\{z_r\}$ such that $z_{r_k} \rightharpoonup z^*$. Therefore, from (3.14) and (3.16), we can have

$$\lim_{k\to\infty} \|J_{\mu_1}^{F_1}(u_{r_k})\| = \|J_{\mu_1}^{F_1}(z^*)\| = 0, \ \lim_{k\to\infty} \|J_{\mu_2}^{F_2}(Mu_{r_k})\| = \|J_{\mu_2}^{F_2}(Mz^*)\| = 0, \ \lim_{k\to\infty} \|z_{r_k} - T(z_{r_k})\| = 0, \tag{3.18}$$

which conclude that $z^* \in \mathbb{S}_1 \cap \mathbb{S}_2$. Next, we show that $z_r \to z^*$. Moreover, letting $v_r = (1-\varsigma_r)z_r + \varsigma_r T(z_r)$, then $u_r = (1-\theta_r)v_r - \varsigma_r\theta_r(z_r - Tz_r) + \epsilon_r(z_r - z_{r-1})$, we estimate

$$\begin{aligned}
\|v_r - p\| &= \|(1-\varsigma_r)z_r + \varsigma_r Tz_r - p\| \\
&= \|(1-\varsigma_r)(z_r - p) + \varsigma_r T(z_r - p)\| \\
&\leq (1-\varsigma_r)\|z_r - p\| + \varsigma_r\|z_r - p\| \\
&\leq \|z_r - p\|.
\end{aligned}$$

In particular,

$$\|v_r - p\|^2 \leq \|z_r - p\|^2.$$

Now,

$$\begin{aligned}
\|u_r - p\|^2 &= \|(1-\theta_r)v_r - \varsigma_r\theta_r(z_r - Tz_r) + \epsilon_r(z_r - z_{r-1}) - p\|^2 \\
&= \|(1-\theta_r)(v_r - p) - \varsigma_r\theta_r(z_r - Tz_r) + \theta_r(-p) + \epsilon_r(z_r - z_{r-1})\|^2 \\
&\leq (1-\theta_r)\|v_r - p\|^2 - 2\Big\langle \varsigma_r\theta_r(z_r - T(z_r) + \theta_r p - \epsilon_r(z_r - z_{r-1}), u_r - p\Big\rangle \\
&\leq (1-\theta_r)\|z_r - p\|^2 - 2\theta_r\Big[\varsigma_r\Big\langle z_r - Tz_r), u_r - p\Big\rangle + \langle p, u_r - p\rangle \\
&\quad - \Big\langle \frac{\epsilon_r}{\theta_r}(z_r - z_{r-1}), u_r - p\Big\rangle\Big].
\end{aligned} \tag{3.19}$$

From (3.9) and (3.19), we get

$$\|z_{r+1} - p\|^2 \leq (1-\theta_r)\|z_r - p\|^2 + \theta_r X_r, \tag{3.20}$$

where $X_r = -2\Big[\varsigma_r\Big\langle z_r - Tz_r), u_r - p\Big\rangle + \Big\langle p, u_r - p\Big\rangle - \Big\langle \frac{\epsilon_r}{\theta_r}(z_r - z_{r-1}), u_r - p\Big\rangle\Big]$.
From the property of the projection operator, we have the following:

$$\liminf_{r\to\infty} \langle p, z_{r+1} - p\rangle \geq \min_{\tilde{z}\in\mathbb{S}_1\cap\mathbb{S}_2} \langle p, \tilde{z} - p\rangle \geq 0. \tag{3.21}$$

Keeping in mind (3.14) and (3.21) and applying Lemma 2.2, we obtain $\|z_r - p\| \to 0$, that is, $z_r \to p = P_{\mathbb{S}_1\cap\mathbb{S}_2}(0)$. Again, by metric projection, it follows that

$$\langle p, z - p\rangle \geq 0, \forall\, z \in \mathbb{S}_1 \cap \mathbb{S}_2,$$

which means that $\langle p, z\rangle \geq \|p\|^2$, hence $\|p\| \leq \|z\|$, that is, p is the solution with minimum norm.
Case II. If Case I is not true, then we can define the function $g : \mathbb{N} :\rightarrow \mathbb{N}$ by $g(r) = \max\{r \geq m : \|z_m - p\| \leq \|z_{m+1} - p\|\}$ to be an increasing sequence, $g(r) \rightarrow \infty$ as $r \rightarrow \infty$, and

$$0 \leq \|z_{g(r)} - p\| \leq \|z_{g(r)+1} - p\|, \ \forall\ r \geq m. \tag{3.22}$$

Applying the arguments used in the proof of Case I, we compare $\|z_{g(r)} - T(z_{g(r)})\| \rightarrow 0$ and $\|z_{g(r)+1} - u_{g(r)}\| \rightarrow 0$ as $r \rightarrow \infty$. Using (3.20) and (3.22), we obtain the following.

$$0 \leq \|z_{g(r)+1} - p\| \leq X_r. \tag{3.23}$$

Thus, we get $\|z_{g(r)} - p\| \rightarrow 0 \quad \text{as} \quad r \rightarrow \infty$. Using Lemma 2.3, we get the following:

$$0 \leq \|z_r - p\| \leq max\Big\{\|z_r - p\|, \|z_{g(r)} - p\|\Big\} \leq \|z_{g(r)+1} - p\|.$$

So, we can say $z_r \rightarrow p$ as $r \rightarrow \infty$. In both cases, we can say that $\{z_r\}$ converges to p, which solves the SVIP and FPP. □

Algorithm 3.2. *Hybrid inertial Yosida approximation iterative scheme-II*

Step 0: *Choose* $\epsilon \in [0, 1)$, $\mu_1 > 0, \mu_2 > 0$ *and* $\mu = \min\{\mu_1, \mu_2\}$ *such that* $\mu > \frac{1}{2}$.

Step 1: *For arbitrary* z_0, z_1 *and* $r \geq 1$, *choose* $0 < \epsilon_r < \bar{\epsilon}_r$, *and estimate*

$$u_r = (1 - \varsigma_r - \theta_r)z_r + \varsigma_r[Tz_r + \epsilon_r(z_r - z_{r-1})], \tag{3.24}$$
$$z_{r+1} = u_r - \sigma_r[J^{F_1}_{\mu_1}(u_r) + M^* J^{F_2}_{\mu_2}(Mu_r)], \tag{3.25}$$

where $\{\epsilon_r\}$ *and* $\{\sigma_r\}$ *are the same as defined in Algorithm 3.1.*
Stopping criteria: *Stop if* $z_{r+1} = z_r = u_r$; *otherwise, go to Step 1.*

Theorem 3.2. *If Assumptions* (A_1)–(A_4) *hold, then the sequence* $\{z_r\}$ *produced by Algorithm 3.2 converges to* $p \in \mathbb{S}_1 \cap \mathbb{S}_2$ *such that* $p = P_{\mathbb{S}_1\cap\mathbb{S}_2}(0)$.

Proof. Initially, we establish that $\{z_r\}$ is bounded. Let $p = P_{\mathbb{S}_1\cap\mathbb{S}_2}(0)$ and $y_r = T(z_r) + \epsilon_r(z_r - z_{r-1})$, then we have

$$\begin{aligned} \|y_r - p\| &= \|T(z_r) + \epsilon_r(z_r - z_{r-1}) - p\| \\ &\leq \|Tz_r - p\| + \epsilon_r(z_r - z_{r-1})\| \\ &\leq \|z_r - p\| + \epsilon_r\|z_r - z_{r-1}\|. \end{aligned} \tag{3.26}$$

Furthermore, from (3.3) and (3.26), we get

$$\begin{aligned} \|u_r - p\| &= \|(1 - \varsigma_r - \theta_r)z_r + \varsigma_r y_r - p\| \\ &= \|(1 - \varsigma_r - \theta_r)(z_r - p) + \varsigma_r(y_r - p) + \theta_r(-p)\| \\ &\leq (1 - \varsigma_r - \theta_r)\|z_r - p\| + \varsigma_r(\|z_r - p\| + \epsilon_r\|z_r - z_{r-1}\|) + \theta_r\|p\| \\ &\leq (1 - \varsigma_r - \theta_r)\|z_r - p\| + \varsigma_r\|z_r - p\| + \epsilon_r\|z_r - z_{r-1}\| + \theta_r\|p\| \\ &\leq (1 - \theta_r)\|z_r - p\| + \theta_r\left[\|p\| + K_1\right] \end{aligned}$$

$$
\begin{aligned}
&\leq \max\{\|z_r - p\|, \|p\| + K_1\},\\
&\leq \max\{\|u_{r-1} - p\|, \|p\| + K_1\},\\
&\quad \vdots\\
&\leq \max\{\|z_0 - p\|, \|p\| + K_1\},
\end{aligned}
$$

that is, $\{u_r\}$ is bounded and hence so is $\{z_r\}$ and $\{y_r\}$. We also have

$$
\begin{aligned}
\|y_r - p\|^2 &= \|T(z_r) + \epsilon_r(z_r - z_{r-1}) - p\|^2\\
&\leq \|T(z_r) - p\|^2 + 2\langle y_r - p,\ \epsilon_r(z_r - z_{r-1})\rangle\\
&\leq \|z_r - p\|^2 + 2\langle y_r - p,\ \epsilon_r(z_r - z_{r-1})\rangle, \qquad (3.27)
\end{aligned}
$$

and

$$
\begin{aligned}
\|y_r - z_r\|^2 &= \|T(z_r) + \epsilon_r(z_r - z_{r-1}) - z_r\|^2\\
&\leq \|T(z_r) - z_r\|^2 + 2\langle y_r - z_r, \epsilon_r(z_r - z_{r-1})\rangle. \qquad (3.28)
\end{aligned}
$$

Using (3.24) and (3.28), we get

$$
\begin{aligned}
\|u_r - p\|^2 &\leq \|(1 - \varsigma_r - \theta_r)z_r + \varsigma_r y_r - p\|^2\\
&= \|(1 - \varsigma_r - \theta_r)(z_r - p) + \varsigma_r(y_r - p) + \theta_r(-p)\|^2\\
&\leq (1 - \varsigma_r - \theta_r)\|z_r - p\|^2 + \varsigma_r\|y_r - p\|^2 + \theta_r\| - p\|^2 - \varsigma_r(1 - \varsigma_r - \theta_r)\|z_r - y_r\|^2\\
&\leq (1 - \varsigma_r - \theta_r)\|z_r - p\|^2 + \varsigma_r\|z_r - p\|^2 + 2\varsigma_r\Big\langle y_r - p, \epsilon_r(z_r - z_{r-1})\Big\rangle + \theta_r\|p\|^2\\
&\quad - \varsigma_r(1 - \varsigma_r - \theta_r)\Big[\|T(z_r) - z_r\|^2 + 2\Big\langle y_r - z_r, \epsilon_r(z_r - z_{r-1})\Big\rangle\Big]\\
&\leq (1 - \theta_r)\|z_r - p\|^2 + \theta_r\|p\|^2 + 2\varsigma_r\Big\langle y_r - p, \epsilon_r(z_r - z_{r-1})\Big\rangle\\
&\quad - \varsigma_r(1 - \varsigma_r - \theta_r)\Big[\|T(z_r) - z_r\|^2 + 2\Big\langle y_r - z_r, \epsilon_r(z_r - z_{r-1})\Big\rangle\Big]. \qquad (3.29)
\end{aligned}
$$

From (3.8) and (3.29), we can express

$$
\begin{aligned}
\|z_{r+1} - p\|^2 &\leq (1 - \theta_r)\|z_r - p\|^2 + \theta_r\|p\|^2 + \varsigma_r\Big[2\Big\langle y_r - p, \epsilon_r(z_r - z_{r-1})\Big\rangle\\
&\quad - (1 - \varsigma_r - \theta_r)\|T(z_r) - z_r\|^2 + 2\Big\langle y_r - z_r, \epsilon_r(z_r - z_{r-1})\Big\rangle\Big]\\
&\quad + (1 - 2\mu)\frac{(\|J^{F_1}_{\mu_1}(u_r)\|^2 + \|J^{F_2}_{\mu_2}(Mu_r)\|^2)^2}{\|J^{F_1}_{\mu_1}(u_r) + M^* J^{F_2}_{\mu_2}(Mu_r)\|^2}. \qquad (3.30)
\end{aligned}
$$

Considering case I and using the same reason as in the proof of Theorem 3.1, we infer that

$$
\lim_{r\to\infty} \frac{(\|J^{F_1}_{\mu_1}(u_r)\|^2 + \|J^{F_2}_{\mu_2}(Mu_r)\|^2)^2}{\|J^{F_1}_{\mu_1}(u_r) + M^* J^{F_2}_{\mu_2}(Mu_r)\|^2} = 0, \quad \text{and} \quad \lim_{r\to\infty} \|T(z_r) - z_r\| = 0. \qquad (3.31)
$$

Hence, from (3.25), we obtain $\lim_{r\to\infty} \|z_{r+1} - u_r\| = 0$. Furthermore, let $v_r = (1 - \varsigma_r)z_r + \varsigma_r y_r$. Using the boundedness of $\{y_r\}$ implies that there exist K_3 such that $\|y_r - p\| \leq K_3$, then we have

$$
\|v_r - p\|^2 \quad = \quad \|(1 - \varsigma_r)z_r + \varsigma_r y_r - p\|^2
$$

$$
\begin{aligned}
&= \|(1-\varsigma_r)(z_r-p)+\varsigma_r(y_r-p)\|^2 \\
&\leq (1-\varsigma_r)\|z_r-p\|^2+\varsigma_r\|y_r-p\|^2 \\
&\leq (1-\varsigma_r)\|z_r-p\|^2+\varsigma_r\Big[\|z_r-p\|^2+2\langle y_r-p,\epsilon_r(z_r-z_{r-1}),\rangle\Big] \\
&= \|z_r-p\|^2+2\varsigma_r\Big\langle y_r-p,\epsilon_r(z_r-z_{r-1})\Big\rangle \\
&\leq \|z_r-p\|^2+2\|y_r-p\|\epsilon_r\|z_r-z_{r-1}\| \\
&\leq \|z_r-p\|^2+2K_3\epsilon_r\|z_r-z_{r-1}\|, \qquad (3.32)
\end{aligned}
$$

and we estimate

$$
\begin{aligned}
\|u_r-p\|^2 &= \|(1-\theta_r)v_r+\varsigma_r\theta_r(y_r-z_r)-p\|^2 \\
&= \|(1-\theta_r)(v_r-p)+\varsigma_r\theta_r(y_r-z_r)-\theta_r p\|^2 \\
&\leq (1-\theta_r)\|v_r-p\|^2+2\Big\langle\varsigma_r\theta_r(y_r-z_r)-\theta_r p,u_r-p\Big\rangle \\
&\leq (1-\theta_r)\Big[\|z_r-p\|^2+2K_3\epsilon_r\|z_r-z_{r-1}\|\Big]+2\theta_r\Big[\langle\varsigma_r(y_r-z_r)-p,u_r-p\rangle\Big] \\
&\leq (1-\theta_r)\|z_r-p\|^2+\theta_r\Big[2K_3\frac{\epsilon_r\|z_r-z_{r-1}\|}{\theta_r}+2\varsigma_r\langle y_r-z_r,u_r-p\rangle-\langle p,u_r-p\rangle\Big] \\
&= (1-\theta_r)\|z_r-p\|^2+\theta_r W_r, \qquad (3.33)
\end{aligned}
$$

where $W_r = 2K_3\frac{\epsilon_r\|z_r-z_{r-1}\|}{\theta_r}+2\varsigma_r\langle y_r-z_r,u_r-p\rangle-\langle p,u_r-p\rangle$. From (3.9) and (3.33), we get

$$
\|z_{r+1}-p\|^2 \quad \leq (1-\theta_r)\|z_r-p\|^2+\theta_r W_r. \qquad (3.34)
$$

In view of (3.31), we see that $\|y_r-z_r\|\leq\|Tz_r-z_r\|+\epsilon_r\|z_r-z_{n-1}\|\to 0$ as $r\to\infty$, and by Remark 3.1, $\frac{\epsilon_r\|z_r-z_{r-1}\|}{\theta_r}\to 0$. Therefore, we have

$$
\liminf_{r\to\infty}\Big[2K_3\frac{\epsilon_r\|z_r-z_{r-1}\|}{\theta_r}+2\varsigma_r\langle y_r-y_r,z_r-p\rangle-\langle p,u_r-p\rangle\Big]\leq 0.
$$

We are currently able to implement Lemma 2.3 in (3.34). The remainder of the proof can be derived by employing analogous steps as those used in the proof of Theorem 3.1. □

Corollary 3.1. *Let $\mathbb{V}_1,\mathbb{V}_2,F_1,F_2,\epsilon_r,\sigma_r,\ \delta_r,\ \mu_1,\mu_2$, and μ be the same as defined in Theorem 3.1. If $\varsigma_r\subset(0,1)$ such that $\lim_{r\to\infty}(1-\varsigma_r)\varsigma_r>0$, then the sequence obtained by*

$$
\begin{aligned}
u_r &= (1-\varsigma_r)z_r+\varsigma_r[Tz_r+\epsilon_r(z_r-z_{r-1})], \\
z_{r+1} &= u_r-\sigma_r[J_{\mu_1}^{F_1}(u_r)+M^*J_{\mu_2}^{F_2}(Mu_r)],
\end{aligned}
$$

converges strongly to $p\in\mathbb{S}_1\cap\mathbb{S}_2$.

Corollary 3.2. *Let $\mathbb{V}_1,\mathbb{V}_2,F_1,F_2,\epsilon_r,\sigma_r,\ \delta_r,\ \mu_1,\mu_2$, and μ be the same as defined in Theorem 3.2. If $\varsigma_r\subset(0,1)$ such that $\lim_{r\to\infty}(1-\varsigma_r)\varsigma_r>0$, then the sequence obtained by*

$$
\begin{aligned}
u_r &= (1-\varsigma_r)z_r+\varsigma_r Tz_r+\epsilon_r(z_r-z_{r-1}), \\
z_{r+1} &= u_r-\sigma_r[J_{\mu_1}^{F_1}(u_r)+M^*J_{\mu_2}^{F_2}(Mu_r)],
\end{aligned}
$$

converges strongly to $p\in\mathbb{S}_1\cap\mathbb{S}_2$.

4. Numerical example

Example 4.1. *Let* $\mathbb{V}_1 = \mathbb{V}_2 = \mathbb{L}^2[0,1]$ *be the infinite dimensional Hilbert space equipped with the inner product*

$$\langle z, u \rangle = \int_0^1 z(s)\, u(s)\, ds, \quad and\ \ norm \quad \|z\| = \left(\int_0^1 |z(s)|^2\, ds \right)^2.$$

The operators $F_1, F_2, M,$ *and* T *are defined by* $F_1(z(s)) = \frac{z(s)-1}{2}$, $F_2(z) = \frac{2z(s)-1}{3}$, $M(z) = \frac{z(s)}{2}$, $T(z) = z(s)$, *and* $f(z(s)) = \frac{z(s)}{4}$. *We select the parameters as* $\delta_r = \frac{1}{(r+1)^2}$, *and* ϵ_r *is selected randomly from* $(0, \overline{\epsilon}_r)$, *where*

$$\overline{\epsilon}_r = \begin{cases} \min\{\frac{1}{(r+1)^2\|z_r - z_{r-1}\|}, \quad 0.25\}, & \text{if } z_r \neq z_{r-1}, \\ 0.25, & \text{otherwise.} \end{cases}$$

The results obtained from Algorithms 3.1 and 3.2 are evaluated and compared with methods studied in [1, 17, 19]. The iterative process is terminated when the stopping condition $Tol_r < D_r$ *is satisfied, where* $Tol_r = \|z_{r+1} - z_r\|$ *and* D_r *is listed Table 1. We choose* $\mu_1 = \frac{1}{2}$, $\mu_2 = 2$, $\varsigma_r = \frac{1}{2} - \frac{1}{r+100}$, *and* $\theta_r = \frac{1}{r+100}$ *and consider the following three different choices of initial points:*
Case (c_1)*:* $z_0 = -2e^{-5s}, \quad z_1 = -e^{-2s}$.
Case (c_2)*:* $z_0(s) = e^{-s}(1 + 5\sin(4s)), \quad z_1(s) = \frac{1}{5} e^{-2s}(\cos(6s))$.
Case (c_3)*:* $z_0(t) = \frac{1}{3}\sin(6\pi s)\, e^{-3s}, \quad z_1(s) = \frac{1}{2}(1+s)^3$.

The convergence is plotted in Figures 1–3, and a comparison is noted in Table 1. It is observed that our methods are simply implemented and approach the solution in fewer numbers of steps in comparison to the methods studied in [1, 17, 19].

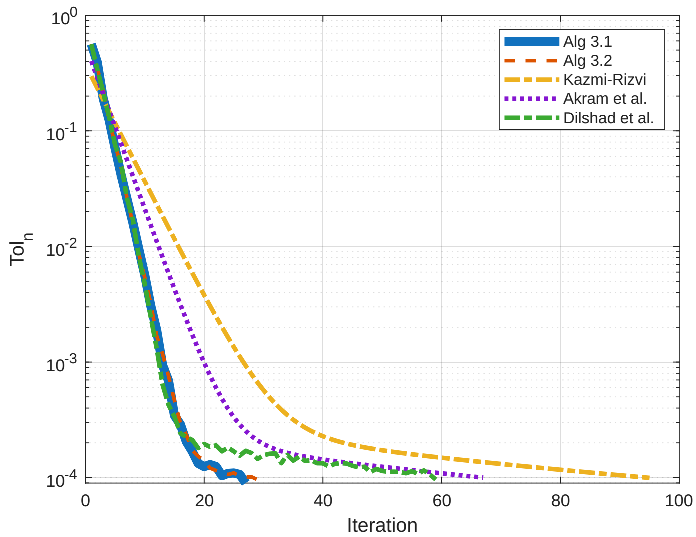

Figure 1. Numerical comparison of algorithms with selected parameters for Case (c_1).

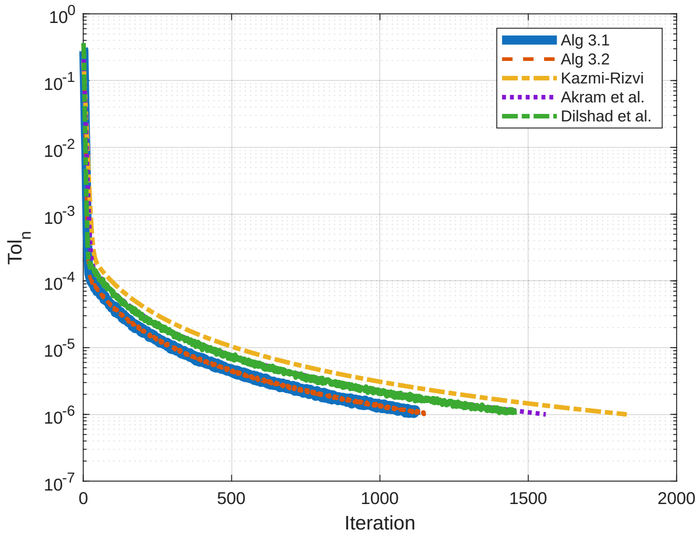

Figure 2. Numerical comparison of algorithms with selected parameters for Case (c_2).

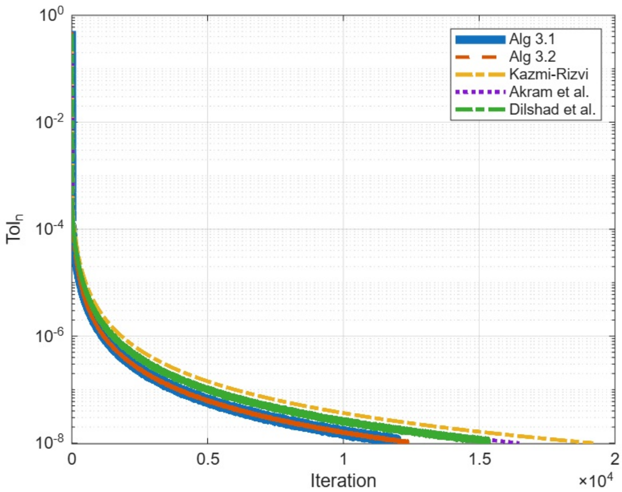

Figure 3. Numerical comparison of algorithms with selected parameters for Case (c_3).

Table 1. Comparison table of Algorithm 3.1 and 3.2 with the methods studied in [1, 17, 19].

Case	D_r		Algorithm 3.1	Algorithm 3.2	Kazmi and Rizvi [19]	Akram et al. [1]	Dilshad et al. [17]
		Step-size	σ_r	σ_r	$\sigma = 0.25$	$\sigma = 0.8$	σ_r
(c_1)	10^{-4}	Iteration	27	29	95	67	59
		Time/Sec	0.000627	0.000478	0.000944	0.000678	0.000842
(c_2)	10^{-6}	Iteration	1121	1156	1832	1559	1454
		Time/Sec	0.011199	0.01083	0.014474	0.014113	0.014559
(c_3)	10^{-8}	Iteration	11979	12385	19206	16479	15351
		Time/Sec	0.073733	0.069731	0.092497	0.088783	0.084847

Example 4.2. *Let $\mathbb{V}_1 = \mathbb{V}_2 = \mathbb{R}^3$ be Real Hilbert spaces. Define the monotone operators F_1 and F_2*

as $F_1(z) = (z_1 + z_2,\ z_2 + z_3,\ z_3 + z_1)$ and $F_2(z) = (z_1 - z_2,\ z_2 - z_3,\ z_3 - z_1)$, $\forall\, z = (z_1, z_2, z_3) \in \mathbb{R}^3$. *The bounded linear operator* $M : \mathbb{V}_1 \to \mathbb{V}_2$ *is defined as* $M(z) = \frac{z}{2}$. *The mapping* $T : \mathbb{V}_1 \to \mathbb{V}_1$ *is defined by* $T(z) = \frac{z}{2}$ *and* $f(z) = \frac{z}{4}$. *It is obvious to see that* F_1 *and* F_2 *are monotone,* M *is bounded,* T *is nonexpansive mapping,* f *is a contraction, and* $\mathbb{S}_1 \cap \mathbb{S}_2 = \{0\}$.

We choose $\varsigma_r = \frac{1}{(2r+10)}$, $\theta_r = \frac{1}{10} - \frac{1}{(2r+10)}$, $\mu_1 = \frac{1}{2}$, *and* $\mu_2 = \frac{1}{3}$. *For numerical computation, we select* ϵ_r *randomly from the interval* $(0, \bar{\epsilon}_r)$, *where*

$$\bar{\epsilon}_r = \begin{cases} \min\left\{\frac{1}{(r+10)^3\|z_r - z_{r-1}\|},\quad 0.55\right\}, & \text{if } z_r \neq z_{r-1}, \\ 0.55, & \text{otherwise.} \end{cases}$$

We consider the following three cases of initial values.
Case $(c_1^{'})$*:* $z_0 = (30, 50, 20)$; $z_1 = (-5, 2, -4)$.
Case $(c_2^{'})$*:* $z_0 = (3, -1, 2)$; $z_1 = (10, 11, 15)$.
Case $(c_3^{'})$*:* $z_0 = (1/2, 0, 1/9)$; $z_1 = (3, 12, 1/5)$.

The behavior of the sequence z_r *is illustrated in Figures 4–6 for three different sets of parameter values, as specified below, and a comparison table of our methods with others is presented in Table 2. It is evident that Algorithms 3.1 and 3.2 are straightforward to implement and demonstrate superior performance compared to the other algorithms, both in execution time and iteration count.*

Table 2. Comparison table of Algorithm 3.1 and 3.2 with the methods studied in [1, 17, 19].

Case	D_r		Algorithm 3.1	Algorithm 3.2	Kazmi and Rizvi [19]	Akram et al. [1]	Dilshad et al. [17]
		Step-size	σ_r	σ_r	$\sigma = 0.15$	$\sigma = 0.8$	σ_r
$(c_1^{'})$	10^{-20}	Iteration	17	15	33	29	18
		Time/Sec	0.003142	0.003058	0.004333	0.003462	0.004441
$(c_2^{'})$	10^{-30}	Iteration	26	22	47	42	26
		Time/Sec	0.000424	0.000397	0.0004111	0.000356	0.000387
$(c_3^{'})$	10^{-40}	Iteration	32	28	62	54	33
		Time/Sec	0.000369	0.000294	0.000486	0.000405	0.000416

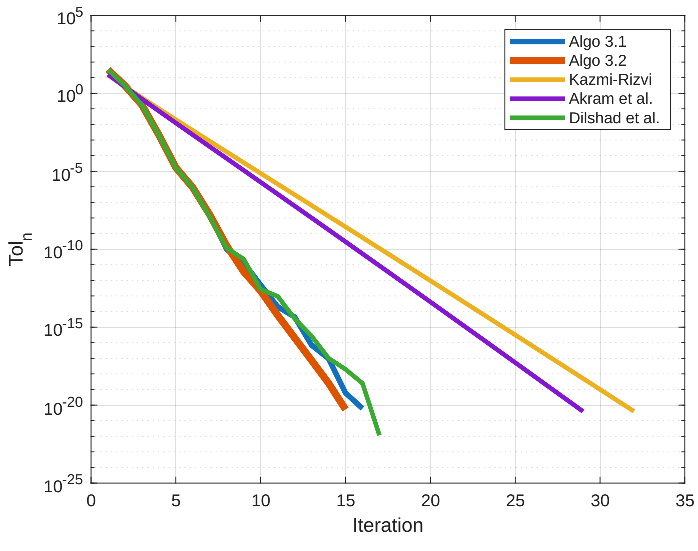

Figure 4. Numerical comparison of algorithms with selected parameters for Case $(c_1^{'})$.

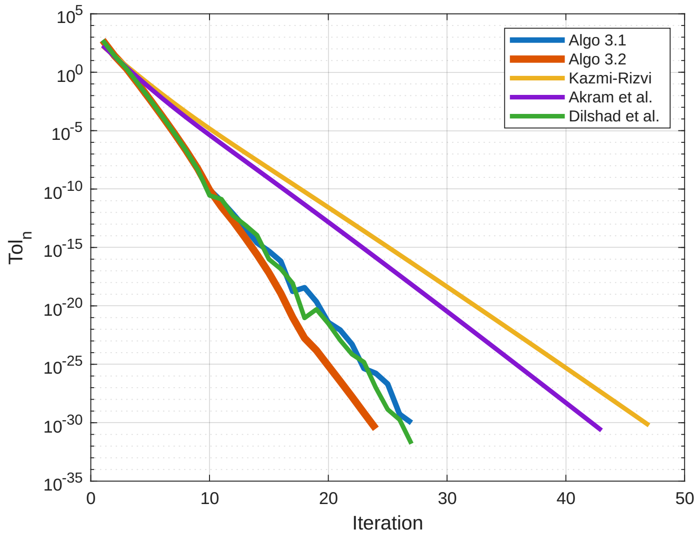

Figure 5. Numerical comparison of algorithms with selected parameters for Case $(c_2^{'})$.

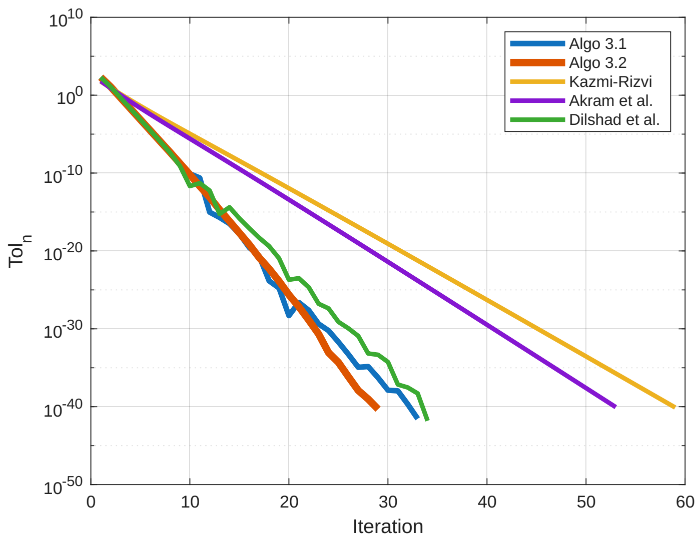

Figure 6. Numerical comparison of algorithms with selected parameters for Case $(c_3^{'})$.

5. Application to solve split common fixed point problem

The SCFPP for the firmly nonexpansive mappings $T_1 : \mathbb{V}_1 \to \mathbb{V}_1$ and $T_2 : \mathbb{V}_2 \to \mathbb{V}_2$ is:

$$\text{Find } z \in Fix(T_1) \text{ such that } Mz \in Fix(T_2),$$

where $M : \mathbb{V}_1 \to \mathbb{V}_2$ is a bounded linear operator. We know that fixed points of mappings T_1 and T_2 are zeros of $(I - T_1)$ and $(I - T_2)$, respectively. Hence, by replacing $J_{\mu_1}^{F_1}$ by $(I - T_1)$, $J_{\mu_2}^{F_2}$ by $(I - T_2)$, and $T = I$, where I is identity mapping, we obtain the Mann-type hybrid inertial iterative schemes for the SCFPP.

Algorithm 5.1. *Hybrid inertial iterative scheme-I for SCFPP*

Step 0: *Choose* $\epsilon \in [0,1)$*, and* $\{\delta_r\}$ *is a positive sequence such that* $\sum_{r=1}^{\infty} \delta_r < \infty$.

Step 1: *For arbitrary* z_0, z_1 *and* $r \geq 1$*, choose* $0 < \epsilon_r < \bar{\epsilon}_r$*, where*

$$\bar{\epsilon}_r = \begin{cases} \min\{\frac{\delta_r}{\|z_r - z_{r-1}\|}, \quad \epsilon\}, & \text{if } z_r \neq z_{r-1}, \\ \epsilon, & \text{otherwise.} \end{cases}$$

Estimate

$$\begin{aligned} u_r &= (1 - \varsigma_r - \theta_r)z_r + \varsigma_r z_r + \epsilon_r(z_r - z_{r-1}), \\ z_{r+1} &= u_r - \sigma_r[(I - T_1)u_r + M^*(I - T_2)Mu_r], \end{aligned}$$

where

$$\sigma_r = \begin{cases} \frac{\|(I-T_1)(u_r)\|^2 + \|(I-T_2)Mu_r\|^2}{\|(I-T_1)u_r + M^*(I-T_2)Mu_r\|^2}, & \text{if } \|(I - T_1)u_r + M^*(I - T_2)Mu_r\| \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Stopping criteria: *Stop if* $z_{r+1} = z_r = u_r$*; otherwise, go to Step 1.*

Algorithm 5.2. *Hybrid inertial iterative scheme-II for SCFPP*

Step 0: *Choose* $\epsilon \in [0,1)$*, and* $\{\delta_r\}$ *is a positive sequence such that* $\sum_{r=1}^{\infty} \delta_r < \infty$.

Step 1: *For arbitrary* z_0, z_1 *and* $r \geq 1$*, choose* $0 < \epsilon_r < \bar{\epsilon}_r$*, where*

$$\bar{\epsilon}_r = \begin{cases} \min\{\frac{\delta_r}{\|z_r - z_{r-1}\|}, \quad \epsilon\}, & \text{if } z_r \neq z_{r-1}, \\ \epsilon, & \text{otherwise.} \end{cases}$$

Estimate

$$\begin{aligned} u_r &= (1 - \varsigma_r - \theta_r)z_r + \varsigma_r[z_r + \epsilon_r(z_r - z_{r-1})], \\ z_{r+1} &= u_r - \sigma_r[(I - T_1)u_r + M^*(I - T_2)Mu_r], \end{aligned}$$

where

$$\sigma_r = \begin{cases} \frac{\|(I-T_1)(u_r)\|^2 + \|(I-T_2)Mu_r\|^2}{\|(I-T_1)u_r + M^*(I-T_2)Mu_r\|^2}, & \text{if } \|(I - T_1)u_r + M^*(I - T_2)Mu_r\| \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Stopping criteria: *Stop if* $z_{r+1} = z_r = u_r$*; otherwise, go to Step 1.*

Theorem 5.1. *Suppose the solution set of the SCFPP is denoted by* Ω *such that* $\Omega \neq \emptyset$*. Let* $T_1 : \mathbb{V}_1 \to \mathbb{V}_1$ *and* $T_2 : \mathbb{V}_2 \to \mathbb{V}_2$ *be firmly nonexpansive mappings. Suppose that Assumptions* (A_2)*–*(A_4) *are satisfied, then the sequence* $\{z_r\}$ *produced by Algorithm 5.1 (Algorithm 5.2) converges to* $q \in \mathbb{S}_1 \cap \mathbb{S}_2$ *such that* $q = P_\Omega(0)$.

Proof. Let $q \in \Omega$. Since T_1 and T_2 are firmly nonexpansive, then by Lemma 3.3, we have that $I - T_1$ and $I - T_2$ are also firmly nonexpansive. Then by replacing $J_{\mu_1}^{F_1}$ by $I - T_1$ and $J_{\mu_2}^{F_2}$ by $I - T_2$ in the calculation of (3.8), we can obtain

$$\|z_{r+1} - q\|^2 \leq \|u_r - q\|^2 - \frac{\left(\|(I - T_1)(u_r)\|^2 + \|(I - T_2)(Mu_r)\|^2\right)^2}{\|(I - T_1)(u_r) + M^*(I - T_2)(Mu_r)\|^2},$$

or

$$\|z_{r+1} - q\| \leq \|u_r - q\|.$$

The rest of the proof can be obtain by applying the similar steps taken in the proof of Theorem 3.1 (Theorem 3.2).

□

6. Conclusions

In this article, we proposed two Mann-type hybrid inertial Yosida approximation iterative schemes for approximating the common solution of SVIP and FPP. Our proposed schemes are traditionally different in the sense that, Mann-type iteration, inertial extrapolation, and fixed point approximation are estimated jointly in the initiation of the iteration process. We proved the strong convergence theorems of the proposed iterative schemes such that the convergence analysis was free from the calculation of $\|M\|$.

We also studied two numerical examples and provided comparisons with related works to illustrate the validity and efficiency of our proposed iterative schemes. It is observed that in the beginning. the Mann step moves the iterate toward the fixed-point region, while the inertial term pushes the sequence further in that direction. The step size σ_r is adaptively computed at each iteration without calculating $\|M\|$. Hence, each iteration moves closer to the solution more efficiently. This improves the initial progress of the algorithm, and the smoothing property of the Yosida approximation reduces the total number of iterations and CPU time compared with some of the related works.

It is shown that our schemes can be applied to solve SCFPP for firmly nonexpansive mappings. Future work will investigate the convergence rates of the proposed algorithms and compare their performance with existing methods for solving the SVIP and FPP.

Author contributions

Doaa Filali: Funding, supervision; Mohammad Dilshad: writing–original draft preparation, writing–review and editing; Mohammad Akram: Conceptualization; Review and editing; Md. Nasiruzzaman: Review and editing; Esmail Alshaban: Supervision. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgements

The first author acknowledges the Princess Nourah bint Abdulrahman University Researchers Supporting Project (Project Number PNURSP2026R174), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia.

Conflict of interest

The authors declare no conflicts of interest.

References

1. M. Akram, M. Dilshad, A. K. Rajpoot, F. Babu, R. Ahmad, J. C. Yao, Modified iterative schemes for a fixed point problem and a split variational inclusion problem, *Mathematics*, **10** (2022), 2098. https://doi.org/10.3390/math10122098
2. M. Akram, M. Dilshad, Generalized system of multi-valued mixed Yosida variation-like inequalities in Hilbert spaces, *J. Math. Anal.*, **16** (2025), 21–34. https://DOI10.54379/JMA-2025-3-2
3. M. Akram, J. W. Chen, M. Dilshad, Generalized Yosida approximation operator with an application to a system of Yosida inclusions, *J. Nonlinear Funct. An.*, **2018** (2018), 1–20. https://doi.org/10.23952/jnfa.2018.17
4. F. Alvarez, H. Attouch, An inertial proximal method for maximal monotone operators via discretization of a nonlinear osculattor with damping, *Set-Valued Anal.*, **9** (2001), 3–11. https://doi.org/10.1023/A:1011253113155
5. M. Akram, M. Dilshad, A. Khan, S. Chandok, I. Ahmad, Convergence analysis for generalized Yosida inclusion problem with applications, *Mathematics*, **11**, (2023), 1409. https://doi.org/10.3390/math11061409
6. T. O. Alakoya, L. O. Jolaoso, A. Taiwo, O. T. Mewomo, Inertial algorithm with self-adaptive step size for split common null point and common fixed point problems for multivalued mappings in Banach spaces, *Optimization*, **71** (2021), 3041–3075. https://doi.org/10.1080/02331934.2021.1895154
7. Arifuzzaman, S. S. Irfan, I. Ahmad, Yosida approximation-based convergence of generalized variational inclusions in ordered Hilbert spaces, *J. Inequal. Appl.*, **2025** (2025), 129. https://doi.org/10.1186/s13660-025-03379-3
8. Arifuzzaman, S. S. Irfan, I. Ahmad, On the convergence of the Yosida–Cayley variational inclusion problem with the XOR operation and inertial extrapolation scheme, *Mathematics*, **13**, (2025), 2447. https://doi.org/10.3390/math13152447
9. H. H. Bauschke, P. L. Combettes, *Convex analysis and monotone operator theory in Hilbert spaces*, Springer, 2017. https://doi.org/10.1007/978-3-319-48311-5
10. F. E. Browder, Nonlinear mappings of nonexpansive and accretive-type in Banach spaces, *B. Am. Math. Soc.*, **73** (1967), 875–882.

11. C. Byrne, Y. Censor, A. Gibali, S. Reich, Weak and strong convergence of algorithms for split common null point problem, *J. Nonlinear Convex Anal.*, **13** (2012), 759–775.

12. C. Byrne, Iterative oblique projection onto convex sets and the split feasibility problems, *Inverse Probl.*, **18** (2002), 441–453. https://doi.org/10.1088/0266-5611/18/2/310

13. R. E. Bruck, S. Reich, Nonexpansive projections and resolvents of accretive operators in Banach spaces, *Houston J. Math.*, **3** (1977), 459–470.

14. P. L. Combettes, The convex feasibility problem in image recovery, *Adv. Imag. Elect. Phys.*, **95** (1996), 155–270. https://doi.org/10.1016/S1076-5670(08)70157-5

15. Y. Censor, T. Elfving, A multi projection algorithm using Bregman projections in a product space, *Numer. Algorithms*, **8** (1994), 221–239. https://doi.org/10.1007/BF02142692

16. Y. Censor, T. Bortfeld, B. Martin, A. Trofimov, A unified approach for inversion problem in intensity modulated radiation therapy, *Phys. Med. Biol.*, **51** (2006), 2353–2365. https://doi.org/10.1088/0031-9155/51/10/001

17. M. Dilshad, M. Akram, M. Nasiruzzaman, D. Filali, A. A. Khidir , Adaptive inertial Yosida approximation iterative algorithms for split variational inclusion and fixed point problems, *AIMS Math.*, **8** (2023), 12922–12942. https://doi.org/10.3934/math.2023651

18. M. Dilshad, A. F. Aljohani, M. Akram, A. Khidir, Yosida approximation iterative methods for split monotone variational inclusion problems, *J. Funct. Spaces*, **2022** (2022), 3665713. https://doi.org/10.1155/2022/3665713

19. K. R. Kazmi, S. H. Rizvi, An iterative method for split variational inclusion problem and fixed point problem for a nonexpansive mapping, *Optim. Lett.*, **8** (2014), 1113–1124. https://doi.org/10.1007/s11590-013-0629-2

20. K. K. Tan, H. K. Xu, Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process, *J. Math. Anal. Appl.*, **178** (1993), 301–308. https://doi.org/10.1006/jmaa.1993.1309

21. G. Lopoz, V. M. Márquez, F. Wang, H. K. Xu, Solving the split feasilibility problem without knowledge of matrix norm, *Inverse Prob.*, **28** (2012), 085004. https://doi.org/10.1088/0266-5611/28/8/085004

22. W. R. Mann, Mean value methods in iteration, *P. Am. Math. Soc.*, **4** (1953), 506–510. https://doi.org/10.2307/2032162

23. P. E. Mainge, Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization, *Set-Valued Anal.*, **16** (2008), 899–912. https://doi.org/10.1007/s11228-008-0102-z

24. A. Moudafi, Split monotone variational inclusions, *J. Optimiz. Theory App.*, **150** (2011), 275–283. https://doi.org/10.1007/s10957-011-9814-6

25. A. Moudafi, B. S. Thakur, Solving proximal split feasibility problem without prior knowledge of operator norm, *Optim. Lett.*, **8** (2014), 2099–2110. https://doi.org/10.1007/s11590-013-0708-4

26. F. O. Nwawuru , O. K. Nrain, M. Dilshad, J. N. Ezeora, Splitting method involving two-step inertial for solving inclusion and fixed point problems with applications, *Fixed Point Theory Algorithms Sci. Eng.*, **2025** (2025), 8. https://doi.org/10.1186/s13663-025-00781-w

27. J. N. Ezeora, C. D. Enyi, F. O. Nwawuru, R. C. Ogbonna, An algorithm for split equilibrium and fixed-point problems using inertial extragradient techniques, *Comput. Appl. Math.*, **4**2 (2023), 103. https://doi.org/10.1007/s40314-023-02244-7

28. G. N. Ogwo, T. O. Alakoya, O. T. Mewomo, Iterative algorithm with self-adaptive step size for approximating the common solution of variational inequality and fixed point problems, *Optimization*, **72** (2023), 677–711. https://doi.org/10.1080/02331934.2021.1981897

29. Z. Opial, Weak covergence of the sequence of successive approximations of nonexpansive mappings, *B. Am. Math. Soc.*, **73** (1976), 591–597.

30. B. T. Polyak, Some methods of speeding up the convergence of iteration methods, *Ussr Comput. Math. Math. Phys.*, **4** (1964), 1–17. https://doi.org/10.1016/0041-5553(64)90137-5

31. B. Tan, X. L. Qin, J. C. Yao, Strong convergence of self-adaptive inertial algorithms for solving split variational inclusion problems with applications, *J. Sci. Comput.*, **87** (2021), 20. https://doi.org/10.1007/s10915-021-01428-9

32. H. K. Xu, Iterative algorithms for nonlinear operators, *J. Lond. Math. Soc.*, **66** (2002), 240–256. https://doi.org/10.1112/S0024610702003332

33. Z. Zhou, B. Tan, S. X. Li, Inertial algorithms with adaptive stepsizes for split variational inclusion problems and their applications to signal recovery problem, *Math. Method. Appl. Sci.*, **47** (2024), 9431–9449. https://doi.org/10.1002/mma.9436

34. Z. Zhou, B. Tan, S. X. Li, Two self-adaptive inertial projection algorithms for solving split variational inclusion problems, *AIMS Math.*, **7** (2022), 4960–4973. https://doi.org/10.3934/math.2022276

35. Z. Zhou, B. Tan, S. X. Li, An inertial shrinking projection algorithm for split common fixed point problems, *J. Appl. Anal. Comput.*, **10** (2020), 2104–2120. https://doi.org/10.11948/20190330

AIMS Press