



Research article

On certain properties of Hybrid Sheffer– λ -type special polynomials and their applications

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Abstract: In this work, a new class of Gould-Hopper Sheffer- λ polynomials was introduced by combining the structural features of Gould-Hopper polynomials with the general framework of Sheffer- λ sequences. The proposed family was defined through an appropriate exponential generating function involving trigonometric factors and was shown to possess rich algebraic and operational properties. Explicit series representations and determinant forms were derived using Riordan array techniques and Cramer's rule. By employing the monomiality principle, the associated multiplicative and derivative operators were constructed, establishing the quasi-monomial character of the introduced polynomials and leading to the corresponding differential equations. Furthermore, several important subclasses, including Gould-Hopper-Bernoulli- λ , Euler- λ , Genocchi- λ , and Laguerre- λ polynomials, were obtained as illustrative examples. These examples demonstrated the unifying nature of the proposed framework and highlighted its potential applicability in operational calculus, special functions, and mathematical physics.

Keywords: Sheffer sequences; λ -polynomials; Gould-Hopper polynomials; monomiality principle; operational rules; determinant approach; applications

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1. Introduction

Special functions and special polynomial families occupy a fundamental position in modern pure and applied mathematics. They serve as structured analytical tools for solving differential, integral, and operator equations and frequently provide compact closed-form representations of complex mathematical models. In many applied problems, the structure of a physical or computational model naturally leads either to new properties of known special functions or to the development of new generalized or hybrid polynomial classes. Such extensions often reveal deeper structural features and interconnections across different branches of mathematical analysis and mathematical physics. Expressing models in terms of special functions not only refines their analytical form but also supports efficient numerical and symbolic treatment, while sometimes exposing hidden links between seemingly unrelated problems.

A powerful general framework for studying polynomial systems of this type is provided by operational techniques together with the monomiality principle. This approach has its origins in the work of Steffensen on poweroids [16] and was later systematically developed by Dattoli and collaborators [4, 5, 7]. It provides an operator-based mechanism for constructing and analyzing polynomial sequences that mimic the behavior of monomials under suitable raising and lowering operators. The basic axioms of the monomiality principle are given by

$$\mathcal{A}_{n+1}(\varphi_1) = \widehat{\mathcal{M}}\{\mathcal{A}_n(\varphi_1)\}, \quad (1.1)$$

and

$$n \mathcal{A}_{n-1}(\varphi_1) = \widehat{\mathcal{P}}\{\mathcal{A}_n(\varphi_1)\}. \quad (1.2)$$

The associated operators satisfy the Weyl-type commutation relation

$$[\widehat{\mathcal{P}}, \widehat{\mathcal{M}}] = \widehat{1}.$$

From this structure, one derives the operational identities

$$\widehat{\mathcal{M}}\widehat{\mathcal{P}}\{\mathcal{A}_n(\varphi_1)\} = n \mathcal{A}_n(\varphi_1), \quad (1.3)$$

$$\mathcal{A}_n(\varphi_1) = \widehat{\mathcal{M}}^n\{1\}, \quad (1.4)$$

$$\exp(\chi \widehat{\mathcal{M}})\{1\} = \sum_{n=0}^{\infty} \mathcal{A}_n(\varphi_1) \frac{\chi^n}{n!}.$$

The Kampé de Fériet polynomials, also known as higher-order Hermite or Gould–Hopper polynomials, are defined by [8]:

$$\mathcal{H}_n^{[j]}(\varphi_1, \varphi_2) = n! \sum_{k=0}^{\lfloor \frac{n}{m} \rfloor} \frac{\varphi_2^k \varphi_1^{n-mk}}{k!(n-mk)!}.$$

The two-variable Hermite (Gould–Hopper) polynomials $\mathcal{H}_n^{[j]}(\varphi_1, \varphi_2)$ admit the generating function representation [1]:

$$e^{\varphi_1 \chi + \varphi_2 \chi^j} = \sum_{n=0}^{\infty} \mathcal{H}_n^{[j]}(\varphi_1, \varphi_2) \frac{\chi^n}{n!}.$$

An important feature of the Gould–Hopper polynomials is their strong algebraic and differential structure, which becomes transparent through operational identities. These identities provide a direct and systematic mechanism for deriving recurrence relations, differential equations, and explicit representations, and they help connect this class with other special polynomial families. In operational form, one has

$$\mathcal{H}_n^{[j]}(\varphi_1, \varphi_2) = \exp\left(\varphi_2 \frac{\partial^j}{\partial \varphi_1^j}\right)\{\varphi_1^n\}.$$

The quasi–monomial structure of $\mathcal{H}_n^{[j]}(\varphi_1, \varphi_2)$ is expressed through the operators

$$\hat{M} := \varphi_1 + m\varphi_2 \frac{\partial^{j-1}}{\partial \varphi_1^{j-1}}, \quad (1.5)$$

$$\hat{P} := \frac{\partial}{\partial \varphi_1}.$$

Another central pillar in the theory of polynomial sequences is the class of Sheffer polynomials introduced by Sheffer [15]. These sequences appear naturally in operational calculus, approximation theory, combinatorics, and mathematical physics. A key advantage of the Sheffer framework is that many classical polynomial families including Hermite, Laguerre, Bernoulli, and Appell polynomials arise as special or limiting cases. Modern algebraic, approximation, and umbral approaches to Sheffer and related polynomial sequences, including those developed by Costabile et al. [2, 3, 14], and several other authors [19–21] further highlight their structural importance and computational usefulness.

A polynomial sequence $\{\mathcal{S}_n(\varphi_1)\}_{n=0}^{\infty}$ is called a Sheffer-type sequence if it is characterized by the exponential generating function

$$\mathcal{A}(\chi) \exp(\varphi_1 \mathcal{H}(\chi)) = \sum_{n=0}^{\infty} \mathcal{S}_n(\varphi_1) \frac{\chi^n}{n!}, \quad (1.6)$$

with $\mathcal{A}(\chi)$ and $\mathcal{H}(\chi)$ given by

$$\mathcal{A}(\chi) = \sum_{n=0}^{\infty} F_n \frac{\chi^n}{n!}, \quad F_0 \neq 0 \quad (1.7)$$

and

$$\mathcal{H}(\chi) = \sum_{n=1}^{\infty} b_n \frac{\chi^n}{n!}, \quad b_1 \neq 0.$$

Following Roman [12], a polynomial sequence $\mathcal{S}_n(\varphi_1)$ can be described in a natural and systematic way by means of two formal power series. More precisely, it is completely determined by the pair

$$\sigma(\chi) = \sum_{n=0}^{\infty} \sigma_n \frac{\chi^n}{n!}, \quad \sigma_0 = 0, \quad \sigma_1 \neq 0,$$

and

$$\mu(\chi) = \sum_{n=0}^{\infty} \mu_n \frac{\chi^n}{n!}, \quad \mu_0 \neq 0,$$

where the first series controls the compositional structure while the second provides the multiplicative component of the construction. The exponential generating function of the sequence $\mathcal{S}_n(\varphi_1)$ then takes the form

$$\frac{1}{\mu(\sigma^{-1}(\chi))} \exp(\varphi_1 \sigma^{-1}(\chi)) = \sum_{n=0}^{\infty} \mathcal{S}_n(\varphi_1) \frac{\chi^n}{n!},$$

for all $x \in \mathbb{C}$, where $\sigma^{-1}(\chi)$ denotes the compositional inverse of $\sigma(\chi)$. This representation makes the dependence on the pair (μ, σ) transparent and is especially convenient for operator and generating-function techniques.

In view of Eqs (1.4)–(1.7), one can directly read off the corresponding structural functions as

$$\mathcal{A}(\chi) = \frac{1}{\mu(\sigma^{-1}(\chi))},$$

and

$$\mathcal{H}(\chi) = \sigma^{-1}(\chi).$$

Hence, the sequence $\mathcal{S}_n(\varphi_1)$ defined through (1.9) is precisely the Sheffer sequence associated with the pair $(\mu(\chi), \sigma(\chi))$.

An important special situation occurs when the pair reduces to $(1, \sigma(\chi))$. In this case one obtains the associated Sheffer sequence, characterized by the generating function

$$\exp(\varphi_1 \mathcal{H}(\chi)) = \sum_{n=0}^{\infty} \mathcal{P}_n(\varphi_1) \frac{\chi^n}{n!}.$$

On the other hand, if the pair is of the form $(\mu(\chi), \chi)$, the Sheffer sequence simplifies to an Appell sequence, whose generating function is given by [15]

$$\mathcal{A}(\chi) \exp(\varphi_1 \chi) = \sum_{n=0}^{\infty} \mathcal{A}_n(\varphi_1) \frac{\chi^n}{n!}.$$

For specific value of $\mathcal{A}(\chi)$, several members of the Appell polynomial family are given in Table 1.

Table 1. Certain members of Appell sequences.

S. No.	APF	Generating function	$\mathcal{A}(\chi)$
I.	The Bernoulli polynomials [10]	$\frac{\chi}{e^\chi - 1} e^{\varphi_1 \chi} = \sum_{n=0}^{\infty} \mathcal{B}_n(\varphi_1) \frac{\chi^n}{n!}$	$\mathcal{A}(\chi) = \frac{\chi}{e^\chi - 1}$
II.	The Euler polynomials [11]	$\frac{2}{e^\chi + 1} e^{\varphi_1 \chi} = \sum_{n=0}^{\infty} \mathcal{E}_n(\varphi_1) \frac{\chi^n}{n!}$	$\mathcal{A}(\chi) = \frac{2}{e^\chi + 1}$
III.	The Genocchi polynomials [11]	$\frac{2\chi}{e^\chi + 1} e^{\varphi_1 \chi} = \sum_{n=0}^{\infty} \mathcal{G}_n(\varphi_1) \frac{\chi^n}{n!}$	$\mathcal{A}(\chi) = \frac{2\chi}{e^\chi + 1}$

These two subclasses illustrate how the general Sheffer framework naturally contains several well-known polynomial families.

Moreover, as established by Roman and Rota [13], a polynomial sequence $\mathcal{S}_n(\varphi_1)$ is a Sheffer sequence relative to a binomial-type sequence $\mathcal{P}_n(\varphi_1)$ if, and only if, it satisfies the addition formula

$$\mathcal{S}_n(\varphi_1 + \varphi_2) = \sum_{k=0}^n \binom{n}{k} \mathcal{S}_k(\varphi_1) \mathcal{P}_{n-k}(\varphi_2),$$

for all $n \geq 0$ and every $\varphi_2 \in \mathbb{R}$, where \mathbb{R} is any field of characteristic zero. This identity plays a central role in characterizing the Sheffer structure through binomial-type convolution.

Let $(\mathcal{S}_n(\varphi_1))_{n \in \mathbb{N}}$ be the Sheffer sequence corresponding to the pair $(\mu(\chi), \sigma(\chi))$ and suppose it admits the expansion

$$\varphi_1^n = \sum_{k=0}^n F_{n,k} \mathcal{S}_k(\varphi_1). \quad (1.8)$$

Determinant methods play an important role in several areas of mathematical physics, including wave propagation, nonlinear evolution equations, and relativistic models. The determinant characterization of Sheffer sequences developed by Wang and Costabile [2, 3, 17] offers a systematic basis for representing structured polynomial families. This framework is compatible with operational techniques and is especially useful for interpolation and symbolic computation. Under the representation (1.8), the polynomials $\mathcal{S}_n(\varphi_1)$ can be written in determinant form, which is particularly useful for explicit computation and structural analysis. According to [2, 3, 17], the determinant form of $(\mathcal{S}_n(\varphi_1))$ polynomials is given as

$$\mathcal{S}_0(\varphi_1) = \frac{1}{F_{0,0}},$$

$$\mathcal{S}_n(\varphi_1) = \frac{(-1)^n}{F_{0,0} F_{1,1} \dots F_{n,n}} \begin{vmatrix} 1 & \varphi_1 & \varphi_1^2 & \cdots & \varphi_1^{n-1} & \varphi_1^n \\ F_{0,0} & F_{1,0} & F_{2,0} & \cdots & F_{n-1,0} & F_{n,0} \\ 0 & F_{1,1} & F_{2,1} & \cdots & F_{n-1,1} & F_{n,1} \\ 0 & 0 & F_{2,2} & \cdots & F_{n-1,2} & F_{n,2} \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & F_{n-1,n-1} & F_{n,n-1} \end{vmatrix},$$

where $F_{n,k}$ denotes the (n, k) -th entry of the Riordan array associated with the pair $(\mu(\chi), \sigma(\chi))$ [18].

A generalized Riordan array linked with a sequence $(\mathcal{C}_n)_{n \in \mathbb{N}_0}$ is determined by the same ordered pair $(\mu(\chi), \sigma(\chi))$ and produces an infinite lower triangular matrix $(F_{n,k})_{0 \leq k \leq n < \infty}$. Its entries are generated through a coefficient-extraction rule that connects each matrix element directly with the generating functions $\mu(\chi)$ and $\sigma(\chi)$. This viewpoint offers a clear and flexible framework for encoding Sheffer-type polynomial sequences and for carrying out coefficient manipulations in an operator and combinatorial setting.

The defining rule of the generalized Riordan array is

$$F_{n,k} = \left[\frac{\chi^n}{\mathcal{C}_n} \right] \mu(\chi) \frac{(\sigma(\chi))^k}{\mathcal{C}_k},$$

where the functions $\mu(\chi)(\sigma(\chi))^k/\mathcal{C}_k$ serve as the column generating functions. Each column is therefore generated in a uniform way, allowing compact representations and efficient computation of the coefficients associated with the pair $(\mu(\chi), \sigma(\chi))$.

Recently, the Sheffer- λ -polynomials ${}_s\Lambda_n(\varphi_1, \varphi_2)$ were introduced through the generating function [6, 11]

$$\mathcal{A}(\chi)e^{\varphi_2\mathcal{H}(\chi)}\cos(\sqrt{\varphi_1}\chi) = \sum_{n=0}^{\infty} {}_s\Lambda_n(\varphi_1, \varphi_2)\frac{\chi^n}{n!},$$

from which one directly derives the explicit series representation

$${}_s\Lambda_n(\varphi_1, \varphi_2) = n! \sum_{k=0}^n \frac{(-1)^k \varphi_1^k \mathcal{S}_{n-k}(\varphi_2)}{(2k)!(n-k)!}.$$

This form clearly shows how the new family blends the Sheffer structure with a cosine-type kernel.

These Sheffer- λ -polynomials also satisfy a quasi-monomial structure with respect to suitable multiplicative and lowering operators [6, 9, 11]. In particular, the multiplicative operator is given by

$$\hat{M}_{s\Lambda} = \frac{\mathcal{A}'(\mathcal{H}^{-1}(\hat{D}_{\varphi_2}))}{\mathcal{A}(\mathcal{H}^{-1}(\hat{D}_{\varphi_2}))} + \varphi_2\mathcal{H}'(\mathcal{H}^{-1}(\hat{D}_{\varphi_2})) - \frac{1}{2}\tan\left(\sqrt{\varphi_1}\mathcal{H}^{-1}(\hat{D}_{\varphi_2})\right)\sqrt{\frac{\varphi_1}{\mathcal{H}^{-1}(\hat{D}_{\varphi_2})}},$$

while the corresponding lowering operator takes the simpler form

$$\hat{P}_{s\Lambda} = \mathcal{H}^{-1}(\hat{D}_{\varphi_2}). \quad (1.9)$$

Together, these operators describe the operational behavior of the Sheffer- λ family and support further algebraic and differential investigations.

The main contribution of this work is the construction of a multiparameter hybrid Gould–Hopper–Sheffer- λ polynomial family through a unified generating function framework combined with operational and monomiality-based methods. In contrast to standard Sheffer or Gould–Hopper extensions, the present class uses a hybrid parameter structure that simultaneously produces several classical families as special cases. Along with the defining generating functions, we develop operator identities, recurrence structures, and determinant representations adapted to this hybrid setting, thereby providing a coherent platform for further analytical and computational developments.

The paper is organized as follows. In Section 2, we introduce the Gould–Hopper–Sheffer- λ polynomials (GHS λ P) via their generating function and establish their main structural properties, including series and determinant representations through Riordan array techniques. Section 3 presents a systematic operational analysis based on the monomiality principle, where the associated multiplicative and derivative operators are constructed and the resulting differential and operational representations are derived. Section 4 contains illustrative subclasses obtained from specific generating functions, such as the Gould–Hopper–Bernoulli- λ , Euler- λ , Genocchi- λ , and Laguerre- λ polynomials, demonstrating the flexibility of the framework. A separate section discusses applications and computational aspects, and the paper concludes with a summary and directions for future work.

2. Gould-Hopper-Sheffer- λ -polynomials

To introduce the GHS λ P, denoted by ${}_{\mathcal{H}}s\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3)$, we first establish a fundamental result that underpins its definition. This result offers a unified operational foundation that connects the generating function, monomiality principle, and operator formalism in a coherent manner. The novelty of this approach lies in the systematic incorporation of the λ -parameter and multivariable structure, which leads to a genuinely new class of hybrid special polynomials. As a consequence, the proposed framework extends several known polynomial families as special cases while opening new directions for analytical and applied investigations.

Theorem 2.1. *The GHS λ P ${}_{\mathcal{H}}s\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3)$ is defined through the following generating function:*

$$\mathcal{A}(\chi) \exp(\varphi_1 \mathcal{H}(\chi) + \varphi_2 (\mathcal{H}(\chi))^j) \cos(\sqrt{\varphi_3 \chi}) = \sum_{n=0}^{\infty} {}_{\mathcal{H}}s\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3) \frac{\chi^n}{n!}. \quad (2.1)$$

Proof. By replacing the variable φ_1 in (1.6) with the multiplicative operator $\hat{M}_{\mathcal{H}_n^{[j]}}$ given by (1.5), we obtain

$$\mathcal{A}(\chi) \cos(\sqrt{\varphi_3 \chi}) \exp(\hat{M}_{\mathcal{H}_n^{[j]}} \mathcal{H}(\chi)) = \sum_{n=0}^{\infty} {}_s\Lambda_n(\hat{M}_{\mathcal{H}_n^{[j]}}) \frac{\chi^n}{n!}. \quad (2.2)$$

Making use of the explicit representation of the operator $\hat{M}_{\mathcal{H}_n^{[j]}}$ given in (1.5), the left-hand side of (2.2) can be rewritten as

$$\mathcal{A}(\chi) \cos(\sqrt{\varphi_3 \chi}) \exp\left(\left(\varphi_1 + m\varphi_2 \frac{\partial^{j-1}}{\partial \varphi_1^{j-1}}\right)\chi\right) = \sum_{n=0}^{\infty} {}_s\Lambda_n(\hat{M}_{\mathcal{H}_n^{[j]}}) \frac{\chi^n}{n!}. \quad (2.3)$$

Next, we decouple the exponential operator appearing on the left-hand side of (2.3) by invoking the Crofton-type identity [7]:

$$f\left(z + m\mu \frac{d^j}{dz^{j-1}}\right)\{1\} = \exp\left(\mu \frac{d^j}{dz^j}\right)\{f(z)\},$$

which allows us to separate the operational components in a convenient form. As a result, we arrive at

$$\mathcal{A}(\chi) \cos(\sqrt{\varphi_3 \chi}) \exp\left(\varphi_2 \frac{\partial^j}{\partial \varphi_3^j}\right) \exp(\varphi_1 \mathcal{H}(\chi)) = \sum_{n=0}^{\infty} {}_s\Lambda_n\left(\varphi_1 + m\varphi_2 \frac{\partial^{j-1}}{\partial \varphi_1^{j-1}}\right) \frac{\chi^n}{n!}. \quad (2.4)$$

Finally, by expanding the first exponential operator on the left-hand side of (2.4) and introducing the notation ${}_{\mathcal{H}}s\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3)$ for the resulting Gould–Hopper–Sheffer- λ polynomials on the righthand side, namely,

$${}_{\mathcal{H}}s\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3) = {}_s\Lambda_n(\hat{M}_{\mathcal{H}_n^{[j]}}) = {}_s\Lambda_n\left(\varphi_1 + m\varphi_2 \frac{\partial^{j-1}}{\partial \varphi_1^{j-1}}\right),$$

the statement in (2.1) follows immediately. This completes the proof. \square

Remark 2.1. *For $\varphi_2 = 0$, the GHS λ P ${}_{\mathcal{H}}s\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3)$ reduces to the Sheffer- λ -polynomials ${}_s\Lambda_n(\varphi_1, \varphi_3)$ and is defined through the generating function:*

$$\mathcal{A}(\chi) \exp(\varphi_1 \mathcal{H}(\chi)) \cos(\sqrt{\varphi_3 \chi}) = \sum_{n=0}^{\infty} {}_s\Lambda_n(\varphi_1, \varphi_3) \frac{\chi^n}{n!}.$$

Remark 2.2. For $\varphi_3 = 0$, the GHS λP ${}_{\mathcal{H}}S\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3)$ reduces to the Gould-Sheffer-polynomials ${}_{\mathcal{H}}S_n^{[j]}(\varphi_1, \varphi_2)$ and is defined through the generating function:

$$\mathcal{A}(\chi) \exp(\varphi_1 \mathcal{H}(\chi) + \varphi_2 (\mathcal{H}(\chi))^j) = \sum_{n=0}^{\infty} {}_{\mathcal{H}}S_n^{[j]}(\varphi_1, \varphi_2) \frac{\chi^n}{n!}.$$

Remark 2.3. For $j = 2$, the GHS λP ${}_{\mathcal{H}}S\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3)$ reduces to the Hermite-Sheffer- λ polynomials ${}_{\mathcal{H}}S\Lambda_n(\varphi_1, \varphi_2, \varphi_3)$ and is defined through the generating function:

$$\mathcal{A}(\chi) \exp(\varphi_1 \mathcal{H}(\chi) + \varphi_2 (\mathcal{H}(\chi))^2) \cos(\sqrt{\varphi_3 \chi}) = \sum_{n=0}^{\infty} {}_{\mathcal{H}}S\Lambda_n(\varphi_1, \varphi_2, \varphi_3) \frac{\chi^n}{n!}.$$

Remark 2.4. For $\mathcal{A}(\chi) = 1$, the GHS λP ${}_{\mathcal{H}}S\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3)$ reduces to the GHAS λP ${}_{\mathcal{H}\mathcal{P}}\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3)$ and is defined through the generating function:

$$\exp(\varphi_1 \mathcal{H}(\chi) + \varphi_2 (\mathcal{H}(\chi))^j) \cos(\sqrt{\varphi_3 \chi}) = \sum_{n=0}^{\infty} {}_{\mathcal{H}\mathcal{P}}\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3) \frac{\chi^n}{n!}. \quad (2.5)$$

Next, we obtain the series definition of the GHS λP ${}_{\mathcal{H}}S\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3)$ by proving following result.

Theorem 2.2. For the GHS λP ${}_{\mathcal{H}}S\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3)$, the following identity holds:

$${}_{\mathcal{H}}S\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3) = n! \sum_{k=0}^{[n]} \frac{(-1)^k \varphi_3^k}{(n-k)!(2k)!} {}_{\mathcal{H}}S_{n-k}^{[j]}(\varphi_1, \varphi_2).$$

Proof. We start from (2.1) and use the cosine expansion

$$\cos(\sqrt{\varphi_3 \chi}) = \sum_{m=0}^{\infty} \frac{(-1)^m (\varphi_3 \chi)^m}{(2m)!}.$$

Substituting this into (2.1), we obtain

$$\mathcal{A}(\chi) \exp(\varphi_1 \mathcal{H}(\chi) + \varphi_2 (\mathcal{H}(\chi))^j) \cos(\sqrt{\varphi_3 \chi}) = \sum_{n=0}^{\infty} {}_{\mathcal{H}}S_n^{[j]}(\varphi_1, \varphi_2) \frac{\chi^n}{n!} \sum_{k=0}^{\infty} \frac{(-1)^k (\varphi_3 \chi)^k}{(2k)!}. \quad (2.6)$$

Now, applying the Cauchy product rule to the righthand side of (2.6), we get

$$\mathcal{A}(\chi) \exp(\varphi_1 \mathcal{H}(\chi) + \varphi_2 (\mathcal{H}(\chi))^j) \cos(\sqrt{\varphi_3 \chi}) = \sum_{n=0}^{\infty} n! \sum_{k=0}^{[n]} {}_{\mathcal{H}}S_{n-k}^{[j]}(\varphi_1, \varphi_2) \frac{(-1)^k (\varphi_3)^k \chi^n}{(n-k)!(2k)! n!}.$$

Inserting the righthand side of the expression (2.1) into the lefthand side of the preceding identity, and comparing the coefficients of like exponents of χ on both sides, yields the required result. \square

Motivated by this approach, we now derive a determinant representation for the GHS λP ${}_{\mathcal{H}}S\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3)$.

Theorem 2.3. The GHS \mathcal{LP} ${}_{\mathcal{H}}S\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3)$ has the following determinant representation

$${}_{\mathcal{H}}S\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3) = \frac{(-1)^n}{(\theta_0)^{n+1}}$$

$$\times \begin{vmatrix} 1 & {}_{\mathcal{H}}\mathcal{P}\Lambda_1^{[j]}(\varphi_1, \varphi_2, \varphi_3) & {}_{\mathcal{H}}\mathcal{P}\Lambda_2^{[j]}(\varphi_1, \varphi_2, \varphi_3) & \dots & {}_{\mathcal{H}}\mathcal{P}\Lambda_{n-1}^{[j]}(\varphi_1, \varphi_2, \varphi_3) & {}_{\mathcal{H}}\mathcal{P}\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3) \\ \theta_0 & \theta_1 & \theta_2 & \dots & \theta_{n-1} & \theta_n \\ 0 & \theta_0 & \binom{2}{1}\theta_1 & \dots & \binom{n-1}{1}\theta_{n-2} & \binom{n}{1}\theta_{n-1} \\ 0 & 0 & \theta_0 & \dots & \binom{n-1}{1}\theta_{n-3} & \binom{n}{2}\theta_{n-2} \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \theta_0 & \binom{n}{n-1}\theta_1 \end{vmatrix}, \quad (2.7)$$

where $\sum_{n=0}^{\infty} {}_{\mathcal{H}}\mathcal{P}\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3) \frac{\chi^n}{n!} = \exp(\varphi_1 \mathcal{H}(\chi) + \varphi_2 (\mathcal{H}(\chi))^j) \cos(\sqrt{\varphi_3 \chi})$, $\frac{1}{\mathcal{A}(\chi)} = \sum_{k=0}^{\infty} \theta_k \frac{\chi^k}{k!}$.

Proof. Using the generation function (2.1), we get

$$\exp(\varphi_1 \mathcal{H}(\chi) + \varphi_2 (\mathcal{H}(\chi))^j) \cos(\sqrt{\varphi_3 \chi}) = \left(\sum_{k=0}^{\infty} \theta_k \frac{\chi^k}{k!} \right) \left(\sum_{n=0}^{\infty} {}_{\mathcal{H}}S\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3) \frac{\chi^n}{n!} \right).$$

Thus, on using (2.5) on the l.h.s. of preceding expression, it follows that

$$\sum_{n=0}^{\infty} {}_{\mathcal{H}}\mathcal{P}\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3) \frac{\chi^n}{n!} = \left(\sum_{k=0}^{\infty} \theta_k \frac{\chi^k}{k!} \right) \left(\sum_{n=0}^{\infty} {}_{\mathcal{H}}S\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3) \frac{\chi^n}{n!} \right).$$

Employing the Cauchy product, we find

$$\sum_{n=0}^{\infty} {}_{\mathcal{H}}\mathcal{P}\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3) \frac{\chi^n}{n!} = \sum_{n=0}^{\infty} \sum_{k=0}^n \binom{n}{k} \theta_k {}_{\mathcal{H}}S\Lambda_{n-k}^{[j]}(\varphi_1, \varphi_2, \varphi_3) \frac{\chi^n}{n!}.$$

By comparing the coefficients of $\frac{\chi^n}{n!}$ from the polynomial equation, we get

$${}_{\mathcal{H}}\mathcal{P}\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3) = \sum_{k=0}^n \binom{n}{k} \theta_k {}_{\mathcal{H}}S\Lambda_{n-k}^{[j]}(\varphi_1, \varphi_2, \varphi_3), \quad n \in \mathbb{N}_0.$$

This leads to the following system of equations:

$$\begin{aligned} {}_{\mathcal{H}}\mathcal{P}\Lambda_0^{[j]}(\varphi_1, \varphi_2, \varphi_3) &= \theta_0 {}_{\mathcal{H}}S\Lambda_0^{[j]}(\varphi_1, \varphi_2, \varphi_3), \\ {}_{\mathcal{H}}\mathcal{P}\Lambda_1^{[j]}(\varphi_1, \varphi_2, \varphi_3) &= \theta_0 {}_{\mathcal{H}}S\Lambda_1^{[j]}(\varphi_1, \varphi_2, \varphi_3) + \theta_1 {}_{\mathcal{H}}S\Lambda_0^{[j]}(\varphi_1, \varphi_2, \varphi_3), \\ {}_{\mathcal{H}}\mathcal{P}\Lambda_2^{[j]}(\varphi_1, \varphi_2, \varphi_3) &= \theta_0 {}_{\mathcal{H}}S\Lambda_2^{[j]}(\varphi_1, \varphi_2, \varphi_3) + \binom{2}{1} \theta_1 {}_{\mathcal{H}}S\Lambda_1^{[j]}(\varphi_1, \varphi_2, \varphi_3) + \theta_2 {}_{\mathcal{H}}S\Lambda_0^{[j]}(\varphi_1, \varphi_2, \varphi_3), \\ &\vdots \end{aligned}$$

$${}_{\mathcal{H}\mathcal{P}}\Lambda_{n-1}^{[j]}(\varphi_1, \varphi_2, \varphi_3) = \theta_0 {}_{\mathcal{H}\mathcal{S}}\Lambda_{n-1}^{[j]}(\varphi_1, \varphi_2, \varphi_3) + \binom{n-1}{1}\theta_1 {}_{\mathcal{H}\mathcal{S}}\Lambda_{n-2}^{[j]}(\varphi_1, \varphi_2, \varphi_3) + \dots + \theta_{n-1} {}_{\mathcal{H}\mathcal{S}}\Lambda_0^{[j]}(\varphi_1, \varphi_2, \varphi_3),$$

$${}_{\mathcal{H}\mathcal{P}}\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3) = \theta_0 {}_{\mathcal{H}\mathcal{S}}\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3) + \binom{n}{1}\theta_1 {}_{\mathcal{H}\mathcal{S}}\Lambda_{n-1}^{[j]}(\varphi_1, \varphi_2, \varphi_3) + \dots + \theta_n {}_{\mathcal{H}\mathcal{S}}\Lambda_0^{[j]}(\varphi_1, \varphi_2, \varphi_3).$$

Applying Cramers' rule, we get

$${}_{\mathcal{H}\mathcal{S}}\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3) = \frac{\begin{vmatrix} \theta_0 & 0 & \dots & 0 & {}_{\mathcal{H}\mathcal{P}}\Lambda_0^{[j]}(\varphi_1, \varphi_2, \varphi_3) \\ \theta_1 & \theta_0 & \dots & 0 & {}_{\mathcal{H}\mathcal{P}}\Lambda_1^{[j]}(\varphi_1, \varphi_2, \varphi_3) \\ \theta_2 & \binom{2}{1}\theta_1 & \dots & 0 & {}_{\mathcal{H}\mathcal{P}}\Lambda_2^{[j]}(\varphi_1, \varphi_2, \varphi_3) \\ \theta_3 & \binom{3}{2}\theta_2 & \dots & 0 & {}_{\mathcal{H}\mathcal{P}}\Lambda_3^{[j]}(\varphi_1, \varphi_2, \varphi_3) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \theta_{n-1} & \binom{n-1}{1}\theta_{n-2} & \dots & \theta_0 & {}_{\mathcal{H}\mathcal{P}}\Lambda_{n-1}^{[j]}(\varphi_1, \varphi_2, \varphi_3) \\ \theta_n & \binom{n}{1}\theta_{n-1} & \dots & \binom{n}{n-1}\theta_1 & {}_{\mathcal{H}\mathcal{P}}\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3) \end{vmatrix}}{\begin{vmatrix} \theta_0 & 0 & \dots & 0 & 0 \\ \theta_1 & \theta_0 & \dots & 0 & 0 \\ \theta_2 & \binom{2}{1}\theta_1 & \dots & 0 & 0 \\ \theta_3 & \binom{3}{2}\theta_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \theta_{n-1} & \binom{n-1}{1}\theta_{n-2} & \dots & \theta_0 & 0 \\ \theta_n & \binom{n}{1}\theta_{n-1} & \dots & \binom{n}{n-1}\theta_1 & \theta_0 \end{vmatrix}}.$$

By taking the transpose in the last equation, we have

$${}_{\mathcal{H}\mathcal{S}}\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3) = \frac{1}{(\theta_0)^{n+1}} \times \begin{vmatrix} \theta_0 & \theta_1 & \dots & \theta_{n-1} & \theta_n \\ 0 & \theta_0 & \dots & \binom{n-1}{1}\theta_{n-2} & \binom{n}{1}\theta_{n-1} \\ 0 & 0 & \dots & \binom{n-1}{1}\theta_{n-3} & \binom{n}{2}\theta_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \theta_0 & \binom{n}{n-1}\theta_1 \\ {}_{\mathcal{H}\mathcal{P}}\Lambda_0^{[j]}(\varphi_1, \varphi_2, \varphi_3) & {}_{\mathcal{H}\mathcal{P}}\Lambda_1^{[j]}(\varphi_1, \varphi_2, \varphi_3) & \dots & {}_{\mathcal{H}\mathcal{P}}\Lambda_{n-1}^{[j]}(\varphi_1, \varphi_2, \varphi_3) & {}_{\mathcal{H}\mathcal{P}}\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3) \end{vmatrix}.$$

Thus, simple row operations are used to finish the proof. □

Note. Although the determinant construction follows the Riordan-array methodology, the present form is specialized to the hybrid Gould–Hopper–Sheffer– λ structure. The parameter coupling and generating-function form lead to modified coefficient arrays and new parameter-dependent determinant entries.

3. Quasi-monomial properties

To place the GHS $\Lambda P_{\mathcal{H}} \Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3)$ in the setting of the monomiality principle, we establish the result stated below. This theorem demonstrates that the proposed polynomial family exhibits a quasi-monomial structure with respect to suitably defined multiplicative and differential operators.

Theorem 3.1. *The GHS $\Lambda P_{\mathcal{H}} \Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3)$ form a quasi-monomial sequence under the action of the following multiplicative and derivative operators:*

$$\hat{M} = \frac{\mathcal{A}'(\mathcal{H}^{-1}(D_{\varphi_2}))}{\mathcal{A}(\mathcal{H}^{-1}(D_{\varphi_2}))} + \left(\varphi_2 + j\varphi_3 [\mathcal{H}(\mathcal{H}^{-1}(D_{\varphi_2}))]^{j-1} \right) \mathcal{H}'(\mathcal{H}^{-1}(D_{\varphi_2})) - \frac{1}{2} \sqrt{\frac{\varphi_1}{\mathcal{H}^{-1}(D_{\varphi_2})}} \tan(\sqrt{\varphi_1 \mathcal{H}^{-1}(D_{\varphi_2})}), \quad (3.1)$$

and

$$\hat{P} = \mathcal{H}^{-1}(D_{\varphi_2}), \quad (3.2)$$

respectively.

Proof. Taking the derivative on each side of generation function (2.1) with respect to χ , we have

$$\begin{aligned} \sum_{n=0}^{\infty} {}_{\mathcal{H}}S\Lambda_{n+1}^{[j]}(\varphi_1, \varphi_2, \varphi_3) \frac{\chi^n}{n!} &= \frac{\mathcal{A}'(\chi)}{\mathcal{A}(\chi)} \mathcal{A}(\chi) \exp(\varphi_2 \mathcal{H}(\chi) + \varphi_3 [\mathcal{H}(\chi)]^j) \cos(\sqrt{\varphi_1 \chi}) + \varphi_2 \mathcal{H}'(\chi) \mathcal{A}(\chi) \exp(\varphi_2 \mathcal{H}(\chi) \\ &+ \varphi_3 [\mathcal{H}(\chi)]^j) \cos(\sqrt{\varphi_1 \chi}) + j\varphi_3 (\mathcal{H}(\chi))^{j-1} \mathcal{H}'(\chi) \mathcal{A}(\chi) \exp(\varphi_2 \mathcal{H}(\chi) + \varphi_3 [\mathcal{H}(\chi)]^j) \cos(\sqrt{\varphi_1 \chi}) \\ &- \frac{1}{2} \sqrt{\frac{\varphi_1}{\chi}} \sin(\sqrt{\varphi_1 \chi}) \mathcal{A}(\chi) \exp(\varphi_2 \mathcal{H}(\chi) + \varphi_3 [\mathcal{H}(\chi)]^j) \cos(\sqrt{\varphi_1 \chi}). \end{aligned}$$

The above expression can be simplified as follows:

$$\begin{aligned} \sum_{n=0}^{\infty} {}_{\mathcal{H}}S\Lambda_{n+1}^{[j]}(\varphi_1, \varphi_2, \varphi_3) \frac{\chi^n}{n!} &= \mathcal{A}(\chi) \exp(\varphi_2 \mathcal{H}(\chi) + \varphi_3 [\mathcal{H}(\chi)]^j) \cos(\sqrt{\varphi_1 \chi}) \left[\frac{\mathcal{A}'(\chi)}{\mathcal{A}(\chi)} + \varphi_2 \mathcal{H}'(\chi) \right. \\ &\left. + j\varphi_3 (\mathcal{H}(\chi))^{j-1} \mathcal{H}'(\chi) - \frac{1}{2} \sqrt{\frac{\varphi_1}{\chi}} \tan(\sqrt{\varphi_1 \chi}) \right]. \quad (3.3) \end{aligned}$$

Also, we have

$$D_{\varphi_2} \left(\mathcal{A}(\chi) \exp(\varphi_2 \mathcal{H}(\chi) + \varphi_3 [\mathcal{H}(\chi)]^j) \cos(\sqrt{\varphi_1 \chi}) \right) = \mathcal{H}(\chi) \mathcal{A}(\chi) \exp(\varphi_2 \mathcal{H}(\chi) + \varphi_3 [\mathcal{H}(\chi)]^j) \cos(\sqrt{\varphi_1 \chi}). \quad (3.4)$$

Thus, on using (3.4) in (3.3), we find

$$\begin{aligned} \sum_{n=0}^{\infty} {}_{\mathcal{H}}S\Lambda_{n+1}^{[j]}(\varphi_1, \varphi_2, \varphi_3) \frac{\chi^n}{n!} &= \mathcal{A}(\chi) \exp(\varphi_2 \mathcal{H}(\chi) + \varphi_3 [\mathcal{H}(\chi)]^j) \cos(\sqrt{\varphi_1 \chi}) \left[\frac{\mathcal{A}'(\mathcal{H}^{-1}(D_{\varphi_2}))}{\mathcal{A}(\mathcal{H}^{-1}(D_{\varphi_2}))} \right. \\ &\left. + \left(\varphi_2 + j\varphi_3 [\mathcal{H}(\mathcal{H}^{-1}(D_{\varphi_2}))]^{j-1} \right) \mathcal{H}'(\mathcal{H}^{-1}(D_{\varphi_2})) - \frac{1}{2} \sqrt{\frac{\varphi_1}{\mathcal{H}^{-1}(D_{\varphi_2})}} \tan(\sqrt{\varphi_1 \mathcal{H}^{-1}(D_{\varphi_2})}) \right]. \end{aligned}$$

Further utilizing (2.1) in the r.h.s. of the above equation, we find

$$\begin{aligned} \sum_{n=0}^{\infty} {}_{\mathcal{H},S}\Lambda_{n+1}^{[j]}(\varphi_1, \varphi_2, \varphi_3) \frac{\chi^n}{n!} &= \sum_{n=0}^{\infty} \left[\frac{\mathcal{A}'(\mathcal{H}^{-1}(D_{\varphi_2}))}{\mathcal{A}(\mathcal{H}^{-1}(D_{\varphi_2}))} \right. \\ &\quad + \left(\varphi_2 + j\varphi_3 [\mathcal{H}(\mathcal{H}^{-1}(D_{\varphi_2}))]^{j-1} \right) \mathcal{H}'(\mathcal{H}^{-1}(D_{\varphi_2})) \\ &\quad \left. - \frac{1}{2} \sqrt{\frac{\varphi_1}{\mathcal{H}^{-1}(D_{\varphi_2})}} \tan\left(\sqrt{\varphi_1 \mathcal{H}^{-1}(D_{\varphi_2})}\right) \right] {}_{\mathcal{H},S}\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3) \frac{\chi^n}{n!}. \end{aligned}$$

By comparing the coefficients of χ on both sides of the above relation, we obtain

$$\begin{aligned} {}_{\mathcal{H},S}\Lambda_{n+1}^{[j]}(\varphi_1, \varphi_2, \varphi_3) &= \left[\frac{\mathcal{A}'(\mathcal{H}^{-1}(D_{\varphi_2}))}{\mathcal{A}(\mathcal{H}^{-1}(D_{\varphi_2}))} + \left(\varphi_2 + j\varphi_3 [\mathcal{H}(\mathcal{H}^{-1}(D_{\varphi_2}))]^{j-1} \right) \right. \\ &\quad \left. \times \mathcal{H}'(\mathcal{H}^{-1}(D_{\varphi_2})) - \frac{1}{2} \sqrt{\frac{\varphi_1}{\mathcal{H}^{-1}(D_{\varphi_2})}} \tan\left(\sqrt{\varphi_1 \mathcal{H}^{-1}(D_{\varphi_2})}\right) \right] {}_{\mathcal{H},S}\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3). \quad (3.5) \end{aligned}$$

In view of (1.1) and (3.5), assertion (3.1) follows.

Next, using Eq (2.1) in (3.4), we find

$$\mathcal{H}^{-1}(D_{\varphi_2}) \sum_{n=0}^{\infty} {}_{\mathcal{H},S}\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3) \frac{\chi^n}{n!} = \sum_{n=0}^{\infty} {}_{\mathcal{H},S}\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3) \frac{\chi^{n+1}}{n!},$$

thus on comparing the coefficients of χ on both sides of the above relation, we obtain

$$\mathcal{H}^{-1}(D_{\varphi_2}) {}_{\mathcal{H},S}\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3) = n {}_{\mathcal{H},S}\Lambda_{n-1}^{[j]}(\varphi_1, \varphi_2, \varphi_3), \quad n \geq 1.$$

In view of (1.2), the preceding equation implies assertion (3.2). \square

In light of Eq (1.4) together with the relations given in (3.1), we derive the following result as a direct consequence of Theorem 3.1.

Corollary 3.1. *The GHS λ P ${}_{\mathcal{H},S}\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3)$ has the following explicit representations:*

$${}_{\mathcal{H},S}\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3) = \hat{M}_{GHS P}^n \{1\},$$

and

$$\begin{aligned} {}_{\mathcal{H},S}\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3) &= \left(\frac{\mathcal{A}'(\mathcal{H}^{-1}(D_{\varphi_2}))}{\mathcal{A}(\mathcal{H}^{-1}(D_{\varphi_2}))} + \left(\varphi_2 + j\varphi_3 [\mathcal{H}(\mathcal{H}^{-1}(D_{\varphi_2}))]^{j-1} \right) \mathcal{H}'(\mathcal{H}^{-1}(D_{\varphi_2})) \right. \\ &\quad \left. - \frac{1}{2} \sqrt{\frac{\varphi_1}{\mathcal{H}^{-1}(D_{\varphi_2})}} \tan\left(\sqrt{\varphi_1 \mathcal{H}^{-1}(D_{\varphi_2})}\right) \right)^n \{1\}. \end{aligned}$$

Since the structural properties of a quasi-monomial sequence are encoded in its multiplicative and derivative operators, these operators can be directly employed to obtain the differential equation satisfied by the GHS λ P ${}_{\mathcal{H},S}\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3)$.

Theorem 3.2. The GHS λP ${}_{\mathcal{H}}S\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3)$ satisfies the following differential equation:

$$\left(\left(\frac{\mathcal{A}'(\mathcal{H}^{-1}(D_{\varphi_2}))}{\mathcal{A}(\mathcal{H}^{-1}(D_{\varphi_2}))} + \left(\varphi_2 + j\varphi_3 \left[\mathcal{H}(\mathcal{H}^{-1}(D_{\varphi_2})) \right]^{j-1} \right) \mathcal{H}'(\mathcal{H}^{-1}(D_{\varphi_2})) \right. \right. \\ \left. \left. - \frac{1}{2} \sqrt{\frac{\varphi_1}{\mathcal{H}^{-1}(D_{\varphi_2})}} \tan(\sqrt{\varphi_1 \mathcal{H}^{-1}(D_{\varphi_2})}) \right) \mathcal{H}^{-1}(D_{\varphi_2}) - n \right) {}_{\mathcal{H}}S\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3) = 0. \quad (3.6)$$

Proof. Using Eqs (3.1) and (3.2) in (1.3), we get assertion (3.6). \square

Theorem 3.3. An explicit operational formula establishing the connection between the GHS λP ${}_{\mathcal{H}}S\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3)$ and the Sheffer- λ polynomials ${}_S\Lambda_n(\varphi_1, \varphi_2)$ is given by

$${}_{\mathcal{H}}S\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3) = \exp\left(\varphi_3 \frac{\partial^j}{\partial \varphi_2^j}\right) \{ {}_S\Lambda_n(\varphi_1, \varphi_2) \}.$$

Proof. The result follows immediately from the defining relation (2.1). Indeed, applying the operational identity given in (2.4) yields the stated representation in a straightforward manner. \square

4. Examples

To present meaningful illustrations, it is helpful to recall that the theory of Sheffer sequences provides a unified framework in which many classical polynomial families can be described in a systematic manner. In the present setting, different choices of the functions $A(\chi)$ and $H(\chi)$ immediately produce distinct hybrid polynomial families. Although these subclasses differ in form, they preserve the same operational foundation and quasi-monomial structure. The examples that follow therefore illustrate how the general construction systematically recovers several classical families as particular cases.

From an applied perspective, these examples underline the practical strength of the framework. A single generating-function mechanism is capable of accommodating both traditional polynomial systems and their hybrid extensions, thereby demonstrating the coherence, flexibility, and extensibility of the proposed approach.

Example 4.1. We consider the special choice $\mathcal{A}(\chi) := \frac{\chi}{e^\chi - 1}$ and $\mathcal{H}(\chi) = 2\chi$ in the left-hand side of the generating function (2.1) associated with the Gould-Hopper-Bernoulli- λ polynomials (GHB λP). Under this specification, one obtains the generating function of the GHB λP ${}_{\mathcal{H}}\mathcal{B}\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3)$ in the form

$$\frac{\chi}{e^\chi - 1} e^{2\varphi_2\chi + 2\varphi_3\chi^j} \cos(\sqrt{\varphi_1\chi}) = \sum_{n=0}^{\infty} {}_{\mathcal{H}}\mathcal{B}\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3) \frac{\chi^n}{n!}.$$

In consideration of the preceding expression, the series representation of the GHB λP ${}_{\mathcal{H}}\mathcal{B}\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3)$ follows as

$${}_{\mathcal{H}}\mathcal{B}\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3) = \sum_{k=0}^n \binom{n}{k} \mathcal{B}_k {}_{\mathcal{H}}\Lambda_{n-k}^{[j]}(\varphi_1, \varphi_2, \varphi_3).$$

Moreover, the GHB λ P ${}_{\mathcal{H}\mathcal{B}}\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3)$ constitutes a quasi-monomial family relative to the operators defined, respectively, by

$$\hat{M} = 2\left(\varphi_2 + j\varphi_3 \frac{\partial^{j-1}}{\partial \varphi_2^{j-1}}\right) - \frac{1}{2} \sqrt{\frac{\varphi_1}{\partial \varphi_2}} \tan(\sqrt{\varphi_1 \partial \varphi_2}) + \frac{e^{\partial \varphi_2} (1 - \partial \varphi_2) - 1}{\partial \varphi_2 (e^{\partial \varphi_2} - 1)},$$

and

$$\hat{P} = \frac{1}{2} \partial \varphi_2.$$

As a consequence of this quasi-monomial structure, the GHB λ P ${}_{\mathcal{H}\mathcal{B}}\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3)$ satisfies the differential equation

$$\left(\left(\varphi_2 + j\varphi_3 \frac{\partial^{j-1}}{\partial \varphi_2^{j-1}} \right) \partial \varphi_2 - \frac{1}{4} \partial \varphi_2 \sqrt{\frac{\varphi_1}{\partial \varphi_2}} \tan(\sqrt{\varphi_1 \partial \varphi_2}) + \frac{e^{\partial \varphi_2} (1 - \partial \varphi_2) - 1}{2(e^{\partial \varphi_2} - 1)} - n \right) {}_{\mathcal{H}\mathcal{B}}\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3) = 0.$$

In addition, a direct connection between the classical Bernoulli polynomials $\mathcal{B}_n(\varphi_1)$ and the GHB λ P ${}_{\mathcal{H}\mathcal{B}}\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3)$ is expressed by

$${}_{\mathcal{H}\mathcal{B}}\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3) = \exp\left(\varphi_3 \frac{\partial^j}{\partial \varphi_2^j}\right) \{\mathcal{B}_n(\varphi_1, \varphi_2)\}.$$

Finally, by taking $\theta_0 = 1$ and $\theta_i = \frac{1}{i+1}$ for $i = 1, 2, \dots, n$ in (2.7), one arrives at the determinantal representation of the GHB λ P ${}_{\mathcal{H}\mathcal{B}}\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3)$. More precisely,

$${}_{\mathcal{H}\mathcal{B}}\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3) = (-1)^n \times \begin{vmatrix} 1 & {}_{\mathcal{H}}\Lambda_1^{[j]}(\varphi_1, \varphi_2, \varphi_3) & {}_{\mathcal{H}}\Lambda_2^{[j]}(\varphi_1, \varphi_2, \varphi_3) & \cdots & {}_{\mathcal{H}}\Lambda_{n-1}^{[j]}(\varphi_1, \varphi_2, \varphi_3) & {}_{\mathcal{H}}\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3) \\ 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n} & \frac{1}{n+1} \\ 0 & 1 & \binom{2}{1} \frac{1}{2} & \cdots & \binom{n-1}{1} \frac{1}{n-1} & \binom{n}{1} \frac{1}{n} \\ 0 & 0 & 1 & \cdots & \binom{n-1}{1} \frac{1}{n-2} & \binom{n}{2} \frac{1}{n-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \binom{n}{n-1} \frac{1}{2} \end{vmatrix},$$

where ${}_{\mathcal{H}}\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3)$, $(n = 0, 1, 2, \dots)$ denote the Gould-Hopper λ -polynomials of degree n .

Next, we present the surface plots of these polynomials to better illustrate their geometric behavior and variation with respect to the parameters. These graphical representations provide a clearer understanding of how the GHB λ P evolves for different orders and parameter choices, highlighting the changes in curvature, growth, and overall surface structure.

Surface plots of GHB λ P for varying orders and parameters are presented from Figures 1–4.

Figure 1

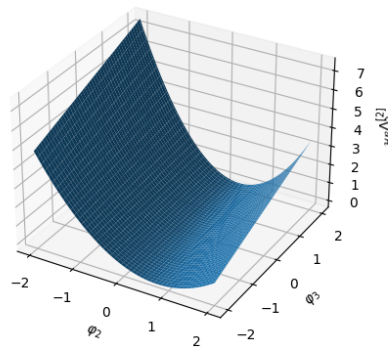


Figure 1. Surface plot of $\mathcal{H}_B\Lambda_2^{[2]}(\varphi_1, \varphi_2, \varphi_3)$.

Figure 2

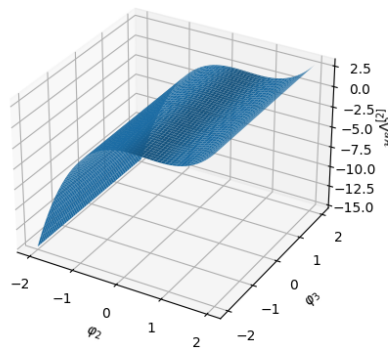


Figure 2. Surface plot of $\mathcal{H}_B\Lambda_3^{[2]}(\varphi_1, \varphi_2, \varphi_3)$.

Figure 3

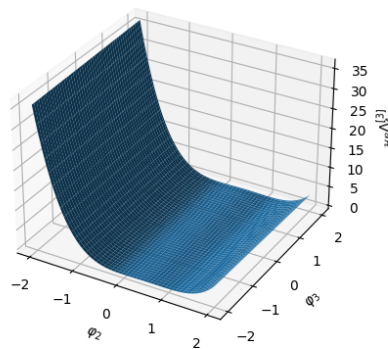


Figure 3. Surface plot of $\mathcal{H}_B\Lambda_4^{[2]}(\varphi_1, \varphi_2, \varphi_3)$.

Figure 4

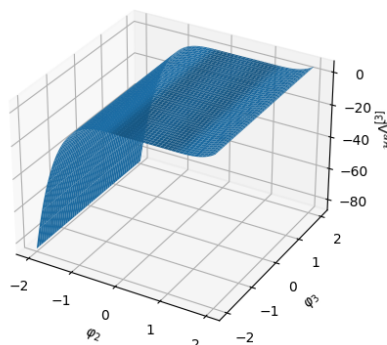


Figure 4. Surface plot of $\mathcal{H}\mathcal{B}\Lambda_5^{[3]}(\varphi_1, \varphi_2, \varphi_3)$.

Example 4.2. Let us choose $\mathcal{A}(\chi) := \frac{2}{e^\chi + 1}$ and $\mathcal{H}(\chi) = 2\chi$ in the left-hand side of the generating function (2.1) associated with the Gould–Hopper–Euler– λ polynomials (GHE λ P). Under this choice, one obtains the following generating function for the GHE λ P ${}_{\mathcal{H}}\mathcal{E}\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3)$:

$$\frac{2}{e^\chi + 1} e^{2\varphi_2\chi + 2\varphi_3\chi^j} \cos(\sqrt{\varphi_1}\chi) = \sum_{n=0}^{\infty} {}_{\mathcal{H}}\mathcal{E}\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3) \frac{\chi^n}{n!}.$$

In consideration of preceding expression, the series representation of the GHE λ P follows in the form

$${}_{\mathcal{H}}\mathcal{E}\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3) = \sum_{k=0}^n \binom{n}{k} \mathcal{E}_k {}_{\mathcal{H}}\Lambda_{n-k}^{[j]}(\varphi_1, \varphi_2, \varphi_3).$$

Moreover, the family of GHE λ P ${}_{\mathcal{H}}\mathcal{E}\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3)$ constitutes a quasi-monomial family relative to the operators defined, respectively, by

$$\hat{M} = 2 \left(\varphi_2 + j\varphi_3 \frac{\partial^{j-1}}{\partial \varphi_2^{j-1}} \right) - \frac{1}{2} \sqrt{\frac{\varphi_1}{\partial \varphi_2}} \tan \left(\sqrt{\varphi_1 \partial \varphi_2} \right) - \frac{e^{\partial \varphi_2}}{e^{\partial \varphi_2} + 1},$$

and

$$\hat{P} = \frac{1}{2} \partial_{\varphi_2}.$$

As a consequence of this operational framework, the GHE λ P satisfies the differential equation

$$\left(\left(\varphi_2 + j\varphi_3 \frac{\partial^{j-1}}{\partial \varphi_2^{j-1}} \right) \partial_{\varphi_2} - \frac{1}{4} \partial_{\varphi_2} \sqrt{\frac{\varphi_1}{\partial \varphi_2}} \tan \left(\sqrt{\varphi_1 \partial \varphi_2} \right) - \frac{e^{\partial \varphi_2}}{2(e^{\partial \varphi_2} + 1)} \partial_{\varphi_2} - n \right) {}_{\mathcal{H}}\mathcal{E}\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3) = 0.$$

In addition, a direct connection between the classical Euler polynomials $\mathcal{E}_n(\varphi_1)$ and the GHE λ P is expressed by

$${}_{\mathcal{H}}\mathcal{E}\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3) = \exp \left(\varphi_3 \frac{\partial^j}{\partial \varphi_2^j} \right) \{ \mathcal{E}\Lambda_n(\varphi_1, \varphi_2) \}.$$

Finally, by taking $\theta_0 = 1$ and $\theta_i = \frac{1}{i+1}$ for $(i = 1, 2, \dots, n)$ in Eq (2.7), one arrives at the determinant representation of the GHEL \mathcal{P} ${}_{\mathcal{H}}\mathcal{E}\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3)$, which is given by

$${}_{\mathcal{H}}\mathcal{E}\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3) = (-1)^n \times \begin{vmatrix} 1 & {}_{\mathcal{H}}\Lambda_1^{[j]}(\varphi_1, \varphi_2, \varphi_3) & {}_{\mathcal{H}}\Lambda_2^{[j]}(\varphi_1, \varphi_2, \varphi_3) & \cdots & {}_{\mathcal{H}}\Lambda_{n-1}^{[j]}(\varphi_1, \varphi_2, \varphi_3) & {}_{\mathcal{H}}\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3) \\ 1 & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \binom{2}{1}\frac{1}{2} & \cdots & \binom{n-1}{1}\frac{1}{2} & \binom{n}{1}\frac{1}{2} \\ 0 & 0 & 1 & \cdots & \binom{n-1}{1}\frac{1}{2} & \binom{n}{2}\frac{1}{2} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \binom{n}{n-1}\frac{1}{2} \end{vmatrix},$$

where ${}_{\mathcal{H}}\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3)$, $(n = 0, 1, 2, \dots)$ denote the Gould–Hopper λ -polynomials of degree n .

Next, we present the colored surface plots of these polynomials to better visualize their structural and geometric behavior. These graphical representations clearly demonstrate how the GHEL \mathcal{P} varies with different orders and parameter choices, revealing noticeable changes in curvature, oscillatory patterns, and surface growth characteristics.

Surface plots of GHEL \mathcal{P} for varying orders and parameters are presented from Figures 5–8.

Figure 5

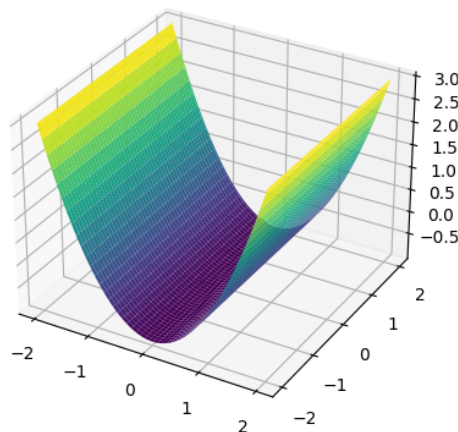


Figure 5. Surface plot of ${}_{\mathcal{H}}\mathcal{E}\Lambda_2^{[2]}(\varphi_1, \varphi_2, \varphi_3)$.

Figure 6

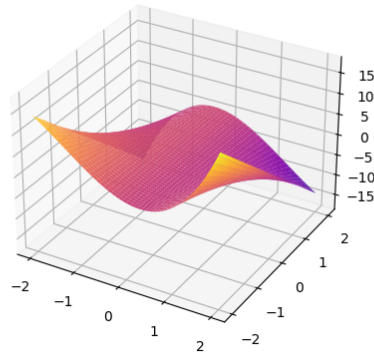


Figure 6. Surface plot of ${}_{\mathcal{H}\mathcal{E}}\Lambda_3^{[2]}(\varphi_1, \varphi_2, \varphi_3)$.

Figure 7

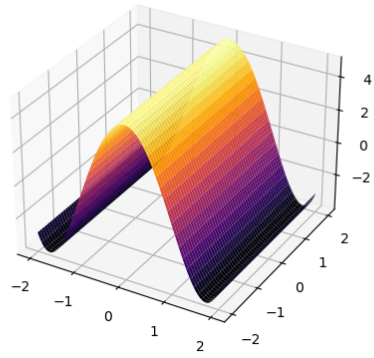


Figure 7. Surface plot of ${}_{\mathcal{H}\mathcal{E}}\Lambda_4^{[3]}(\varphi_1, \varphi_2, \varphi_3)$.

Figure 8

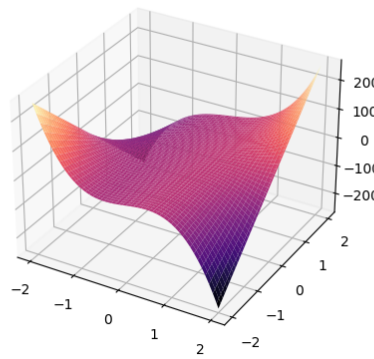


Figure 8. Surface plot of ${}_{\mathcal{H}\mathcal{E}}\Lambda_5^{[3]}(\varphi_1, \varphi_2, \varphi_3)$.

Example 4.3. As a concrete illustration, we choose $\mathcal{A}(\chi) := \frac{2\chi}{e^\chi + 1}$ and $\mathcal{H}(\chi) = 2\chi$ in the left-hand side of the generating function (2.1) corresponding to the Gould–Hopper–Genocchi- λ polynomials (GHG λ P). With this choice, one obtains the following generating function for the GHG λ P ${}_{\mathcal{H}}\mathcal{G}\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3)$:

$$\frac{2\chi}{e^\chi + 1} e^{2\varphi_2\chi + 2\varphi_3\chi^j} \cos(\sqrt{\varphi_1}\chi) = \sum_{n=0}^{\infty} {}_{\mathcal{H}}\mathcal{G}\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3) \frac{\chi^n}{n!}.$$

In consideration of preceding expression, the series representation of the GHG λ P ${}_{\mathcal{H}}\mathcal{G}\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3)$ follows as

$${}_{\mathcal{H}}\mathcal{G}\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3) = \sum_{k=0}^n \binom{n}{k} \mathcal{G}_k {}_{\mathcal{H}}\Lambda_{n-k}^{[j]}(\varphi_1, \varphi_2, \varphi_3).$$

Moreover, the GHG λ P ${}_{\mathcal{H}}\mathcal{G}\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3)$ constitutes a quasi-monomial family relative to the multiplication and differentiation operators defined, respectively, by

$$\hat{M} = 2 \left(\varphi_2 + j\varphi_3 \frac{\partial^{j-1}}{\partial \varphi_2^{j-1}} \right) - \frac{1}{2} \sqrt{\frac{\varphi_1}{\partial \varphi_2}} \tan(\sqrt{\varphi_1 \partial \varphi_2}) + \frac{e^{\partial \varphi_2} (1 - \partial \varphi_2) - 1}{\partial \varphi_2 (e^{\partial \varphi_2} + 1)},$$

and

$$\hat{P} = \frac{1}{2} \partial \varphi_2,$$

respectively.

As a consequence of this quasi-monomial structure, the GHG λ P ${}_{\mathcal{H}}\mathcal{G}\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3)$ satisfies the following differential equation:

$$\left(\left(\varphi_2 + j\varphi_3 \frac{\partial^{j-1}}{\partial \varphi_2^{j-1}} \right) \partial \varphi_2 - \frac{1}{4} \partial \varphi_2 \sqrt{\frac{\varphi_1}{\partial \varphi_2}} \tan(\sqrt{\varphi_1 \partial \varphi_2}) + \frac{e^{\partial \varphi_2} (1 - \partial \varphi_2) - 1}{2(e^{\partial \varphi_2} + 1)} - n \right) {}_{\mathcal{H}}\mathcal{G}\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3) = 0.$$

In addition, a direct operational connection between the classical Genocchi polynomials $\mathcal{G}_n(\varphi_1)$ and the GHG λ P ${}_{\mathcal{H}}\mathcal{G}\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3)$ is expressed by

$${}_{\mathcal{H}}\mathcal{G}\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3) = \exp\left(\varphi_3 \frac{\partial^j}{\partial \varphi_2^j}\right) \{\mathcal{G}\Lambda_n(\varphi_1, \varphi_2)\}.$$

Finally, by taking $\theta_0 = 1$ and $\theta_i = \frac{1}{i+1}$, ($i = 1, 2, \dots, n$) in Eq (2.7), one arrives at the determinant representation of the GHG λ P ${}_{\mathcal{H}}\mathcal{G}\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3)$. Specifically, these polynomials admit the determinant form

$${}_{\mathcal{H}}\mathcal{G}\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3) = (-1)^n \times \begin{vmatrix} 1 & {}_{\mathcal{H}}\Lambda_1^{[j]}(\varphi_1, \varphi_2, \varphi_3) & {}_{\mathcal{H}}\Lambda_2^{[j]}(\varphi_1, \varphi_2, \varphi_3) & \cdots & {}_{\mathcal{H}}\Lambda_{n-1}^{[j]}(\varphi_1, \varphi_2, \varphi_3) & {}_{\mathcal{H}}\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3) \\ \frac{1}{2} & \frac{(1)_2}{4} & \frac{(1)_3}{6} & \cdots & \frac{(1)_n}{2n} & \frac{(1)_{n+1}}{2(n+1)} \\ 0 & \frac{1}{2} & \binom{2}{1} \frac{(1)_2}{4} & \cdots & \binom{n-1}{1} \frac{(1)_{n-1}}{2(n-1)} & \binom{n}{1} \frac{(1)_n}{2} \\ 0 & 0 & \frac{1}{2} & \cdots & \binom{n-1}{1} \frac{(1)_{n-2}}{2(n-2)} & \binom{n}{2} \frac{(1)_{n-1}}{2(n-1)} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{2} & \binom{n}{n-1} \frac{(1)_2}{4} \end{vmatrix},$$

where ${}_{\mathcal{H}}\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3)$, ($n = 0, 1, 2, \dots$), are referred to as the Gould–Hopper λ -polynomials of degree n .

Example 4.4. Let us choose $\mathcal{A}(\chi) := \frac{1}{1-\chi}$ and $\mathcal{H}(\chi) = \frac{-\chi}{1-\chi}$, which corresponds to the special case where the Sheffer polynomials $\mathcal{S}_n(\varphi_1)$ reduce to the classical Laguerre polynomials $\mathcal{L}_n(\varphi_1)$. Substituting these choices into the left-hand side of the generating function (2.1), we arrive at the following generating function for the Gould-Hopper-Laguerre λ -polynomials ${}_{\mathcal{H}\mathcal{L}}\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3)$:

$$\frac{1}{1-\chi} \exp\left(\frac{-\varphi_2\chi}{1-\chi} + \varphi_3\left(\frac{-\chi}{1-\chi}\right)^j\right) \cos(\sqrt{\varphi_1\chi}) = \sum_{n=0}^{\infty} {}_{\mathcal{H}\mathcal{L}}\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3) \frac{\chi^n}{n!}.$$

In consideration of preceding expression, the series representation of the Gould-Hopper-Laguerre λ -polynomials follows immediately and can be written as

$${}_{\mathcal{H}\mathcal{L}}\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3) = \sum_{k=0}^n \binom{n}{k} \mathcal{L}_k {}_{\mathcal{H}}\Lambda_{n-k}^{[j]}(\varphi_1, \varphi_2, \varphi_3).$$

Moreover, the Gould-Hopper-Laguerre λ -polynomials ${}_{\mathcal{H}\mathcal{L}}\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3)$ constitute a quasi-monomial family relative to the multiplication and differentiation operators defined, respectively, by

$$\hat{M} = 2\left(\varphi_2 + j\varphi_3 \frac{\partial^{j-1}}{\partial \varphi_2^{j-1}}\right) - \frac{1}{2} \sqrt{\frac{\varphi_1}{\partial \varphi_2}} \tan(\sqrt{\varphi_1 \partial \varphi_2}) + \frac{e^{\partial \varphi_2} (1 - \partial \varphi_2) - 1}{\partial \varphi_2 (e^{\partial \varphi_2} + 1)},$$

and

$$\hat{P} = \frac{1}{2} \partial_{\varphi_2},$$

respectively.

As a consequence of their quasi-monomial structure, these polynomials satisfy the differential equation

$$\left(\left(\varphi_2 + j\varphi_3 \frac{\partial^{j-1}}{\partial \varphi_2^{j-1}}\right) \partial_{\varphi_2} - \frac{1}{4} \partial_{\varphi_2} \sqrt{\frac{\varphi_1}{\partial \varphi_2}} \tan(\sqrt{\varphi_1 \partial \varphi_2}) + \frac{e^{\partial \varphi_2} (1 - \partial \varphi_2) - 1}{2(e^{\partial \varphi_2} + 1)} - n\right) {}_{\mathcal{H}\mathcal{L}}\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3) = 0.$$

Finally, the Gould-Hopper-Laguerre λ -polynomials are connected to the classical Laguerre polynomials $\mathcal{L}_n(\varphi_1)$ through the operational identity

$${}_{\mathcal{H}\mathcal{L}}\Lambda_n^{[j]}(\varphi_1, \varphi_2, \varphi_3) = \exp\left(\varphi_3 \frac{\partial^j}{\partial \varphi_2^j}\right) \{\mathcal{L}_n(\varphi_1, \varphi_2)\}.$$

These examples illustrates how the general hybrid generating structure reduces to the classical-polynomial family under the indicated parameter choice, confirming the consistency and extensibility of the framework.

4.1. Applications and applicability of the proposed polynomial family

The polynomial family introduced in this work is not only of theoretical relevance, but also offers a flexible and constructive framework that can be used in a variety of applied and computational contexts. Because the polynomials are defined through generating functions and operational techniques, they fit naturally into several analytical settings. In this subsection, we briefly describe some practical directions in which the proposed Gould–Hopper–Sheffer– λ type polynomials can be used.

Operational calculus and differential equations. The proposed polynomials arise from generating functions together with multiplicative and derivative-type operators, which makes them well suited for use in operational calculus. Their monomiality-based structure allows operator relations to be translated into explicit polynomial identities in a systematic way. Consequently, they can be used to express and manipulate solutions of linear as well as certain nonlinear differential equations. In particular, operator factorizations and ladder-type constructions can be handled conveniently within this polynomial framework, leading to clearer and more structured solution procedures.

Basis construction for higher-order differential models. Due to their recurrence relations and operator characterizations, these polynomials form a convenient basis for representing solutions of higher-order differential and integro-differential equations. Expanding solutions in this polynomial basis often leads to expressions that are both stable and algebraically manageable. This approach is especially helpful in parameter-dependent models, where the hybrid nature of the polynomials provides additional freedom to accommodate boundary and initial conditions.

Hybrid special-function expansions in mathematical physics. Because of their hybrid structure, the proposed polynomials are suitable for building mixed special-function expansions that occur in mathematical physics and applied analysis. They can be used in the study of generalized heat-type equations, evolution problems, and operator-based models. Moreover, since many well-known polynomial families (including Bernoulli-, Euler-, Laguerre-type, and Genocchi-type cases) appear as special subclasses, the present framework remains compatible with classical special-function methods and existing analytical tools.

Determinant representations and computational schemes. The determinant representations obtained in this work give compact and practical formulas for computing the polynomial sequences. These forms are well suited for both symbolic and numerical implementation. Furthermore, the connection with Riordan-array-type structures supports efficient coefficient extraction, recursive computation, and algorithmic generation of the polynomials. This makes the family particularly useful for computer algebra systems and for the development of symbolic computation routines.

Taken together, these aspects show that the proposed polynomial class is not only of theoretical interest, but also serves as a practical and adaptable tool for applications in differential equations, operational methods, and computational mathematics.

5. Conclusions

This paper presented a systematic investigation of the Gould-Hopper-Sheffer- λ polynomials by embedding the Gould-Hopper framework within the general theory of Sheffer- λ sequences. Through an appropriate generating function approach, explicit series representations and determinant forms were derived, providing a transparent algebraic and operational characterization of the newly introduced polynomial family.

The monomiality principle was shown to play a fundamental role in establishing the quasi-monomial structure of these polynomials. The explicit formulation of the corresponding multiplicative and derivative operators naturally gives rise to the associated differential equations, thereby establishing the internal coherence as well as the analytical soundness of the proposed framework. In addition, the derived operational representations that relate the Gould-Hopper-Sheffer- λ polynomials to the classical Sheffer- λ polynomials clearly emphasize the unifying character and the inherent flexibility of the present approach.

Several important subclasses emerge as particular instances of the general theory, including the Gould-Hopper-Bernoulli- λ , Euler- λ , Genocchi- λ , and Laguerre- λ polynomials. These illustrative cases not only corroborate the validity of the theoretical development but also reveal how a wide range of well-known polynomial families can be systematically retrieved within the proposed operational framework.

The results obtained in this work contribute to the growing theory of hybrid special polynomials and suggest several promising directions for future research. Potential extensions include the investigation of orthogonality and bi-orthogonality properties, the development of fractional and q -analogues, and the application of the proposed polynomials to fractional differential equations and boundary value problems. Further exploration of combinatorial interpretations, probabilistic models, and applications in quantum mechanics, signal processing, and mathematical physics may also yield fruitful outcomes. The hybrid Gould-Hopper-Sheffer- λ framework developed here provides a unified and extensible structure that connects operational calculus, Sheffer theory, and hybrid special polynomial systems. The added application directions and computational representations indicate further potential developments in both analytical and applied settings.

Author contributions

All authors of this article have been contributed equally. All authors have read and approved the final version of the manuscript for publication

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

This work does not have any conflicts of interest.

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