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Research article

A new generalized differential transform method for fractional ODEs with statistical applications

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Abstract: This paper presents a new generalized differential transform method (NGDTM) in the solution of fractional-order differential equations. The technique is based on the generalized Taylor formula and the Riemann–Liouville fractional derivative. Theorems of fundamental transformation are developed based on rigorous proofs, and the convergence and uniqueness of the solutions obtained are proven. A number of linear and nonlinear examples such as models of statistical relevance are provided to demonstrate the accuracy and efficiency of the proposed approach. Besides, the classical exponential distribution is derived using the proposed NGDTM, and a new fractional exponential distribution is proposed on the same basis with the use of the same framework. The findings reveal that the technique provides very good approximate solutions and, in few instances, the exact solution by only few iterations, thus validating it as a tool of fractional differential equations as well as use in statistics.

Keywords: fractional differential equations; Riemann–Liouville fractional derivative; generalized Taylor formula; differential transform method; probability distribution
Mathematics Subject Classification: 60E05, 34A30

1. Introduction

Fractional-order mathematical models have gained substantial popularity as useful mechanisms in explaining systems whose dynamics are not only based on the present state but also on past dynamics. In contrast to classical integer-order differential equations, in fractional differential equations (FDEs), nonlocal operators are used by nature to capture memory and hereditary properties. These properties are especially useful in the modeling of phenomena in viscoelasticity, anomalous transport, biology, signal analysis, and new applications of applied statistics. FDEs are much harder to analyze than integer-order equations. Exact solutions are not available in most real-world scenarios, particularly nonlinear equations, and this requires approximate or semi-analytical solutions [1]. The broad spectrum of techniques suggested over the last decades comprises of homotopy-based schemes, decomposition schemes, variational iteration schemes, and weighted residual formulations [2–4]. Even though these techniques have been very successful, they can be characterized by complex calculations, auxiliary parameters, or correction cycles, which can influence convergence efficiency. One such strategy based on series solutions is the differential transform method (DTM), which has been found to be an attractive method because of its simplicity of implementation and algebraic character. Initially developed for the analysis of nonlinear electrochemical systems by Zhou and later refined by Bukhov [5, 6], DTM converts differential equations into recursive algebraic relations without discretization.This property permits the construction of approximate solutions in the form of a polynomial with relatively low computational cost and rapid convergence [7]. With the growing popularity of fractional modeling, a number of extensions to DTM were made to allow fractional-order operators. Among the more interesting extensions is the fractional differential transform method developed by Arikoglu and Ozkol; it uses fractional power series and the Caputo derivative, which is based on the Caputo derivative [8,9]. The other advancement is the generalised differentiated transform method, which was proposed by Odibat and Shawagfeh, in which the generalized Taylor expansion is used to expand the applicability of DTM to the FDEs [10]. Fractional calculus was applied in many statistical applications, including time series analysis, where persistent autocorrelations in financial and economic data using fractional differencing are captured [11], and spatial statistics that model long-range spatial dependency using fractional Laplacian operators [12]. The estimation of fractional order parameters from empirical data is commonly carried out using the maximum likelihood method and Bayesian techniques [13, 14]. The significance of the modeling of these phenomena illustrates why new methods can be applied and used to model the data and estimate different statistical parameters and measures [15, 16]. Other applications of fractional calculus in probability theory and stochastic processes are discussed in [17, 18]. Although useful, these methods can face constraints in terms of the selection of a fractional derivative or in the complexity of recurrence relations obtained. In the present work, we suggest a new generalized differential transform method (NGDTM) as another framework to solve the fractional ordinary differential equations (ODEs). The presented approach is built on the generalized Taylor expansion together with the Riemann–Liouville fractional derivation, resulting in an elastic and unified scheme of transformation. The NGDTM formulation allows deriving recurrence formulas of nonlinear and linear problems in a systematic manner without disrupting the analytic form of the solution. A group of transformation properties is deduced and strictly proven to determine the validity of the proposed approach. In addition, convergence and uniqueness issues are resolved to provide the mathematical validity of the solutions produced by the NGDTM. A number of illustrative examples are provided in order to illustrate how the method performs. The findings indicate that the NGDTM is very accurate using few terms, and in most instances, it recreates precise solutions. To obtain the classical exponential distribution, the proposed NGDTM has been employed. Moreover, a new fractional exponential distribution was developed with the help of the NGDTM. All these aspects suggest that the NGDTM is a valid and effective tool of fractional-order modeling that can potentially be used in applied mathematics and statistical analysis.

2. Basic fractional calculus definitions

In this section, there are some basic definitions of fractional calculus.

Definition 2.1. Let $\alpha \geq 0$, $\gamma \in \mathbb{R}$ and let $f(t)$ be continuous on $[\gamma, \tau]$. The *Riemann–Liouville fractional integral* of order α with lower limit γ is defined by [31]

$$J_{\gamma}^{\alpha} f(t) = J^{\alpha} f(t) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{\gamma}^{t} (t-\tau)^{\alpha-1} f(\tau) d\tau & \text{if } \alpha > 0, t > \gamma, \\ f(t) & \text{if } \alpha = 0. \end{cases} \tag{2.1}$$

Proposition 2.2. *Let $\alpha, \beta \geq 0$, $\gamma \in \mathbb{R}$ and let $f(t)$ be continuous on $[\gamma, \tau]$. Then the operator J_{γ}^{α} satisfies the following properties [31]*

$$\begin{aligned} &(i) \quad J_{\gamma}^{\alpha} J_{\gamma}^{\beta} f(t) = J_{\gamma}^{\alpha+\beta} f(t), \\ &(ii) \quad J_{\gamma}^{\alpha} J_{\gamma}^{\beta} f(t) = J_{\gamma}^{\beta} J_{\gamma}^{\alpha} f(t), \\ &(iii) \quad J_{\gamma}^{\alpha} (t-\gamma)^{n} = \frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)} (t-\gamma)^{n+\alpha}, \qquad n > -1. \end{aligned} \tag{2.2}$$

Definition 2.3. Let v be a real number, let m be the smallest integer such that $m - 1 < v \leq m$, and let $f(t)$ be continuous on $[\gamma, \tau]$. The Riemann–Liouville fractional derivative of order v with lower limit γ is defined by [31]

$$D_{\gamma}^{v} f(t) = D^{m} J_{\gamma}^{m-v} f(t), \tag{2.3}$$

where J_{γ}^{m-v} denotes the Riemann–Liouville fractional integral of order $m - v$ with lower limit γ. Equivalently, it can be expressed in integral form as

$$D_{\gamma}^{v} f(t) = \begin{cases} \dfrac{\mathrm{d}^{m}}{\mathrm{d}t^{m}} \left[\dfrac{1}{\Gamma(m-v)} \displaystyle\int_{\gamma}^{t} \dfrac{f(\tau) d\tau}{(t-\tau)^{v+1-m}} \right] & \text{if } m-1 < v < m, \\ \dfrac{\mathrm{d}^{m}}{\mathrm{d}t^{m}} f(t) & \text{if } v = m. \end{cases} \tag{2.4}$$

Property 2.4. *Let $\alpha > 0$ and $n > -1$. If $f(x) = (x-\gamma)^{n}$, then the Riemann–Liouville fractional derivative of order α with lower limit γ is given by [31]*

$$D_{\gamma}^{\alpha} (x-\gamma)^{n} = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} (x-\gamma)^{n-\alpha} \tag{2.5}$$

for $x > \gamma$.

Theorem 2.5. *(Generalized fractional Taylor formula [20]) Let $0 < \alpha \leq 1$ and assume that the fractional derivatives $D_{\gamma}^{j\alpha} f$ exist and are continuous on $(\gamma, \tau]$ for $j = 0, 1, \ldots, N+1$. If $x \in [\gamma, \tau]$, then*

$$f(x) \cong \sum_{j=0}^{N} \frac{C_j}{\Gamma(j\alpha + \alpha)} (x-\gamma)^{j\alpha+\alpha-1}, \tag{2.6}$$

where C_j defined by

$$C_j = \Gamma(\alpha) \left[(x-\gamma)^{1-\alpha} D_{\gamma}^{j\alpha} f(x) \right]_{x \to \gamma^{+}}. \tag{2.7}$$

3. New generalized differential transform method

Using Theorem 2.5, the NGDTM for the k–th derivative of a function $f(x)$ is defined as follows:

$$F(k) = \frac{C_k}{\Gamma(\alpha k + \alpha)} = \frac{\Gamma(\alpha)\Big[(x-\gamma)^{1-\alpha} D_{\gamma}^{k\alpha} f(x)\Big]_{x=\gamma^+}}{\Gamma(\alpha k + \alpha)}, \tag{3.1}$$

where $D_{\gamma}^{k\alpha}$ denotes the Riemann–Liouville fractional derivative of order $k\alpha$ and the expression is evaluated at $x = \gamma$ from the right-hand side, i.e.,

$$\Big[(x-\gamma)^{1-\alpha} D_{\gamma}^{k\alpha} f(x)\Big]_{x=\gamma^+} = \lim_{x\to\gamma^+} (x-\gamma)^{1-\alpha} D_{\gamma}^{k\alpha} f(x),$$

which is required since the Riemann–Liouville fractional derivative is defined for $x > \gamma$.

The corresponding inverse transform is given as

$$f(x) = \sum_{k=0}^{\infty} F_{\alpha}(k)(x-\gamma)^{\alpha k+\alpha-1} = \sum_{k=0}^{\infty} \frac{C_k}{\Gamma(\alpha k+\alpha)}(x-\gamma)^{\alpha k+\alpha-1}. \tag{3.2}$$

Hence, Eq (3.2) represents the inverse of the NGDTM. In applications, using Theorem 2.5, the solution $f(x)$ is approximated by the series

$$f(x) = \sum_{k=0}^{n} F_{\alpha}(k)(x-\gamma)^{\alpha k+\alpha-1}. \tag{3.3}$$

Some fundamental properties of the proposed method are presented in the following.

Theorem 3.1. *(Linearity) If $f(x) = g(x) + h(x)$, then the corresponding NGDTMs satisfy*

$$F(k) = G(k) + H(k), \quad k = 0, 1, 2, \ldots. \tag{3.4}$$

Proof. Using the inverse generalized differential transform Eq (3.2), the relation $f(x) = g(x)+h(x)$ can be written as

$$\sum_{k=0}^{\infty} F(k)(x-\gamma)^{\alpha k+\alpha-1} = g(x) = \sum_{k=0}^{\infty} G(k)(x-\gamma)^{\alpha k+\alpha-1} + \sum_{k=0}^{\infty} H(k)(x-\gamma)^{\alpha k+\alpha-1}. \tag{3.5}$$

That is,

$$\sum_{k=0}^{\infty} F(k)(x-\gamma)^{\alpha k+\alpha-1} = \sum_{k=0}^{\infty} (G(k)+H(k))(x-\gamma)^{\alpha k+\alpha-1}. \tag{3.6}$$

Equating the coefficients of like powers of $(x-\gamma)$ yields

$$F(k) = G(k) + H(k), \quad k = 0, 1, 2, \ldots. \tag{3.7}$$

□

Theorem 3.2. *(Scalar multiplication) If $f(x) = c, g(x)$, where c is a constant, then*

$$F(k) = c\,G(k), \quad k \geq 0. \tag{3.8}$$

Proof. Using the inverse generalized differential transform Eq (3.2), we write

$$\sum_{k=0}^{\infty} F(k)(x-\gamma)^{\alpha k+\alpha-1} = c\sum_{k=0}^{\infty} G(k)(x-\gamma)^{\alpha k+\alpha-1}. \tag{3.9}$$

Equating the coefficients of like powers of $(x-\gamma)$ yields

$$F(k) = c\,G(k), \quad k = 0, 1, 2, \ldots. \tag{3.10}$$

□

Theorem 3.3. *(Product) If $f(x) = g(x)\,h(x)$, then the NGDTM coefficients satisfy*

$$F(k) = \sum_{i=0}^{k} G(i)\,H(k-i), \quad k \geq 0. \tag{3.11}$$

Proof. Applying the inverse generalized differential transform Eq (3.2) to $f(x) = g(x)\,h(x)$, we obtain

$$f(x) = \left(\sum_{k=0}^{\infty} G(k)(x-\gamma)^{\alpha k+\alpha-1}\right)\left(\sum_{k=0}^{\infty} H(k)(x-\gamma)^{\alpha k+\alpha-1}\right). \tag{3.12}$$

Hence,

$$f(x) = (x-\gamma)^{\alpha-1}\left(\sum_{k=0}^{\infty} G(k)(x-\gamma)^{\alpha k}\right)\left(\sum_{k=0}^{\infty} H(k)(x-\gamma)^{\alpha k}\right). \tag{3.13}$$

Applying the Cauchy product of series yields

$$f(x) = (x-\gamma)^{\alpha-1}\sum_{k=0}^{\infty}\left(\sum_{i=0}^{k} G(i)H(k-i)\right)(x-\gamma)^{\alpha k}. \tag{3.14}$$

Comparing coefficients with the inverse transform Eq (3.2) gives

$$F(k) = \sum_{i=0}^{k} G(i)H(k-i), \quad k \geq 0. \tag{3.15}$$

□

Theorem 3.4. *(Triple product) If $f(x) = g_1(x)g_2(x)g_3(x)$, then the corresponding NGDTM coefficients satisfy*

$$F(k) = \sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2} G_1(k_1)G_2(k_2-k_1)G_3(k-k_2), \quad k \geq 0. \tag{3.16}$$

Proof. Using the inverse generalized differential transform Eq (3.2) for each function and applying the Cauchy product of three series, the result follows by direct coefficient comparison in a manner analogous to Theorem 3.3. □

General case. If

$$f(x) = \prod_{j=1}^{n} g_j(x), \tag{3.17}$$

then the NGDTM coefficients are given by

$$F(k) = \sum_{k_{n-1}=0}^{k} \sum_{k_{n-2}=0}^{k_{n-1}} \cdots \sum_{k_1=0}^{k_2} \prod_{j=1}^{n} G_j\left(k_j - k_{j-1}\right), \tag{3.18}$$

where $k_0 = 0$ and $k_n = k$.

Theorem 3.5. *(Power function) If $f(x) = (x-\gamma)^{\alpha m}$, $\quad m \in \mathbb{N}_0$, then the NGDTM coefficients are given by*

$$F(k) = \Delta_{k,m}, \tag{3.19}$$

where $\Delta_{k,m}$ is the Kronecker-type delta defined as

$$\Delta_{k,m} = \begin{cases} \dfrac{\Gamma(\alpha)\Gamma(\alpha m+1)}{\Gamma(\alpha m+\alpha)}(x-\gamma)^{1-\alpha}, & k = m, \\ 0, & k \neq m. \end{cases} \tag{3.20}$$

Proof. From the NGDTM definition (3.1), for $f(x) = (x-\gamma)^{\alpha m}$,

$$F(k) = \frac{\Gamma(\alpha)\left[(x-\gamma)^{1-\alpha} D_{\gamma}^{k\alpha}(x-\gamma)^{\alpha m}\right]}{\Gamma(\alpha k+\alpha)}. \tag{3.21}$$

Using the Riemann–Liouville fractional derivative formula

$$D^{\nu}\gamma(x-\gamma)^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\nu+1)}(x-\gamma)^{\mu-\nu}, \tag{3.22}$$

we obtain

$$F(k) = \frac{\Gamma(\alpha)\Gamma(\alpha m+1)}{\Gamma(\alpha k+\alpha)\Gamma(\alpha m-\alpha k+1)}(x-\gamma)^{1-\alpha+\alpha(m-k)}. \tag{3.23}$$

Evaluating at $x = \gamma^{+}$, the expression vanishes for all $k \neq m$, while for $k = m$, it reduces to

$$F(m) = \frac{\Gamma(\alpha)\Gamma(\alpha m+1)}{\Gamma(\alpha m+\alpha)}(x-\gamma)^{1-\alpha}. \tag{3.24}$$

Hence,

$$F(k) = \Delta_{k,m}. \tag{3.25}$$

□

Theorem 3.6. *(Fractional derivative) If $f(x) = D_{\gamma}^{\alpha} g(x)$, then the NGDTM coefficients satisfy*

$$F(k) = \frac{\Gamma(\alpha k+2\alpha)}{\Gamma(\alpha k+\alpha)}, G(k+1), \quad k \geq 0. \tag{3.26}$$

Proof. By the NGDTM definition Eq (3.1),

$$F(k) = \frac{\Gamma(\alpha)\Big[(x-\gamma)^{1-\alpha} D_\gamma^{k\alpha} D_\gamma^{\alpha} g(x)\Big]}{\Gamma(\alpha k+\alpha)}. \tag{3.27}$$

Using the semigroup property of the Riemann–Liouville derivative

$$D_\gamma^{k\alpha} D_\gamma^{\alpha} g(x) = D_\gamma^{(k+1)\alpha} g(x), \tag{3.28}$$

we have

$$F(k) = \frac{\Gamma(\alpha)\Big[(x-\gamma)^{1-\alpha} D_\gamma^{(k+1)\alpha} g(x)\Big]}{\Gamma(\alpha k+\alpha)}. \tag{3.29}$$

Recognizing the definition of $G(k+1)$ in Eq (3.1), we obtain

$$F(k) = \frac{\Gamma(\alpha k+2\alpha)}{\Gamma(\alpha k+\alpha)} G(k+1). \tag{3.30}$$

□

Theorem 3.7. *(Fractional derivative of order β) If $f(x) = D_\gamma^{\beta} g(x), \beta > 0$, then the NGDTM coefficients satisfy*

$$F(k) = \frac{\Gamma(\alpha k+\beta+\alpha)}{\Gamma(\alpha k+\alpha)} G\Big(k+\frac{\beta}{\alpha}\Big), \quad k \geq 0. \tag{3.31}$$

Proof. By the NGDTM definition Eq (3.1),

$$F(k) = \frac{\Gamma(\alpha)\Big[(x-\gamma)^{1-\alpha} D_\gamma^{k\alpha} D_\gamma^{\beta} g(x)\Big]}{\Gamma(\alpha k+\alpha)}. \tag{3.32}$$

Using the semigroup property of Riemann–Liouville derivatives

$$D_\gamma^{k\alpha} D_\gamma^{\beta} g(x) = D_\gamma^{k\alpha+\beta} g(x), \tag{3.33}$$

we have

$$F(k) = \frac{\Gamma(\alpha)\Big[(x-\gamma)^{1-\alpha} D_\gamma^{k\alpha+\beta} g(x)\Big]}{\Gamma(\alpha k+\alpha)}. \tag{3.34}$$

Recognizing the definition of $G\left(k+\frac{\beta}{\alpha}\right)$ from Eq (3.1) gives

$$F(k) = \frac{\Gamma(\alpha k+\beta+\alpha)}{\Gamma(\alpha k+\alpha)} G\Big(k+\frac{\beta}{\alpha}\Big). \tag{3.35}$$

□

4. Convergence and uniqueness of the NGDTM

4.1. Convergence of the NGDTM

Lemma 4.1. *(Fractional Cauchy-type estimate) Let $f(x)$ be analytic in a neighborhood of γ. Then there exist constants $M, R > 0$ such that*

$$\left|(x-\gamma)^{1-\alpha} D_{\gamma}^{k\alpha} f(x)\right|_{x=\gamma^+} \leq M R^k \Gamma(\alpha k + \alpha), \qquad k \geq 0. \tag{4.1}$$

Proof. Since $f(x)$ is analytic at γ, it admits the Taylor expansion

$$f(x) = \sum_{n=0}^{\infty} a_n (x-\gamma)^n, \qquad |x-\gamma| < \rho \tag{4.2}$$

for some $\rho > 0$.

Using the Riemann–Liouville fractional derivative formula

$$D_{\gamma}^{\mu} (x-\gamma)^n = \frac{\Gamma(n+1)}{\Gamma(n-\mu+1)} (x-\gamma)^{n-\mu}, \tag{4.3}$$

we obtain

$$D_{\gamma}^{k\alpha} f(x) = \sum_{n=0}^{\infty} a_n \frac{\Gamma(n+1)}{\Gamma(n-k\alpha+1)} (x-\gamma)^{n-k\alpha}. \tag{4.4}$$

Multiplying by $(x-\gamma)^{1-\alpha}$ gives

$$(x-\gamma)^{1-\alpha} D_{\gamma}^{k\alpha} f(x) = \sum_{n=0}^{\infty} a_n \frac{\Gamma(n+1)}{\Gamma(n-k\alpha+1)} (x-\gamma)^{n-k\alpha+1-\alpha}. \tag{4.5}$$

Evaluating at $x = \gamma^+$ and using the analyticity of f, the coefficients a_n satisfy the Cauchy estimate

$$|a_n| \leq \frac{M_\rho}{\rho^n}.$$

Hence,

$$\left|(x-\gamma)^{1-\alpha} D_{\gamma}^{k\alpha} f(x)\right|_{x=\gamma^+} \leq \sum_{n=0}^{\infty} \frac{M_\rho}{\rho^n} \frac{\Gamma(n+1)}{\Gamma(n-k\alpha+1)}.$$

Using standard bounds for the Gamma function, the dominant growth with respect to k is proportional to $\Gamma(\alpha k + \alpha)$. Therefore, there exist constants $M, R > 0$ such that

$$\left|(x-\gamma)^{1-\alpha} D_{\gamma}^{k\alpha} f(x)\right|_{x=\gamma^+} \leq M R^k \Gamma(\alpha k + \alpha).$$

□

Theorem 4.2. *(Convergence) Let $f(x)$ be analytic in a neighborhood of γ. Then the NGDTM series*

$$f(x) = \sum_{k=0}^{\infty} F_\alpha(k) (x-\gamma)^{\alpha k + \alpha - 1} \tag{4.6}$$

converges absolutely for $|x-\gamma| < \frac{1}{R}$, where R is the constant given in Lemma 4.1.

Proof. From the definition of the NGDTM coefficients,

$$F_{\alpha}(k) = \frac{\Gamma(\alpha)\left[(x-\gamma)^{1-\alpha} D_{\gamma}^{k\alpha} f(x)\right]_{x=\gamma^{+}}}{\Gamma(\alpha k + \alpha)}. \tag{4.7}$$

Using Lemma 4.1, we obtain

$$\left|(x-\gamma)^{1-\alpha} D_{\gamma}^{k\alpha} f(x)\right|_{x=\gamma^{+}} \leq MR^{k}\Gamma(\alpha k + \alpha). \tag{4.8}$$

Therefore,

$$|F_{\alpha}(k)| \leq \frac{\Gamma(\alpha) MR^{k}\Gamma(\alpha k + \alpha)}{\Gamma(\alpha k + \alpha)} = CR^{k}, \tag{4.9}$$

where $C = \Gamma(\alpha)M$.

Hence, the general term of the NGDTM series satisfies

$$|F_{\alpha}(k)(x-\gamma)^{\alpha k + \alpha - 1}| \leq CR^{k}|x-\gamma|^{\alpha k + \alpha - 1}. \tag{4.10}$$

Since

$$R^{k}|x-\gamma|^{\alpha k} = (R|x-\gamma|^{\alpha})^{k},$$

the series is dominated by a geometric series

$$\sum_{k=0}^{\infty} C(R|x-\gamma|^{\alpha})^{k}.$$

This series converges whenever

$$R|x-\gamma|^{\alpha} < 1.$$

In particular, this condition holds if

$$|x-\gamma| < \frac{1}{R^{\alpha}} < \frac{1}{R} \sim |x-\gamma| < \frac{1}{R}.$$

□

4.2. Uniqueness of the NGDTM solution

Consider the FDE

$$D_{\gamma}^{\alpha} f(x) = F(x, f(x)), \quad x \in [\gamma, \tau], \quad 0 < \alpha \leq 1, \tag{4.11}$$

with the initial condition

$$\left[(x-\gamma)^{1-\alpha} f(x)\right]_{x=\gamma^{+}} = f_{0}. \tag{4.12}$$

Assume that $F(x, y)$ is Lipschitz continuous in y with constant $L > 0$.

Theorem 4.3. *(Uniqueness) If* $|x-\gamma| \leq h$ *and* $\frac{Lh^{\alpha}}{\Gamma(\alpha+1)} < 1$, *then the solution generated by the NGDTM is unique in* $[\gamma, \gamma + h]$.

Proof. Suppose $f_1(x)$ and $f_2(x)$ are two NGDTM solutions with the same initial condition. Using the NGDTM series representation:

$$f_i(x) = \sum_{k=0}^{\infty} F_{\alpha}^{(i)}(k)(x-\gamma)^{\alpha k+\alpha-1}, \quad i = 1, 2,$$

the FDE can be rewritten in the integral form (Riemann–Liouville):

$$f_i(x) = f_0 + \frac{1}{\Gamma(\alpha)} \int_{\gamma}^{x} (x-t)^{\alpha-1} F(t, f_i(t))\, dt, \quad i = 1, 2.$$

Subtracting the two solutions and applying the Lipschitz condition yields

$$\left| f_1(x) - f_2(x) \right| \le \frac{L}{\Gamma(\alpha)} \int_{\gamma}^{x} (x-t)^{\alpha-1} \left| f_1(t) - f_2(t) \right| dt.$$

By the fractional Grönwall inequality, it follows that

$$\left| f_1(x) - f_2(x) \right| = 0 \quad \text{for all } x \in [\gamma, \gamma + h],$$

hence $f_1(x) \equiv f_2(x)$. Therefore, the NGDTM series represents a unique solution under the given Lipschitz condition. □

5. Application

Example 5.1. *Consider the linear FDE [29]*

$$D_x^{\alpha} u(x) - u(x) = 1, \quad m < \alpha \le m+1, \quad x \ge 0, \tag{5.1}$$

with the initial condition

$$u(0) = 0. \tag{5.2}$$

The exact solution is

$$u(x) = x^{\alpha} E_{\alpha,\alpha+1}(x^{\alpha}), \tag{5.3}$$

where $E_{\alpha,\alpha+1}(x^{\alpha})$ is the generalized Mittag–Leffler function.

Using Theorems 3.1, 3.5, and 3.6 and applying the NGDTM to Eq (5.1), we obtain the recurrence relation

$$U_{\alpha}(k+1) = \frac{\Gamma(\alpha k + \alpha)}{\Gamma(\alpha k + 2\alpha)} \Big(U(k) + \Delta(k) \Big), \tag{5.4}$$

with the transformed initial condition

$$U_{\alpha}(0) = 0. \tag{5.5}$$

Using the recurrence relation and the inverse NGDTM transformation (Eq (3.3)), the approximate solution is

$$u(x) = \sum_{k=0}^{\infty} \frac{\Gamma(\alpha)\, x^{\alpha(k+1)}}{\Gamma(\alpha(k+1)+\alpha)} = \frac{\Gamma(\alpha) x^{\alpha}}{\Gamma(2\alpha)} + \frac{\Gamma(\alpha) x^{2\alpha}}{\Gamma(3\alpha)} + \frac{\Gamma(\alpha) x^{3\alpha}}{\Gamma(4\alpha)} + \cdots. \tag{5.6}$$

For specific values of α:
$\alpha = 0.8$:

$$u(x) = 1.11917\, x^{0.8} + 0.69148\, x^{1.4} + 0.30437\, x^{2.0} + 0.10562\, x^{2.6} + 0.03051\, x^{3.2} + \cdots .$$

$\alpha = 0.9$:

$$u(x) = 1.07367\, x^{0.9} + 0.60581\, x^{1.7} + 0.23559\, x^{2.5} + 0.07045\, x^{3.3} + 0.01719\, x^{4.1} + \cdots .$$

$\alpha = 1$:

$$u(x) = x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \cdots .$$

$\alpha = 1.1$:

$$u(x) = 0.90760\, x^{1.1} + 0.39171\, x^{2.3} + 0.10901\, x^{3.5} + 0.02219\, x^{4.7} + 0.00354\, x^{5.9} + \cdots .$$

$\alpha = 1.2$:

$$u(x) = 0.80504\, x^{1.2} + 0.29301\, x^{2.6} + 0.06650\, x^{4.0} + 0.01077\, x^{5.4} + 0.00134\, x^{6.8} + \cdots .$$

Table 1 presents the NGDTM approximate values for various values of α. The consistency of these results across the table highlights the accuracy and robustness of the NGDTM approximations. Figure 1 depicts the corresponding solution profiles for different values of α. It is observed that all curves exhibit similar qualitative behavior, which further confirms the effectiveness and reliability of the NGDTM.

Table 1. Comparison of the approximate solutions obtained by the NGDTM for various values of α with the exact solution when $\alpha = 1.0$, in Example 5.1.

x	$u(x)_{\alpha=0.8}$	$u(x)_{\alpha=0.9}$	$u(x)_{\alpha=1.0}$	$u(x)_{Exact}$	$u(x)_{\alpha=1.1}$	$u(x)_{\alpha=1.2}$	Absolute error ($\alpha = 1.0$)
0.0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.0
0.1	0.208234	0.148036	0.105171	0.105171	0.074092	0.051537	0.0
0.2	0.395440	0.296088	0.221403	0.221403	0.164606	0.121266	0.0
0.3	0.587980	0.454617	0.349858	0.349859	0.267655	0.203191	2.86e-6
0.4	0.789503	0.625941	0.491819	0.491820	0.383593	0.296931	2.03e-6
0.5	1.001652	0.811626	0.648698	0.648701	0.513503	0.403167	4.63e-6
0.6	1.225459	1.013040	0.822048	0.822110	0.658847	0.523094	7.54e-5
0.7	1.461696	1.231507	1.013572	1.013713	0.821388	0.658299	1.39e-4
0.8	1.711011	1.468356	1.225131	1.225541	1.003165	0.810711	3.34e-4
0.9	1.973986	1.724954	1.458759	1.459603	1.206510	0.982610	5.78e-4
1.0	2.251153	2.002706	1.716667	1.718282	1.434052	1.176660	9.40e-4

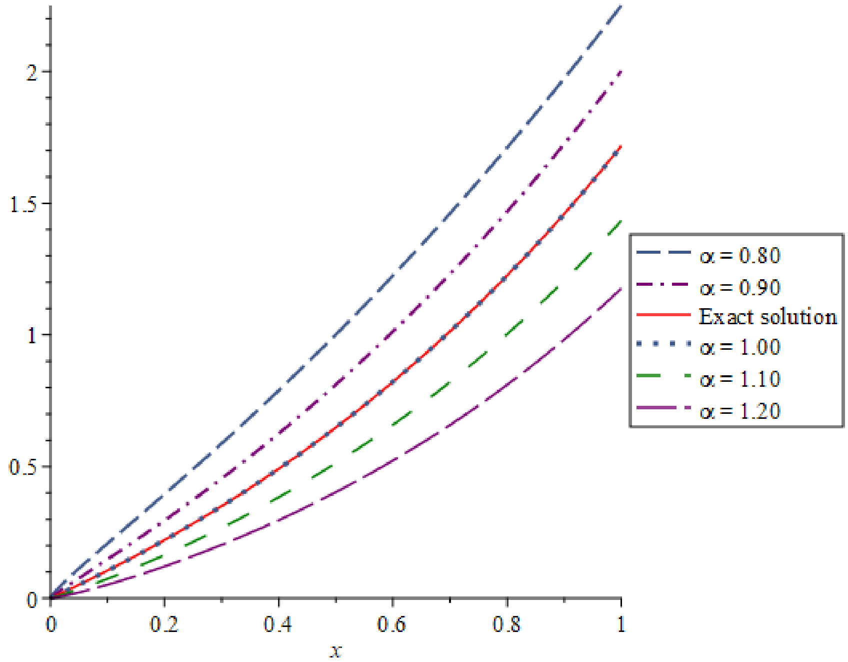

Figure 1. NGDTM approximate solutions for different values of α in Example 5.1.

Example 5.2. *Consider the nonlinear FDE [30]:*

$$D^{\alpha} y(x) = y^2(x) + 1, \quad m-1 < \alpha \le m, \quad 0 < x < 1, \tag{5.7}$$

subject to the initial conditions

$$y^{(k)}(0) = 0, \quad k = 0, 1, \ldots, m-1. \tag{5.8}$$

The exact solution of this equation for $\alpha = 1$ *is*

$$y(x) = \tan(x). \tag{5.9}$$

By applying Theorems 3.3, 3.5, and 3.6 and implementing the NGDTM on both sides of Eq (5.7), we obtain the following recurrence relation:

$$Y_{\alpha}(k+1) = \frac{\Gamma(\alpha k + \alpha)}{\Gamma(\alpha k + 2\alpha)} \left(\sum_{i=0}^{k} Y_{\alpha}(i)\, Y_{\alpha}(k-i) + \Delta(k) \right). \tag{5.10}$$

The generalized differential transform of the initial conditions (5.8) is given by

$$Y_{\alpha}^{(k)}(0) = 0, \quad k = 0, 1, \ldots, m-1. \tag{5.11}$$

Using the recurrence relation (5.10), the transformed initial conditions (5.11), and the inverse NGDTM formula (3.3), the approximate solution of problem (5.7) can be expressed as the following series:

$$y(x) = \frac{x^{\alpha}}{\Gamma(2\alpha)} + \frac{\Gamma(3\alpha)}{\Gamma(\alpha)\Gamma(4\alpha)(\Gamma(2\alpha))^2}\, x^{3\alpha} + \frac{2\,\Gamma(5\alpha)\Gamma(3\alpha)}{(\Gamma(\alpha))^2\Gamma(6\alpha)(\Gamma(2\alpha))^3\Gamma(4\alpha)}\, x^{5\alpha} + O\left(x^{7\alpha}\right). \tag{5.12}$$

Table 2 shows that the NGDTM solutions closely match the exact values. Figure 2 displays the approximate solutions for various α, all exhibiting consistent behavior, which underscores the accuracy and reliability of NGDTM.

Table 2. Comparison of the NGDTM and exact solutions for $\alpha = 1$ in Example 5.2, along with the absolute error on [0, 1].

x	NGDTM	Exact	Absolute error
0	0.000000	0.000000	0.000000
0.1	0.100335	0.100335	0.000000
0.2	0.202710	0.202710	0.000000
0.3	0.309336	0.309336	0.000000
0.4	0.422793	0.422793	0.000000
0.5	0.546298	0.546302	0.000004
0.6	0.684099	0.684137	0.000038
0.7	0.842070	0.842288	0.000218
0.8	1.028611	1.029639	0.001028
0.9	1.256018	1.260158	0.004140
1.0	1.542504	1.557408	0.014904

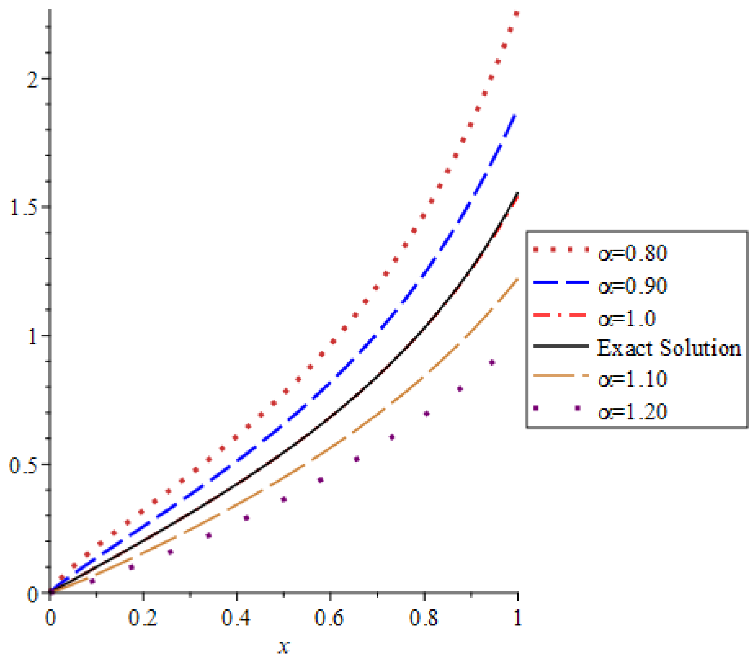

Figure 2. NGDTM approximate solutions for different values of α in Example 5.2.

Example 5.3. *Consider the composite fractional oscillation equation [10]*

$$\frac{d^2y}{dx^2} + \frac{d^{0.5}y}{dx^{0.5}} + y(x) = 8, \quad x > 0, \tag{5.13}$$

with the initial conditions

$$y(0) = 0, \quad y'(0) = 0. \tag{5.14}$$

Applying the NGDTM to both sides of Eq (5.13) and using Theorems 3.5–3.7, the equation transforms to

$$Y(k+4) = -\frac{\Gamma(\alpha k+\alpha)}{\Gamma(\alpha)\Gamma(\alpha k+\alpha+\beta)}\left[\frac{\Gamma(\alpha)\Gamma(\alpha k+2\alpha)}{\Gamma(\alpha k+\alpha)}Y(k+1)+Y(k)-8\Delta(k)\right]. \tag{5.15}$$

The NGDTM of the initial conditions (5.14) is

$$Y(0) = 0, \quad Y(1) = 0, \quad Y(2) = 0, \quad Y(3) = 0. \tag{5.16}$$

Using the recurrence relation (5.15) together with the initial conditions (5.16), the solution is computed up to iteration (N=30). Table 3 presents a comparison of the NGDTM solution with the exact solution and the results obtained by variation iteration method (VIM) [32] for $\alpha = 0.5$. It is observed that the NGDTM provides higher accuracy than VIM, and the approximate results are in excellent agreement with the exact solution.

Table 3. Comparison between VIM, NGDTM, and exact solution value when $\alpha = 0.5$.

	$\alpha = 0.5$		
x	*VIM*	*NGDTM*	*Exact*
0	0	0	0
0.1	0.039874	0.039695	0.039750
0.2	0.158512	0.156456	0.157036
0.3	0.353625	0.345122	0.347370
0.4	0.622083	0.598914	0.604695
0.5	0.960047	0.909919	0.921768
0.6	1.363093	1.2694814	1.290457
0.7	1.826257	1.668546	1.702008
0.8	2.344224	2.097996	2.147287
0.9	2.911278	2.548996	2.617001
1.0	3.527462	3.0133244	3.101906

Example 5.4. *Consider the following FDE:*

$$D^{\frac{1}{2}}u(t) = -u(t) + t^2 + \frac{2}{\Gamma(\frac{5}{2})}t^{\frac{3}{2}}, \tag{5.17}$$

with the initial condition

$$u(0) = 0. \tag{5.18}$$

The exact solution of Eq (5.17) *is*

$$u(t) = t^2. \tag{5.19}$$

Applying the NGDTM to both sides of Eq. (5.17) and using Theorems 3.1, 3.5, and 3.6, we obtain

$$U_{\frac{1}{2}}(k) = \frac{\Gamma(\frac{k}{2}+\frac{1}{2})}{\Gamma(\frac{k}{2}+1)}\Big(- U_{\frac{1}{2}}(k) + \Delta(k-4) + \frac{2}{\Gamma(\frac{5}{2})}\Delta(k-3)\Big). \tag{5.20}$$

The NGDTM of the initial condition (5.18) is

$$U_{\frac{1}{2}}(0) = 0. \tag{5.21}$$

Using Eqs (5.20) and (5.21), we find

$$U_{\frac{1}{2}}(1) = U_{\frac{1}{2}}(2) = U_{\frac{1}{2}}(3) = 0, \quad U_{\frac{1}{2}}(4) = 1. \tag{5.22}$$

Finally, applying the inverse transformation in Eq (3.3), the approximate solution $u(t)$ is

$$u(t) = t^2. \tag{5.23}$$

This example illustrates that the NGDTM reproduces the exact solution perfectly. Remarkably, the exact solution is achieved with only $n = 4$ terms, highlighting the efficiency and accuracy of the method.

6. Approximation of probability distributions using NGDTM

Probability modeling is fundamental in various scientific fields, such as medicine, geography, environment, and agriculture. The probability distributions can be expressed as solutions to specific differential equations. In the current study, the NGDTM is used to generate analytical series approximations for statistical distributions. This connection allows us to generate or approximate distributions by solving these equations or employing them as a fractional series expansion that yields an approximation of the probability density functions (PDFs). To demonstrate the technique, we apply the NGDTM to derive the well-known exponential distribution, confirming that the method recovers its exact PDF. In addition to this, we generate another case for $0 < \alpha \leq 1$ as a novel PDF to provide the applicability in generating new novel PDFs.

6.1. Approximation of exponential density using NGDTM

The exponential distribution with rate parameter $\lambda > 0$ has PDF $f(x) = \lambda e^{-\lambda x}, x \geq 0$. This PDF satisfies a simple ODE. Although this first-order equation admits a trivial solution, we consider its fractional generalization to demonstrate the flexibility of the NGDTM.

$$D^{\alpha} f(x) + \lambda f(x) = 0, \quad 0 < \alpha \leq 1, \quad f(0) = \lambda. \tag{6.1}$$

We apply the NGDTM to Eq (6.1) using Theorem 4.6 for the fractional derivative and Theorem 4.2 for scalar multiplication:

$$\frac{\Gamma(\alpha k + 2\alpha)}{\Gamma(\alpha k + \alpha)} F(k+1) + \lambda F(k) = 0, \quad k \geq 0, \tag{6.2}$$

where $F(k)$ is the NGDTM coefficient of $f(x)$. This solves the recurrence relation:

$$F(k+1) = -\lambda \frac{\Gamma(\alpha k + \alpha)}{\Gamma(\alpha k + 2\alpha)} F(k), \quad k \geq 0. \tag{6.3}$$

The initial coefficient $F(0)$ is determined from the initial condition. For consistency with the classical exponential distribution, we take $F(0) = \lambda$. For $\alpha = 1$, Eq (6.3) simplifies to:

$$F(k+1) = -\lambda \frac{\Gamma(k+1)}{\Gamma(k+2)} F(k) = -\frac{\lambda}{k+1} F(k), \quad k \geq 0. \tag{6.4}$$

This yields

$$F(k) = \lambda \frac{(-\lambda)^k}{k!}, \quad k = 0, 1, 2, \ldots. \tag{6.5}$$

The inverse transform provides the series approximation

$$f_\alpha(x) = \sum_{k=0}^{N} F(k)\, x^{\alpha k + \alpha - 1}, \tag{6.6}$$

where N is the truncation order. For the classical case at $\alpha = 1$, the coefficients simplify to $F(k) = \lambda(-\lambda)^k/k!$, yielding the exact exponential series

$$f_1(x) = \lambda \sum_{k=0}^{\infty} \frac{(-\lambda x)^k}{k!} = \lambda e^{-\lambda x}. \tag{6.7}$$

The moments of the fractional exponential distribution can be derived from the series representation. The $m^t h$ moment is

$$E[X^m] = \int_0^\infty x^m f_\alpha(x)\, dx = \sum_{k=0}^{\infty} F(k) \frac{\Gamma(\alpha k + \alpha + m)}{\Gamma(\alpha k + \alpha)}, \tag{6.8}$$

provided the integral converges. For $\alpha = 1$, this yields the well-known moments of the exponential distribution

$$E[X^m] = \frac{m!}{\lambda^m}, \quad m = 1, 2, \ldots. \tag{6.9}$$

6.2. *New fractional exponential distribution* ($\alpha < 1$)

New statistical distributions can be generated by solving FDEs with $0 < \alpha \leq 1$. Fractional orders present the memory effects. This section demonstrates two specific examples of distributions with $\alpha < 1$. Consider the FDE

$$D^\alpha f(x) + \lambda f(x) = 0, \quad x \geq 0, \tag{6.10}$$

with the initial condition chosen for proper normalization:

$$\left[J^{\alpha-1} f\right](0) = \frac{\alpha\lambda}{\Gamma(\alpha)}. \tag{6.11}$$

Here, $0 < \alpha \leq 1$ is the fractional order, and $\lambda > 0$ is a rate parameter. By applying the NGDTM to Eq (6.10)

$$\frac{\Gamma(\alpha k + 2\alpha)}{\Gamma(\alpha k + \alpha)} F(k+1) + \lambda F(k) = 0. \tag{6.12}$$

This solves the recurrence

$$F(k+1) = -\lambda \frac{\Gamma(\alpha k+\alpha)}{\Gamma(\alpha k+2\alpha)} F(k), \quad k \geq 0. \tag{6.13}$$

Using the initial condition in (6.11), $F(0) = \frac{\alpha\lambda}{\Gamma(\alpha)}$. The general coefficient is

$$F(k) = \frac{\alpha\lambda}{\Gamma(\alpha)} (-\lambda)^k \prod_{j=0}^{k-1} \frac{\Gamma(\alpha j+\alpha)}{\Gamma(\alpha j+2\alpha)}, \quad k \geq 0. \tag{6.14}$$

The product telescopes

$$\prod_{j=0}^{k-1} \frac{\Gamma(\alpha j+\alpha)}{\Gamma(\alpha j+2\alpha)} = \frac{\Gamma(\alpha)}{\Gamma(\alpha k+\alpha)}, \quad k \geq 1. \tag{6.15}$$

Thus,

$$F(k) = \frac{\alpha\lambda(-\lambda)^k}{\Gamma(\alpha k+\alpha)}, \quad k \geq 0. \tag{6.16}$$

The distribution is

$$\begin{aligned} f_\alpha(x) &= \sum_{k=0}^{\infty} F(k)\, x^{\alpha k+\alpha-1} = \alpha\lambda x^{\alpha-1} E_{\alpha,\alpha}(-\lambda x^\alpha) \\ &= \alpha\lambda x^{\alpha-1} \sum_{k=0}^{\infty} \frac{(-\lambda x^\alpha)^k}{\Gamma(\alpha k+\alpha)}, \quad x \geq 0, \end{aligned} \tag{6.17}$$

where, $E_{\alpha,\alpha}(-\lambda x^\alpha) = \sum_{k=0}^{\infty} \frac{(-\lambda x^\alpha)^k}{\Gamma(\alpha k+\alpha)}$. represent the two parameters of the Mittag-Leffler function [19] and $\int_0^\infty f_\alpha(x)\,dx = 1$ for all $0 < \alpha \leq 1$. As a special case, when $\alpha = 1$, the density represents the exponential distribution $f_1(x) = \lambda e^{-\lambda x}$. For $\alpha = 0.5$, the density becomes

$$\begin{aligned} f_{0.5}(x) &= 0.5\,\lambda\, x^{-0.5}\, E_{0.5,0.5}(-\lambda x^{0.5}) \\ &= 0.5\,\lambda\, x^{-0.5} \sum_{k=0}^{\infty} \frac{(-\lambda x^{0.5})^k}{\Gamma(0.5k+0.5)}, \quad x \geq 0. \end{aligned} \tag{6.18}$$

For $m > \alpha - 1$, the $m^t h$ moments of the random variable are

$$E[X^m] = \int_0^\alpha \lambda x^{\alpha+m-1} E_{\alpha,\alpha}(-\lambda x^\alpha)\,dx, \quad m = 1, 2, \ldots. \tag{6.19}$$

More cases for fractional exponential densities (FED) are presented in Figure 3.

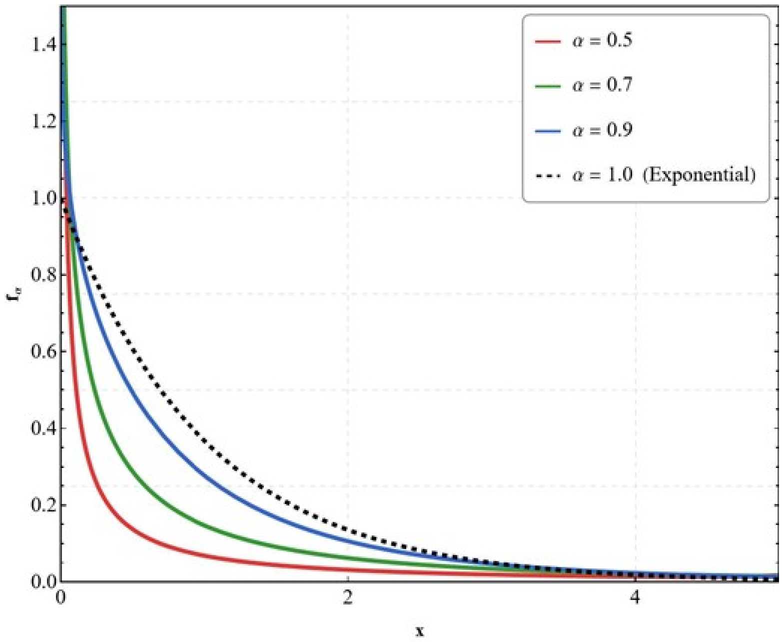

Figure 3. Fractional exponential PDFs for different α values.

6.3. Statistical properties and inference

In this subsection, we discuss important statistical properties of the FED derived in Section 6.2.

6.3.1. Survival function

The survival function of the FED is defined as

$$S_\alpha(x) = \int_x^\infty f_\alpha(t)\,dt = \int_x^\infty \alpha\lambda t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha)\,dt = E_\alpha(-\lambda x^\alpha), \qquad 0 < \alpha \le 1, \tag{6.20}$$

where again $E_\alpha(\cdot)$ denotes the Mittag–Leffler function [19].

For the classical case $\alpha = 1$, the survival function is

$$S_1(x) = e^{-\lambda x}, \tag{6.21}$$

which corresponds to the survival function of the classical exponential random variable. However, when $0 < \alpha < 1$, the Mittag–Leffler function does not exhibit exponential decay. Its asymptotic behavior for large x satisfies

$$E_\alpha(-\lambda x^\alpha) \sim \frac{1}{\lambda\Gamma(1-\alpha)} x^{-\alpha}, \qquad x \to \infty. \tag{6.22}$$

Thus, the survival function decreases at a polynomial rate rather than an exponential rate. Consequently, the FED possesses heavier tails than the classical exponential distribution.

6.3.2. Hazard rate function

The hazard rate function is defined as

$$h_\alpha(x) = \frac{f_\alpha(x)}{S_\alpha(x)}. \tag{6.23}$$

For $\alpha = 1$, we obtain

$$h_1(x) = \lambda, \tag{6.24}$$

which is constant and reflects the memoryless property of the exponential distribution.

For $0 < \alpha < 1$, the hazard rate is no longer constant. In fact, it is typically decreasing in x, indicating a non-memoryless structure and the presence of aging effects. This behavior makes the fractional exponential model appropriate for reliability and survival applications where failure rates decline over time.

6.3.3. Moment existence

For the classical exponential case ($\alpha = 1$), all positive moments exist and are finite. For $0 < \alpha < 1$, the heavier tail behavior implies that the growth of higher-order moments is slower compared to the exponential case. Lower-order moments exist and are finite, while the existence and magnitude of higher-order moments depend on the asymptotic behavior of the Mittag–Leffler function. This characteristic reflects the long-memory structure inherent in fractional models.

6.3.4. Parameter estimation

Let $x_1, x_2, \ldots, x_n$ be a random sample from the fractional exponential distribution with parameters $\lambda > 0$ and $0 < \alpha \leq 1$. The log-likelihood function is given by

$$\ell(\lambda, \alpha) = n \log \alpha + n \log \lambda + (\alpha - 1) \sum_{i=1}^{n} \log x_i + \sum_{i=1}^{n} \log E_{\alpha,\alpha}(-\lambda x_i^{\alpha}). \tag{6.25}$$

The closed-form solutions are not available; therefore, the maximum likelihood estimators are obtained numerically using any optimization techniques.

The NGDTM provides a method for approximating statistical distributions through FDEs. The exponential distribution is discussed as a clear example (at $\alpha = 1$). In addition, other probability distributions can be generated when $0 < \alpha < 1$. These offer bridges for distribution theory and fractional calculus by producing series expansions. Future research could create methods for estimating the fractional order α from empirical data.

7. Conclusions

In this paper, a new numerical–analytical approach, the NGDTM, was developed for solving linear and nonlinear FDEs. The method is based on the generalized Taylor expansion and the Riemann–Liouville fractional derivative, and its convergence and uniqueness are rigorously proven. Several examples show that the NGDTM provides highly accurate approximations with few iterations. The method was also used to derive the classical exponential distribution and a fractional exponential distribution was derived within the NGDTM framework.

Author contributions

Ammar Abuualshaikh: Conceptualization, methodology, formal analysis, investigation, writing-original draft; Mahmoud Z. Aldrabseh: Conceptualization, methodology, supervision, validation, writing-review & editing; Tariq S. Alshammari: Data curation, software, investigation, writing-review & editing, funding acquisition; Khudhayr A. Rashedi: Formal analysis, visualization, validation, funding acquisition; Khalid M. K. Alshammari: Resources, validation, funding acquisition. All authors have read and agreed to publish the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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