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Research article

Caterpillars with diameter at least 5 are k-shifted antimagic for any integer k

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Abstract: Antimagic labeling is a type of graph labeling, and k-shifted antimagic labeling is a natural generalization of antimagic labeling. In this paper, we extended the results on antimagic labeling to k-shifted antimagic labeling for caterpillar graphs and proved that every caterpillar of diameter at least five is absolutely antimagic.

Keywords: antimagic labeling; k-shifted antimagic labeling; absolutely antimagic; caterpillar
Mathematics Subject Classification: 05C78

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1. Introduction

Most graph labelings trace their origins to the one, later termed graceful labeling by Golomb [1], introduced by Rosa [2] in 1967, where he showed that all paths and caterpillars are graceful. In the intervening years, a large number of graph labeling techniques were studied. Gallian's survey [3], frequently updated, is essential for tracking the rapid development of graph labeling, an area of immense theoretical interest and diverse practical applications.

Antimagic labeling, a type of graph labeling, has been a captivating area of research for decades. A study [4] revealed the versatility and practicality of antimagic labeling in various fields, from enhancing security systems to optimizing communication networks and urban planning.

An antimagic labeling of a graph G with the vertex set $V(G)$ and edge set $E(G)$ is a bijection $f\colon E(G) \to \{1, 2, \cdots, |E(G)|\}$ such that the induced vertex sums are pairwise distinct, where the sum of a vertex is defined as the sum of the labels on all edges incident to it. A graph is called antimagic if it has antimagic labeling. This makes every tree except K_2 an intriguing subject of study, as originally conjectured by Hartsfield and Ringel [5] in 1990. Furthermore, they conjectured that every connected graph except K_2 is antimagic.

Many classes of connected graphs have been proven antimagic, including paths, cycles, wheels, stars, complete graphs, complete bipartite graphs [5], cubic graphs [6], dense graphs [7], and regular graphs [8]. However, proving this property for trees in general remains an open challenge. Noteworthy advancements include the work of Kaplan et al. [9], who showed that trees with at most one vertex of degree two are antimagic; Shang's proof [10] for spider graphs; the proof for double spider graphs by Chang et al. [11]; and the work concerning caterpillar graphs by Lozano, et al. [12].

Expanding on this, the concept of k-shifted antimagic labeling was introduced. Let $[a, b]$ denote the set of integers x with $a \leq x \leq b$. The k-shifted antimagic labeling involves labeling the edges of a graph with the set $[k+1, k+|E(G)|]$ such that the sums of the labels incident to each vertex are distinct. Wang and Hsiao [13] introduced the concept for the Cartesian product and the lexicographic product of sparse graphs, and Chang et al. [14] formalized the definition as follows.

Definition 1.1. Let G be a graph with m edges. Given an integer k, if there is a bijective function f from $E(G)$ to $[1+k, m+k]$ such that the vertex sums

$$\phi(v) := \sum_{uv \in E(G)} f(uv)$$

for all vertices $v \in V(G)$ are distinct, then f is a *k-shifted antimagic labeling* for G, and G is *k-shifted antimagic*. Moreover, if G is k-shifted antimagic for any integer k, then G is *absolutely antimagic*.

The study of k-shifted antimagic labeling of graphs has evolved into an intriguing area of research in its own right. It extends the labels for antimagicness from consecutive integers starting at 1 to starting at any integer, and thus serves as a natural extension of traditional antimagic labeling, which is 0-shifted antimagic labeling. The labelings can be negative.

Despite substantial progress, determining whether trees and connected graphs can be k-shifted antimagic for some k remains a complex problem. Chang et al. resolved this for trees with sufficiently large k, proving that every tree with at least three vertices is k-shifted antimagic for infinitely many k. On the other hand, not all trees are absolutely antimagic. As indicated in [14], the smallest such example is P_3, which is k-shifted antimagic if and only if $k \neq -1$ or -2. Things become much more complicated when the considered graph is large. Only a few classes of graphs have been shown to be absolutely antimagic, including linear forests, star forests, and odd tree forests with a few sporadic exceptions [15]. In general, it is difficult to determine for which values of k a graph is not k-shifted antimagic.

The k-shifted antimagicness for trees with a diameter at most five has been characterized. In [14], the authors fully characterized k-shifted antimagicness for some classes of trees, including paths, stars, and double stars (tree of diameter 3). Later, Li and Wang [16] characterized the k-shifted antimagicness for trees with diameter 4 and established that a tree with diameter 5 is k-shifted antimagic for every integer k.

In this study, we extend the results on antimagic labeling to k-shifted antimagic labeling for caterpillar graphs and prove the following main result: Every caterpillar of diameter at least five is absolutely antimagic, that is, k-shifted antimagic for any integer k. Building on the algorithm for antimagic labeling of caterpillars by Lozano et al. [12], we generalize it to accommodate k-shifted antimagic labeling for $k \geq 0$ and elaborate it to fit the case of $k < 0$.

The rest of this paper is organized as follows: In Section 2, we introduce a k-shifted antimagic labeling algorithm for caterpillars for $k \geq 0$ and deduce some straightforward consequence for certain $k < 0$. In Section 3, we resolve the remaining cases of $k < 0$. Finally, some concluding remarks with open questions are provided in Section 4.

2. A k-shifted antimagic labeling algorithm

Given a caterpillar C, we designate a longest path in C as the *spine* of C. The edges not on the spine are called the *legs* of C. Let p denote the diameter of C, that is, the number of edges on the spine. The vertices and edges on the spine are denoted in order as $v_0e_1, v_1e_2, \cdots, v_{p-1}e_pv_p$, defining the spine as the path from v_0 to v_p. The *degree* of a vertex v, denoted by $deg(v)$, is the number of edges incident to v. We firstly deal with the case of $k \geq 0$. Let m denote the number of edges of C. Our algorithm will define an edge labeling f: $E(C) \to [1+k, m+k]$, and the process starts by assigning consecutive labels to every other edge on the spine, except for some cutting-in-line legs. More explicitly, the edges e_i for $i \not\equiv p \pmod 2$ receive labels from $L_1 = [m+k-\lfloor p/2 \rfloor + 1, m+k]$ in increasing order, and the edges e_i for $i \equiv p \pmod 2$ receive labels from $L_0 = [1+k, m+k-\lfloor p/2 \rfloor]$ in increasing order with the following exception; when assigning a label to the edge e_i with $3 < i \leq p$ and $i \equiv p \pmod 2$, we check whether the following condition holds:

$$Q(i) : f(e_{i-1}) = f(e_l) + f(e_{l+1}) \text{ for some } l \in [1, i-3] \text{ with } \deg(v_l) = 3.$$

If $Q(i)$ does not hold, then e_i is assigned with the next unused label of L_0. Otherwise, the next unused label of L_0 is assigned to the leg g_l incident to v_l, satisfying the criterion $Q(i)$, and the next one to e_i. Such a vertex v_l is referred to as a *light vertex* and the leg g_l a cutting-in-line leg. Remark that a cutting-in-line leg g_l will receive a relatively small label, and thus the light vertex v_l has a relatively small vertex sum. The remaining vertices of degree at least 3 are called *heavy vertices*.

According to the criterion $Q(i)$, we recursively define U_i, for $i \equiv p \pmod 2$, to be the set consisting of all light vertices v_l with $l \leq i-3$ as follows:

$$U_i = \begin{cases} \emptyset, & \text{if } 1 \leq i \leq 3; \\ U_{i-2}, & \text{if } i > 3 \text{ and } Q(i) \text{ does not hold}; \\ U_{i-2} \cup \{v_l\}, & \text{if } i > 3 \text{ and } Q(i) \text{ holds for some } l. \end{cases}$$

Now we introduce the algorithm CSAL$^+$ for caterpillars. In practical use, to describe certain parameters, we write it as CSAL$^+(m, p, k)$.

CSAL$^+(m, p, k)$, where $k \geq 0$

Step 0: Find the spine of caterpillar C with p edges.

Step 1: Label e_i for $i \not\equiv p \pmod 2$ in increasing order by

$$f(e_i) = m + k - \left\lfloor \frac{p}{2} \right\rfloor + \left\lceil \frac{i}{2} \right\rceil. \tag{2.1}$$

Step 2: Label e_i for $i \equiv p \pmod 2$ in increasing order by

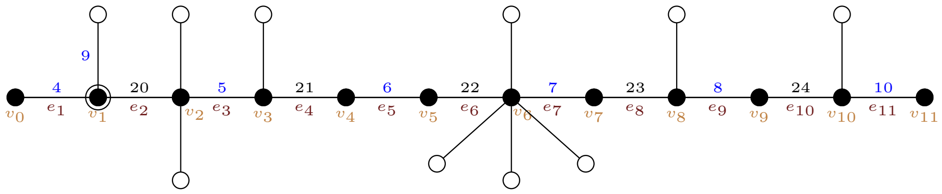
$$f(e_i) = \left\lceil \frac{i}{2} \right\rceil + |U_i| + k. \tag{2.2}$$

In addition, if $Q(i)$ holds for some l, label g_l by

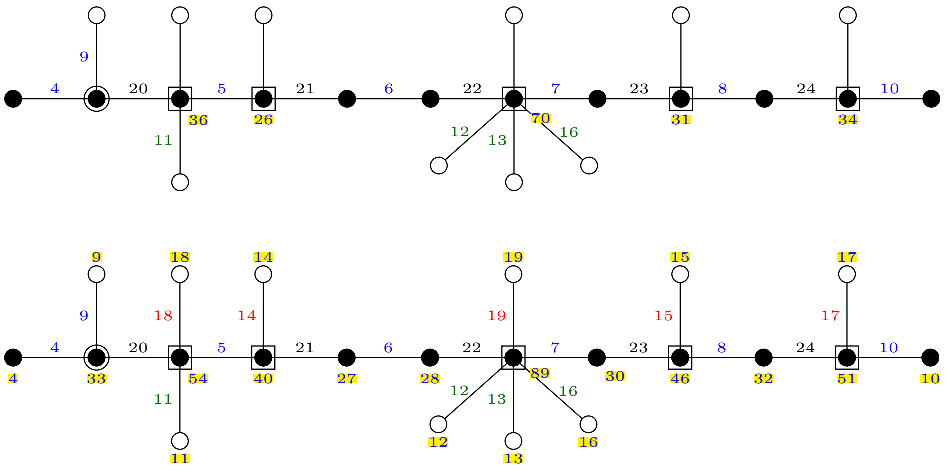
$$f(g_l) = \left\lceil \frac{i}{2} \right\rceil + |U_i| + k - 1. \tag{2.3}$$

Step 3: For each heavy vertex v, randomly assign unused labels to all but one of the legs incident to v. The unlabeled leg incident to v is denoted by h_v. Next, sort these unlabeled legs h_v's according to the current vertex sum $\tilde{\phi}(v)$ of these heavy vertices v's, and assign the remaining unused labels to the sorted legs h_v's such that $f(h_v)$ is smaller whenever $\tilde{\phi}(v)$ is smaller.

Figures 1 and 2 demonstrate two labelings produced by CSAL$^+$. The circled vertices represent the light vertices, while the squared vertices represent the heavy vertices.



(a) Steps 1 and 2 of CSAL$^+(21, 11, 3)$.



(b) Step 3 of CSAL$^+(21, 11, 3)$.

Figure 1. Example of applying the algorithm CSAL$^+(21, 11, 3)$.

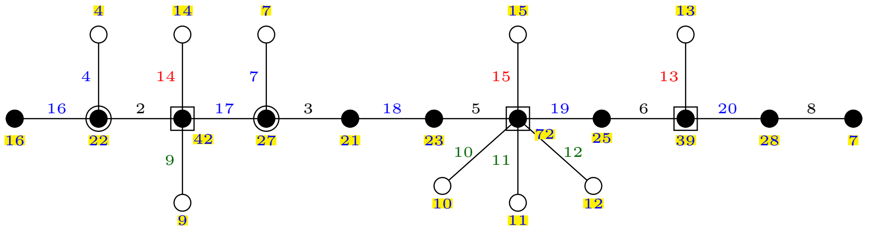


Figure 2. Example of applying the algorithm CSAL$^+(19, 10, 1)$.

Next, we prove that labeling f produced by CSAL$^+$ is a k-shifted antimagic labeling for C. To support the proof, we state the following lemma:

Lemma 2.1. *For any positive integers* i, j *with* $1 \leq i < j \leq p$*, we have*

(1) If $i \equiv j \pmod 2$*, then* $f(e_i) < f(e_j)$*;*
(2) $f(e_i) + f(e_{i+1}) < f(e_j) + f(e_{j+1})$*;*
(3) If v_i *and* v_j *are light vertices, then* $\phi(v_i) < \phi(v_j)$*.*

Proof. For (1), if $i \equiv j \pmod 2$, then e_i and e_j are both labeled with Eq (2.1) or with Eq (2.2). Since $i \leq j-2$ implies $\lceil \frac{i}{2} \rceil < \lceil \frac{j}{2} \rceil$ and $|U_i| \leq |U_j|$, it follows that $f(e_i) < f(e_j)$. The proof of (2) is a consequence of (1) by discussing the parities of i, $i+1$, j and $j+1$. Finally, by Eq (2.3), the value of $f(g_l)$ increases on increasing the value of l. Therefore, if v_i and v_j are light vertices, then combining (2),

$$\phi(v_i) = f(e_i) + f(e_{i+1}) + f(g_i) < f(e_j) + f(e_{j+1}) + f(g_j) = \phi(v_j),$$

proving (3). □

Proposition 2.2. *Let* $k \geq 0$*. The labeling* f *produced by CSAL*$^+$ *is a* k*-shifted antimagic labeling for* C*.*

Proof. Partition $V(C)$ into V_1, V_2, and V_3 such that V_1 is the set of vertices of degree 1, V_2 is the set of vertices of degree 2 and light vertices, and V_3 is the set of heavy vertices. To prove f is a k-shifted antimagic labeling for C, it is sufficient to prove that the vertices in V_i, for each $i = 1, 2, 3$, have pairwise distinct vertex sums, and the sets of vertex sums of vertices in V_1, V_2, V_3, respectively, are pairwise disjoint.

We begin by showing that the vertices in V_1 have pairwise distinct vertex sums. The vertex sum of a vertex of degree 1 is equal to the label of the only incident edge. Since all edges receive different labels, the vertex sums of vertices of degree 1 are pairwise distinct. In addition, notice that a pendant edge can be a leg, e_1 or e_p. It is immediately obtained from the algorithm that for any v with $\deg(v) = 1$, we have $\phi(v)$ bounded by $f(e_1)$ and $f(e_2)$; precisely,

$$1 + k = f(e_1) \leq \phi(v) < f(e_2) = m - \lfloor \tfrac{p}{2} \rfloor + 1 + k \tag{2.4}$$

when p is odd;

$$1 + k = f(e_2) \leq \phi(v) \leq f(e_1) = m - \lfloor \tfrac{p}{2} \rfloor + 1 + k \tag{2.5}$$

when p is even.

Next, we show that the vertices in V_2 have pairwise distinct vertex sums. Consider two arbitrary distinct vertices, v_j and v_l in V_2. There are three cases: Both vertices are of degree 2, both vertices are light vertices, and one vertex has degree 2 while the other is a light vertex. The first two cases follow immediately from Lemma 2.1 (2) and (3). For the remaining cases, suppose that v_j is of degree 2, and v_l is a light vertex, which implies that v_l satisfies $Q(i)$ for some i with $1 \leq l < i-2$, that is

$$f(e_l) + f(e_{l+1}) = f(e_{i-1}). \tag{2.6}$$

Moreover,

$$f(g_l) = f(e_i) - 1 = f(e_{i-2}) + 1 \text{ and } \phi(v_l) = f(e_l) + f(e_{l+1}) + f(g_l). \tag{2.7}$$

If $1 \leq j < l$, then

$$\phi(v_j) = f(e_j) + f(e_{j+1}) < f(e_l) + f(e_{l+1}) < f(e_l) + f(e_{l+1}) + f(g_l) = \phi(v_l);$$

if $l < j \le i-2$, then

$$\phi(v_j) = f(e_j) + f(e_{j+1}) < 1 + f(e_{i-2}) + f(e_{i-1}) = f(g_l) + f(e_l) + f(e_{l+1}) = \phi(v_l);$$

and if $l < i-2 < j$, then

$$\phi(v_j) = f(e_j) + f(e_{j+1}) \ge f(e_{i-1}) + f(e_i) = f(e_l) + f(e_{l+1}) + f(g_l) + 1 > \phi(v_l),$$

where we have applied Lemma 2.1 (2) and equalities (2.6) and (2.7). Hence, all vertices in V_2 have different vertex sums. Let $v_j \in V_2$. For the lower bound of $\phi(v_j)$, by Lemma 2.1 (2), we have

$$\phi(v_j) \ge f(e_1) + f(e_2) = (m - \left\lfloor \tfrac{p}{2} \right\rfloor + 1 + k) + (1 + k) = m - \left\lfloor \tfrac{p}{2} \right\rfloor + 2 + 2k.$$

For the upper bound, if v_j is not a light vertex, by Lemma 2.1 (2) again, we have

$$\phi(v_j) = f(e_j) + f(e_{j+1}) \le f(e_{p-1}) + f(e_p).$$

If v_j is a light vertex, that is v_j satisfies $Q(i)$ for some i with $1 \le j < i-2$, then we have

$$\begin{aligned}\phi(v_j) &= f(e_j) + f(e_{j+1}) + f(g_j)\\ &= f(e_{i-1}) + f(e_{i-2}) + 1\\ &\le f(e_{p-1}) + f(e_{p-2}) + 1\\ &\le f(e_{p-1}) + f(e_p).\end{aligned}$$

Also, from the algorithm, we know that

$$f(e_{p-1}) + f(e_p) = (m + k) + (\lceil \frac{p}{2} \rceil + |U_p| + k).$$

Therefore, for any vertex $v_j \in V_2$,

$$m - \lfloor \tfrac{p}{2} \rfloor + 2 + 2k \le \phi(v_j) \le m + \lceil \tfrac{p}{2} \rceil + |U_p| + 2k. \tag{2.8}$$

It can be seen from Step 3 of the algorithm that all vertex sums of heavy vertices are pairwise distinct. To determine the range of these vertex sums, we first consider a heavy vertex v_j with $\deg(v_j) = 3$. We infer that the partial sum of v_j after Step 2 must be at least $m + k + 1$; otherwise, it would be equal to the labeling of some edge, that is, $Q(i)$: $f(e_j) + f(e_{j+1}) = f(e_{i-1})$ would hold for some i, and then v_j would be a light vertex. Hence, $f(e_j) + f(e_{j+1}) \ge m + k + 1$. Additionally, according to the algorithm, the label of the only leg is at least $\phi(e_p) + 1 = \lceil \frac{p}{2} \rceil + |U_p| + k + 1$. Therefore,

$$\phi(v_j) \ge m + \left\lceil \tfrac{p}{2} \right\rceil + |U_p| + 2k + 2.$$

In the case where $\deg(v_j) \ge 4$, since there are at least two legs incident to v_j, we have

$$\begin{aligned}\phi(v_j) &\ge f(e_1) + f(e_2) + 2(\phi(e_p) + 1)\\ &= (1 + k) + (m - \lfloor \frac{p}{2} \rfloor + 1 + k) + 2(\lceil \frac{p}{2} \rceil + |U_p| + k + 1) = m + \lceil \frac{p}{2} \rceil + 2|U_p| + 4k + 4.\end{aligned}$$

We summarize that for each $v_j \in V_3$,

$$\phi(v_j) \ge m + \lceil \tfrac{p}{2} \rceil + |U_p| + 2k + 2. \tag{2.9}$$

By Inequalities (2.4), (2.5), (2.8), and (2.9), the vertex sum of a vertex in V_i is not equal to that of a vertex in V_j for $i \ne j$. We conclude from the discussion above that labeling f produced by CSAL$^+$ is a k-shifted antimagic labeling for any caterpillar C. □

In the following, we deduce from the previous algorithm the constructions of k-shifted antimagic labelings for some negative integers k. For the case $k < -m$, which implies all labels are negative, we apply CSAL$^+$ by symmetry. More explicitly, we use the absolute values of the labels, that is, the numbers of $[-(m+k), -(1+k)]$, and apply CSAL$^+(m, p, -(m+k)-1)$ to label caterpillar C. In this way, we obtain a $(-(m+k+1))$-shifted antimagic labeling f' for C and a consequent k-shifted antimagic labeling f for C by defining $f(e) = -f'(e)$ for all edges e. We name the procedures described above as CSAL$^-$, which will be used in the next section. We modify the inequalities (2.4), (2.5), (2.8), and (2.9) derived for C using CSAL$^+(m, p, k)$, $k \geq 0$, to obtain the following result for C using CSAL$^-(m, p, k)$ for $k < -m$:

$$\phi(v) \in \begin{cases} \left[\left\lfloor \frac{p}{2} \right\rfloor + k, m + k\right], & \text{if } v \in V_1, \\ \left[m - \left\lceil \frac{p}{2} \right\rceil - |U_p| + 2 + 2k, m + \left\lfloor \frac{p}{2} \right\rfloor + 2k\right], & \text{if } v \in V_2, \\ \left(-\infty, m - \left\lceil \frac{p}{2} \right\rceil - |U_p| + 2k\right], & \text{if } v \in V_3. \end{cases} \tag{2.10}$$

When $k = -1$, the labels are the numbers of $[0, m-1]$. We simply assign 0 to e_1, and then assign $1, 2, \cdots, m-1$ to the edges of $C - v_0$ with CSAL$^+$. This constitutes a (-1)-shifted antimagic labeling except for the case where $m = 2$, and that case does not occur because, otherwise, C would be P_3 with a diameter less than 5 ; actually, P_3 is not (-1)-shifted antimagic. When $k = -m$, we apply case $k = -1$ by symmetry.

Thus, case $-m < k < -1$ remains, where the set of labels $[1 + k, m + k]$ contains positive and negative numbers. As such, if there is an edge e in C such that we can assign the positive and negative labels to the two components of $C - e$ using CSAL$^+$ and CSAL$^-$, respectively, then the labeling, after assigning 0 to e, provides a k-shifted antimagic labeling for C as long as the number of positive labels and that of negative labels are both more than one (see Figure 3). However, such an edge does not always exist, in general. We will solve the remaining case in the next section.

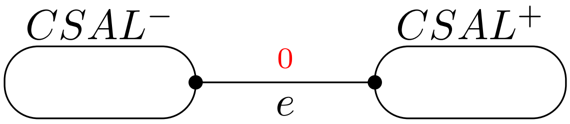


Figure 3. The existence of a special edge.

3. Labeling a caterpillar with labels of mixed signs

We are now ready to complete the proof of the main result.

Theorem 3.1. *Every caterpillar of diameter at least five is absolutely antimagic, that is, k-shifted antimagic for any integer k.*

Proof. Let C be a caterpillar with m edges and let k be an integer. With the discussions at the end of Section 2, it remains to show that every caterpillar is k-shifted antimagic for the case where $-m < k < -1$. By symmetry, it suffices to prove the subcase $-\left\lceil \frac{m}{2} \right\rceil \leq k < -1$, which implies that the positive labels are not less than the negative ones. To begin with, we pair up legs that are incident to the same vertex along the spine. Let x be the maximum number of such pairs we can collect in C. In the sequel, we will assign a pair of opposite numbers to each pair of paired legs. In this way, the labels on the paired legs will have no influence on the vertex sums of the vertices on the spine.

If $x \geq -(1+k)$, we first assign $\pm i$ to a pair of paired legs for $i = 1, 2, \cdots, -(1+k)$ sequentially. Observe that the subgraph of C formed by the unlabeled edges, denoted by C', is a caterpillar with a spine $v_0 e_1 v_1 e_2 \cdots v_{p-1} e_p v_p$ of length p. Next, assign 0 to e_1 and then assign the numbers of $[-k, m+k]$ to the edges of $C' - e_1$ with CSAL$^+$. It is clear that this labeling process produces a k-shifted antimagic labeling for C except for the case where $-k = m + k$, and that case does not occur because otherwise C' would be P_3 and C would be the star S_{-2k} with a diameter less than 5; the star S_{2n} is known to not be $(-n)$-shifted antimagic for any positive integer n.

If $x < -(1+k)$, unless mentioned otherwise, we start by assigning $\pm i$ to each pair of paired legs for $i = 1, 2, \cdots, x$; if $x = 0$, skip this step. The remaining negative labels are $1+k, 2+k, \cdots, -(x+1)$, while the remaining positive labels are $x+1, x+2, \cdots, m+k$. For convenience, denote the numbers of the remaining negative and positive labels by m^- and m^+, respectively; hence, $m^- = -x-1-k$, $m^+ = m+k-x$ and $m^+ \geq m^- \geq 1$. Furthermore, 0 is also a remaining label. As before, the unlabeled edges of C form a caterpillar C' that has a spine $v_0 e_1 v_1 e_2 \cdots v_{p-1} e_p v_p$ of length p. Moreover, every vertex of C' has degree at most 3. If a vertex v_i on the spine has degree 3, then we define $g_i = v_i u_i$ to be the leg incident to v_i; otherwise g_i does not exist. By adjoining the edges e_1, g_1, e_2, $g_2, \cdots$ one by one, we obtain a connected subgraph C'' of C' that contains exactly $m^- + 1$ edges. The last edge adjoined to C'' is either some e_i or some g_i.

Suppose that the last adjoined edge is some e_i. Note that

$$|E(C' - C'')| = m^+ \quad \text{and} \quad |E(C'' - e_i)| = m^-.$$

Suppose that $i \geq 3$ or $i = 2$ and $m^- = 2$. By the arguments at the end of Section 2, we label e_i with 0 and assign the labels of $[x+1, m+k]$ and those of $[1+k, -(x+1)]$ to $C' - C''$ and $C'' - e_i$ using CSAL$^+$ and CSAL$^-$, respectively, to derive a k-shifted antimagic labeling for C. For the remaining special case $i = 2$ and $m^- = 1$, the method above does not work and we need to consider whether or not g_2 exists. If g_2 exists, then we label e_1, e_2, and g_2 with $0, -(x+1)$ and $x+1$, respectively; otherwise, we label e_1, e_2, and e_3 with $-(x+1), x+1$ and 0, respectively (see Figure 4). At this stage, the vertex sums are not all distinct, but the labeling assignment is not finished yet; in $C' - C''$, we have at least two unlabeled edges since the diameter $p \geq 5$. Next, we assign the labels of $[x+2, m+k]$ to the remaining unlabeled edges of $C' - C''$ using CSAL$^+$. This labeling process gives a k-shifted antimagic labeling for C.

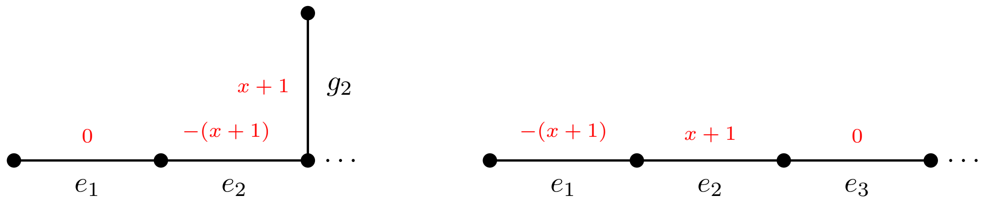


Figure 4. (left) g_2 exists; (right) g_2 does not exist.

Suppose now that the last adjoined edge is some g_i. We separate the discussion based on the parity of i. Suppose that i is odd. We first label g_i and e_{i+1} with $x+1$ and 0, respectively. Next, let the spine of $C'' - g_i$ be the path from v_i to v_0, and assign the labels of $[1+k, -(x+1)]$ to its edges using CSAL$^-$.

Since the number of edges on the spine of $C'' - g_i$ is odd, it follows that $f(e_i) = -(x+1)$, and therefore $\phi(v_i) = 0$ (see Figure 5). Note that there are at least two unlabeled edges in $C' - C'' - e_{i+1}$, which has exactly $m^+ - 1$ edges, since $p \geq 5$ and $m^+ \geq m^-$, and label them with the numbers of $[x+2, m+k]$ using CSAL$^+$. We conclude that this labeling process gives a k-shifted antimagic labeling for C. Suppose now that i is even. We instead assign $\pm i$ to each pair of paired legs for $i = 2, \cdots, x+1$ and consider whether or not g_{i-1} exists.

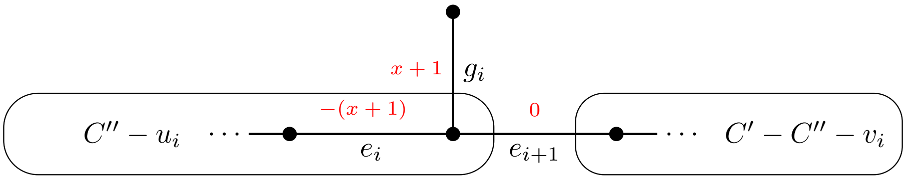


Figure 5. i is odd.

Case 1. g_{i-1} exists. Notice that, in this case, $m^- \geq 3$. We start by labeling the edges e_i, e_{i+1}, and g_i with $1, 0$, and -1, respectively, and assign the labels of $[x+2, m+k]$ to the edges of $C' - C'' - e_{i+1}$ using CSAL$^+$. Next, let the spine of $C'' - v_i$ be the path from u_{i-1} to v_0, and assign the labels of $[1+k, -(x+2)]$ to its edges using CSAL$^-(m^- - 1, i, k)$. Since i is even, it follows that

$$f(g_{i-1}) = i/2 + k \quad \text{and} \quad f(e_{i-1}) = m^- - 1 + k = -(x+2).$$

Figure 6 shows this in detail. To show that this gives a k-shifted antimagic labeling for C, it suffices to show that

$$\phi(v_{i-1}) = -x - 1 + i/2 + k$$

is distinct from $\phi(v)$ for all $v \in C'' - v_i$, which is true since by Eq (2.10), $\phi(v) \in [i/2 + k, -x - 2]$ if $v \in V_1$, the set of vertices of degree 1 in $C'' - v_i$, and $\phi(v) \leq -x - 2 + i/2 + k$ otherwise.

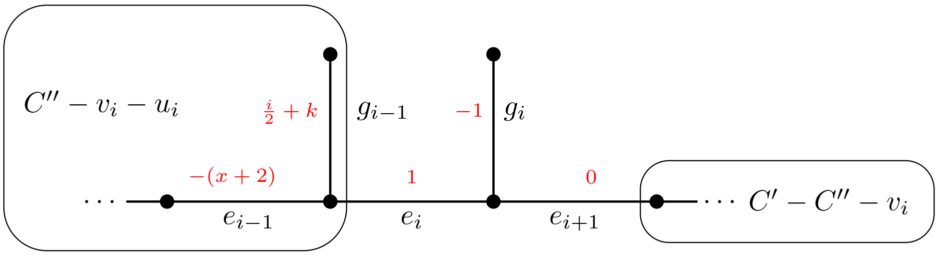


Figure 6. i is even and g_{i-1} exists.

Case 2. g_{i-1} does not exist. We start by labeling e_i, e_{i+1}, and g_i with $-1, 0$, and 1, respectively, and then assign the labels of $[x+2, m+k]$ to the edges of $C' - C'' - e_{i+1}$ using CSAL$^+$. Suppose that $i \geq 6$. Let the spine of $C'' - v_i$ be the path from v_{i-1} to v_0, and assign the labels of $[1+k, -(x+2)]$ to its edges using CSAL$^-(m^- - 1, i-1, k)$. Since $i - 1$ is odd, it follows that $f(e_{i-1}) = -x - 2$ and $f(e_{i-3}) = -x - 3$ (see Figure 7). We need to check only that $\phi(v_{i-1}) = -(x+3)$ is distinct from $\phi(v)$ for all $v \in C'' - v_i$. We conclude that this gives a k-shifted antimagic labeling for C. A similar argument can be applied to the case $i = 2$, where $C'' - v_i$ contains exactly one edge, namely $e_{i-1} = e_1$, and $f(e_1) = -x - 2$ works fine. Notice also that in this subcase, there are at least two unlabeled edges in $C' - C'' - e_{i+1}$ since

$p \geq 5$. As for the case $i = 4$, we label the edges e_1, e_2, and e_3 with $-(x+2)$, $-(x+4)$ and $-(x+3)$, respectively. Next, if one of g_1 or g_2 exists, assign $-(x+5)$ to the existing one. If both g_1 and g_2 exist, assign $-(x+5)$ and $-(x+6)$ to them, respectively (see Figure 8). It is clear that this gives a k-shifted antimagic labeling for C.

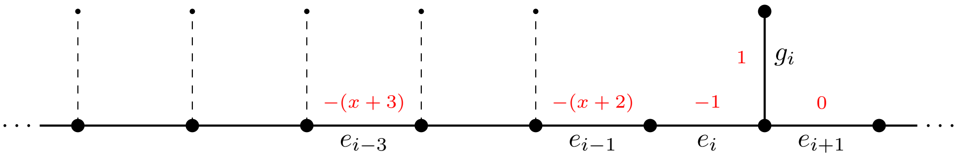


Figure 7. i is even, $i \geq 6$ and g_{i-1} does not exist (g_j, $j < i-1$, could exist).

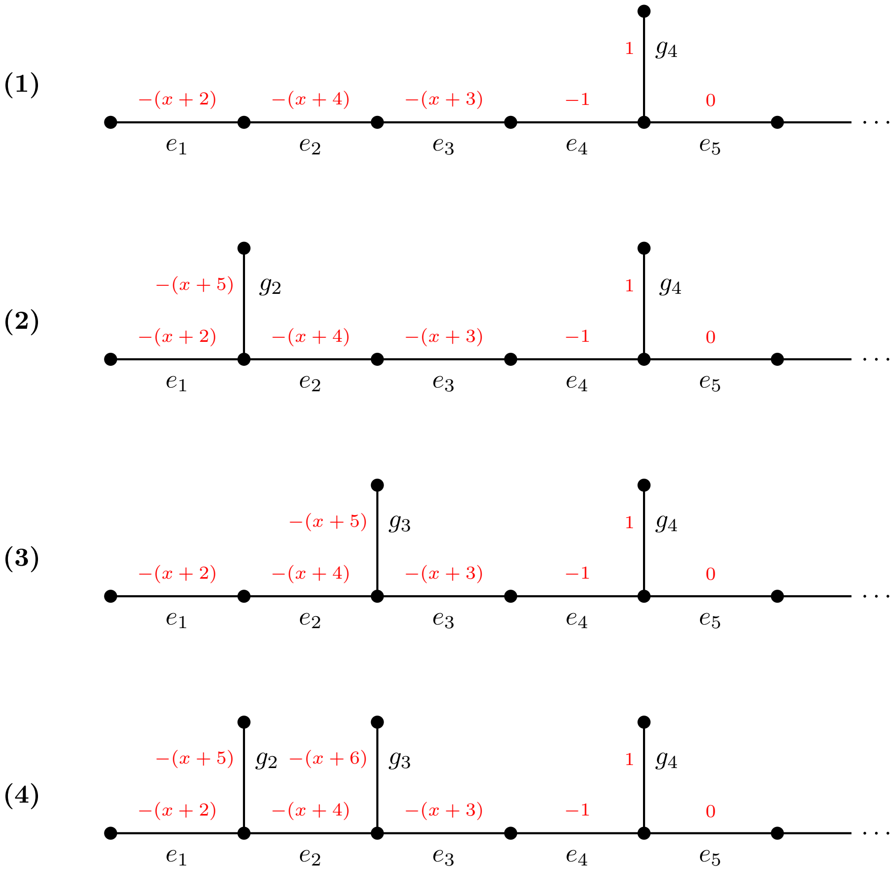


Figure 8. $i = 4$ and g_{i-1} does not exist.

$\square$

4. Concluding remarks

In the proof of Theorem 3.1, we apply the assumption $p \geq 5$ several times when the labels are of mixed signs. One can see the algorithm might not work in those cases if $p < 5$ and might wonder which caterpillars of diameter at most four are absolutely antimagic. As mentioned in the introduction, the k-shifted antimagicness for trees with a diameter at most five has been characterized [14, 16]. To sum up, below we list caterpillars lacking k-shifted antimagic labeling for specific k values:

$p = 2$: the stars S_{2n} with $k = -n-1, -n$ and S_{2n+1} with $k = -n-1$;
$p = 3$: the double stars $S_{2,1}$ with $k = -2, -3$ and $S_{2n-1,1}$ with $k = -n-1$;
$p = 4$: the path P_5 with $k = -2, -3$ and the tree P'_5 with $k = -3$;

where n is any integer and P'_5 is the caterpillar obtained by attaching an edge to the central vertex of P_5.

The concept of strongly antimagic for a graph is introduced in [11]; a graph G is strongly antimagic if it has an antimagic labeling f that additionally satisfies the following vertex sum condition: $\phi(u) < \phi(v)$ whenever $\deg(u) < \deg(v)$. It can be shown that if a graph is strongly antimagic, then it is k-shifted antimagic for all $k \geq 0$. It is then natural to ask the following question:

Question 4.1. *Are caterpillars strongly antimagic?*

Besides, lobsters, defined as trees such that the removal of their leaves results in caterpillars, are a natural class of trees that one would next ask whether they are absolutely antimagic.

Question 4.2. *Are lobsters absolutely antimagic?*

One can first try to apply the technique presented in this paper to study whether lobsters are absolutely antimagic. It is also interesting to answer the following question:

Question 4.3. *For each integer k, which tree of diameter more than five is not k-shfited antimagic?*

We end this section with the following conjecture:

Conjecture 4.4. *Every tree of diameter at least five is absolutely antimagic.*

Author contributions

Wei-Ting Chen: investigation, methodology, writing-original draft, validation, writing-review and editing; Jung-Miao Kuo: investigation, methodology, writing-original draft, validation, writing-review and editing; All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors have no conflicts of interest to declare.

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