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Research article

Application of decision method based on aggregation operators of nPIVFNs and graph structure in supplier selection

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Abstract: The n-polygonal interval-valued fuzzy set is an effective tool to deal with multi-attribute group decision-making problems, which can represent fuzzy information. Aggregating multiple evaluation values in the decision-making process is a key challenge. In this study, new operations are proposed for n-polygonal interval-valued fuzzy numbers based on the algebraic sum and the Einstein sum. The arithmetic mean operators and geometric mean operators are constructed, and the related properties of the constructed operator are explored. The newly proposed aggregation operators have clear fuzzy logic semantics, which can accurately express the "or" logic and compensation between attributes, and reflects the decision intention by which "any attribute can improve the overall evaluation". In view of the fact that graph theory can intuitively depict various attributes, it has significant practical value in promoting more accurate evaluation of alternatives. The path analysis can abstract a complex system into a graph structure and simplify the problem. We develop a decision-making method that combines path and graph strategies. This method is relatively simple to operate and can represent the path between the scheme and all attributes. In order to illustrate the effectiveness of the method, it is applied to supplier selection. Finally, the practicability and stability of the method are verified by comparative experiments and a sensitivity analysis.

Keywords: n-polygonal interval-valued fuzzy number; weighted arithmetic averaging operator; weighted geometric averaging operator; path graph-based decision-making method
Mathematics Subject Classification: 90B50, 91A35

1. Introduction

With the rapid advancement of technology, people are facing more and more complex scenarios when dealing with multi-attribute decision-making problems, which will lead to an increase in uncertainties and ambiguities. Thus the concept of fuzzy sets was introduced and subsequently widely applied in various fields. Due to the fact that fuzzy sets only utilize membership degrees to describe fuzzy information, they cannot effectively reflect people's subjective intentions. Numerous scholars have carried out a series of extensions to fuzzy sets, such as intuitionistic fuzzy sets (IFSs) [1–3], interval-valued fuzzy sets [4, 5], hesitant fuzzy sets [6, 7], and so on. Given the computational complexity of fuzzy sets and their extended forms, this characteristic, to some extent, restricts the in-depth theoretical research and expansion of practical applications. To mitigate the complexity of operations on fuzzy sets, n-polygonal fuzzy sets [8,9] have been developed. Moreover, the complexity of operations in the extension form of fuzzy sets was streamlined by intuitionistic polygonal fuzzy sets [10, 11] and n-polygonal interval-valued fuzzy sets [12, 13]. The aforementioned fuzzy sets exhibit linearity and closure with respect to arithmetic operations, thereby transitioning theoretical investigations from fuzzy metrics to Euclidean spaces.

In multi-attribute group decision-making (MAGDM) problems, the aggregation of existing decision information via a particular model is instrumental in the ranking of limited schemes. Aggregation operators are frequently used in the decision-making process to integrate information. Currently, aggregation operators have been extensively researched by numerous scholars. The mathematical structure and properties of several aggregation operators in fuzzy sets were investigated by Beliakov [14], mainly including the arithmetic mean and geometric mean operators. In 2006, some geometric mean operators of intuitionistic fuzzy numbers were presented by Xu and Yager [15], and their related properties were discussed. Boczen [16] introduced interval-valued seminormed fuzzy operators (ISFOs), and avoided situations where unreasonable circumstances enter the upper and lower limits of interval values in a simple way. To address the limitation of certain aggregation operators on intuitionistic fuzzy sets(IFSs) and T-spherical fuzzy sets that cannot aggregate the value of 0, Zeng et al. [17] researched the properties and applications of geometric mean operators on spherical fuzzy sets. Heronian mean operators were established in the environment of q-rung orthopair fuzzy sets by Liu et al. [18]. Aczel-Alsina power Bonferroni mean aggregation operators were investigated in picture fuzzy sets by Ma et al. [19], combining the studied score function to solve a multi-attribute decision-making problem. Geometric mean aggregation operators for IFSs were improved by [20–22] and their application to MAGDM problems were investigated. Zhang et al. [23] introduced several Hamacher power aggregation operators and integrated them with entropy measures to address the MAGDM problem within the context of spherical fuzzy sets. Aggregation operators for two types of T-spherical fuzzy sets were provided by [24,25] and combined with the combinative distance-based assessment (CODAS) method for software selection. Inspired by concepts of power geometry and power mean, Verma et al. [26] developed a new aggregation operator to aggregate the two-dimensional language of intuitively fuzzy variables to solve the problem of ranking alternatives. Debbarma et al. [27] introduced two aggregation operators, namely the Fermatean fuzzy double Hamy mean operator and the Fermatean fuzzy weighted double Hamy mean operator, and analyzed their properties. The MAGDM problem was solved by using the proposed aggregation operator. The trapezoidal-valued intuitionistic fuzzy Dombi weighted geometric operator, the trapezoidal-valued intuitionistic fuzzy Dombi order weighted geometric operator, and the trapezoidal-valued intuitionistic fuzzy Dombi hybrid geometric operator were developed by Meher et al. [28] and used to solve MAGDM problems involving photovoltaic site selection.

Table 1. Symbol explanation.

Abbreviation	Explanation
R	The set of real numbers
R^+	The set of positive real numbers
N	The set of natural numbers
N^+	The set of positive integers
nPIVFN	n-Polygonal interval-valued fuzzy number
IFSs	Intuitionistic fuzzy sets
nPIVFN(R)	n-Polygonal interval-valued fuzzy numbers on R
ISFOs	Introduced interval-valued seminormed fuzzy operators
CODAS	Combinative distance-based assessment
TOPSIS	Technique for Order Preference by Similarity to an Ideal Solution
VIKOR	VlseKriterijumska Optimizacija I Kompromisno Resenje
MAGDM	Multi-attribute group decision-making
SRCC	Spearman rank correlation coefficient
nPIVFWAA	n-Polygonal interval-valued fuzzy weighted arithmetic averaging
nPIVFOWAA	n-Polygonal interval-valued fuzzy ordered weighted arithmetic averaging
nPIVFHAA	n-Polygonal interval-valued fuzzy hybrid arithmetic averaging
nPIVFWGA	n-Polygonal interval-valued fuzzy weighted geometric averaging
nPIVFOWGA	n-Polygonal interval-valued fuzzy ordered weighted geometric averaging
nPIVFHGA	n-Polygonal interval-valued fuzzy hybrid geometric averaging
Symbol	Explanation
$\widetilde{A}, \widetilde{B}$	Two nPIVFNs
$\widetilde{A}^L, \widetilde{A}^U$	Upper membership function and lower membership function of $\widetilde{A}$
$\lvert\widetilde{A}\rvert$	The distance $\widetilde{A}$ between and $\widetilde{0}$
$D(\star, \star)$	The distance of nPIVFNs
$\star \oplus \star$	The addition operation of nPIVFNs
$\star \otimes \star$	The multiplication of nPIVFNs
X	The set of all schemes
C	The set of all attributes
DM	The set of all decision-makers
ϑ	The weight vector of decision-makers
δ	The weight vector of attributes
H	Decision matrix
$\overline{H}$	Standardized decision matrix
$\widetilde{H}_\iota$	n-Intuitionistic polygonal fuzzy difference matrix
$\widetilde{H}$	Comprehensive decision matrix

In order to solve the ambiguity of traditional depression and anxiety questionnaires and the shortcomings of existing multi-attribute decision-making techniques, a multi-attribute decision-making method for large-scale data-sets based on discrete Z-number theory and the Aczel-Alsina aggregation operator was proposed by Ren et al. [29]. Because a few scholars have studied aggregation operators for *n*PIVFNs, two types of aggregation operators are presented, including weighted arithmetic averaging operators and weighted geometric averaging operators.

As decision-making problems in our daily lives grow increasingly complex, traditional MAGDM methods often fall short in handling these pressing challenges. To capture the human state of hesitation in decision-making, fuzzy decision-making was born. Interval-valued fuzzy sets have a wide range of applications in dealing with MAGDM problems, for which the commonly used decision-making methods include approximate methods for ideal solutions include the Technique for Order Preference by Similarity to an Ideal Solution(TOPSIS) [30] method, the VlseKriterijumska Optimizacija I Kompromisno Resenje, multi-criteria optimization and compromise solution(VIKOR) [31] method, and so on. Graph theory abstracts the structure of nodes and edges, and its advantage lies in transforming complex relationships into computable models. Albagli-Kim et al. [32] proposed using knowledge graphs to connect experts and end-users and applied them to solve medical diagnosis problems. A dual intent network was introduced by Jin et al. [33], which learns the users' intent from attention mechanisms and the distribution of historical data to simulate the decision-making process of users when interacting with new items. In order to explore the relationship between alternative solutions, the strong expressive power of graph neural networks was utilized by Meng et al. [34] to mine decision information and determine the optimal solution. Dutta et al. [35] designed a new arithmetic operation and sorting method for trapezoidal generalized interval-valued fuzzy numbers, and applied it to medical decision-making. Other relevant studies [36, 37] also assist us in conducting in-depth research on mathematical properties. In light of the substantial computational complexity inherent in existing graph-based decision-making applications, this study introduces a novel path-oriented graph-theoretic decision-making method. Specifically, we construct path graphs wherein the alternatives and their respective attributes are represented as vertices, while the relationships between solutions and attributes are modeled as edges. The optimal alternative is subsequently identified through determination of the shortest path.

The remaining sections of this paper are organized as follows. Section 2 reviews the relevant concepts and properties of *n*-polygonal interval-valued fuzzy numbers(*n*PIVFNs). In Section 3, novel laws of operation for *n*PIVFNs are proposed and two kinds of new operators are constructed according to operational laws. In addition, the characteristics of these operators are discussed in depth. In Section 4, a new decision-making method that considers both paths and graphs is introduced. In Section 5, taking supplier selection as a specific example, the feasibility and effectiveness of the improved method are verified through a comparative analysis and a sensitivity analysis. In Section 6, the conclusion and potential future work are presented.

2. Preliminaries

In this section, we briefly review the concepts, properties, and operational laws of *n*PIVFNs, as well as the distance between *n*PIVFNs. Let R, R^+, N, and N^+ represent the set of real numbers, the set of positive real numbers, the set of natural numbers, and the set of positive integers, respectively.

Definition 1. [13] Assume that $\widetilde{A}$ is an interval-valued fuzzy number on R. If there are two groups of $2n$+2 order real numbers $(a_0^1, a_1^1, \ldots, a_n^1, a_n^2, \ldots, a_1^2, a_0^2)$ and $(a_0^3, a_1^3, \ldots, a_n^3, a_n^4, \ldots, a_1^4, a_0^4)$, such that the upper membership function $\widetilde{A}^L$ and the lower membership function $\widetilde{A}^U$ of $\widetilde{A}$ have the following forms, then $\widetilde{A}$ is called an n-polygonal interval-valued fuzzy number (nPIVFN):

$$\widetilde{A}^L(x) = \begin{cases} \frac{\xi}{n} + \frac{x - a_\xi^1}{n(a_{\xi+1}^1 - a_\xi^1)}, & x \in [a_\xi^1, a_{\xi+1}^1], \\ 1, & x \in [a_n^1, a_n^2], \\ \frac{\xi}{n} + \frac{a_\xi^2 - x}{n(a_\xi^2 - a_{\xi+1}^2)}, & x \in [a_{\xi+1}^2, a_\xi^2], \\ 0, & \text{Otherwise} \end{cases}$$

$$\widetilde{A}^U(x) = \begin{cases} \frac{\xi}{n} + \frac{x - a_\xi^3}{n(a_{\xi+1}^3 - a_\xi^3)}, & x \in [a_\xi^3, a_{\xi+1}^{P_1}], \\ 1, & x \in [a_n^3, a_n^4], \\ \frac{\xi}{n} + \frac{a_\xi^4 - x}{n(a_\xi^4 - a_{\xi+1}^4)}, & x \in [a_{\xi+1}^4, a_\xi^4], \\ 0, & \text{Otherwise} \end{cases}$$

where $\xi = 0, 1, 2, \ldots, n-1$. See Figure 1.

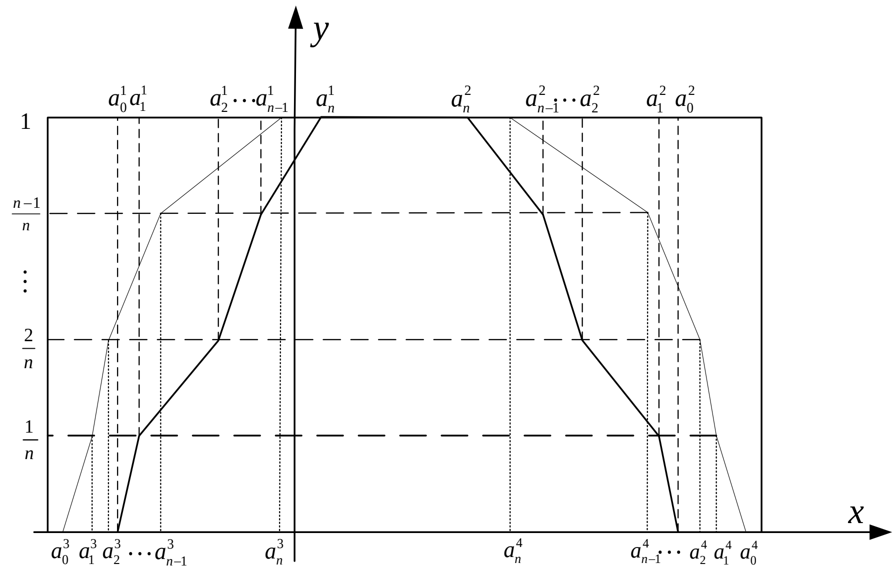


Figure 1. n-Polygonal interval-valued fuzzy number $\widetilde{A}$.

An nPIVFN $\widetilde{A}$ can be written as

$$\widetilde{A} = [(a_0^1, \ldots, a_n^1, a_n^2, \ldots, a_0^2), (a_0^3, \ldots, a_n^3, a_n^4, \ldots, a_0^4)].$$

For convenience, nPIVFN(R) expresses all n-polygonal interval-valued fuzzy numbers on R. Several a series of special nPIVFNs can be abbreviated as

$$\widetilde{P} = [(\underbrace{P, P, \ldots, P}_{2n+2}), (\underbrace{P, P, \ldots, P}_{2n+2})],$$

where $P \in N$.

Definition 2. [13] Let $\widetilde{A}, \widetilde{B} \in n$PIVFN(R), then the distance between $\widetilde{A}$ and $\widetilde{B}$ can be expressed as:

$$D(\widetilde{A}, \widetilde{B}) = \bigvee_{\xi=0}^{n} |a_\xi^1 - b_\xi^1| \vee |a_\xi^2 - b_\xi^2| \vee |a_\xi^3 - b_\xi^3| \vee |a_\xi^4 - b_\xi^4|. \tag{2.1}$$

Obviously, $|\widetilde{A}|$ can be seen as the distance $\widetilde{A}$ between and $\widetilde{0}$, denoted as

$$\begin{aligned}|\widetilde{A}| = D(\widetilde{A}, \widetilde{0}) &= \bigvee_{\xi=0}^{n} |a_\xi^1 - 0| \vee |a_\xi^2 - 0| \vee |a_\xi^3 - 0| \vee |a_\xi^4 - 0| \\ &= \max\{|a_\xi^1|, |a_\xi^2|, |a_\xi^3|, |a_\xi^4|\}.\end{aligned}$$

Definition 3. [9] If we let the closed intervals be $a = [a^-, a^+]$ and $[b^-, b^+]$, then $a \subseteq b$ is only if $b^- \leq a^- \leq a^+ \leq b^+$.

Theorem 1. [12] Let $\widetilde{A}, \widetilde{B} \in n$PIVFN(R), then the inclusion relationship between $\widetilde{A}$ and $\widetilde{B}$ is as follows:

$$\widetilde{A} \subseteq \widetilde{B} \Leftrightarrow [a_\xi^1, a_\xi^2] \subseteq [b_\xi^1, b_\xi^2] \text{ and } [a_\xi^3, a_\xi^4] \subseteq [b_\xi^3, b_\xi^4],$$

where $\xi = 0, 1, \ldots, n$.

3. The proposed *n*-polygonal interval-valued averaging operators

In this section, new operational laws for nPIVFNs are established. Furthermore, to facilitate the direct aggregation of multiple nPIVFNs into a single nPIVFN using mathematical models, we propose two types of aggregation operators: Weighted arithmetic averaging operators and weighted geometric averaging operators. Additionally, the properties and proof methods associated with these aggregation operators are provided. For the convenience of the following description, let $\{a_\xi\}_{\xi=0}^n = \{a_0, a_1, \ldots, a_n\}$.

3.1. Operations on nPIVFNs

Definition 4. Let $\widetilde{A}, \widetilde{B} \in n$PIVFN(R), for which the laws of operation of $\widetilde{A}$ and $\widetilde{B}$ are defined as follows:

$$\begin{aligned}\widetilde{A} \oplus \widetilde{B} =& \Bigg[\Bigg(\Big\{\frac{a_\xi^1}{|\widetilde{A}|} + \frac{b_\xi^1}{|\widetilde{B}|} - \frac{a_\xi^1}{|\widetilde{A}|}\frac{b_\xi^1}{|\widetilde{B}|}\Big\}_{\xi=0}^n, \Big\{\frac{a_{n-\xi}^2}{|\widetilde{A}|} + \frac{b_{n-\xi}^2}{|\widetilde{B}|} - \frac{a_{n-\xi}^2}{|\widetilde{A}|}\frac{b_{n-\xi}^2}{|\widetilde{B}|}\Big\}_{\xi=0}^n\Bigg), \\ &\Bigg(\Big\{\frac{a_\xi^3}{|\widetilde{A}|} + \frac{b_\xi^3}{|\widetilde{B}|} - \frac{a_\xi^3}{|\widetilde{A}|}\frac{b_\xi^3}{|\widetilde{B}|}\Big\}_{\xi=0}^n, \Big\{\frac{a_{n-\xi}^4}{|\widetilde{A}|} + \frac{b_{n-\xi}^4}{|\widetilde{B}|} - \frac{a_{n-\xi}^4}{|\widetilde{A}|}\frac{b_{n-\xi}^4}{|\widetilde{B}|}\Big\}_{\xi=0}^n\Bigg)\Bigg],\end{aligned} \tag{3.1}$$

$$\begin{aligned}\widetilde{A} \otimes \widetilde{B} =& \Bigg[\Bigg(\Big\{\frac{\frac{a_\xi^1}{|\widetilde{A}|} + \frac{b_\xi^1}{|\widetilde{B}|}}{1 + \frac{a_\xi^1}{|\widetilde{A}|}\frac{b_\xi^1}{|\widetilde{B}|}}\Big\}_{\xi=0}^n, \Big\{\frac{\frac{a_{n-\xi}^2}{|\widetilde{A}|} + \frac{b_{n-\xi}^2}{|\widetilde{B}|}}{1 + \frac{a_{n-\xi}^2}{|\widetilde{A}|}\frac{b_{n-\xi}^2}{|\widetilde{B}|}}\Big\}_{\xi=0}^n\Bigg), \\ &\Bigg(\Big\{\frac{\frac{a_\xi^3}{|\widetilde{A}|} + \frac{b_\xi^3}{|\widetilde{B}|}}{1 + \frac{a_\xi^3}{|\widetilde{A}|}\frac{b_\xi^3}{|\widetilde{B}|}}\Big\}_{\xi=0}^n, \Big\{\frac{\frac{a_{n-\xi}^4}{|\widetilde{A}|} + \frac{b_{n-\xi}^4}{|\widetilde{B}|}}{1 + \frac{a_{n-\xi}^4}{|\widetilde{A}|}\frac{b_{n-\xi}^4}{|\widetilde{B}|}}\Big\}_{\xi=0}^n\Bigg)\Bigg],\end{aligned} \tag{3.2}$$

Theorem 2. If we let $A, B \in n$PIVFN(R), then $\widetilde{A} \oplus \widetilde{B}, \widetilde{A} \otimes \widetilde{B} \in n$PIVFN(R).

Proof. Let

$$f(x,y) = \frac{x}{|\widetilde{A}|} + \frac{y}{|\widetilde{B}|} - \frac{xy}{|\widetilde{A}||\widetilde{B}|}, x \in [0, |\widetilde{A}|], y \in [0, |\widetilde{B}|],$$

$$\frac{\partial f(x,y)}{\partial x} = \frac{1}{|\widetilde{A}|} - \frac{y}{|\widetilde{A}||\widetilde{B}|}(|\widetilde{B}| - y) \geq 0,$$

$$\frac{\partial f(x,y)}{\partial y} = \frac{1}{|\widetilde{B}|} - \frac{x}{|\widetilde{A}||\widetilde{B}|}(|\widetilde{A}| - y) \geq 0,$$

and $f(x,y)$ monotonically increases with respect to x and y.

That is, we assume that

$$\forall x_1, x_2 \in [0, |\widetilde{A}|] \text{ and } x_1 \leq x_2, \forall y_1, y_2 \in [0, |\widetilde{B}|] \text{ and } y_1 \leq y_2,$$

then $f(x_1, y_1) \leq f(x_2, y_1) \leq f(x_2, y_2)$, that is

$$\frac{x_1}{|\widetilde{A}|} + \frac{y_1}{|\widetilde{B}|} - \frac{x_1 y_1}{|\widetilde{A}||\widetilde{B}|} \leq \frac{x_2}{|\widetilde{A}|} + \frac{y_2}{|\widetilde{B}|} - \frac{x_2 y_2}{|\widetilde{A}||\widetilde{B}|}.$$

Therefore, a finite sequence $\left\{\frac{a_\xi^1}{|\widetilde{A}|} + \frac{b_\xi^1}{|\widetilde{B}|} - \frac{a_\xi^1}{|\widetilde{A}|}\frac{b_\xi^1}{|\widetilde{B}|}\right\}_{\xi=0}^n$ is monotonically increasing with respect to a_ξ^1 and b_ξ^1.

Similarly, the finite sequences $\left\{\frac{a_{n-\xi}^2}{|\widetilde{A}|} + \frac{b_{n-\xi}^2}{|\widetilde{B}|} - \frac{a_{n-\xi}^2}{|\widetilde{A}|}\frac{b_{n-\xi}^2}{|\widetilde{B}|}\right\}_{\xi=0}^n$, $\left\{\frac{a_\xi^3}{|\widetilde{A}|} + \frac{b_\xi^3}{|\widetilde{B}|} - \frac{a_\xi^3}{|\widetilde{A}|}\frac{b_\xi^3}{|\widetilde{B}|}\right\}_{\xi=0}^n$ and $\left\{\frac{a_{n-\xi}^4}{|\widetilde{A}|} + \frac{b_{n-\xi}^4}{|\widetilde{B}|} - \frac{a_{n-\xi}^4}{|\widetilde{A}|}\frac{b_{n-\xi}^4}{|\widetilde{B}|}\right\}_{\xi=0}^n$ also exhibit an increasing property. Hence, $\widetilde{A} \oplus \widetilde{B}$ is still an nPIVFN(R). It can also be proven that $\widetilde{A} \otimes \widetilde{B}$ is an nPIVFN(R).

For the convenience of later expression, $\widetilde{A}_\xi^{\star t}$, $\widetilde{B}_\xi^{\star t}$, $\widetilde{A}_{n-\xi}^{\star r}$ and $\widetilde{B}_{n-\xi}^{\star r}$($t = 1, 3, r = 2, 4$) denote $\frac{a_\xi^t}{|\widetilde{A}|}$, $\frac{b_\xi^t}{|\widetilde{B}|}$, $\frac{a_{n-\xi}^r}{|\widetilde{A}|}$ and $\frac{b_{n-\xi}^r}{|\widetilde{B}|}$.

We consider two 3PIVFNs given by

$$\widetilde{A} = [(5, 5.5, 6, 6.8, 7, 8, 8.8, 9), (4.5, 5, 5.8, 6.5, 7.2, 8, 9, 10)],$$

$$\widetilde{B} = [(4, 5, 5.5, 6, 6.5, 6.8, 7, 8), (3, 4, 4.2, 5, 6.7, 7, 8, 8.5)],$$

as shown in **Figures 2 and 3**. According to the definition, it can be seen that

$$|\widetilde{A}| = 10, |\widetilde{B}| = 8.5,$$

in which case

$$\begin{aligned}
\widetilde{A} \oplus \widetilde{B} =& [(0.74, 0.81, 0.86, 0.91, 0.93, 0.96, 0.98, 0.99),\\
& (0.64, 0.74, 0.79, 0.86, 0.94, 0.96, 0.99, 1.00)],\\
\widetilde{A} \otimes \widetilde{B} =& [(0.79, 0.86, 0.90, 0.94, 0.95, 0.98, 0.99, 1),\\
& (0.69, 0.79, 0.83, 0.90, 0.96, 0.99, 1.00, 1.00)].
\end{aligned}$$

$\widetilde{A} \oplus \widetilde{B}$ and $\widetilde{A} \otimes \widetilde{B}$ are shown in **Figures 4 and 5**.

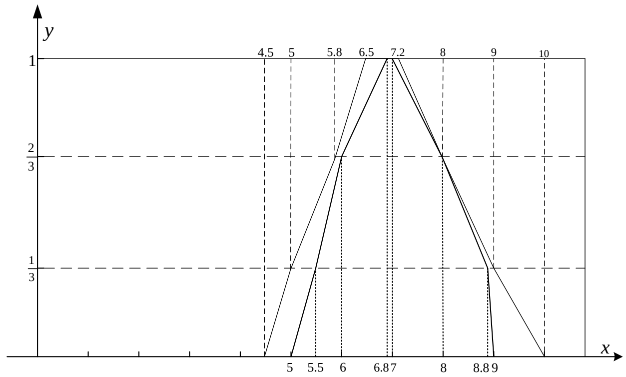


Figure 2. Graph of $\widetilde{A}$.

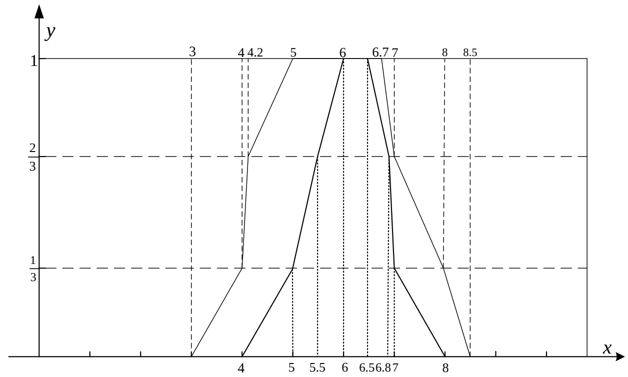


Figure 3. Graph of $\widetilde{B}$.

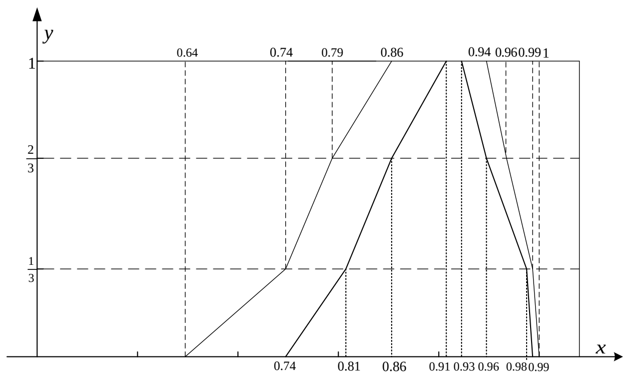


Figure 4. Graph of $\widetilde{A} \oplus \widetilde{B}$.

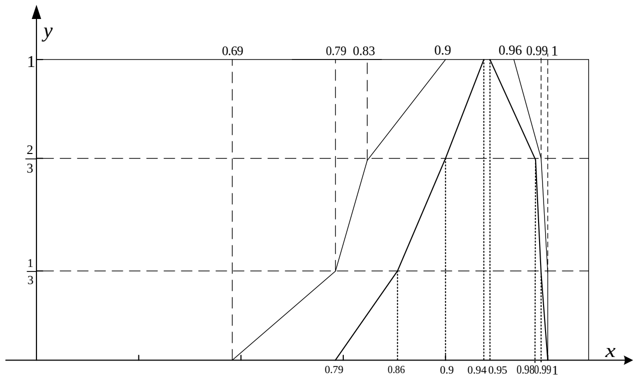


Figure 5. Graph of $\widetilde{A} \otimes \widetilde{B}$.

Theorem 3. If we let $\widetilde{A}, \widetilde{B} \in n$PIVFN(R) and $\lambda \in N^+$, then

$$\begin{aligned}\lambda\widetilde{A} =&\Big[\Big(\Big\{1-(1-\widetilde{A}_{\xi}^{\star 1})^{\lambda}\Big\}_{\xi=0}^{n}, \Big\{1-(1-\widetilde{A}_{n-\xi}^{\star 2})^{\lambda}\Big\}_{\xi=0}^{n}\Big),\\ &\Big(\Big\{1-(1-\widetilde{A}_{\xi}^{\star 3})^{\lambda}\Big\}_{\xi=0}^{n}, \Big\{1-(1-\widetilde{A}_{n-\xi}^{\star 4})^{\lambda}\Big\}_{\xi=0}^{n}\Big)\Big],\end{aligned} \tag{3.3}$$

$$\begin{aligned}\widetilde{A}^{\lambda} =&\Big[\Big(\Big\{\frac{(1+\widetilde{A}_{\xi}^{\star 1})^{\lambda}-(1-\widetilde{A}_{\xi}^{\star 1})^{\lambda}}{(1+\widetilde{A}_{\xi}^{\star 1})^{\lambda}+(1-\widetilde{A}_{\xi}^{\star 1})^{\lambda}}\Big\}_{\xi=0}^{n}, \Big\{\frac{(1+\widetilde{A}_{n-\xi}^{\star 2})^{\lambda}-(1-\widetilde{A}_{n-\xi}^{\star 2})^{\lambda}}{(1+\widetilde{A}_{n-\xi}^{\star 2})^{\lambda}+(1-\widetilde{A}_{n-\xi}^{\star 2})^{\lambda}}\Big\}_{\xi=0}^{n}\Big),\\ &\Big(\Big\{\frac{(1+\widetilde{A}_{\xi}^{\star 3})^{\lambda}-(1-\widetilde{A}_{\xi}^{\star 3})^{\lambda}}{(1+\widetilde{A}_{\xi}^{\star 3})^{\lambda}+(1-\widetilde{A}_{\xi}^{\star 3})^{\lambda}}\Big\}_{\xi=0}^{n}, \Big\{\frac{(1+\widetilde{A}_{n-\xi}^{\star 4})^{\lambda}-(1-\widetilde{A}_{n-\xi}^{\star 4})^{\lambda}}{(1+\widetilde{A}_{n-\xi}^{\star 4})^{\lambda}+(1-\widetilde{A}_{n-\xi}^{\star 4})^{\lambda}}\Big\}_{\xi=0}^{n}\Big)\Big],\end{aligned} \tag{3.4}$$

where $\lambda\widetilde{A} = \underbrace{\widetilde{A} \oplus \widetilde{A} \oplus \cdots \oplus \widetilde{A}}_{\lambda}$, $\widetilde{A}^{\lambda} = \underbrace{\widetilde{A} \otimes \widetilde{A} \otimes \cdots \otimes \widetilde{A}}_{\lambda}$.

Proof. (i) By the mathematical induction, we prove that Eq.(4) holds for $\lambda = 2$

$$\begin{aligned}
2\widetilde{A} =& \widetilde{A} \oplus \widetilde{A} \\
=& \Bigg[\bigg(\Big\{\widetilde{A}^{\star 1}_{\xi} + 1 - (1 - \widetilde{A}^{\star 1}_{\xi}) - (1 - (1 - \widetilde{A}^{\star 1}_{\xi}))\widetilde{A}^{\star 1}_{\xi}\Big\}_{\xi=0}^{n}, \\
& \Big\{\widetilde{A}^{\star 2}_{n-\xi} + 1 - (1 - \widetilde{A}^{\star 2}_{n-\xi}) - (1 - (1 - \widetilde{A}^{\star 2}_{n-\xi}))\widetilde{A}^{\star 2}_{n-\xi}\Big\}_{\xi=0}^{n}\bigg), \\
& \bigg(\Big\{\widetilde{A}^{\star 3}_{\xi} + 1 - (1 - \widetilde{A}^{\star 3}_{\xi}) - (1 - (1 - \widetilde{A}^{\star 3}_{\xi}))\widetilde{A}^{\star 3}_{\xi}\Big\}_{\xi=0}^{n}, \\
& \Big\{\widetilde{A}^{\star 4}_{n-\xi} + 1 - (1 - \widetilde{A}^{\star 4}_{n-\xi}) - (1 - (1 - \widetilde{A}^{\star 4}_{n-\xi}))\widetilde{A}^{\star 4}_{n-\xi}\Big\}_{\xi=0}^{n}\bigg)\Bigg] \\
=& \Bigg[\bigg(\Big\{1 - (1 - \widetilde{A}^{\star 1}_{\xi})^{2}\Big\}_{\xi=0}^{n}, \Big\{1 - (1 - \widetilde{A}^{\star 2}_{n-\xi})^{2}\Big\}_{\xi=0}^{n}\bigg), \\
& \bigg(\Big\{1 - (1 - \widetilde{A}^{\star 3}_{\xi})^{2}\Big\}_{\xi=0}^{n}, \Big\{1 - (1 - \widetilde{A}^{\star 4}_{n-\xi})^{2}\Big\}_{\xi=0}^{n}\bigg)\Bigg].
\end{aligned}$$

If Eq (4) holds for $\lambda = k$, then we have

$$\begin{aligned}
k\widetilde{A} =& \Bigg[\bigg(\Big\{1 - (1 - \widetilde{A}^{\star 1}_{\xi})^{k}\Big\}_{\xi=0}^{n}, \Big\{1 - (1 - \widetilde{A}^{\star 2}_{n-\xi})^{k}\Big\}_{\xi=0}^{n}\bigg), \\
& \Big\{1 - (1 - \widetilde{A}^{\star 3}_{\xi})^{k}\Big\}_{\xi=0}^{n}, \Big\{1 - (1 - \widetilde{A}^{\star 4}_{n-\xi})^{k}\Big\}_{\xi=0}^{n}\bigg)\Bigg].
\end{aligned}$$

Now, if we let $\lambda = k + 1$, it can be inferred that

$$\begin{aligned}
(k+1)\widetilde{A} =& \widetilde{A} \oplus k\widetilde{A} \\
=& \Bigg[\bigg(\Big\{1 - (1 - \widetilde{A}^{\star 1}_{\xi})(1 - (1 - (1 - \widetilde{A}^{\star 1}_{\xi})^{k})\Big\}_{\xi=0}^{n}, \\
& \Big\{1 - (1 - \widetilde{A}^{\star 2}_{n-\xi})(1 - (1 - (1 - \widetilde{A}^{\star 2}_{n-\xi})^{k})\Big\}_{\xi=0}^{n}\bigg), \\
& \bigg(\Big\{1 - (1 - \widetilde{A}^{\star 3}_{\xi})(1 - (1 - (1 - \widetilde{A}^{\star 3}_{\xi})^{k})\Big\}_{\xi=0}^{n}, \\
& \Big\{1 - (1 - \widetilde{A}^{\star 4}_{n-\xi})(1 - (1 - (1 - \widetilde{A}^{\star 4}_{n-\xi})^{k})\Big\}_{\xi=0}^{n}\bigg)\Bigg] \\
=& \Bigg[\bigg(\Big\{1 - (1 - \widetilde{A}^{\star 1}_{\xi})^{k+1}\Big\}_{\xi=0}^{n}, \Big\{1 - (1 - \widetilde{A}^{\star 2}_{n-\xi})^{k+1}\Big\}_{\xi=0}^{n}\bigg), \\
& \bigg(\Big\{1 - (1 - \widetilde{A}^{\star 3}_{\xi})^{k+1}\Big\}_{\xi=0}^{n}, \Big\{1 - (1 - \widetilde{A}^{\star 4}_{n-\xi})^{k+1}\Big\}_{\xi=0}^{n}\bigg)\Bigg].
\end{aligned}$$

(ii) According to the mathematical induction, Eq (5) is shown to hold for $\lambda = 2$

$$
\begin{aligned}
\widetilde{A}^2 =& \widetilde{A} \otimes \widetilde{A} \\
=& \Bigg[\Bigg(\Bigg\{\frac{\widetilde{A}_\xi^{\star 1} + \frac{(1+\widetilde{A}_\xi^{\star 1})-(1-\widetilde{A}_\xi^{\star 1})}{(1+\widetilde{A}_\xi^{\star 1})+(1-\widetilde{A}_\xi^{\star 1})}}{1 + \frac{(1+\widetilde{A}_\xi^{\star 1})-(1-\widetilde{A}_\xi^{\star 1})}{(1+\widetilde{A}_\xi^{\star 1})+(1-\widetilde{A}_\xi^{\star 1})}\widetilde{A}_\xi^{\star 1}}\Bigg\}_{\xi=0}^{n}, \Bigg(\Bigg\{\frac{\widetilde{A}_{n-\xi}^{\star 2} + \frac{(1+\widetilde{A}_{n-\xi}^{\star 2})-(1-\widetilde{A}_{n-\xi}^{\star 2})}{(1+\widetilde{A}_{n-\xi}^{\star 2})+(1-\widetilde{A}_{n-\xi}^{\star 2})}}{1 + \frac{(1+\widetilde{A}_{n-\xi}^{\star 2})-(1-\widetilde{A}_{n-\xi}^{\star 2})}{(1+\widetilde{A}_{n-\xi}^{\star 2})+(1-\widetilde{A}_{n-\xi}^{\star 2})}\widetilde{A}_{n-\xi}^{\star 2}}\Bigg\}_{\xi=0}^{n}\Bigg), \\
& \Bigg(\Bigg\{\frac{\widetilde{A}_\xi^{\star 1} + \frac{(1+\widetilde{A}_\xi^{\star 3})-(1-\widetilde{A}_\xi^{\star 3})}{(1+\widetilde{A}_\xi^{\star 3})+(1-\widetilde{A}_\xi^{\star 3})}}{1 + \frac{(1+\widetilde{A}_\xi^{\star 3})-(1-\widetilde{A}_\xi^{\star 3})}{(1+\widetilde{A}_\xi^{\star 3})+(1-\widetilde{A}_\xi^{\star 3})}\widetilde{A}_\xi^{\star 3}}\Bigg\}_{\xi=0}^{n}, \Bigg(\Bigg\{\frac{\widetilde{A}_{n-\xi}^{\star 4} + \frac{(1+\widetilde{A}_{n-\xi}^{\star 4})-(1-\widetilde{A}_{n-\xi}^{\star 4})}{(1+\widetilde{A}_{n-\xi}^{\star 4})+(1-\widetilde{A}_{n-\xi}^{\star 4})}}{1 + \frac{(1+\widetilde{A}_{n-\xi}^{\star 4})-(1-\widetilde{A}_{n-\xi}^{\star 4})}{(1+\widetilde{A}_{n-\xi}^{\star 4})+(1-\widetilde{A}_{n-\xi}^{\star 4})}\widetilde{A}_{n-\xi}^{\star 4}}\Bigg\}_{\xi=0}^{n}\Bigg)\Bigg] \\
=& \Bigg[\Bigg(\Bigg\{\frac{(1+\widetilde{A}_\xi^{\star 1})^2 - (1-\widetilde{A}_\xi^{\star 1})^2}{(1+\widetilde{A}_\xi^{\star 1})^2 + (1-\widetilde{A}_\xi^{\star 1})^2}\Bigg\}_{\xi=0}^{n}, \Bigg\{\frac{(1+\widetilde{A}_{n-\xi}^{\star 2})^2 - (1-\widetilde{A}_{n-\xi}^{\star 2})^2}{(1+\widetilde{A}_{n-\xi}^{\star 2})^2 + (1-\widetilde{A}_{n-\xi}^{\star 2})^2}\Bigg\}_{\xi=0}^{n}\Bigg), \\
& \Bigg(\Bigg\{\frac{(1+\widetilde{A}_\xi^{\star 3})^2 - (1-\widetilde{A}_\xi^{\star 3})^2}{(1+\widetilde{A}_\xi^{\star 3})^2 + (1-\widetilde{A}_\xi^{\star 3})^2}\Bigg\}_{\xi=0}^{n}, \Bigg\{\frac{(1+\widetilde{A}_{n-\xi}^{\star 4})^2 - (1-\widetilde{A}_{n-\xi}^{\star 4})^2}{(1+\widetilde{A}_{n-\xi}^{\star 4})^2 + (1-\widetilde{A}_{n-\xi}^{\star 4})^2}\Bigg\}_{\xi=0}^{n}\Bigg)\Bigg].
\end{aligned}
$$

If Eq (5) holds for $\lambda = k$, then

$$
\begin{aligned}
\widetilde{A}^k =& \Bigg[\Bigg(\Bigg\{\frac{(1+\widetilde{A}_\xi^{\star 1})^k - (1-\widetilde{A}_\xi^{\star 1})^k}{(1+\widetilde{A}_\xi^{\star 1})^k + (1-\widetilde{A}_\xi^{\star 1})^k}\Bigg\}_{\xi=0}^{n}, \Bigg\{\frac{(1+\widetilde{A}_{n-\xi}^{\star 2})^k - (1-\widetilde{A}_{n-\xi}^{\star 2})^k}{(1+\widetilde{A}_{n-\xi}^{\star 2})^k + (1-\widetilde{A}_{n-\xi}^{\star 2})^k}\Bigg\}_{\xi=0}^{n}\Bigg), \\
& \Bigg(\Bigg\{\frac{(1+\widetilde{A}_\xi^{\star 3})^k - (1-\widetilde{A}_\xi^{\star 3})^k}{(1+\widetilde{A}_\xi^{\star 3})^k + (1-\widetilde{A}_\xi^{\star 3})^k}\Bigg\}_{\xi=0}^{n}, \Bigg\{\frac{(1+\widetilde{A}_{n-\xi}^{\star 4})^k - (1-\widetilde{A}_{n-\xi}^{\star 4})^k}{(1+\widetilde{A}_{n-\xi}^{\star 4})^k + (1-\widetilde{A}_{n-\xi}^{\star 4})^k}\Bigg\}_{\xi=0}^{n}\Bigg)\Bigg].
\end{aligned}
$$

When $\lambda = k + 1$, it is found that

$$
\begin{aligned}
\widetilde{A}^{k+1} =& \widetilde{A} \otimes \widetilde{A}^k \\
=& \Bigg[\Bigg(\Bigg\{\frac{\widetilde{A}_\xi^{\star 1} + \frac{(1+\widetilde{A}_\xi^{\star 1})^k-(1-\widetilde{A}_\xi^{\star 1})^k}{(1+\widetilde{A}_\xi^{\star 1})^k+(1-\widetilde{A}_\xi^{\star 1})^k}}{1 + \frac{(1+\widetilde{A}_\xi^{\star 1})^k-(1-\widetilde{A}_\xi^{\star 1})^k}{(1+\widetilde{A}_\xi^{\star 1})^k+(1-\widetilde{A}_\xi^{\star 1})^k}\widetilde{A}_\xi^{\star 1}}\Bigg\}_{\xi=0}^{n}, \Bigg\{\frac{\widetilde{A}_{n-\xi}^{\star 2} + \frac{(1+\widetilde{A}_{n-\xi}^{\star 2})^k-(1-\widetilde{A}_{n-\xi}^{\star 2})^k}{(1+\widetilde{A}_{n-\xi}^{\star 2})^k+(1-\widetilde{A}_{n-\xi}^{\star 2})^k}}{1 + \frac{(1+\widetilde{A}_{n-\xi}^{\star 2})^k-(1-\widetilde{A}_{n-\xi}^{\star 2})^k}{(1+\widetilde{A}_{n-\xi}^{\star 2})^k+(1-\widetilde{A}_{n-\xi}^{\star 2})^k}\widetilde{A}_{n-\xi}^{\star 2}}\Bigg\}_{\xi=0}^{n}\Bigg), \\
& \Bigg(\Bigg\{\frac{\widetilde{A}_\xi^{\star 3} + \frac{(1+\widetilde{A}_\xi^{\star 3})^k-(1-\widetilde{A}_\xi^{\star 3})^k}{(1+\widetilde{A}_\xi^{\star 3})^k+(1-\widetilde{A}_\xi^{\star 3})^k}}{1 + \frac{(1+\widetilde{A}_\xi^{\star 3})^k-(1-\widetilde{A}_\xi^{\star 3})^k}{(1+\widetilde{A}_\xi^{\star 3})^k+(1-\widetilde{A}_\xi^{\star 3})^k}\widetilde{A}_\xi^{\star 3}}\Bigg\}_{\xi=0}^{n}, \Bigg\{\frac{\widetilde{A}_{n-\xi}^{\star 4} + \frac{(1+\widetilde{A}_{n-\xi}^{\star 4})^k-(1-\widetilde{A}_{n-\xi}^{\star 4})^k}{(1+\widetilde{A}_{n-\xi}^{\star 4})^k+(1-\widetilde{A}_{n-\xi}^{\star 4})^k}}{1 + \frac{(1+\widetilde{A}_{n-\xi}^{\star 4})^k-(1-\widetilde{A}_{n-\xi}^{\star 4})^k}{(1+\widetilde{A}_{n-\xi}^{\star 4})^k+(1-\widetilde{A}_{n-\xi}^{\star 4})^k}\widetilde{A}_{n-\xi}^{\star 4}}\Bigg\}_{\xi=0}^{n}\Bigg)\Bigg] \\
=& \Bigg[\Bigg(\Bigg\{\frac{(1+\widetilde{A}_\xi^{\star 1})^{k+1} - (1-\widetilde{A}_\xi^{\star 1})^{k+1}}{(1+\widetilde{A}_\xi^{\star 1})^{k+1} + (1-\widetilde{A}_\xi^{\star 1})^{k+1}}\Bigg\}_{\xi=0}^{n}, \Bigg\{\frac{(1+\widetilde{A}_{n-\xi}^{\star 2})^{k+1} - (1-\widetilde{A}_{n-\xi}^{\star 2})^{k+1}}{(1+\widetilde{A}_{n-\xi}^{\star 2})^{k+1} + (1-\widetilde{A}_{n-\xi}^{\star 2})^{k+1}}\Bigg\}_{\xi=0}^{n}\Bigg), \\
& \Bigg(\Bigg\{\frac{(1+\widetilde{A}_\xi^{\star 3})^{k+1} - (1-\widetilde{A}_\xi^{\star 3})^{k+1}}{(1+\widetilde{A}_\xi^{\star 3})^{k+1} + (1-\widetilde{A}_\xi^{\star 3})^{k+1}}\Bigg\}_{\xi=0}^{n}, \Bigg\{\frac{(1+\widetilde{A}_{n-\xi}^{\star 4})^{k+1} - (1-\widetilde{A}_{n-\xi}^{\star 4})^{k+1}}{(1+\widetilde{A}_{n-\xi}^{\star 4})^{k+1} + (1-\widetilde{A}_{n-\xi}^{\star 4})^{k+1}}\Bigg\}_{\xi=0}^{n}\Bigg)\Bigg].
\end{aligned}
$$

Lemma 1. If $\lambda \in R^{+}$, then

$$\begin{aligned}\lambda\widetilde{A} =&\Big[\Big(\Big\{1-(1-\widetilde{A}_{\xi}^{\star 1})^{\lambda}\Big\}_{\xi=0}^{n},\Big\{1-(1-\widetilde{A}_{n-\xi}^{\star 2})^{\lambda}\Big\}_{\xi=0}^{n}\Big),\\ &\Big(\Big\{1-(1-\widetilde{A}_{\xi}^{\star 3})^{\lambda}\Big\}_{\xi=0}^{n},\Big\{1-(1-\widetilde{A}_{n-\xi}^{\star 4})^{\lambda}\Big\}_{\xi=0}^{n}\Big)\Big],\end{aligned}$$

$$\begin{aligned}\widetilde{A}^{\lambda} =&\Big[\Big(\Big\{\frac{(1+\widetilde{A}_{\xi}^{\star 1})^{\lambda}-(1-\widetilde{A}_{\xi}^{\star 1})^{\lambda}}{(1+\widetilde{A}_{\xi}^{\star 1})^{\lambda}+(1-\widetilde{A}_{\xi}^{\star 1})^{\lambda}}\Big\}_{\xi=0}^{n},\Big\{\frac{(1+\widetilde{A}_{n-\xi}^{\star 2})^{\lambda}-(1-\widetilde{A}_{n-\xi}^{\star 2})^{\lambda}}{(1+\widetilde{A}_{n-\xi}^{\star 2})^{\lambda}+(1-\widetilde{A}_{n-\xi}^{\star 2})^{\lambda}}\Big\}_{\xi=0}^{n}\Big),\\ &\Big(\Big\{\frac{(1+\widetilde{A}_{\xi}^{\star 3})^{\lambda}-(1-\widetilde{A}_{\xi}^{\star 3})^{\lambda}}{(1+\widetilde{A}_{\xi}^{\star 3})^{\lambda}+(1-\widetilde{A}_{\xi}^{\star 3})^{\lambda}}\Big\}_{\xi=0}^{n},\Big\{\frac{(1+\widetilde{A}_{n-\xi}^{\star 4})^{\lambda}-(1-\widetilde{A}_{n-\xi}^{\star 4})^{\lambda}}{(1+\widetilde{A}_{n-\xi}^{\star 4})^{\lambda}+(1-\widetilde{A}_{n-\xi}^{\star 4})^{\lambda}}\Big\}_{\xi=0}^{n}\Big)\Big].\end{aligned}$$

Theorem 4. Let $\widetilde{A}, \widetilde{B} \in n$PIVFN(R) and $\lambda, \lambda_1, \lambda_2 \in N^{+}$. The operational properties of $\widetilde{A}$ and $\widetilde{B}$ are defined as follows:

(i) $\widetilde{A} \oplus \widetilde{B} = \widetilde{B} \oplus \widetilde{A}$; (ii) $\widetilde{A} \otimes \widetilde{B} = \widetilde{B} \otimes \widetilde{A}$; (iii) $\lambda(\widetilde{A} \oplus \widetilde{B}) = \lambda\widetilde{A} \oplus \lambda\widetilde{B}$; (iv) $\widetilde{A}^{\lambda} \otimes \widetilde{B}^{\lambda} = (\widetilde{A} \otimes \widetilde{B})^{\lambda}$;

(v) $\lambda_1\widetilde{A} \oplus \lambda_2\widetilde{A} = (\lambda_1 + \lambda_2)\tilde{A}$; (vi) $(\widetilde{A})^{\lambda_1} \otimes (\widetilde{A})^{\lambda_2} = (\widetilde{A})^{\lambda_1+\lambda_2}$.

Proof. (i) and (ii) are easily proven.

(iii) According to Eqs (2) and (4), it can be deduced that

$$\begin{aligned}\lambda(\widetilde{A}\oplus\widetilde{B}) =&\Big[\Big(\Big\{1-(1-\widetilde{A}_{\xi}^{\star 1})^{\lambda}(1-\widetilde{B}_{\xi}^{\star 1})^{\lambda}\Big\}_{\xi=0}^{n},\Big\{1-(1-\widetilde{A}_{n-\xi}^{\star 2})^{\lambda}(1-\widetilde{B}_{n-\xi}^{\star 2})^{\lambda}\Big\}_{\xi=0}^{n}\Big),\\ &\Big(\Big\{1-(1-\widetilde{A}_{\xi}^{\star 3})^{\lambda}(1-\widetilde{B}_{\xi}^{\star 3})^{\lambda}\Big\}_{\xi=0}^{n},\Big\{1-(1-\widetilde{A}_{n-\xi}^{\star 4})^{\lambda}(1-\widetilde{B}_{n-\xi}^{\star 4})^{\lambda}\Big\}_{\xi=0}^{n}\Big)\Big]\\ =&\lambda\widetilde{A}\oplus\lambda\widetilde{B}.\end{aligned}$$

(iv) By Eqs (3) and (5), the following can be obtained:

$$\begin{aligned}\widetilde{A}^{\lambda}\otimes\widetilde{B}^{\lambda} =&\Big[\Big(\Big\{\frac{(1+\widetilde{A}_{\xi}^{\star 1})^{\lambda}(1+\widetilde{B}_{\xi}^{\star 1})^{\lambda}-(1-\widetilde{A}_{\xi}^{\star 1})^{\lambda}(1-\widetilde{B}_{\xi}^{\star 1})^{\lambda}}{(1+\widetilde{A}_{\xi}^{\star 1})^{\lambda}(1+\widetilde{B}_{\xi}^{\star 1})^{\lambda}+(1-\widetilde{A}_{\xi}^{\star 1})^{\lambda}(1-\widetilde{B}_{\xi}^{\star 1})^{\lambda}}\Big\}_{\xi=0}^{n},\\ &\Big\{\frac{(1+\widetilde{A}_{n-\xi}^{\star 2})^{\lambda}(1+\widetilde{B}_{n-\xi}^{\star 2})^{\lambda}-(1-\widetilde{A}_{n-\xi}^{\star 2})^{\lambda}(1-\widetilde{B}_{n-\xi}^{\star 2})^{\lambda}}{(1+\widetilde{A}_{n-\xi}^{\star 2})^{\lambda}(1+\widetilde{B}_{n-\xi}^{\star 2})^{\lambda}+(1-\widetilde{A}_{n-\xi}^{\star 2})^{\lambda}(1-\widetilde{B}_{n-\xi}^{\star 2})^{\lambda}}\Big\}_{\xi=0}^{n}\Big),\\ &\Big(\Big\{\frac{(1+\widetilde{A}_{\xi}^{\star 3})^{\lambda}(1+\widetilde{B}_{\xi}^{\star 3})^{\lambda}-(1-\widetilde{A}_{\xi}^{\star 3})^{\lambda}(1-\widetilde{B}_{\xi}^{\star 3})^{\lambda}}{(1+\widetilde{A}_{\xi}^{\star 3})^{\lambda}(1+\widetilde{B}_{\xi}^{\star 3})^{\lambda}+(1-\widetilde{A}_{\xi}^{\star 3})^{\lambda}(1-\widetilde{B}_{\xi}^{\star 3})^{\lambda}}\Big\}_{\xi=0}^{n},\\ &\Big\{\frac{(1+\widetilde{A}_{n-\xi}^{\star 4})^{\lambda}(1+\widetilde{B}_{n-\xi}^{\star 4})^{\lambda}-(1-\widetilde{A}_{n-\xi}^{\star 4})^{\lambda}(1-\widetilde{B}_{n-\xi}^{\star 4})^{\lambda}}{(1+\widetilde{A}_{n-\xi}^{\star 4})^{\lambda}(1+\widetilde{B}_{n-\xi}^{\star 4})^{\lambda}+(1-\widetilde{A}_{n-\xi}^{\star 4})^{\lambda}(1-\widetilde{B}_{n-\xi}^{\star 4})^{\lambda}}\Big\}_{\xi=0}^{n}\Big)\Big]\\ =&(\widetilde{A}\otimes\widetilde{B})^{\lambda}.\end{aligned}$$

From Eqs (4) and (5), (v) and (vi) are clearly established.

Subsequently, we propose the nPIVFNs' aggregation operators in accordance with the operational laws in **Definition 4**.

3.2. Weighted arithmetic averaging aggregation operators for nPIVFNs

In this section, n-polygonal interval-valued arithmetic averaging operators and their properties are presented. For the sake of simplicity, the unification of $\widetilde{A}_i (i = 1, 2, \ldots, m) \in n$PIVFN(R) is denoted as

$$\begin{aligned}\widetilde{A}_i =& \left[\left(\left\{\widetilde{A}^{\star 1}_{i\xi}\right\}^n_{\xi=0}, \left\{\widetilde{A}^{\star 2}_{i(n-\xi)}\right\}^n_{\xi=0}\right), \left(\left\{\widetilde{A}^{\star 3}_{i\xi}\right\}^n_{\xi=0}, \left\{\widetilde{A}^{\star 4}_{i(n-\xi)}\right\}^n_{\xi=0}\right)\right]\\ =& [(a^1_{i0}, \ldots, a^1_{in}, a^2_{in}, \ldots, a^2_{i0}), (a^3_{i0}, \ldots, a^3_{in}, a^4_{in}, \ldots, a^4_{i0})].\end{aligned}$$

Definition 5. Suppose that $(\widetilde{A}_1, \widetilde{A}_2, \ldots, \widetilde{A}_m) \in n\text{PIVFN(R)}^m$. In this case, a function *nPIVFWAA*: $n\text{PIVFN(R)}^m \rightarrow n$PIVFN(R) is called an n-polygonal interval-valued fuzzy weighted arithmetic averaging (nPIVFWAA) operator, which is defined as

$$nPIVFWAA(\widetilde{A}_1, \widetilde{A}_2, \ldots, \widetilde{A}_m) = \theta_1(\widetilde{A}_1 + \widetilde{\epsilon}) \oplus \theta_2(\widetilde{A}_2 + \widetilde{\epsilon}) \oplus \cdots \oplus \theta_m(\widetilde{A}_m + \widetilde{\epsilon}),$$

where $\widetilde{\epsilon} = [(\epsilon, \ldots, \epsilon, \epsilon, \ldots, \epsilon), (\epsilon, \ldots, \epsilon, \epsilon, \ldots, \epsilon)]$, ϵ is an arbitrarily small real number, θ_i is the weight of $\widetilde{A}_i$ with $0 \le \theta_i \le 1$, and $\sum_{i=1}^m \theta_i = 1$.

Theorem 5. If we let $(\widetilde{A}_1, \widetilde{A}_2, \ldots, \widetilde{A}_m) \in n\text{PIVFN(R)}^m$, then the nPIVFWAA operator of nPIVFNs has the following formula:

$$\begin{aligned}&nPIVWAA(\widetilde{A}_1, \widetilde{A}_2, \ldots, \widetilde{A}_m)\\ =& \left[\left(\left\{1 - \prod_{i=1}^m (1 - (\widetilde{A}^{\star 1}_{i\xi} + \epsilon))^{\theta_i}\right\}^n_{\xi=0}, \left\{1 - \prod_{i=1}^m (1 - (\widetilde{A}^{\star 2}_{i(n-\xi)} + \epsilon))^{\theta_i}\right\}^n_{\xi=0}\right),\right.\\ & \left.\left\{1 - \prod_{i=1}^m (1 - (\widetilde{A}^{\star 3}_{i\xi} + \epsilon))^{\theta_i}\right\}^n_{\xi=0}, \left\{1 - \prod_{i=1}^m (1 - (\widetilde{A}^{\star 4}_{i(n-\xi)} + \epsilon))^{\theta_i}\right\}^n_{\xi=0}\right)\Bigg],\end{aligned} \tag{3.5}$$

where $0 \le \theta_i \le 1$ and $\sum_{i=1}^m \theta_i = 1$.

Proof. On the basis of Eqs (2) and (4), we can obtain

$$\begin{aligned}&\theta_1(\widetilde{A}_1 + \widetilde{\epsilon}) \oplus \theta_2(\widetilde{A}_2 + \epsilon) \oplus \cdots \oplus \theta_m(\widetilde{A}_m + \epsilon)\\ =& \left[\left(\left\{1 - (1 - (\widetilde{A}^{\star 1}_{1\xi} + \epsilon))^{\theta_1}(1 - (\widetilde{A}^{\star 1}_{2\xi} + \epsilon))^{\theta_2}\right\}^n_{\xi=0},\right.\right.\\ & \left.\left\{1 - (1 - (\widetilde{A}^{\star 2}_{1(n-\xi)} + \epsilon))^{\theta_1}(1 - (\widetilde{A}^{\star 2}_{2(n-\xi)} + \epsilon))^{\theta_2}\right\}^n_{\xi=0}\right),\\ & \left(\left\{1 - (1 - (\widetilde{A}^{\star 3}_{1\xi} + \epsilon))^{\theta_1}(1 - (\widetilde{A}^{\star 3}_{2\xi} + \epsilon))^{\theta_2}\right\}^n_{\xi=0},\right.\\ & \left.\left.\left\{1 - (1 - (\widetilde{A}^{\star 4}_{1(n-\xi)} + \epsilon))^{\theta_1}(1 - (\widetilde{A}^{\star 4}_{2(n-\xi)} + \epsilon))^{\theta_2}\right\}^n_{\xi=0}\right)\right]\\ & \oplus \cdots \oplus \left[\left(\left\{1 - (1 - (\widetilde{A}^{\star 1}_{m\xi} + \epsilon))^{\theta_m}\right\}^n_{\xi=0}, \left\{1 - (1 - (\widetilde{A}^{\star 2}_{m(n-\xi)} + \epsilon))^{\theta_m}\right\}^n_{\xi=0}\right),\right.\\ & \left.\left(\left\{1 - (1 - (\widetilde{A}^{\star 3}_{m\xi} + \epsilon))^{\theta_m}\right\}^n_{\xi=0}, \left\{1 - (1 - (\widetilde{A}^{\star 4}_{m(n-\xi)} + \epsilon))^{\theta_m}\right\}^n_{\xi=0}\right)\right]\\ =& \left[\left(\left\{1 - \prod_{i=1}^m (1 - (\widetilde{A}^{\star 1}_{i\xi} + \epsilon))^{\theta_i}\right\}^n_{\xi=0}, \left\{1 - \prod_{i=1}^m (1 - (\widetilde{A}^{\star 2}_{i(n-\xi)} + \epsilon))^{\theta_i}\right\}^n_{\xi=0}\right),\right.\\ & \left.\left(\left\{1 - \prod_{i=1}^m (1 - \widetilde{A}^{\star 3}_{i\xi} + \epsilon))^{\theta_i}\right\}^n_{\xi=0}, \left\{1 - \prod_{i=1}^m (1 - (\widetilde{A}^{\star 4}_{i(n-\xi)} + \epsilon))^{\theta_i}\right\}^n_{\xi=0}\right)\right].\end{aligned}$$

The proposed nPIVFWAA operator avoids the occurrence of meaningless values such as 0^0 in this paper.

Theorem 6. If we let $\widetilde{A}_i, \widetilde{A}_i^*(i = 1, 2, \ldots, m) \in n$PIVFN(R), the nPIVFWAA operator possesses following properties:
(i) Boundedness: $\widetilde{0} \subseteq nPIVFWAA(\widetilde{A}_1, \widetilde{A}_2, \ldots, \widetilde{A}_m) \subseteq \widetilde{1}$;
(ii) Monotonicity: If $\widetilde{A}_i^* \subseteq \widetilde{A}_i$, i.e., $[a_\xi^{*1}, a_\xi^{*2}] \subseteq [a_\xi^1, a_\xi^2]$ and $[a_\xi^{*3}, a_\xi^{*4}] \subseteq [a_\xi^3, a_\xi^4]$, then

$$nPIVFWAA(\widetilde{A}_1^*, \widetilde{A}_2^*, \ldots, \widetilde{A}_m^*) \subseteq nPIVFWAA(\widetilde{A}_1, \widetilde{A}_2, \ldots, \widetilde{A}_m).$$

Proof. (i) It is easy to know that

$$1 - \prod_{i=1}^{m} (1 - (\widetilde{A}_{i\xi}^{\star z} - \epsilon))^{\theta_i} \in [0, 1], \xi = 0, 1, \ldots, n, z = 1, 2, 3, 4.$$

It can be deduced that

$$\widetilde{0} \subseteq nPIVFWAA(\widetilde{A}_1, \widetilde{A}_2, \ldots, \widetilde{A}_m) \subseteq \widetilde{1}.$$

(ii) If $\widetilde{A}_i^* \subseteq \widetilde{A}_i$, i.e. $\widetilde{A}_{i\xi}^{\star 1} \leq \widetilde{A}_{i\xi}^{*\star 1} \leq \widetilde{A}_{i\xi}^{*\star 2} \leq \widetilde{A}_{i\xi}^{\star 2}$ and $\widetilde{A}_{i\xi}^{\star 3} \leq \widetilde{A}_{i\xi}^{*\star 3} \leq \widetilde{A}_{i\xi}^{*\star 4} \leq \widetilde{A}_{i\xi}^{\star 4}$, then it is easy to prove that

$$\begin{aligned} &1 - \prod_{i=1}^{m}(1 - \widetilde{A}_{i\xi}^{\star 1})^{\theta_i} \leq 1 - \prod_{i=1}^{m}(1 - \widetilde{A}_{i\xi}^{*\star 1})^{\theta_i} \leq 1 - \prod_{i=1}^{m}(1 - \widetilde{A}_{i\xi}^{*\star 2})^{\theta_i} \leq 1 - \prod_{i=1}^{m}(1 - \widetilde{A}_{i\xi}^{\star 2})^{\theta_i}, \\ &1 - \prod_{i=1}^{m}(1 - \widetilde{A}_{i\xi}^{\star 3})^{\theta_i} \leq 1 - \prod_{i=1}^{m}(1 - \widetilde{A}_{i\xi}^{*\star 3})^{\theta_i} \leq 1 - \prod_{i=1}^{m}(1 - \widetilde{A}_{i\xi}^{*\star 4})^{\theta_i} \leq 1 - \prod_{i=1}^{m}(1 - \widetilde{A}_{i\xi}^{\star 4})^{\theta_i}. \end{aligned} \tag{3.6}$$

According to **Theorem 1**, Eq (7) can be derived as follows:

$$nPIVFWAA(\widetilde{A}_1^*, \widetilde{A}_2^*, \ldots, \widetilde{A}_m^*) \subseteq nPIVFWAA(\widetilde{A}_1, \widetilde{A}_2, \ldots, \widetilde{A}_m).$$

Definition 6. If we let $(\widetilde{A}_1, \widetilde{A}_2, \ldots, \widetilde{A}_m) \in n$PIVFN(R)^m, then an n-polygonal interval-valued fuzzy ordered weighted arithmetic averaging (nPIVFOWAA) operator is a mapping *nPIVFOWAA*: nPIVFN(R)$^m \to n$PIVFN(R)

$$nPIVFOWAA(\widetilde{A}_1, \widetilde{A}_2, \ldots, \widetilde{A}_m) = \zeta_1(\widetilde{A}_{\sigma_{(1)}} + \widetilde{\epsilon}) \oplus \zeta_2(\widetilde{A}_{\sigma_{(2)}} + \widetilde{\epsilon}) \oplus \cdots \oplus \zeta_m(\widetilde{A}_{\sigma_{(m)}} + \widetilde{\epsilon}),$$

where $\widetilde{\epsilon} = [(\epsilon, \ldots, \epsilon, \epsilon, \ldots, \epsilon), (\epsilon, \ldots, \epsilon, \epsilon, \ldots, \epsilon)]$, ϵ is an arbitrarily small real number, and ζ_i is the corresponding weight of $\widetilde{A}_{\sigma(i)}$ with $0 \leq \zeta_i \leq 1, \sum_{i=1}^m \zeta_i = 1$. There is a permutation function σ : $\{1, 2, \ldots, m\} \to \{1, 2, \ldots, m\}$, such that $\widetilde{A}_{\sigma(r)}(r = 1, 2, \ldots, m)$ is the rth largest of all $\widetilde{A}_i(i = 1, 2, \ldots, m)$.

Theorem 7. If we let $(\widetilde{A}_1, \widetilde{A}_2, \ldots, \widetilde{A}_m) \in n$PIVFN(R)^m, then the nPIVFOWAA operator of nPIVFNs has the following formula:

$$\begin{aligned} &nPIVFOWAA(\widetilde{A}_1, \widetilde{A}_2, \ldots, \widetilde{A}_m) \\ =&\Bigg[\Bigg(\Big\{1 - \prod_{i=1}^{m} (1 - (\widetilde{A}_{\sigma(i)\xi}^{\star 1} + \epsilon))^{\zeta_i}\Big\}_{\xi=0}^{n}, \Big\{1 - \prod_{i=1}^{m} (1 - (\widetilde{A}_{\sigma(i)(n-\xi)}^{\star 2} + \epsilon))^{\zeta_i}\Big\}_{\xi=0}^{n}\Bigg), \\ &\Big\{1 - \prod_{i=1}^{m} (1 - (\widetilde{A}_{\sigma(i)\xi}^{\star 3} + \epsilon))^{\zeta_i}\Big\}_{\xi=0}^{n}, \Big\{1 - \prod_{i=1}^{m} (1 - (\widetilde{A}_{\sigma(i)(n-\xi)}^{\star 4} + \epsilon))^{\zeta_i}\Big\}_{\xi=0}^{n}\Bigg)\Bigg], \end{aligned} \tag{3.7}$$

where ζ_i is the homologous weight of $\widetilde{A}_{\sigma(i)}$ with $0 \leq \zeta_i \leq 1$ and $\sum_{i=1}^m \zeta_i = 1$.

Theorem 8. Let $\widetilde{A}_i, \widetilde{A}_i^*(i = 1, 2, \ldots, m) \in n$PIVFN(R), the nPIVFOWAA operator contains following properties:
(i) Boundedness: $\widetilde{0} \subseteq nPIVFOWAA(\widetilde{A}_1, \widetilde{A}_2, \ldots, \widetilde{A}_m) \subseteq \tilde{1}$;
(ii) Monotonicity: If $\widetilde{A}_i^* \subseteq \widetilde{A}_i$ (that is, $[a_\xi^{*1}, a_\xi^{*2}] \subseteq [a_\xi^1, a_\xi^2]$ and $[a_\xi^{*3}, a_\xi^{*4}] \subseteq [a_\xi^3, a_\xi^4]$), then

$$nPIVFOWAA(\widetilde{A}_1^*, \widetilde{A}_2^*, \ldots, \widetilde{A}_m^*) \subseteq nPIVFOWAA(\widetilde{A}_1, \widetilde{A}_2, \ldots, \widetilde{A}_m).$$

Definition 7. Let $(\widetilde{A}_1, \widetilde{A}_2, \ldots, \widetilde{A}_m) \in n\text{PIVFN(R)}^m$ and $\overline{A}_i = m\theta_i\widetilde{A}_i(\theta_i$ is the weight of $\widetilde{A}_i)$, then a function *nPIVFHAA*: $n\text{PIVFN(R)}^m \rightarrow n$PIVFN(R) is called an n-polygonal interval-valued fuzzy hybrid arithmetic averaging (nPIVFHAA) operator

$$nPIVFHAA(\widetilde{A}_1, \widetilde{A}_2, \ldots, \widetilde{A}_m) = \eta_1(\overline{A}_{\sigma(1)} + \widetilde{\epsilon}) \otimes \eta_2(\overline{A}_{\sigma(2)} + \widetilde{\epsilon}) \otimes \cdots \otimes \eta_m(\overline{A}_{\sigma(m)} + \widetilde{\epsilon}),$$

where $\widetilde{\epsilon} = [(\epsilon, \ldots, \epsilon, \epsilon, \ldots, \epsilon), (\epsilon, \ldots, \epsilon, \epsilon, \ldots, \epsilon)]$ and ϵ is an arbitrarily small real number. There is a mapping $\sigma : \{1, 2, \ldots, m\} \rightarrow \{1, 2, \ldots, m\}$, such that $\overline{A}_{\sigma(r)}(r = 1, 2, \ldots, m)$ is the rth largest of all $\overline{A}_i(i = 1, 2, \ldots, m)$. Moreover, η_i is the weight of $\overline{A}_{\sigma(i)}$, satisfying $0 \leq \eta_i \leq 0$ and $\sum_{i=1}^m \eta_i = 1$.

Theorem 9. If we let $(\widetilde{A}_1, \widetilde{A}_2, \ldots, \widetilde{A}_m) \in n\text{PIVFN(R)}^m$, the nPIVFHAA operator of nPIVFNs can be expressed as follows:

$$\begin{aligned} &nPIVFHAA(\widetilde{A}_1, \widetilde{A}_2, \ldots, \widetilde{A}_m) \\ =&\Bigg[\Bigg(\Big\{1 - \prod_{i=1}^m (1 - (\overline{A}^{\star 1}_{\sigma(i)\xi} + \epsilon))^{\eta_i}\Big\}_{\xi=0}^n, \Big\{1 - \prod_{i=1}^m (1 - (\overline{A}^{\star 2}_{\sigma(i)(n-\xi)} + \epsilon))^{\eta_i}\Big\}_{\xi=0}^n\Bigg), \\ &\Bigg(\Big\{1 - \prod_{i=1}^m (1 - (\overline{A}^{\star 3}_{\sigma(i)\xi} + \epsilon))^{\eta_i}\Big\}_{\xi=0}^n, \Big\{1 - \prod_{i=1}^m (1 - (\overline{A}^{\star 4}_{\sigma(i)(n-\xi)} + \epsilon))^{\eta_i}\Big\}_{\xi=0}^n\Bigg)\Bigg], \end{aligned} \tag{3.8}$$

where η_i is the weight of $\overline{A}_{\sigma(i)}$ with $0 \leq \eta_i \leq 1$, and $\sum_{i=1}^m \eta_i = 1$.

The nPIVFHAA operator has the same properties as **Theorem 5**, so the specific content will not be repeated.

3.3. Weighted geometric averaging aggregation operators for nPIVFNs

The geometric mean operator is not affected by outliers in the dataset and enhances the elasticity of extreme values, thus more accurately reflecting the overall state of the data. Due to the advantage of the geometric mean operator, this section describes the aggregation operator of nPIVFN and studies their excellent properties.

Definition 8. If we let $(\widetilde{A}_1, \widetilde{A}_2, \ldots, \widetilde{A}_m) \in n\text{PIVFN(R)}^m$, then an n-polygonal interval-valued fuzzy weighted geometric averaging (nPIVFWGA) operator is a mapping *nPIVFWGA*: $n\text{PIVFN(R)}^m \rightarrow$ nPIVFN(R),

$$nPIVFWGA(\widetilde{A}_1, \widetilde{A}_2, \ldots, \widetilde{A}_m) = (\widetilde{A}_1 + \widetilde{\epsilon})^{w_1} \otimes (\widetilde{A}_2 + \widetilde{\epsilon})^{w_2} \otimes \cdots \otimes (\widetilde{A}_m + \widetilde{\epsilon})^{w_m},$$

where $\widetilde{\epsilon} = [(\epsilon, \ldots, \epsilon, \epsilon, \ldots, \epsilon), (\epsilon, \ldots, \epsilon, \epsilon, \ldots, \epsilon)]$, ϵ is an arbitrarily small real number, and w_i is the corresponding weight of $\widetilde{A}_i$ with $0 \leq w_i \leq 1$ and $\sum_{i=1}^m w_i = 1$.

Theorem 10. If we let $(\widetilde{A}_1, \widetilde{A}_2, \ldots, \widetilde{A}_m) \in n$PIVFN(R)^m, then the nPIVFWGA operator of nPIVFNs has the following formula:

$$
\begin{aligned}
&nPIVFWGA(\widetilde{A}_1, \widetilde{A}_2, \ldots, \widetilde{A}_m)\\
=&\Big[\Big(\Big\{\frac{\prod_{i=1}^{m}(1+(\widetilde{A}^{\star 1}_{i\xi}+\epsilon))^{w_i}-\prod_{i=1}^{m}(1-(\widetilde{A}^{\star 1}_{i\xi}+\epsilon))^{w_i}}{\prod_{i=1}^{m}(1+(\widetilde{A}^{\star 1}_{i\xi}+\epsilon))^{w_i}+\prod_{i=1}^{m}(1-(\widetilde{A}^{\star 1}_{i\xi}+\epsilon))^{w_i}}\Big\}_{\xi=0}^{n},\\
&\Big\{\frac{\prod_{i=1}^{m}(1+(\widetilde{A}^{\star 2}_{i(n-\xi)}+\epsilon))^{w_i}-\prod_{i=1}^{m}(1-(\widetilde{A}^{\star 2}_{i(n-\xi)}+\epsilon))^{w_i}}{\prod_{i=1}^{m}(1+(\widetilde{A}^{\star 2}_{i(n-\xi)}+\epsilon))^{w_i}+\prod_{i=1}^{m}(1-(\widetilde{A}^{\star 2}_{i(n-\xi)}+\epsilon))^{w_i}}\Big\}_{\xi=0}^{n}\Big),\\
&\Big(\Big\{\frac{\prod_{i=1}^{m}(1+(\widetilde{A}^{\star 3}_{i\xi}+\epsilon))^{w_i}-\prod_{i=1}^{m}(1-(\widetilde{A}^{\star 3}_{i\xi}+\widetilde{\epsilon}))^{w_i}}{\prod_{i=1}^{m}(1+(\widetilde{A}^{\star 3}_{i\xi}+\epsilon))^{w_i}+\prod_{i=1}^{m}(1-(\widetilde{A}^{\star 3}_{i\xi}+\epsilon))^{w_i}}\Big\}_{\xi=0}^{n},\\
&\Big\{\frac{\prod_{i=1}^{m}(1+(\widetilde{A}^{\star 4}_{i(n-\xi)}+\epsilon))^{w_i}-\prod_{i=1}^{m}(1-(\widetilde{A}^{\star 4}_{i(n-\xi)}+\epsilon))^{w_i}}{\prod_{i=1}^{m}(1+(\widetilde{A}^{\star 4}_{i(n-\xi)}+\epsilon))^{w_i}+\prod_{i=1}^{m}(1-(\widetilde{A}^{\star 4}_{i(n-\xi)}+\epsilon))^{w_i}}\Big\}_{\xi=0}^{n}\Big)\Big],
\end{aligned}
\tag{3.9}
$$

where $0 \le w_i \le 1$ and $\sum_{i=1}^{m} w_i = 1$.

Proof. By Eqs (3) and (5), it can be found that

$$
\begin{aligned}
&\widetilde{A}_1^{w_1} \otimes \widetilde{A}_2^{w_2} \otimes \cdots \otimes \widetilde{A}_m^{w_m}\\
=&\Big[\Big(\Big\{\frac{(1+(\widetilde{A}^{\star 1}_{1\xi}+\epsilon))^{w_1}(1+(\widetilde{A}^{\star 1}_{2\xi}+\epsilon))^{w_2}-(1-(\widetilde{A}^{\star 1}_{1\xi}+\epsilon))^{w_1}(1-(\widetilde{A}^{\star 1}_{2\xi}+\epsilon))^{w_2}}{(1+(\widetilde{A}^{\star 1}_{1\xi}+\epsilon))^{w_1}(1+(\widetilde{A}^{\star 1}_{2\xi}+\epsilon))^{w_2}+(1-(\widetilde{A}^{\star 1}_{1\xi}+\epsilon))^{w_1}(1-(\widetilde{A}^{\star 1}_{2\xi}+\epsilon))^{w_2}}\Big\}_{\xi=0}^{n},\\
&\Big\{\frac{(1+(\widetilde{A}^{\star 2}_{1(n-\xi)}+\epsilon))^{w_1}(1+(\widetilde{A}^{\star 2}_{2(n-\xi)}+\epsilon))^{w_2}-(1-(\widetilde{A}^{\star 2}_{1(n-\xi)}+\epsilon))^{w_1}(1-(\widetilde{A}^{\star 2}_{2(n-\xi)}+\epsilon))^{w_2}}{(1+(\widetilde{A}^{\star 2}_{1(n-\xi)}+\epsilon))^{w_1}(1+(\widetilde{A}^{\star 2}_{2(n-\xi)}+\epsilon))^{w_2}+(1-(\widetilde{A}^{\star 2}_{1(n-\xi)}+\epsilon))^{w_1}(1-(\widetilde{A}^{\star 2}_{2(n-\xi)}+\epsilon))^{w_2}}\Big\}_{\xi=0}^{n}\Big),\\
&\Big(\Big\{\frac{(1+(\widetilde{A}^{\star 3}_{1\xi}+\epsilon))^{w_1}(1+(\widetilde{A}^{\star 3}_{2\xi}+\epsilon))^{\theta_2}-(1-(\widetilde{A}^{\star 3}_{1\xi}+\epsilon))^{w_1}(1-(\widetilde{A}^{\star 3}_{2\xi}+\epsilon))^{\theta_2}}{(1+(\widetilde{A}^{\star 3}_{1\xi}+\epsilon))^{w_1}(1+(\widetilde{A}^{\star 3}_{2\xi}+\epsilon))^{\theta_2}+(1-(\widetilde{A}^{\star 3}_{1\xi}+\epsilon))^{w_1}(1-(\widetilde{A}^{\star 3}_{2\xi}+\epsilon))^{\theta_2}}\Big\}_{\xi=0}^{n},\\
&\Big\{\frac{(1+(\widetilde{A}^{\star 4}_{1(n-\xi)}+\epsilon))^{w_1}(1+(\widetilde{A}^{\star 4}_{2(n-\xi)}+\epsilon))^{\theta_2}-(1-(\widetilde{A}^{\star 4}_{1(n-\xi)}+\epsilon))^{w_1}(1-(\widetilde{A}^{\star 4}_{2(n-\xi)}+\epsilon))^{\theta_2}}{(1+(\widetilde{A}^{\star 4}_{1(n-\xi)}+\epsilon))^{w_1}(1+(\widetilde{A}^{\star 4}_{2(n-\xi)}+\epsilon))^{\theta_2}+(1-(\widetilde{A}^{\star 4}_{1(n-\xi)}+\epsilon))^{w_1}(1-(\widetilde{A}^{\star 4}_{2(n-\xi)}+\epsilon))^{\theta_2}}\Big\}_{\xi=0}^{n}\Big)\Big] \oplus \cdots \oplus\\
&\Big[\Big(\Big(\Big(\Big\{\frac{(1+(\widetilde{A}^{\star 1}_{m\xi}+\epsilon))^{w_m}-(1-(\widetilde{A}^{\star 1}_{m\xi}+\epsilon))^{w_m}}{(1+(\widetilde{A}^{\star 1}_{m\xi}+\epsilon))^{w_m}+(1-(\widetilde{A}^{\star 1}_{m\xi}+\epsilon))^{w_m}}\Big\}_{\xi=0}^{n},\Big\{\frac{(1+(\widetilde{A}^{\star 2}_{m(n-\xi)}+\epsilon))^{w_m}-(1-(\widetilde{A}^{\star 2}_{m(n-\xi)}+\epsilon))^{w_m}}{(1+(\widetilde{A}^{\star 2}_{m(n-\xi)}+\epsilon))^{w_m}+(1-(\widetilde{A}^{\star 2}_{m(n-\xi)}+\epsilon))^{w_m}}\Big\}_{\xi=0}^{n}\Big),\\
&\Big(\Big\{\frac{(1+(\widetilde{A}^{\star 3}_{m\xi}+\epsilon))^{w_m}-(1-(\widetilde{A}^{\star 3}_{m\xi}+\epsilon))^{w_m}}{(1+(\widetilde{A}^{\star 3}_{m\xi}+\epsilon))^{w_m}+(1-(\widetilde{A}^{\star 3}_{m\xi}+\epsilon))^{w_m}}\Big\}_{\xi=0}^{n},\Big\{\frac{(1+(\widetilde{A}^{\star 4}_{m(n-\xi)}+\epsilon))^{w_m}-(1-(\widetilde{A}^{\star 4}_{m(n-\xi)}+\epsilon))^{w_m}}{(1+(\widetilde{A}^{\star 4}_{m(n-\xi)}+\epsilon))^{w_m}+(1-(\widetilde{A}^{\star 4}_{m(n-\xi)}+\epsilon))^{w_m}}\Big\}_{\xi=0}^{n}\Big)\Big]\\
=&\Big[\Big(\Big\{\frac{\prod_{i=1}^{m}(1+(\widetilde{A}^{\star 1}_{i\xi}+\epsilon))^{w_i}-\prod_{i=1}^{m}(1-(\widetilde{A}^{\star 1}_{i\xi}+\epsilon))^{w_i}}{\prod_{i=1}^{m}(1+(\widetilde{A}^{\star 1}_{i\xi}+\epsilon))^{w_i}+\prod_{i=1}^{m}(1-(\widetilde{A}^{\star 1}_{i\xi}+\epsilon))^{w_i}}\Big\}_{\xi=0}^{n},\\
&\Big\{\frac{\prod_{i=1}^{m}(1+(\widetilde{A}^{\star 2}_{i(n-\xi)}+\epsilon))^{w_i}-\prod_{i=1}^{m}(1-(\widetilde{A}^{\star 2}_{i(n-\xi)}+\epsilon))^{w_i}}{\prod_{i=1}^{m}(1+(\widetilde{A}^{\star 2}_{i(n-\xi)}+\epsilon))^{w_i}+\prod_{i=1}^{m}(1-(\widetilde{A}^{\star 2}_{i(n-\xi)}+\epsilon))^{w_i}}\Big\}_{\xi=0}^{n}\Big),\\
&\Big(\Big\{\frac{\prod_{i=1}^{m}(1+(\widetilde{A}^{\star 3}_{i\xi}+\epsilon))^{w_i}-\prod_{i=1}^{m}(1-(\widetilde{A}^{\star 3}_{i\xi}+\widetilde{\epsilon}))^{w_i}}{\prod_{i=1}^{m}(1+(\widetilde{A}^{\star 3}_{i\xi}+\epsilon))^{w_i}+\prod_{i=1}^{m}(1-(\widetilde{A}^{\star 3}_{i\xi}+\epsilon))^{w_i}}\Big\}_{\xi=0}^{n},\\
&\Big\{\frac{\prod_{i=1}^{m}(1+(\widetilde{A}^{\star 4}_{i(n-\xi)}+\epsilon))^{w_i}-\prod_{i=1}^{m}(1-(\widetilde{A}^{\star 4}_{i(n-\xi)}+\epsilon))^{w_i}}{\prod_{i=1}^{m}(1+(\widetilde{A}^{\star 4}_{i(n-\xi)}+\epsilon))^{w_i}+\prod_{i=1}^{m}(1-(\widetilde{A}^{\star 4}_{i(n-\xi)}+\epsilon))^{w_i}}\Big\}_{\xi=0}^{n}\Big)\Big].
\end{aligned}
$$

The proposed nPIFWGA operator also effectively prevents the failure form of 0^0 in this paper.

Theorem 11. Let $\widetilde{A}_i, \widetilde{A}_i^*(i = 1, \ldots, m) \in n\text{PIVFN(R)}$. The following properties of the nPIVFWGA operator are easily verified.
(i) Boundedness: $\widetilde{0} \subseteq nPIVFWGA(\widetilde{A}_1, \widetilde{A}_2, \ldots, \widetilde{A}_m) \subseteq \widetilde{1}$;
(ii) Monotonicity: If $\widetilde{A}_i^* \subseteq \widetilde{A}_i$(that is, $[a_\xi^{*1}, a_\xi^{*2}] \subseteq [a_\xi^1, a_\xi^2]$ and $[a_\xi^{*3}, a_\xi^{*4}] \subseteq [a_\xi^3, a_\xi^4]$), then

$$nPIVFWGA(\widetilde{A}_1^*, \widetilde{A}_2^*, \ldots, \widetilde{A}_m^*) \subseteq nPIVFWGA(\widetilde{A}_1, \widetilde{A}_2, \ldots, \widetilde{A}_m).$$

Proof. (i) For any $(\widetilde{A}_{i\xi}^{\star z} + \epsilon) \in [0, 1](i = 0, 1, \ldots, m, \xi = 0, 1, \ldots, n, z = 1, 2, 3, 4)$, we have

$$1 + (\widetilde{A}_{i\xi}^{\star z} + \epsilon) \geq 1 - (\widetilde{A}_{i\xi}^{\star z} + \epsilon) \geq 0.$$

Since $w_i \geq 0$, it follows that

$$(1 + (\widetilde{A}_{i\xi}^{\star z} + \epsilon))^{w_i} \geq (1 - (\widetilde{A}_{i\xi}^{\star z} + \epsilon))^{w_i}.$$

Therefore

$$\prod_{i=1}^{m} (1 + (\widetilde{A}_{i\xi}^{\star z} + \epsilon))^{w_i} \geq \prod_{i=1}^{m} (1 - (\widetilde{A}_{i\xi}^{\star z} + \epsilon))^{w_i} \geq 0.$$

It can be deduced that

$$(1 + (\widetilde{A}_{i\xi}^{\star z} + \epsilon))^{w_i} + \prod_{i=1}^{m} (1 - (\widetilde{A}_{i\xi}^{\star z} + \epsilon))^{w_i} \geq 0,$$

$$(1 + (\widetilde{A}_{i\xi}^{\star z} + \epsilon))^{w_i} - \prod_{i=1}^{m} (1 - (\widetilde{A}_{i\xi}^{\star z} + \epsilon))^{w_i} > 0.$$

Thus

$$\frac{\prod_{i=1}^{m} (1 + (\widetilde{A}_{i\xi}^{\star z} + \epsilon))^{w_i} - \prod_{i=1}^{m} (1 - (\widetilde{A}_{i\xi}^{\star z} + \epsilon))^{w_i}}{\prod_{i=1}^{m} (1 + (\widetilde{A}_{i\xi}^{\star z} + \epsilon))^{w_i} + \prod_{i=1}^{m} (1 - (\widetilde{A}_{i\xi}^{\star z} + \epsilon))^{w_i}} > 0.$$

We show that

$$\frac{\prod_{i=1}^{m} (1 + (\widetilde{A}_{i\xi}^{\star z} + \epsilon))^{w_i} - \prod_{i=1}^{m} (1 - (\widetilde{A}_{i\xi}^{\star z} + \epsilon))^{w_i}}{\prod_{i=1}^{m} (1 + (\widetilde{A}_{i\xi}^{\star z} + \epsilon))^{w_i} + \prod_{i=1}^{m} (1 - (\widetilde{A}_{i\xi}^{\star z} + \epsilon))^{w_i}} \leq 1.$$

This inequality is equivalent to

$$(1 + (\widetilde{A}_{i\xi}^{\star z} + \epsilon))^{w_i} - \prod_{i=1}^{m} (1 - (\widetilde{A}_{i\xi}^{\star z} + \epsilon))^{w_i} \leq (1 + (\widetilde{A}_{i\xi}^{\star z} + \epsilon))^{w_i} + \prod_{i=1}^{m} (1 - (\widetilde{A}_{i\xi}^{\star z} + \epsilon))^{w_i},$$

which simplifies to

$$2 \prod_{i=1}^{m} (1 - (\widetilde{A}_{i\xi}^{\star z} + \epsilon))^{w_i} \geq 0.$$

Since $\prod_{i=1}^{m} (1 - (\widetilde{A}_{i\xi}^{\star z} + \epsilon))^{w_i} \geq 0$, the inequality holds.
Therefore, it can be concluded that

$$0 \leq \frac{\prod_{i=1}^{m} (1 + (\widetilde{A}_{i\xi}^{\star z} + \epsilon))^{w_i} - \prod_{i=1}^{m} (1 - (\widetilde{A}_{i\xi}^{\star z} + \epsilon))^{w_i}}{\prod_{i=1}^{m} (1 + (\widetilde{A}_{i\xi}^{\star z} + \epsilon))^{w_i} + \prod_{i=1}^{m} (1 - (\widetilde{A}_{i\xi}^{\star z} + \epsilon))^{w_i}} \leq 1.$$

(iii) Let $g(x)=\frac{1-x}{1+x}$, $x\in[0,1]$, $f'(x)<0$. Therefore, $f(x)$ is a decreasing function. If $\widetilde{A}_i^{*}\subseteq\widetilde{A}_i$ for all $i=1,2,\ldots,m$, namely $\widetilde{A}_{i\xi}^{\star 1}\leq\widetilde{A}_{i\xi}^{*\star 1}\leq\widetilde{A}_{i\xi}^{*\star 2}\leq\widetilde{A}_{i\xi}^{\star 2}$ and $\widetilde{A}_{i\xi}^{\star 3}\leq\widetilde{A}_{i\xi}^{*\star 3}\leq\widetilde{A}_{i\xi}^{*\star 4}\leq\widetilde{A}_{i\xi}^{\star 4}$, then we can get

$$\frac{1-\widetilde{A}_{i\xi}^{\star 4}}{1+\widetilde{A}_{i\xi}^{\star 4}}\leq\frac{1-\widetilde{A}_{i\xi}^{*\star 4}}{1+\widetilde{A}_{i\xi}^{*\star 4}}\Leftrightarrow\left(\frac{1-\widetilde{A}_{i\xi}^{\star 4}}{1+\widetilde{A}_{i\xi}^{\star 4}}\right)^{w_i}\leq\left(\frac{1-\widetilde{A}_{i\xi}^{*\star 4}}{1+\widetilde{A}_{i\xi}^{*\star 4}}\right)^{w_i}\Leftrightarrow$$

$$\prod_{i=1}^{m}\left(\frac{1-\widetilde{A}_{i\xi}^{\star 4}}{1+\widetilde{A}_{i\xi}^{\star 4}}\right)^{w_i}\leq\prod_{i=1}^{m}\left(\frac{1-\widetilde{A}_{i\xi}^{*\star 4}}{1+\widetilde{A}_{i\xi}^{*\star 4}}\right)^{w_i}\Leftrightarrow$$

$$1+\prod_{i=1}^{m}\left(\frac{1-\widetilde{A}_{i\xi}^{\star 4}}{1+\widetilde{A}_{i\xi}^{\star 4}}\right)^{w_i}\leq 1+\prod_{i=1}^{m}\left(\frac{1-\widetilde{A}_{i\xi}^{*\star 4}}{1+\widetilde{A}_{i\xi}^{*\star 4}}\right)^{w_i}\Leftrightarrow$$

$$\frac{1}{1+\prod_{i=1}^{m}\left(\frac{1-\widetilde{A}_{i\xi}^{*\star 4}}{1+\widetilde{A}_{i\xi}^{*\star 4}}\right)^{w_i}}\leq\frac{1}{1+\prod_{i=1}^{m}\left(\frac{1-\widetilde{A}_{i\xi}^{\star 4}}{1+\widetilde{A}_{i\xi}^{\star 4}}\right)^{w_i}}\Leftrightarrow$$

$$\frac{2}{1+\prod_{i=1}^{m}\left(\frac{1-\widetilde{A}_{i\xi}^{*\star 4}}{1+\widetilde{A}_{i\xi}^{*\star 4}}\right)^{w_i}}\leq\frac{2}{1+\prod_{i=1}^{m}\left(\frac{1-\widetilde{A}_{i\xi}^{\star 4}}{1+\widetilde{A}_{i\xi}^{\star 4}}\right)^{w_i}}\Leftrightarrow$$

$$\frac{2}{1+\prod_{i=1}^{m}\left(\frac{1-\widetilde{A}_{i\xi}^{*\star 4}}{1+\widetilde{A}_{i\xi}^{*\star 4}}\right)^{w_i}}-1\leq\frac{2}{1+\prod_{i=1}^{m}\left(\frac{1-\widetilde{A}_{i\xi}^{\star 4}}{1+\widetilde{A}_{i\xi}^{\star 4}}\right)^{w_i}}-1.$$

$$\frac{\prod_{i=1}^{m}(1+\widetilde{A}_{i\xi}^{*\star 4})^{\theta_i}-\prod_{i=1}^{m}(1-\widetilde{A}_{i\xi}^{*\star 4})^{\theta_i}}{\prod_{i=1}^{m}(1+\widetilde{A}_{i\xi}^{*\star 4})^{\theta_i}+\prod_{i=1}^{m}(1-\widetilde{A}_{i\xi}^{*\star 4})^{\theta_i}}\leq\frac{\prod_{i=1}^{m}(1+\widetilde{A}_{i\xi}^{\star 4})^{\theta_i}-\prod_{i=1}^{m}(1-\widetilde{A}_{i\xi}^{\star 4})^{\theta_i}}{\prod_{i=1}^{m}(1+\widetilde{A}_{i\xi}^{\star 4})^{\theta_i}+\prod_{i=1}^{m}(1-\widetilde{A}_{i\xi}^{\star 4})^{\theta_i}}.$$

Therefore, we can also speculate that

$$\frac{\prod_{i=1}^{m}(1+\widetilde{A}_{i\xi}^{\star 1})^{w_i}-\prod_{i=1}^{m}(1-\widetilde{A}_{i\xi}^{\star 1})^{w_i}}{\prod_{i=1}^{m}(1+\widetilde{A}_{i\xi}^{\star 1})^{w_i}+\prod_{i=1}^{m}(1-\widetilde{A}_{i\xi}^{\star 1})^{w_i}}\leq\frac{\prod_{i=1}^{m}(1+\widetilde{A}_{i\xi}^{*\star 1})^{w_i}-\prod_{i=1}^{m}(1-\widetilde{A}_{i\xi}^{*\star 1})^{w_i}}{\prod_{i=1}^{m}(1+\widetilde{A}_{i\xi}^{*\star 1})^{w_i}+\prod_{i=1}^{m}(1-\widetilde{A}_{i\xi}^{*\star 1})^{w_i}}\leq$$
$$\frac{\prod_{i=1}^{m}(1+\widetilde{A}_{i\xi}^{*\star 2})^{w_i}-\prod_{i=1}^{m}(1-\widetilde{A}_{i\xi}^{*\star 2})^{w_i}}{\prod_{i=1}^{m}(1+\widetilde{A}_{i\xi}^{*\star 2})^{w_i}+\prod_{i=1}^{m}(1-\widetilde{A}_{i\xi}^{*\star 2})^{w_i}}\leq\frac{\prod_{i=1}^{m}(1+\widetilde{A}_{i\xi}^{\star 2})^{w_i}-\prod_{i=1}^{m}(1-\widetilde{A}_{i\xi}^{\star 2})^{w_i}}{\prod_{i=1}^{m}(1+\widetilde{A}_{i\xi}^{\star 2})^{w_i}+\prod_{i=1}^{m}(1-\widetilde{A}_{i\xi}^{\star 2})^{w_i}},$$
$$\frac{\prod_{i=1}^{m}(1+\widetilde{A}_{i\xi}^{\star 3})^{w_i}-\prod_{i=1}^{m}(1-\widetilde{A}_{i\xi}^{\star 3})^{w_i}}{\prod_{i=1}^{m}(1+\widetilde{A}_{i\xi}^{\star 3})^{w_i}+\prod_{i=1}^{m}(1-\widetilde{A}_{i\xi}^{\star 3})^{w_i}}\leq\frac{\prod_{i=1}^{m}(1+\widetilde{A}_{i\xi}^{*\star 3})^{w_i}-\prod_{i=1}^{m}(1-\widetilde{A}_{i\xi}^{*\star 3})^{w_i}}{\prod_{i=1}^{m}(1+\widetilde{A}_{i\xi}^{*\star 3})^{w_i}+\prod_{i=1}^{m}(1-\widetilde{A}_{i\xi}^{*\star 3})^{w_i}}\leq$$
$$\frac{\prod_{i=1}^{m}(1+\widetilde{A}_{i\xi}^{*\star 4})^{w_i}-\prod_{i=1}^{m}(1-\widetilde{A}_{i\xi}^{*\star 4})^{w_i}}{\prod_{i=1}^{m}(1+\widetilde{A}_{i\xi}^{*\star 4})^{w_i}+\prod_{i=1}^{m}(1-\widetilde{A}_{i\xi}^{*\star 4})^{w_i}}\leq\frac{\prod_{i=1}^{m}(1+\widetilde{A}_{i\xi}^{\star 4})^{w_i}-\prod_{i=1}^{m}(1-\widetilde{A}_{i\xi}^{\star 4})^{w_i}}{\prod_{i=1}^{m}(1+\widetilde{A}_{i\xi}^{\star 4})^{w_i}+\prod_{i=1}^{m}(1-\widetilde{A}_{i\xi}^{\star 4})^{w_i}}.$$

If we take $\widetilde{A}_i$ as $(\widetilde{A}_i+\widetilde{\epsilon})$, it can be seen that

$$nPIVFWGA(\widetilde{A}_1^{*},\widetilde{A}_2^{*},\ldots,\widetilde{A}_m^{*})\subseteq nPIVFWGA(\widetilde{A}_1,\widetilde{A}_2,\ldots,\widetilde{A}_m).$$

Definition 9. If $(\widetilde{A}_1, \widetilde{A}_2, \ldots, \widetilde{A}_m) \in n\text{PIVFN(R)}^m$, then an n-polygonal interval-valued fuzzy ordered weighted geometric averaging (nPIVFOWGA) operator is a mapping *nPIVFOWGA*: $n\text{PIVFN(R)}^m \to n\text{PIVFN(R)}$

$$nPIVFOWGA(\widetilde{A}_1, \widetilde{A}_2, \ldots, \widetilde{A}_m) = (\widetilde{A}_{\gamma(1)} + \epsilon)^{\omega_1} \otimes (\widetilde{A}_{\gamma(2)} + \epsilon)^{\omega_2} \otimes \cdots \otimes (\widetilde{A}_{\gamma(m)} + \epsilon)^{\omega_m},$$

where $\widetilde{\epsilon} = [(\epsilon, \ldots \epsilon, \epsilon, \ldots, \epsilon), (\epsilon, \ldots \epsilon, \epsilon, \ldots, \epsilon)]$, ϵ is an arbitrarily small real number, and ω_i is the corresponding weight of $\widetilde{A}_i$ with $0 \le \omega_i \le 1, \sum_{i=1}^m \omega_i = 1$. Moreover, there is a mapping $\gamma : \{1, 2, \ldots, m\} \to \{1, 2, \ldots, m\}$, such that $\widetilde{A}_{\gamma(r)} (r = 1, 2, \ldots, m)$ is the rth largest of all $\widetilde{A}_i (i = 1, 2, \ldots, m)$.

Theorem 12. If we let $(\widetilde{A}_1, \widetilde{A}_2, \ldots, \widetilde{A}_m) \in n\text{PIVFN(R)}^m$, the nPIVFOWGA operator of nPIVFNs can be represented as follows:

$$\begin{aligned}
&nPIVFOWGA(\widetilde{A}_1, \widetilde{A}_2, \ldots, \widetilde{A}_m)\\
=&\Bigg[\Bigg(\Big\{\frac{\prod_{i=1}^m (1 + (\widetilde{A}^{\star 1}_{\gamma(i)\xi} + \epsilon))^{\omega_i} - \prod_{i=1}^m (1 - (\widetilde{A}^{\star 1}_{\gamma(i)\xi} + \epsilon))^{\omega_i}}{\prod_{i=1}^m (1 + (\widetilde{A}^{\star 1}_{\gamma(i)\xi} + \epsilon))^{\omega_i} + \prod_{i=1}^m (1 - (\widetilde{A}^{\star 1}_{\gamma(i)\xi} + \epsilon))^{\omega_i}}\Big\}_{\xi=0}^n,\\
&\Big\{\frac{\prod_{i=1}^m (1 + (\widetilde{A}^{\star 2}_{\gamma(i)(n-\xi)} + \epsilon))^{\omega_i} - \prod_{i=1}^m (1 - (\widetilde{A}^{\star 2}_{\gamma(i)(n-\xi)} + \epsilon))^{\omega_i}}{\prod_{i=1}^m (1 + (\widetilde{A}^{\star 2}_{\gamma(i)(n-\xi)} + \epsilon))^{\omega_i} + \prod_{i=1}^m (1 - (\widetilde{A}^{\star 2}_{\gamma(i)(n-\xi)} + \epsilon))^{\omega_i}}\Big\}_{\xi=0}^n\Bigg),\\
&\Bigg(\Big\{\frac{\prod_{i=1}^m (1 + (\widetilde{A}^{\star 3}_{i\xi} + \epsilon))^{\omega_i} - \prod_{i=1}^m (1 - (\widetilde{A}^{\star 3}_{\gamma(i)\xi} + \epsilon))^{\omega_i}}{\prod_{i=1}^m (1 + (\widetilde{A}^{\star 3}_{\gamma(i)\xi} + \epsilon))^{\omega_i} + \prod_{i=1}^m (1 - (\widetilde{A}^{\star 3}_{\gamma(i)\xi} + \epsilon))^{\omega_i}}\Big\}_{\xi=0}^n,\\
&\Big\{\frac{\prod_{i=1}^m (1 + (\widetilde{A}^{\star 4}_{\gamma(i)(n-\xi)} + \epsilon))^{\omega_i} - \prod_{i=1}^m (1 - (\widetilde{A}^{\star 4}_{\gamma(i)(n-\xi)} + \epsilon))^{\omega_i}}{\prod_{i=1}^m (1 + (\widetilde{A}^{\star 4}_{\gamma(i)(n-\xi)} + \epsilon))^{\omega_i} + \prod_{i=1}^m (1 - (\widetilde{A}^{\star 4}_{\gamma(i)(n-\xi)} + \epsilon))^{\omega_i}}\Big\}_{\xi=0}^n\Bigg)\Bigg],
\end{aligned} \tag{3.10}$$

where ω_i is the weight of $\widetilde{A}_{\gamma(i)}$ with $0 \le \omega_i \le 1$ and $\sum_{i=1}^m \omega_i = 1$.

Theorem 13. Let $\widetilde{A}_i, \widetilde{A}^*_i (i = 1, 2, \ldots, m) \in n\text{PIVFN(R)}$. The properties of the nPIVFOWGA operator are as follows:
(i) Boundedness: $\widetilde{0} \subseteq nPIVFOWGA(\widetilde{A}_1, \widetilde{A}_2, \ldots, \widetilde{A}_m) \subseteq \widetilde{1}$;
(ii) Monotonicity: If $\widetilde{A}^*_i \subseteq \widetilde{A}_i$, i.e., $[a^{*1}_\xi, a^{*2}_\xi] \subseteq [a^1_\xi, a^2_\xi]$ and $[a^{*3}_\xi, a^{*4}_\xi] \subseteq [a^3_\xi, a^4_\xi]$, then

$$nPIVFOWGA(\widetilde{A}^*_1, \widetilde{A}^*_2, \ldots, \widetilde{A}^*_m) \subseteq nPIVFOWG(\widetilde{A}_1, \widetilde{A}_2, \ldots, \widetilde{A}_m).$$

Definition 10. If we assume that $(\widetilde{A}_1, \widetilde{A}_2, \ldots, \widetilde{A}_m) \in n\text{PIVFN(R)}^m$, and $\overline{A}_{\gamma(i)} = \widetilde{A}_i^{mw_i}$($w_i$ is the weight of $\widetilde{A}_i$), then a function *nPIVHGA*: $n\text{PIVFN(R)}^m \to n\text{PIVFN(R)}$ is called an n-polygonal interval-valued fuzzy hybrid geometric averaging (nPIVFHGA) operator, which is defined as

$$nPIVFHGA(\widetilde{A}_1, \widetilde{A}_2, \ldots, \widetilde{A}_m) = (\overline{A}_{\gamma(1)} + \epsilon)^{\tau_1} \otimes (\overline{A}_{\gamma(2)} + \epsilon)^{\tau_2} \otimes \cdots \otimes (\overline{A}_{\gamma(m)} + \epsilon)^{\tau_m},$$

where $\widetilde{\epsilon} = [(\epsilon, \ldots, \epsilon, \epsilon \ldots, \epsilon), (\epsilon, \ldots, \epsilon, \epsilon \ldots, \epsilon)]$ and ϵ is an arbitrarily small real number. There is a mapping $\gamma : \{1, 2, \ldots, m\} \to \{1, 2, \ldots, m\}$, such that $\overline{A}_{\gamma(r)}$ is the rth largest of all $\overline{A}_i (i = 1, 2, \ldots, m)$. τ_i is the weight of $\overline{A}_{\gamma(i)}$, satisfying $0 \le \tau_i \le 1$ and $\sum_{i=1}^m \tau_i = 1$.

Theorem 14. If we let $(\widetilde{A}_1, \widetilde{A}_2, \ldots, \widetilde{A}_m) \in n\text{PIVFN(R)}^m$, the nPIVFHGA operator of nPIVFNs has the following formula:

$$
\begin{aligned}
& nPIVFHGA(\widetilde{A}_1, \widetilde{A}_2, \ldots, \widetilde{A}_m) \\
=& \Bigg[\Bigg(\Big\{\frac{\prod_{i=1}^{m}(1+(\widetilde{A}^{\star 1}_{\gamma(i)\xi}+\epsilon))^{\tau_i}-\prod_{i=1}^{m}(1-(\widetilde{A}^{\star 1}_{\gamma(i)\xi}+\epsilon))^{\tau_i}}{\prod_{i=1}^{m}(1+(\widetilde{A}^{\star 1}_{\gamma(i)\xi}+\epsilon))^{\tau_i}+\prod_{i=1}^{m}(1-(\widetilde{A}^{\star 1}_{\gamma(i)\xi}+\epsilon))^{\tau_i}}\Big\}_{\xi=0}^{n}, \\
& \Big\{\frac{\prod_{i=1}^{m}(1+(\widetilde{A}^{\star 2}_{\gamma(i)(n-\xi)}+\epsilon))^{\tau_i}-\prod_{i=1}^{m}(1-(\widetilde{A}^{\star 2}_{\gamma(i)(n-\xi)}+\epsilon))^{\tau_i}}{\prod_{i=1}^{m}(1+(\widetilde{A}^{\star 2}_{\gamma(i)(n-\xi)}+\epsilon))^{\tau_i}+\prod_{i=1}^{m}(1-(\widetilde{A}^{\star 2}_{\gamma(i)(n-\xi)}+\epsilon))^{\tau_i}}\Big\}_{\xi=0}^{n}\Bigg), \\
& \Bigg(\Big\{\frac{\prod_{i=1}^{m}(1+(\widetilde{A}^{\star 3}_{i\xi}+\epsilon))^{\tau_i}-\prod_{i=1}^{m}(1-(\widetilde{A}^{\star 3}_{\gamma(i)\xi}+\epsilon))^{\tau_i}}{\prod_{i=1}^{m}(1+(\widetilde{A}^{\star 3}_{\gamma(i)\xi}+\epsilon))^{\tau_i}+\prod_{i=1}^{m}(1-(\widetilde{A}^{\star 3}_{\gamma(i)\xi}+\epsilon))^{\tau_i}}\Big\}_{\xi=0}^{n}, \\
& \Big\{\frac{\prod_{i=1}^{m}(1+(\widetilde{A}^{\star 4}_{\gamma(i)(n-\xi)}+\epsilon))^{\tau_i}-\prod_{i=1}^{m}(1-(\widetilde{A}^{\star 4}_{\gamma(i)(n-\xi)}+\epsilon))^{\tau_i}}{\prod_{i=1}^{m}(1+(\widetilde{A}^{\star 4}_{\gamma(i)(n-\xi)}+\epsilon))^{\tau_i}+\prod_{i=1}^{m}(1-(\widetilde{A}^{\star 4}_{\gamma(i)(n-\xi)}+\epsilon))^{\tau_i}}\Big\}_{\xi=0}^{n}\Bigg)\Bigg],
\end{aligned}
\tag{3.11}
$$

where $\tau = (\tau_1, \tau_2, \ldots, \tau_m)^T$ is the weight vector of $\overline{A}_{\gamma(i)}$ with $0 \leq \tau_i \leq 1 (i = 1, 2, \ldots, m)$ and $\sum_{i=1}^{m} \tau_i = 1$.

The properties of the nPIVFHGA operator are consistent with **Theorem 10**, so the specific content will not be explained in detail.

4. A decision-making method based on graph and path

In MAGDM problems, graphs can abstract complex entity relationships into nodes and edges, enabling decision-makers to intuitively visualize the system's structure. By converting edges into paths and treating the attributes and alternatives as nodes, we construct paths between the attributes and alternatives. The scheme with a smaller total path length is regarded as superior. The detailed steps are outlined below, and a flowchart of the decision-making procedure is presented in Figure 6.

Consider building a path graph $G = \{V, E\}$ for each alternative, where $V = \{v_1, v_2, \ldots, v_q, v_{q+1}\}$ is a set of nodes, each consisting of the solution and all its attributes. $E \subseteq V \times V$ represents a set of edges, and the edges between the scheme and the attributes are considered to be paths. The paths' lengths are calculated via the norm.

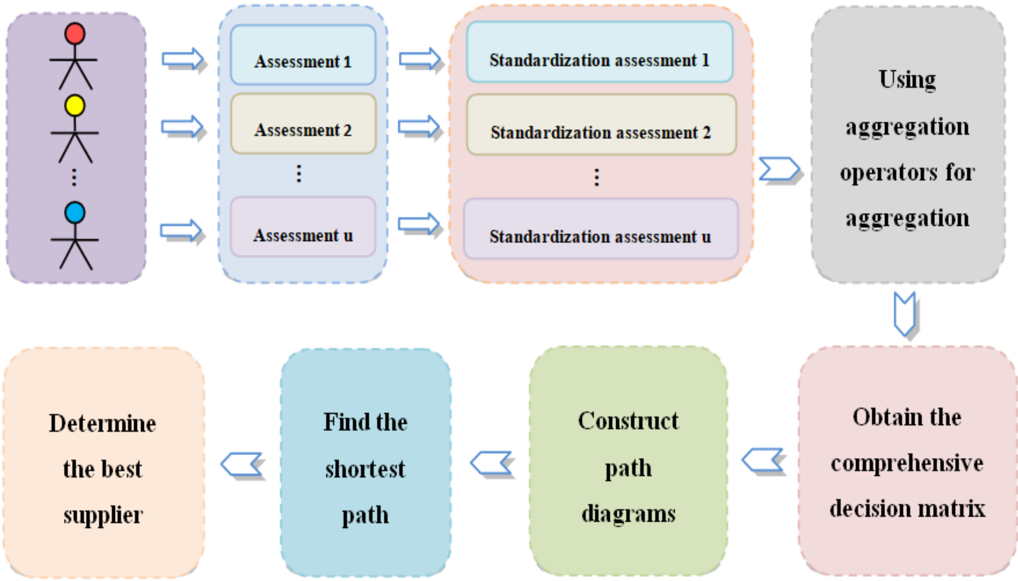


Figure 6. A flowchart of the decision-making process.

Assume that there are φ schemes $X = (X_\iota|\iota = 1, 2, \ldots, \varphi)$, u decision-makers $DM = (D_k|k = 1, 2, \ldots, u)$, and the decision-maker weight vector is $\vartheta = (\vartheta_1, \vartheta_2, \ldots, \vartheta_u)^T$. There are q attributes $C = (C_j|j = 1, 2, \ldots, q)$, the weight vector of the attributes is $\delta = (\delta_1, \delta_2, \ldots, \delta_q)^T$. The evaluation values of a scheme X_ι of the attributes C_j are provided by the decision-makers D_k in the form of

$$\begin{aligned} h^k_{\iota j} =&[(a^1_{0\iota}(jk), a^1_{1\iota}(jk), \ldots, a^1_{n\iota}(jk), a^2_{n\iota}(jk), \ldots, a^2_{1\iota}(jk), a^2_{0\iota}(jk)),\\ &(a^3_{0\iota}(jk), a^3_{1\iota}(jk), \ldots, a^3_{n\iota}(jk), a^4_{n\iota}(jk), \ldots, a^4_{1\iota}(jk), a^4_{0\iota}(jk))]. \end{aligned}$$

According to the given conversion criteria for language variables, an n-polygonal interval-valued fuzzy decision matrix H_k can be found.

$$H_k = \begin{array}{cc} & \begin{array}{cccc} C_1 & C_2 & \cdots & C_q \end{array} \\ & \left(\begin{array}{cccc} h^k_{11} & h^k_{12} & \cdots & h^k_{1q} \\ h^k_{21} & h^k_{22} & \cdots & h^k_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ h^k_{\varphi 1} & h^k_{\varphi 2} & \cdots & h^k_{\varphi q} \end{array}\right) \begin{array}{c} X_1 \\ X_2 \\ \vdots \\ X_\varphi \end{array} \end{array},$$

where $k = 1, 2, \ldots, u$.

Here, G^+ denotes the group of risk attributes and G^- represents the group of return attributes. Let $\alpha^k_j = \max\limits_{1\leq\iota\leq\varphi} a^4_{0\iota}(jk), \beta^k_j = \min\limits_{1\leq\iota\leq\varphi} a^3_{0\iota}(jk)$. The risk attributes ($j \in G^+$) are expressed as:

$$\begin{array}{l} \begin{cases} \overline{a}^1_{\xi\iota}(jk) = \frac{a^1_{\xi\iota}(jk) - \beta^k_j}{\alpha^k_j - \beta^k_j} \\ \overline{a}^2_{\xi\iota}(jk) = \frac{a^2_{\xi\iota}(jk) - \beta^k_j}{\alpha^k_j - \beta^k_j} \end{cases} , \xi = 0, 1, 2, \ldots, n, \\ \begin{cases} \overline{a}^3_{\xi\iota}(jk) = \frac{a^3_{\xi\iota}(jk) - \beta^k_j}{\alpha^k_j - \beta^k_j} \\ \overline{a}^4_{\xi\iota}(jk) = \frac{a^4_{\xi\iota}(jk) - \beta^k_j}{\alpha^k_j - \beta^k_j} \end{cases} , \xi = 0, 1, 2, \ldots, n. \end{array} \tag{4.1}$$

Return attributes ($j \in G^-$) are expressed as

$$\begin{cases} \overline{a}^1_{\xi\iota}(jk) = \frac{\alpha^k_j - a^2_{\xi\iota}(jk)}{\alpha^k_j - \beta^k_j} \\ \overline{a}^2_{\xi\iota}(jk) = \frac{\alpha^k_j - a^1_{\xi\iota}(jk)}{\alpha^k_j - \beta^k_j} \end{cases}, \xi = 0, 1, 2, \ldots, n,$$
$$\begin{cases} \overline{a}^3_{\xi\iota}(jk) = \frac{\alpha^k_j - a^4_{\xi\iota}(jk)}{\alpha^k_j - \beta^k_j} \\ \overline{a}^4_{\xi\iota}(jk) = \frac{\alpha^k_j - a^3_{\xi\iota}(jk)}{\alpha^k_j - \beta^k_j} \end{cases}, \xi = 0, 1, 2, \ldots, n. \tag{4.2}$$

The decision matrix H_k is normalized to eliminate the influence of the dimensions. The normalized decision matrix is represented by $\overline{H}_k$:

$$\overline{H}_k = \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \cdots & C_q \end{array} \\ \begin{pmatrix} \overline{h}^k_{11} & \overline{h}^k_{12} & \cdots & \overline{h}^k_{1q} \\ \overline{h}^k_{21} & \overline{h}^k_{22} & \cdots & \overline{h}^k_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{h}^k_{\varphi 1} & \overline{h}^k_{\varphi 2} & \cdots & \overline{h}^k_{\varphi q} \end{pmatrix} \end{array} \begin{array}{c} \\ X_1 \\ X_2 \\ \vdots \\ X_\varphi \end{array},$$

where the element $\overline{h}^k_{\iota j}$ is as follows:

$$\begin{aligned} \overline{h}^k_{\iota j} =& [(\overline{a}^1_{0\iota}(jk), \overline{a}^1_{1\iota}(jk), \ldots, \overline{a}^1_{n\iota}(jk), \overline{a}^2_{n\iota}(jk), \ldots, \overline{a}^2_{1\iota}(jk), \overline{a}^2_{0\iota}(jk)), \\ & (\overline{a}^3_{0\iota}(jk), \overline{a}^3_{1\iota}(jk), \ldots, \overline{a}^3_{n\iota}(jk), \overline{a}^4_{n\iota}(jk), \ldots, \overline{a}^4_{1\iota}(jk), \overline{a}^4_{0\iota}(jk))], \end{aligned}$$

$\iota = 1, 2, \ldots, \varphi$, $j = 1, 2, \ldots, q, k = 1, 2, \ldots, u$.

According to the aggregation operator introduced in Section 3, a comprehensive decision matrix $\widetilde{H} = (\widetilde{h}_{\iota j})_{\varphi \times q}$ is derived through Eq (6), and the element $\widetilde{h}_{\iota j}$ can be expressed as follows:

$$\widetilde{h}_{\iota j} = nPIVFWAA(\overline{h}^1_{\iota j}, \overline{h}^2_{\iota j}, \ldots, \overline{h}^u_{\iota j}) = [\widetilde{h}^L_{\iota j}, \widetilde{h}^U_{\iota j}],$$

where

$$\widetilde{h}^L_{\iota j} = \left(\left\{ 1 - (1 - \prod_{k=1}^{u} (\overline{A}^{\star 1}_{\xi\iota}(jk) + \epsilon)^{\vartheta_k} \right\}_{\xi=0}^{n}, \left\{ 1 - (1 - \prod_{k=1}^{u} (\overline{A}^{\star 2}_{(n-\xi)\iota}(jk) + \epsilon)^{\vartheta_k} \right\}_{\xi=0}^{n} \right),$$

$$\widetilde{h}^U_{\iota j} = \left(\left\{ 1 - (1 - \prod_{k=1}^{u} (\overline{A}^{\star 3}_{\xi\iota}(jk) + \epsilon)^{\vartheta_k} \right\}_{\xi=0}^{n}, \left\{ 1 - (1 - \prod_{k=1}^{u} (\overline{A}^{\star 4}_{(n-\xi)\iota}(jk) + \epsilon)^{\vartheta_k} \right\}_{\xi=0}^{n} \right).$$

Then, the comprehensive decision matrix $\widetilde{H}$ can be represented as

$$\widetilde{H} = \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \cdots & C_q \end{array} \\ \begin{pmatrix} \widetilde{h}_{11} & \widetilde{h}_{12} & \cdots & \widetilde{h}_{1q} \\ \widetilde{h}_{21} & \widetilde{h}_{22} & \cdots & \widetilde{h}_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ \widetilde{h}_{\varphi 1} & \widetilde{h}_{\varphi 2} & \cdots & \widetilde{h}_{\varphi q} \end{pmatrix} \end{array} \begin{array}{c} \\ X_1 \\ X_2 \\ \vdots \\ X_\varphi \end{array}.$$

At this stage, path diagrams for all schemes are constructed based on the comprehensive evaluation matrix, where the path $T_{\iota j}(\iota = 1, 2, \ldots, \phi, j = 1, 2, \ldots, q)$ are calculated by the $L - p$ norm.

$$T_{\iota j} = \left(\sum_{\xi=0}^{n} ((a_{\xi}^{1})^{p} + (a_{\xi}^{2})^{p} + (a_{\xi}^{3})^{p} + (a_{\xi}^{4})^{p}) \right)^{1/p}, p \geq 1. \tag{4.3}$$

Finally, we multiply all paths of each scheme by the weights of the corresponding attributes and sum them up. The minimum sum S_{ι} is the shortest path that we want to obtain. Therefore, the optimal alternative X_{ι} can be determined.

$$S_{\iota} = \sum_{j=1}^{q} T_{\iota j} * \delta_{j} \tag{4.4}$$

A two-layer aggregation mechanism is proposed: First, the $L - p$ norm is used to internally fuse n-polygonal interval-valued fuzzy numbers into characteristic values while preserving their morphological features. Then weighted summation integrates these values across attributes. This mechanism not only avoids information loss but also flexibly captures the compensation among the attributes by adjusting the parameter p.

5. Supplier selection

For an company to achieve success, one of the paramount aspects is the execution of effective supplier management. This practice is essential in enterprises of all sizes and categories. It is not uncommon for companies to engage in the procurement of diverse goods from numerous suppliers on an annual basis, where the selection process exerts a profound influence on the efficacy of business operations. A judicious choice can facilitate operations significantly, whereas a poor selection can pose impediments to the entity's progress. Consequently, the task of sifting through numerous potential suppliers and identifying the most appropriate partner is critical for enterprises. In addressing this challenge, we combine the newly introduced aggregation operators with with the newly proposed decision-making method for optimal supplier selection.

5.1. Application of the proposed path graph-based decision-making method

A daily necessities production company is desirous of selecting the most appropriate among three prospective suppliers, $X_{\iota}(\iota = 1, 2, 3, 4)$. Consequently, the company selects three employees from the investment department as the decision- makers $DM_k(k = 1, 2, 3)$, whose weight vector is $\vartheta = (0.33, 0.28, 0.39)^T$. To evaluate these suppliers, there are four attributes C_1(product quality), C_2 (reputation), C_3(delivery capacity), C_4(price), C_5 (after-sale service), C_6(risk management) are considered, where $C_j(j = 1, 2, 3, 5, 6)$ are the return attributes and C_4 is the risk attribute. The corresponding weight vector for the attributes $C_j(j = 1, 2, \ldots, 6)$ is $\delta = (0.3, 0.1, 0.15, 0.2, 0.15, 0.1)^T$. First, the evaluation values of experts are presented in the form of linguistic variables (Table 3). Second, the evaluation values in the shape of language variables are transformed into nPIVFNs for n=3 (Table 2).

The decision matrices proffered by the trio of experts are denoted as H_1, H_2, and H_3.

$$H_1 = \begin{array}{c} \\ X_1 \\ X_2 \\ X_3 \\ X_4 \end{array} \begin{array}{c} \begin{array}{cccccc} C_1 & C_2 & C_3 & C_4 & C_5 & C_6 \end{array} \\ \begin{pmatrix} h_{11}^1 & h_{12}^1 & h_{13}^1 & h_{14}^1 & h_{15}^1 & h_{16}^1 \\ h_{21}^1 & h_{22}^1 & h_{23}^1 & h_{24}^1 & h_{25}^1 & h_{26}^1 \\ h_{31}^1 & h_{32}^1 & h_{33}^1 & h_{34}^1 & h_{35}^1 & h_{36}^1 \\ h_{41}^1 & h_{42}^1 & h_{43}^1 & h_{44}^1 & h_{45}^1 & h_{46}^1 \end{pmatrix} \end{array},$$

$$H_2 = \begin{array}{c} \\ X_1 \\ X_2 \\ X_3 \\ X_4 \end{array} \begin{array}{c} \begin{array}{cccccc} C_1 & C_2 & C_3 & C_4 & C_5 & C_6 \end{array} \\ \begin{pmatrix} h_{11}^2 & h_{12}^2 & h_{13}^2 & h_{14}^2 & h_{15}^2 & h_{16}^2 \\ h_{21}^2 & h_{22}^2 & h_{23}^2 & h_{24}^2 & h_{25}^2 & h_{26}^2 \\ h_{31}^2 & h_{32}^2 & h_{33}^2 & h_{34}^2 & h_{35}^2 & h_{36}^2 \\ h_{41}^2 & h_{42}^2 & h_{43}^2 & h_{44}^2 & h_{45}^2 & h_{46}^2 \end{pmatrix} \end{array},$$

$$H_3 = \begin{array}{c} \\ X_1 \\ X_2 \\ X_3 \\ X_4 \end{array} \begin{array}{c} \begin{array}{cccccc} C_1 & C_2 & C_3 & C_4 & C_5 & C_6 \end{array} \\ \begin{pmatrix} h_{11}^3 & h_{12}^3 & h_{13}^3 & h_{14}^3 & h_{15}^3 & h_{16}^3 \\ h_{21}^3 & h_{22}^3 & h_{23}^3 & h_{24}^3 & h_{25}^3 & h_{26}^3 \\ h_{31}^3 & h_{32}^3 & h_{33}^3 & h_{34}^3 & h_{35}^3 & h_{36}^3 \\ h_{41}^3 & h_{42}^3 & h_{43}^3 & h_{44}^3 & h_{45}^3 & h_{46}^3 \end{pmatrix} \end{array}.$$

Table 2. The corresponding language variables at $n = 3$.

Linguistic variables	Polygonal interval-valued fuzzy numbers
Very bad (VB)	[(0,0,0,0,0,0,0,0.05),(0,0,0,0,0,0,0,0.1)]
Bad (B)	[(0.05,0.07,0.08,0.1,0.1,0.11,0.17,0.25),(0,0.02,0.04,0.1,0.11,0.25,0.32,0.35)]
Medium bad (MB)	[(0.15,0.28,0.29,0.3,0.31,0.35,0.38,0.4),(0,0.04,0.12,0.3,0.38,0.4,0.43,0.45)]
Medium (M)	[(0.4,0.42,0.45,0.45,0.47,0.48,0.5,0.55),(0.3,0.32,0.35,0.4,0.48,0.5,0.56,0.6)]
Medium good (MG)	[(0.5,0.52,0.55,0.58,0.6,0.62,0.65,0.7),(0.45,0.48,0.5,0.5,0.6,0.67,0.7,0.75)]
Good (G)	[(0.7,0.72,0.75,0.75,0.77,0.78,0.8,0.85),(0.55,0.58,0.6,0.7,0.78,0.8,0.9,0.94)]
Very good (VG)	[(0.95,0.97,0.98,1,1,1,1,1),(0.85,0.89,0.9,1,1,1,1,1)]

Subsequently, in accordance with the decision-making process delineated in Section 4, the decision matrix $H_k (k = 1, 2, 3)$ is normalized. It is easy to see that

$$\begin{cases} \alpha_1^1 = \max\limits_{1\le\iota\le3} a_{0\iota}^4(11) = 1; \\ \alpha_2^1 = \max\limits_{1\le\iota\le3} a_{0\iota}^4(21) = 1; \\ \alpha_3^1 = \max\limits_{1\le\iota\le3} a_{0\iota}^4(31) = 1; \\ \alpha_4^1 = \max\limits_{1\le\iota\le3} a_{0\iota}^4(41) = 1; \\ \alpha_5^1 = \max\limits_{1\le\iota\le3} a_{0\iota}^4(51) = 1; \\ \alpha_6^1 = \max\limits_{1\le\iota\le3} a_{0\iota}^4(61) = 0.94; \end{cases} \quad \begin{cases} \alpha_1^2 = \max\limits_{1\le\iota\le3} a_{0\iota}^4(12) = 0.94; \\ \alpha_2^2 = \max\limits_{1\le\iota\le3} a_{0\iota}^4(22) = 1; \\ \alpha_3^2 = \max\limits_{1\le\iota\le3} a_{0\iota}^4(32) = 1; \\ \alpha_4^2 = \max\limits_{1\le\iota\le3} a_{0\iota}^4(42) = 1; \\ \alpha_5^2 = \max\limits_{1\le\iota\le3} a_{0\iota}^4(52) = 0.94; \\ \alpha_6^2 = \max\limits_{1\le\iota\le3} a_{0\iota}^4(62) = 1; \end{cases} \quad \begin{cases} \alpha_1^3 = \max\limits_{1\le\iota\le3} a_{0\iota}^4(13) = 1; \\ \alpha_2^3 = \max\limits_{1\le\iota\le3} a_{0\iota}^4(23) = 1; \\ \alpha_3^3 = \max\limits_{1\le\iota\le3} a_{0\iota}^4(33) = 1; \\ \alpha_4^3 = \max\limits_{1\le\iota\le3} a_{0\iota}^4(43) = 1; \\ \alpha_5^3 = \max\limits_{1\le\iota\le3} a_{0\iota}^4(53) = 0.94; \\ \alpha_6^3 = \max\limits_{1\le\iota\le3} a_{0\iota}^4(63) = 0.94; \end{cases}$$

$$\begin{cases}\beta_1^1 = \max\limits_{1\le\iota\le3} a_{0\iota}^3(11) = 0.45;\\ \beta_2^1 = \max\limits_{1\le\iota\le3} a_{0\iota}^3(21) = 0.55;\\ \beta_3^1 = \max\limits_{1\le\iota\le3} a_{0\iota}^3(31) = 0.3;\\ \beta_4^1 = \max\limits_{1\le\iota\le3} a_{0\iota}^3(41) = 0.45;\\ \beta_5^1 = \max\limits_{1\le\iota\le3} a_{0\iota}^3(51) = 0.55;\\ \beta_5^1 = \max\limits_{1\le\iota\le3} a_{0\iota}^3(61) = 0.45;\end{cases}\quad \begin{cases}\beta_1^2 = \max\limits_{1\le\iota\le3} a_{0\iota}^3(12) = 0.55;\\ \beta_2^2 = \max\limits_{1\le\iota\le3} a_{0\iota}^3(22) = 0.55;\\ \beta_3^2 = \max\limits_{1\le\iota\le3} a_{0\iota}^3(32) = 0.45;\\ \beta_4^2 = \max\limits_{1\le\iota\le3} a_{0\iota}^3(42) = 0.55;\\ \beta_5^2 = \max\limits_{1\le\iota\le3} a_{0\iota}^3(52) = 0.45;\\ \beta_6^2 = \max\limits_{1\le\iota\le3} a_{0\iota}^3(62) = 0.45;\end{cases}\quad \begin{cases}\beta_1^3 = \max\limits_{1\le\iota\le3} a_{0\iota}^3(13) = 0.45;\\ \beta_2^3 = \max\limits_{1\le\iota\le3} a_{0\iota}^3(23) = 0.55;\\ \beta_3^3 = \max\limits_{1\le\iota\le3} a_{0\iota}^3(33) = 0.45;\\ \beta_4^3 = \max\limits_{1\le\iota\le3} a_{0\iota}^3(43) = 0.55.\\ \beta_5^3 = \max\limits_{1\le\iota\le3} a_{0\iota}^3(53) = 0.55.\\ \beta_6^3 = \max\limits_{1\le\iota\le3} a_{0\iota}^3(53) = 0.45.\end{cases}$$

Table 3. Assessments of decision-makers based on each criterion.

Decision-makers	Suppliers	C_1	C_2	C_3	C_4	C_5	C_6
DM_1	X_1	MG	VG	VG	G	G	MG
	X_2	G	G	G	MG	VG	G
	X_3	VG	G	M	VG	G	MG
	X_4	G	G	MG	MG	VG	G
DM_2	X_1	G	G	G	VG	MG	G
	X_2	G	VG	VG	G	G	VG
	X_3	G	VG	MG	VG	G	G
	X_4	G	G	MG	G	G	MG
DM_3	X_1	MG	G	G	G	G	MG
	X_2	MG	G	VG	VG	G	G
	X_3	VG	VG	MG	G	G	G
	X_4	G	VG	G	G	G	G

Normalization matrix $\overline{H}_k = (\overline{h}_{\iota j}^k)_{3\times6}$ can be obtained with Eqs (13) and (14). Next, we provide two examples, the calculation steps for the benefit evaluation value $\overline{h}_{11}^1$, and the cost evaluation value $\overline{h}_{14}^1$.

$$\overline{h}_{11}^1 = \Big[\big(\frac{1-0.7}{1-0.45}, \frac{1-0.65}{1-0.45}, \frac{1-0.62}{1-0.45}, \frac{1-0.6}{1-0.45}, \frac{1-0.58}{1-0.45}, \frac{1-0.55}{1-0.45}, \frac{1-0.52}{1-0.45}, \frac{1-0.5}{1-0.45}\big),\\ \big(\frac{1-0.75}{1-0.45}, \frac{1-0.7}{1-0.45}, \frac{1-0.67}{1-0.45}, \frac{1-0.6}{1-0.45}, \frac{1-0.5}{1-0.45}, \frac{1-0.5}{1-0.45}, \frac{1-0.48}{1-0.45}, \frac{1-0.45}{1-0.45}\big)\Big],$$

$$\overline{h}_{14}^1 = \Big[\big(\frac{0.7-0.45}{1-0.45}, \frac{0.72-0.45}{1-0.45}, \frac{0.75-0.45}{1-0.45}, \frac{0.75-0.45}{1-0.45}, \frac{0.77-0.45}{1-0.45}, \frac{0.78-0.45}{1-0.45},\\ \frac{0.8-0.45}{1-0.45}, \frac{0.85-0.45}{1-0.45}\big), \big(\frac{0.55-0.45}{1-0.45}, \frac{0.58-0.45}{1-0.45}, \frac{0.6-0.45}{1-0.45}, \frac{0.7-0.45}{1-0.45},\\ \frac{0.78-0.45}{1-0.45}, \frac{0.8-0.45}{1-0.45}, \frac{0.9-0.45}{1-0.45}, \frac{0.94-0.45}{1-0.45}\big)\Big].$$

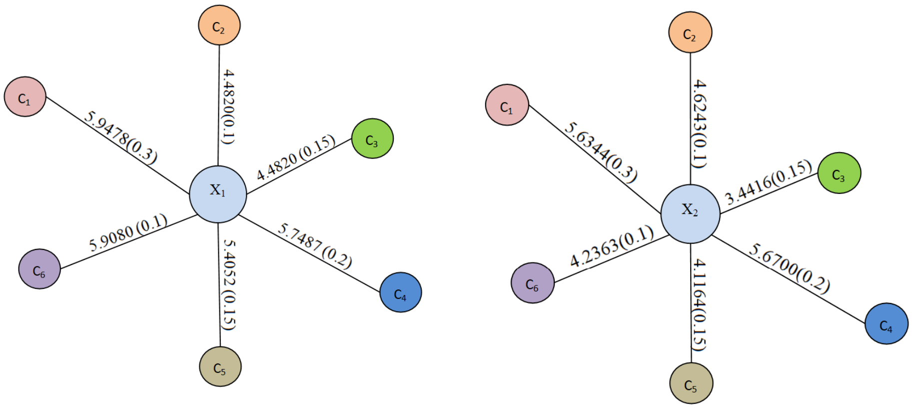
Similarly, other elements in the standardized decision matrix can be calculated; refer to **Appendix A (3)**.

$\overline{H}_k (k = 1, 2, 3)$ can be aggregated by Eq (8), and a comprehensive decision matrix $\widetilde{H}$ is obtained.

Specific values are detailed in **Appendix A (4)**.

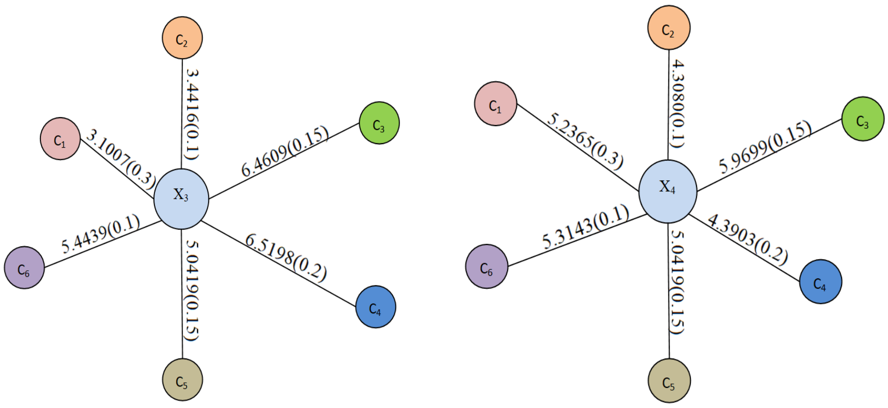
$$\widetilde{H} = \begin{array}{c} \\ X_1 \\ X_2 \\ X_3 \\ X_4 \end{array} \begin{array}{c} \begin{array}{cccccc} C_1 & C_2 & C_3 & C_4 & C_5 & C_6 \end{array} \\ \begin{pmatrix} \widetilde{h}_{11} & \widetilde{h}_{12} & \widetilde{h}_{13} & \widetilde{h}_{14} & \widetilde{h}_{15} & \widetilde{h}_{16} \\ \widetilde{h}_{21} & \widetilde{h}_{22} & \widetilde{h}_{23} & \widetilde{h}_{24} & \widetilde{h}_{25} & \widetilde{h}_{26} \\ \widetilde{h}_{31} & \widetilde{h}_{32} & \widetilde{h}_{33} & \widetilde{h}_{34} & \widetilde{h}_{35} & \widetilde{h}_{36} \\ \widetilde{h}_{41} & \widetilde{h}_{42} & \widetilde{h}_{43} & \widetilde{h}_{44} & \widetilde{h}_{45} & \widetilde{h}_{46} \end{pmatrix} \end{array}.$$

On the basis on the comprehensive evaluation matrix $\widetilde{H}$, the sum of paths for each supplier can be calculated using Eqs (15) and (16) (parameter $p = 1$). The calculated results of $T_{\iota j}(\iota = 1, 2, 3, 4, j = 1, 2, \ldots, 6)$ and S_ι are shown in Table 4. Figures 7 and 8 display the path diagrams of each supplier. From Table 4, it can be seen that $S_2 < S_3 < S_1 < S_4$; therefore, $X_2 \succ X_3 \succ X_1 \succ X_4$. The shortest path can result in the identification of the optimal alternative. Consequently, the optimal supplier is determined to be X_2.



(a) Path diagram of supplier X_1 (b) Path diagram of supplier X_2

Figure 7. Path diagrams of the suppliers.
Note. The numbers in parentheses on the line in the figure represent the weight of the attribute.



(a) Path diagram of supplier X_3. (b) Path diagram of supplier X_4.

Figure 8. Path diagrams of the suppliers.
Note: The numbers in parentheses on the line in the figure represent the weight of the attribute.

Table 4. Sum of the path values, and path values of the alternatives.

T_{11}	T_{12}	T_{13}	T_{14}	T_{15}	T_{16}	S_1
5.9478	4.4820	4.4820	5.7487	5.4052	5.9080	5.46
T_{21}	T_{22}	T_{23}	T_{24}	T_{25}	T_{26}	S_2
5.6344	4.6243	3.4416	5.6700	4.1164	4.2364	4.84
T_{31}	T_{32}	T_{33}	T_{34}	T_{35}	T_{36}	S_3
3.1077	3.4416	6.4609	6.5198	5.0419	5.4439	4.85
T_{41}	T_{42}	T_{43}	T_{44}	T_{45}	T_{46}	S_3
5.2365	4.3080	5.9699	4.3903	5.0419	5.3143	6.58

5.2. Comparative analysis

In this section, we will study the newly introduced decision-making methods from three aspects. First, we verify the impact of different values of n on the ranking of schemes. Second, the ranking of alternatives generated by the proposed decision-making method is compared and analyzed with the ranking of alternatives generated by the decision-making methods developed by Suo et al. [12], Yin et al. [31] and Dutta et al. [35]. Finally, a comparative analysis is performed to assess the discrimination ability of the proposed novel decision-making method versus the other two existing methods, thereby verifying its effectiveness and superiority. Next, we will discuss these three aspects in detail.

As the operational complexity of the nPIVFWAA operator increases with an increase in the n value, it is important to choose the appropriate n value. Without loss of generality, we can consider calculating the sum of paths for each scheme in the case of $n = 1, 2, 3, 4$, discussing and comparing supplier rankings in different n scenarios. Meanwhile, the corresponding language variable values are provided in Appendix A (1). The detailed calculation results are shown in Table 5. It is easy to see from Table 5 that as n changes, the total path of X_2 is always the shortest, and the suppliers' rankings remain at $X_2 \succ X_3 \succ X_1 \succ X_4$. Figure 9 shows that as the value of n increases, the degree of differentiation between suppliers also increases. However, the larger the value of n, the greater the computational complexity, so it is necessary to choose an appropriate value of n.

Table 5. The values of each alternative evaluation matrix for n =1,2,3,4.

	X_1	X_2	X_3	X_4
$n = 1$	2.81	2.56	2.61	3.37
$n = 2$	4.15	3.74	3.81	4.93
$n = 3$	5.46	4.84	4.85	6.58
$n = 4$	7.01	6.22	6.29	8.36

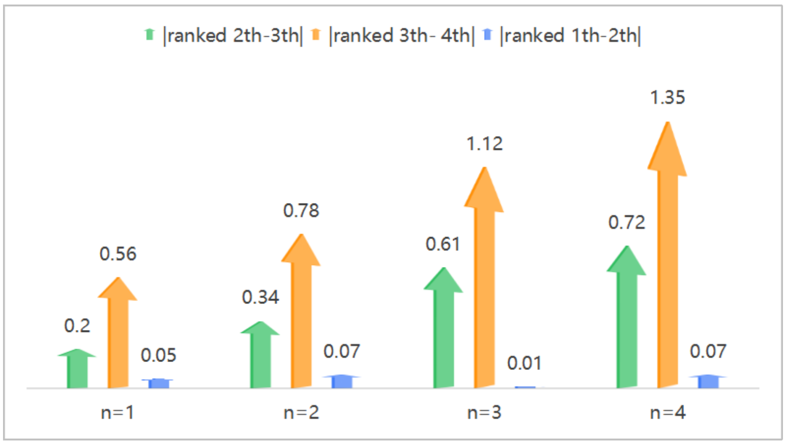


Figure 9. Differentiation comparison at n =1,2,3,4.

To validate the rationality of the proposed decision-making method based on nPIVFWAA and nPIVFWGA operators, compare the supplier selection results obtained by the proposed method with the results sorted by other methods, as shown in Table 6. It can be seen that X_2 is the best supplier; this validates the effectiveness of our proposed method. The data visualization in Table 6 in Figure 10 provides a more intuitive demonstration of the enhanced differentiation capability of the newly proposed method compared with the others, where the larger the value calculated by the proposed methods of Suo et al. [12] and Dutta et al. [35], the better the solution.

To measure the correlation between the method proposed in this paper and other decision-making methods, we adopt the Spearman rank correlation coefficient (SRCC) to quantify the similarity. On the basis on the calculation results in Table 6, we visualize the SRCC values of any two ranking methods, as shown in Figure 11. The SRCC value between the decision-making method based on nPIVFWAA and nPIVFWGA operators in this paper and the decision-making methods proposed by Yue et al. [31] and Dutta et al. [35] is 0.8, indicating that our ranking results are highly similar to those of Yue et al. [31] and Dutta et al. [35]. By contrast, the SRCC value between our method and the decision-making method proposed by Suo et al. [12] is 0.4, which suggests a low similarity between the two ranking results. Nevertheless, the optimal supplier is consistently X_2. The reason for this result may be due to the small sample size. Increasing the number of alternatives may increase the SRCC.

Table 6. Comparison of different methods for n = 3.

Methods		Suppliers' order
Suo et al. [12]	$D_1 < D_3 < D_4 < D_2$	$X_2 > X_4 > X_3 > X_1$
Yue et al. [31]	$\xi_2 < \xi_3 < \xi_4 < \xi_1$	$X_2 > X_3 > X_4 > X_1$
Dutta et al. [35]	$\eta_1 < \eta_4 < \eta_3 < \eta_2$	$X_2 > X_3 > X_4 > X_1$
nPIVFWAA operator	$S_2 < S_3 < S_1 < S_4$	$X_2 > X_3 > X_1 > X_4$
nPIVFWGA operator	$S^*_2 < S^*_3 < S^*_1 < S^*_4$	$X_2 > X_3 > X_1 > X_4$

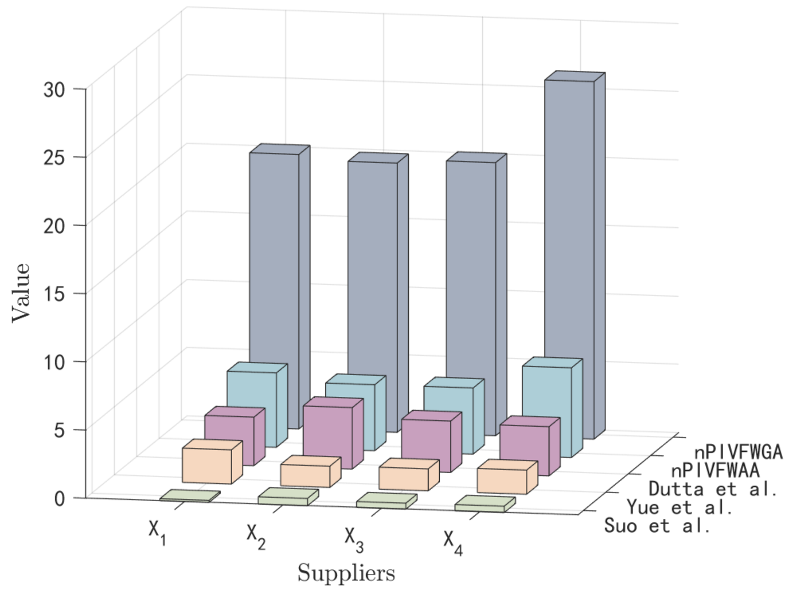


Figure 10. Ranking of suppliers for different methods.

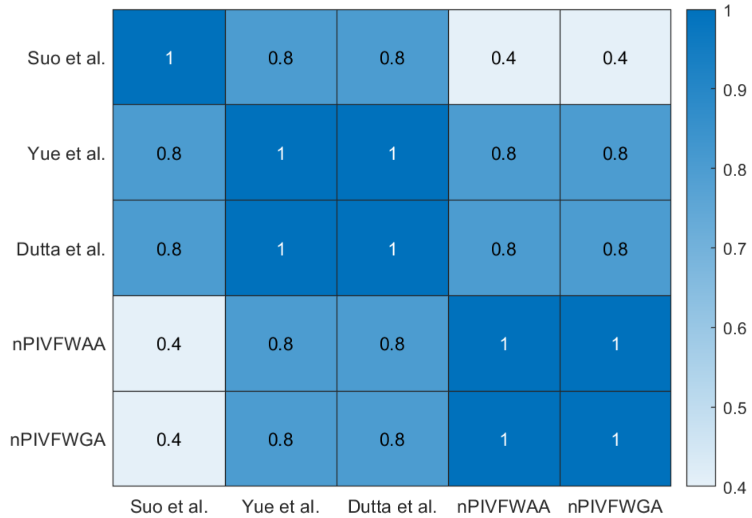


Figure 11. Heatmap of Spearman rank correlation coefficient between our decision method and the other decision-making methods.

In order to compare the discrimination between two suppliers more scientifically, the difference between the paths $S_\iota(\iota = 1, 2, 3, 4)$ of the two suppliers that are close to each other. The larger the difference, the better the discrimination. As shown in Figure 12, the blue bar represents the degree of differentiation between the first-ranked supplier of and the second-ranked supplier, while the red bar represents the degree of differentiation between the suppliers ranked second and third. Obviously, the discriminative effect of the method we introduced is better, thus further verifying the practicality of the newly proposed path graph-based method.

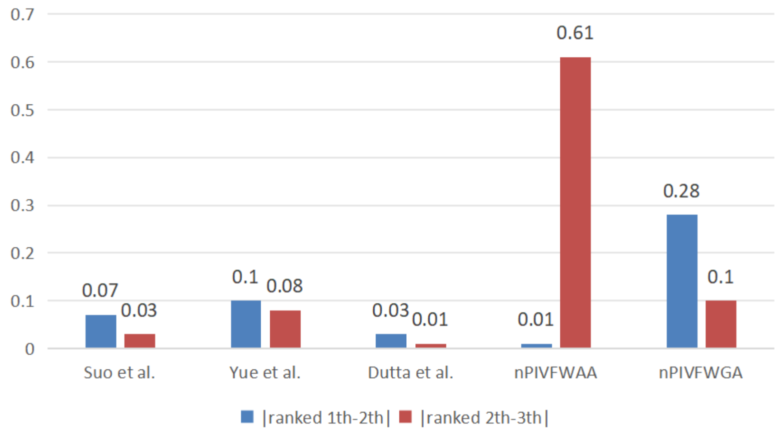


Figure 12. Differentiation comparison of different methods.

5.3. Sensitivity analysis

Considering that changes in the parameters can lead to changes in the suppliers' ranking, our subsequent investigation will focus on the sensitivity of the parameter p.

As shown in Figure 13, when $1 \leq p \leq 4$, the path value S_ι of the suppliers shows a downward trend; S_ι remains stable for $p > 4$, and the differentiation effect among suppliers is significantly weakened (with the highest degree of differentiation at $p = 1$). The underlying mechanism of this phenomenon lies in the gradual saturation of the extremum-dominant characteristic of the $L-p$ norm as p increases: The result of the $L-p$ norm gradually converges to the absolute value of the maximum component of the vector (i.e., the L_∞ norm) with an increase in p. The calculation of the path value is dominated by the maximum component in the attributes for $p > 4$, and the contribution of the other components is compressed, so S_ι no longer changes with p.

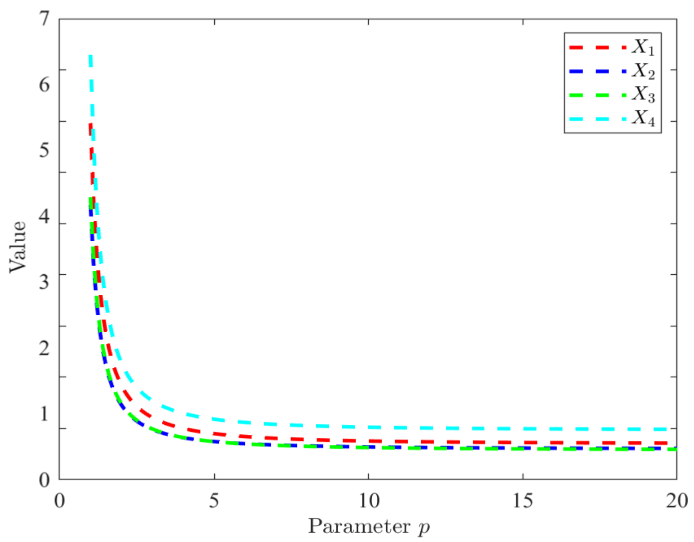


Figure 13. Sensitivity analysis chart.

It is worth noting that although the differentiation degree weakens when $p > 4$, the ranking of the suppliers does not change, and always remains $X_2 \succ X_3 \succ X_1 \succ X_4$. This indicates that the change in the parameter p does not affect the ranking result; that is, the ranking is insensitive to the parameter p, which is an extension of the extremum-dominant characteristic of the L - p norm: The ranking is determined by the key (maximum) components in the attributes and is not affected by the further adjustment of p. Although the ranking is insensitive to the change in p, selecting an appropriate p value (such as $p = 1$) can achieve a better differentiation effect among suppliers; at the same time, this also verifies that the decision-making method proposed in this paper has strong robustness.

6. Conclusions

In practical applications, MAGDM problems are widespread, which highlights the key role of aggregation operators in the group decision-making process. Therefore, on the basis on triangular norms, this paper proposes new operational rules for nPIVFNs and constructs two different types of aggregation operators: The weighted arithmetic averaging operator and the weighted geometric averaging operator. The advantages of the newly proposed aggregation operators are as follows: They avoid situations such as meaningless forms like 0^0, overcome the defects existing in the existing aggregation operators, are more capable of capturing the compensatory relationships among attributes, and are also easy to extend to other polygonal fuzzy sets in subsequent studies. Aiming at the problem of the high computational complexity of the existing graph-based decision-making methods, this paper proposes a new decision-making method combining paths and graphs. The primary constraint of the present study is rooted in the choice of n values. For the sake of computational expediency, the paper exclusively delves into the instances where n=1,2,3,4, without exploring additional selections. In the future, we hope to apply the newly proposed decision-making method to other extended fuzzy sets such as n-polygonal interval-valued fuzzy sets when selecting the appropriate n value. Furthermore, it is hope that the newly presented information measure will find application in domains including pattern recognition and machine learning.

Author contributions

Chunfeng Suo: Writing-review and editing, Supervision, Project administration, Funding acquisition. Lili Zhang: Conceptualization, Methodology, Software, Validation, Writing-original draft, Writing-review and editing. Shuang Guo: Investigation, Supervision.

Use of Generative AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest in the publication of this paper.

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Appendix A

(1) When n =1, 2, 4, the corresponding language variables are as shown Tables A1–A3.

Table A1. The corresponding language variable at n = 1.

Linguistic variables	Polygonal interval-valued fuzzy numbers
Very bad (VB)	[(0,0,0,0.05),(0,0,0,0.1)]
Bad (B)	[(0.05,0.07,0.11,0.25),(0,0.02,0.25,0.35)]
Medium bad (MB)	[(0.15,0.3,0.35,0.4),(0,0.04,0.4,0.45)]
Medium (M)	[(0.4,0.42,0.48,0.55),(0.3,0.32,0.5,0.6)]
Medium good (MG)	[(0.5,0.52,0.62,0.7),(0.45,0.48,0.67,0.75)]
Good (G)	[(0.7,0.72,0.78,0.85),(0.55,0.58,0.8,0.94)]
Very good (VG)	[(0.95,0.97,1,1),(0.85,0.89,1,1)]

Table A2. The corresponding language variable at $n = 2$.

Linguistic variables	Polygonal interval-valued fuzzy numbers
Very bad (VB)	[(0,0,0,0,0,0.05),(0,0,0,0,0,0.1)]
Bad (B)	[(0.05,0.07,0.08,0.11,0.17,0.25),(0,0.02,0.04,0.25,0.32,0.35)]
Medium bad (MB)	[(0.15,0.28,0.29,0.35,0.38,0.4),(0,0.04,0.12,0.4,0.43,0.45)]
Medium (M)	[(0.4,0.42,0.45,0.48,0.5,0.55),(0.3,0.32,0.35,0.5,0.56,0.6)]
Medium good (MG)	[(0.5,0.52,0.55,0.62,0.65,0.7),(0.45,0.48,0.5,0.67,0.7,0.75)]
Good (G)	[(0.7,0.72,0.75,0.78,0.8,0.85),(0.55,0.58,0.6,0.8,0.9,0.94)]
Very good (VG)	[(0.95,0.97,0.98,1,1,1),(0.85,0.89,0.9,1,1,1)]

Table A3. The corresponding language variable at $n = 4$.

Linguistic variables	Polygonal interval-valued fuzzy numbers
Very bad (VB)	[(0,0,0,0,0,0,0,0,0,0.05),(0,0,0,0,0,0,0,0,0,0.1)]
Bad (B)	[(0.05,0.06,0.07,0.08,0.1,0.1,0.1,0.11,0.17,0.25),(0,0.02,0.03,0.04,0.1,0.1,0.11,0.25,0.32,0.35)]
Medium bad (MB)	[(0.15,0.2,0.28,0.29,0.3,0.31,0.32,0.35,0.38,0.4),(0,0.04,0.08,0.12,0.3,0.32,0.38,0.4,0.43,0.45)]
Medium (M)	[(0.4,0.41,0.42,0.45,0.45,0.47,0.47,0.48,0.5,0.55),(0.3,0.31,0.32,0.35,0.4,0.45,0.48,0.5,0.56,0.6)]
Medium good (MG)	[(0.5,0.51,0.52,0.55,0.58,0.6,0.61,0.62,0.65,0.7),(0.45,0.47,0.48,0.5,0.5,0.6,0.62,0.67,0.7,0.75)]
Good (G)	[(0.7,0.71,0.72,0.75,0.75,0.77,0.77,0.78,0.8,0.85),(0.55,0.56,0.58,0.6,0.7,0.78,0.79,0.8,0.9,0.94)]
Very good (VG)	[(0.95,0.96,0.97,0.98,1,1,1,1,1,1),(0.85,0.87,0.89,0.9,1,1,1,1,1,1)]

(2) The elements $h_{\iota j}^{k}$ are shown as follows:

$h_{11}^{1} = h_{16}^{1} = h_{24}^{1} = h_{26}^{1} = h_{43}^{1} = h_{43}^{1} = h_{15}^{2} = h_{33}^{2} = h_{43}^{2} = h_{44}^{2} = h_{11}^{3} = h_{16}^{3} = h_{21}^{3} = h_{33}^{3} = [(0.5, 0.52, 0.55, 0.58, 0.6, 0.62, 0.65, 0.7), (0.45, 0.48, 0.5, 0.5, 0.6, 0.67, 0.7, 0.75)]$,

$h_{33}^{1} = [(0.4, 0.42, 0.45, 0.45, 0.47, 0.48, 0.5, 0.55), (0.3, 0.32, 0.35, 0.4, 0.48, 0.5, 0.56, 0.6)]$,

$h_{12}^{1} = h_{13}^{1} = h_{25}^{1} = h_{31}^{1} = h_{34}^{1} = h_{45}^{1} = h_{14}^{2} = h_{22}^{2} = h_{23}^{2} = h_{26}^{2} = h_{32}^{2} = h_{34}^{2} = h_{23}^{3} = h_{24}^{3} = h_{31}^{3} = h_{32}^{3} = h_{32}^{3} = [(0.95, 0.97, 0.98, 1, 1, 1, 1, 1), (0.85, 0.89, 0.9, 1, 1, 1, 1, 1)]$,

$h_{14}^{1} = h_{15}^{1} = h_{21}^{1} = h_{22}^{1} = h_{23}^{1} = h_{26}^{1} = h_{32}^{1} = h_{35}^{1} = h_{41}^{1} = h_{42}^{1} = h_{46}^{1} = h_{11}^{2} = h_{12}^{2} = h_{13}^{2} = h_{16}^{2} = h_{21}^{2} = h_{24}^{2} = h_{25}^{2} = h_{31}^{2} = h_{35}^{2} = h_{41}^{2} = h_{42}^{2} = h_{44}^{2} = h_{45}^{2} = h_{12}^{3} = h_{13}^{3} = h_{14}^{3} = h_{15}^{3} = h_{22}^{3} = h_{25}^{3} = h_{26}^{3} = h_{34}^{3} = h_{35}^{3} = h_{36}^{3} = h_{41}^{3} = h_{43}^{3} = h_{44}^{3} = h_{45}^{3} = h_{46}^{3} = [(0.7, 0.72, 0.75, 0.75, 0.77, 0.78, 0.8, 0.85), (0.55, 0.58, 0.6, 0.7, 0.78, 0.8, 0.9, 0.94)]$.

(3) The elements $\overline{h}_{\iota j}^{k}$ are shown as follows:

$\overline{h}_{11}^{1} = \overline{h}_{33}^{2} = \overline{h}_{43}^{2} = \overline{h}_{46}^{2} = \overline{h}_{11}^{3} = \overline{h}_{21}^{3} = \overline{h}_{33}^{3} = [(0.5455, 0.6364, 0.6909, 0.7273, 0.7636, 0.8182, 0.8727, 0.9091), (0.4545, 0.5455, 0.6000, 0.7273, 0.9091, 0.9091, 0.9455, 1.0000)]$,

$\overline{h}_{16}^{1} = \overline{h}_{36}^{1} = \overline{h}_{15}^{2} = \overline{h}_{16}^{3} = [(0.4898, 0.5918, 0.6531, 0.6939, 0.7347, 0.7959, 0.8571, 0.8980), (0.3878, 0.4898, 0.5510, 0.6939, 0.8980, 0.8980, 0.9388, 1.0000)]$,

$\overline{h}_{43}^{1} = [(0.4286, 0.5000, 0.5429, 0.5714, 0.6000, 0.6429, 0.6857, 0.7143), (0.3571, 0.4286, 0.4714, 0.5714, 0.7143, 0.7143, 0.7429, 0.7857)]$,

$\overline{h}_{12}^{1} = \overline{h}_{25}^{1} = \overline{h}_{45}^{1} = \overline{h}_{22}^{2} = \overline{h}_{32}^{2} = \overline{h}_{32}^{3} = \overline{h}_{42}^{3} = [(0, 0, 0, 0, 0, 0.0444, 0.0667, 0.1111), (0, 0, 0, 0, 0, 0.2222, 0.2444, 0.3333)]$,

$\overline{h}_{13}^{1} = [(0, 0, 0, 0, 0, 0.0286, 0.0429, 0.0714), (0, 0, 0, 0, 0, 0.1429, 0.1571, 0.2143)]$,

$\overline{h}_{31}^{1} = \overline{h}_{23}^{2} = \overline{h}_{26}^{2} = \overline{h}_{23}^{3} = \overline{h}_{31}^{3} = [(0, 0, 0, 0, 0, 0.0364, 0.0545, 0.0909), (0, 0, 0, 0, 0, 0.1818, 0.2000, 0.2727)]$,

$\overline{h}_{15}^{1} = \overline{h}_{22}^{1} = \overline{h}_{32}^{1} = \overline{h}_{35}^{1} = \overline{h}_{42}^{1} = \overline{h}_{12}^{2} = \overline{h}_{42}^{2} = \overline{h}_{12}^{3} = \overline{h}_{22}^{3} = [(0.3333, 0.4444, 0.4889, 0.5111, 0.5556, 0.5556, 0.6222, 0.6667), (0.1333, 0.2222, 0.4444, 0.4889, 0.6667, 0.8889, 0.9333, 1.0000)]$,

$\overline{h}_{21}^{1} = \overline{h}_{41}^{1} = \overline{h}_{13}^{2} = \overline{h}_{16}^{2} = \overline{h}_{36}^{2} = \overline{h}_{13}^{3} = \overline{h}_{41}^{3} = \overline{h}_{43}^{3} = [(0.2727, 0.3636, 0.4000, 0.4182, 0.4545, 0.4545, 0.5091, 0.5455), (0.1091, 0.1818, 0.3636, 0.4000, 0.5455, 0.7273, 0.7636, 0.8182)]$,

$\overline{h}_{23}^{1} = [(0.2143, 0.2857, 0.3143, 0.3286, 0.3571, 0.3571, 0.4000, 0.4286), (0.0857, 0.1429, 0.2857, 0.3143, 0.4286, 0.5714, 0.6000, 0.6429)]$,

$\overline{h}_{26}^{1} = \overline{h}_{46}^{1} = \overline{h}_{25}^{2} = \overline{h}_{35}^{2} = \overline{h}_{45}^{2} = \overline{h}_{26}^{3} = \overline{h}_{36}^{3} = \overline{h}_{46}^{3} = [(0.1837, 0.2857, 0.3265, 0.3469, 0.3878, 0.3878, 0.4490, 0.4898), (0, 0.0816, 0.2857, 0.3265, 0.4898, 0.6939, 0.7347, 0.7959)]$,

$\overline{h}_{11}^{2} = \overline{h}_{21}^{2} = \overline{h}_{31}^{2} = \overline{h}_{41}^{2} = \overline{h}_{15}^{3} = \overline{h}_{25}^{3} = \overline{h}_{35}^{3} = \overline{h}_{45}^{3} = [(0.2308, 0.3590, 0.4103, 0.4359, 0.4872, 0.4872, 0.5641, 0.6154), (0, 0.1026, 0.3590, 0.4103, 0.6154, 0.8718, 0.9231, 1)]$,

$\overline{h}_{33}^{1} = [(0.6429, 0.7143, 0.7429, 0.7571, 0.7857, 0.7857, 0.8286, 0.8571), (0.5714, 0.6286, 0.7143, 0.7429, 0.8571, 0.9286, 0.9714, 1.0000)]$

$\overline{h}_{14}^{1} = [(0.4545, 0.4909, 0.5455, 0.5455, 0.5818, 0.6000, 0.6364, 0.7273), (0.1818, 0.2364, 0.2727, 0.4545, 0.6000, 0.6364, 0.8182, 0.8909)],$

$\overline{h}_{24}^{1} = \overline{h}_{44}^{1} = [(0.0909, 0.1273, 0.1818, 0.2364, 0.2727, 0.3091, 0.3636, 0.4545), (0, 0.0545, 0.0909, 0.0909, 0.2727, 0.4000, 0.4545, 0.5455)],$

$\overline{h}_{34}^{1} = [(0.9091, 0.9455, 0.9636, 1.0000, 1.0000, 1.0000, 1.0000, 1.0000), (0.7273, 0.8000, 0.8182, 1.0000, 1.0000, 1.0000, 1.0000, 1.0000)],$

$\overline{h}_{14}^{2} = \overline{h}_{34}^{2} = \overline{h}_{24}^{3} = [(0.8889, 0.9333, 0.9556, 1.0000, 1.0000, 1.0000, 1.0000, 1.0000), (0.6667, 0.7556, 0.7778, 1.0000, 1.0000, 1.0000, 1.0000, 1.0000)],$

$\overline{h}_{24}^{2} = \overline{h}_{44}^{2} = \overline{h}_{14}^{3} = \overline{h}_{34}^{3} = \overline{h}_{44}^{3} = [(0.3333, 0.3778, 0.4444, 0.4444, 0.4889, 0.5111, 0.5556, 0.6667), (0, 0.0667, 0.1111, 0.3333, 0.5111, 0.5556, 0.7778, 0.8667)].$

(4) The elements $\widetilde{h}_{\iota j}^{k}$ for n=3 are shown as follows :

$\widetilde{h}_{11} = [(0.2483, 0.3039, 0.3334, 0.3517, 0.3739, 0.3955, 0.4289, 0.4511), (0.1778, 0.2294, 0.2897, 0.3478, 0.4511, 0.4948, 0.5178, 0.5527)],$

$\widetilde{h}_{12} = [(0.1389, 0.1876, 0.2075, 0.2176, 0.2379, 0.2639, 0.3070, 0.3542), (0.0543, 0.0914, 0.1876, 0.2075, 0.2898, 0.5226, 0.5567, 0.6575)],$

$\widetilde{h}_{13} = [(0.1389, 0.1876, 0.2075, 0.2176, 0.2379, 0.2639, 0.3070, 0.3542), (0.0543, 0.0914, 0.1876, 0.2075, 0.2898, 0.5226, 0.5567, 0.6575)],$

$\widetilde{h}_{14} = [(0.3086, 0.3334, 0.3630, 0.3708, 0.3879, 0.3966, 0.4140, 0.4586), (0.1418, 0.1831, 0.2040, 0.3287, 0.3966, 0.414, 0.5047, 0.5429)],$

$\widetilde{h}_{15} = [(0.1953, 0.2629, 0.2931, 0.3099, 0.3369, 0.3469, 0.3874, 0.4143), (0.0874, 0.1448, 0.2565, 0.2997, 0.4143, 0.5198, 0.5473, 0.5887)],$

$\widetilde{h}_{16} = [(0.2422, 0.2995, 0.3303, 0.3496, 0.3725, 0.3965, 0.4307, 0.4536), (0.1713, 0.2251, 0.2838, 0.3461, 0.4536, 0.4907, 0.5139, 0.5486)],$

$\widetilde{h}_{21} = [(0.2197, 0.2817, 0.3105, 0.3270, 0.3518, 0.3640, 0.4012, 0.4259), (0.1250, 0.1787, 0.2738, 0.3186, 0.4259, 0.5137, 0.5392, 0.5777)],$

$\widetilde{h}_{22} = [(0.1484, 0.2001, 0.2212, 0.2318, 0.2532, 0.2749, 0.3175, 0.3613), (0.0583, 0.0979, 0.2001, 0.2212, 0.3077, 0.5242, 0.5572, 0.6493)],$

$\widetilde{h}_{23} = [(0.0710, 0.0973, 0.1083, 0.1138, 0.1252, 0.1849, 0.2313, 0.3032), (0.0271, 0.0461, 0.0973, 0.1083, 0.1551, 0.5113, 0.5533, 0.7081)],$

$\widetilde{h}_{24} = [(0.2806, 0.3088, 0.3399, 0.3657, 0.3830, 0.3970, 0.4204, 0.4675), (0.1463, 0.1943, 0.2170, 0.3096, 0.3867, 0.4316, 0.4875, 0.5341)],$

$\widetilde{h}_{25} = [(0.0981, 0.1550, 0.1782, 0.1900, 0.2138, 0.2406, 0.2893, 0.3405), (0, 0.0430, 0.1550, 0.1782, 0.2748, 0.5269, 0.5645, 0.6685)]$,

$\widetilde{h}_{26} = [(0.1051, 0.1655, 0.1902, 0.2026, 0.2277, 0.2502, 0.2987, 0.3468), (0, 0.0461, 0.1655, 0.1902, 0.2920, 0.5289, 0.5655, 0.6614)]$,

$\widetilde{h}_{31} = [(0.0422, 0.0679, 0.0788, 0.0843, 0.09560.1617, 0.2112, 0.2892), (0, 0.0182, 0.0679, 0.0788, 0.1257, 0.5114, 0.5561, 0.7187)]$,

$\widetilde{h}_{32} = [(0.0710, 0.0973, 0.1083, 0.1138, 0.1252, 0.1849, 0.2313, 0.3032), (0.0271, 0.0461, 0.0973, 0.1083, 0.1551, 0.5113, 0.5533, 0.7081)]$,

$\widetilde{h}_{33} = [(0.3063, 0.3507, 0.3747, 0.3899, 0.4077, 0.4268, 0.4535, 0.4714), (0.2618, 0.3036, 0.3383, 0.3873, 0.4714, 0.4842, 0.5046, 0.5287)]$,

$\widetilde{h}_{34} = [(0.3688, 0.3920, 0.4128, 0.4267, 0.4359, 0.4406, 0.4502, 0.4755), (0.2310, 0.2729, 0.2888, 0.4045, 0.4406, 0.4502, 0.5028, 0.5265)]$.

$\widetilde{h}_{35} = [(0.1622, 0.2372, 0.2672, 0.2822, 0.3122, 0.3122, 0.3573, 0.3873), (0.0271, 0.0871, 0.2372, 0.2672, 0.3873, 0.5376, 0.5677, 0.6129)]$.

$\widetilde{h}_{36} = [(0.1984, 0.2652, 0.2958, 0.3130, 0.3397, 0.3514, 0.3914, 0.4181), (0.0940, 0.1514, 0.2577, 0.3035, 0.4181, 0.5169, 0.5441, 0.5852)]$.

$\widetilde{h}_{41} = [(0.1844, 0.2545, 0.2825, 0.2965, 0.3246, 0.3246, 0.3667, 0.3947), (0.0583, 0.1143, 0.2545, 0.2825, 0.3947, 0.5351, 0.5632, 0.6054)]$,

$\widetilde{h}_{42} = [(0.1273, 0.1724, 0.1909, 0.2002, 0.2191, 0.2506, 0.2942, 0.3454), (0.0496, 0.0836, 0.1724, 0.1909, 0.2677, 0.5206, 0.5561, 0.6670)]$,

$\widetilde{h}_{43} = [(0.2534, 0.3081, 0.3360, 0.3530, 0.3748, 0.3932, 0.4259, 0.4477), (0.1800, 0.2298, 0.2961, 0.3482, 0.4477, 0.4993, 0.5216, 0.5551)]$,

$\widetilde{h}_{44} = [(0.1777, 0.2090, 0.2559, 0.2726, 0.3040, 0.3254, 0.3628, 0.4414), (0, 0.0471, 0.0786, 0.1777, 0.3139, 0.3750, 0.4904, 0.5588)]$,

$\widetilde{h}_{45} = [(0.0981, 0.1550, 0.1782, 0.1900, 0.2138, 0.2406, 0.2893, 0.3405), (0, 0.0430, 0.1550, 0.1782, 0.2748, 0.5269, 0.5645, 0.6685)]$,

$\widetilde{h}_{46} = [(0.1855, 0.2551, 0.2859, 0.3028, 0.3307, 0.3397, 0.3814, 0.4092), (0.0733, 0.1319, 0.2492, 0.2919, 0.4092, 0.5227, 0.5513, 0.5945)]$,