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*Research article*

## Lyapunov stability of a heavy charged gyrostat in a central attractive potential

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**Abstract:** This study investigated the stability of certain motions of a heavy, electrically charged gyrostat moving in a central attractive field. The Euler–Poisson equations were derived and then reformulated within the Lie–Poisson framework. We identified and analyzed the system’s permanent rotations, presenting a mechanical interpretation of these motions. To assess stability, the Energy–Casimir method is employed to establish sufficient stability criteria. Moreover, the necessary conditions for the stability of the corresponding equilibrium configurations were also presented.

**Keywords:** Lyapunov stability; energy Casimir method; rigid body; gyrostat

**Mathematics Subject Classification:** 70E55, 37J25

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### 1. Introduction

Rigid-body dynamics is among the earliest and most challenging subjects in classical mechanics. The first integrable example was presented by Euler in 1750 through his analysis of the torque-free rigid body, which laid the foundation for subsequent studies. Later contributions by d’Alembert, Poinsot, Lagrange, and Poisson culminated in the formulation of the Euler–Poisson equations, governing the motion of a rigid body with a fixed point under external forces. This system comprises six coupled first-order differential equations. Further notable integrable cases were later introduced by Lagrange and Kowalevskaya [1].

A gyrostat represents a simple composite mechanical system consisting of a rigid body  $\mathcal{P}_1$ , referred to as the platform, which contains an internal body  $\mathcal{P}_2$  that is not rigidly fixed. The relative motion of  $\mathcal{P}_2$  with respect to  $\mathcal{P}_1$  is designed so that the overall center of mass remains unchanged. The combined dynamics of  $\mathcal{P}_1$  and  $\mathcal{P}_2$  are commonly described as a dual-spin system [2]. The notion of a gyrostat was originally proposed by Zhukovskii [3] and later employed by Volterra to model the Earth’s rotation by representing it as a cavity filled with an inviscid homogeneous fluid [4]. Over recent decades, gyrostat

dynamics has attracted extensive attention and has been applied in numerous engineering contexts, including satellites, spacecraft, aircraft, and robotic systems. In particular, gyrostats play a crucial role in attitude control and rotational stabilization of spacecraft [5–8]. A classical practical example is the use of gyrostats in torpedoes to preserve their initial direction of motion.

Research problems related to rigid-body motion and its gyrostat generalizations typically fall into three main categories. The first concerns integrability and the construction of first integrals of motion, as discussed in [9, 10]. The second focuses on the identification of periodic solutions and the analysis of chaotic dynamics in rigid-body gyrostat systems. Recent studies have extensively explored the dynamics and stability of gyrostat systems. Multistability and coexisting attractors in chaotic gyrostats were analyzed using basins of attraction, bifurcation diagrams, Lyapunov exponents, and Poincaré sections, along with synchronization of QAV systems via Lyapunov-based control [11]. The motion and equilibrium of gyrostat satellites under the Newtonian forces of coplanar bodies and small tri-axial bodies in planar rings were investigated, with equations of motion derived and stability analyzed [12, 13]. Chaotic instability of spinning gyrostat satellites under gravity-gradient torques was examined using Hamiltonian equations in Deprit’s canonical variables and the Poincaré–Arnold–Melnikov integral [14], while low-order models represented as coupled gyrostats led to “minimal chaotic models” highlighting energy conservation and fixed-point arrangements for chaos [15]. Rotational dynamics of charged symmetric gyrostats under electromagnetic fields and gyrostatic moments were studied analytically and numerically, with stability assessed via phase-plane diagrams, the Poincaré small parameter method, and fourth-order Runge-Kutta simulations, offering insights for satellites, submarines, autopilot systems, and planetary applications [16–18]. Additionally, theoretical connections between free gyrostats and parametric quadratic Hamiltonians were established [19]. The third category addresses the stability of particular motions of rigid bodies and gyrostats, either in orbital motion or about a fixed point. Recent studies have extensively investigated the dynamics and stability of gyrostat and rigid body systems within Hamiltonian frameworks. In [20], non-canonical Hamiltonian dynamics of a gyrostat interacting with  $n$  bodies were reduced, yielding conditions for relative equilibria and generalizing classical celestial mechanics problems. Stability of permanent rotations of heavy gyrostats was analyzed via the energy-Casimir method in [21–23], where sufficient and necessary conditions were established, highlighting the influence of geometry and gyrostatic moments. In [24–26], rigid-body gyrostat systems under gravitational and magnetic fields were formulated within Lie–Poisson Hamiltonian structures, and their equilibrium and stability were studied using linearization and energy-Casimir approaches. Fluid interaction effects were considered in [27], showing equivalence with classical rigid body motion. Additionally, non-integrability and stability of galactic star motion were examined in [28] using analytical and numerical methods. Finally, [29] derived integrability conditions and analyzed Lyapunov stability of gyrostat motion using Chetaev’s method.

It is worth emphasizing that the present work considers only the attitude dynamics of a gyrostat about a fixed point, deliberately omitting the interaction between rotational and translational motion. In practical spacecraft settings, however, spin–orbit coupling can have a substantial impact on the overall dynamics, particularly for orbiting bodies subject to gravity–gradient effects and other orbital perturbations. A full description of roto–orbital motion must therefore incorporate translational degrees of freedom and often relies on geometric reduction techniques to handle the resulting increase in complexity. For example, Crespo and Turner [30] constructed a Poisson structure for the complete

gravitational  $N$ -body problem and applied reduction by stages to study the coupled rotational–translational dynamics. Likewise, Zapata et al. [31] analyzed relative equilibria within an intermediary model of roto–orbital motion, focusing on the regime of slow rotation.

The present study falls within this third research direction. The stability of rigid-body gyrostat motion has received increasing attention in recent decades, largely driven by advances and demands in space exploration. In 1956, Rumiantsev derived sufficient conditions for the stability of a rigid body rotating about a principal axis through a fixed point, without imposing restrictions on the remaining principal moments of inertia [32]. Subsequent investigations extended these results to systems influenced by various force fields, including arbitrary potential fields [33] and Newtonian gravitational fields [34]. Rumiantsev and his collaborators employed Lyapunov-based techniques [35] to analyze the stability of the associated systems of differential equations.

Earlier, Routh applied linearization methods to study the stability of the steady rotation of a heavy asymmetric top [36]; however, such linear approaches generally fail to capture the full nonlinear behavior of the system [37]. In 1961, Rumiantsev used Lyapunov’s direct method to investigate the stability of a heavy gyrostat rotating about a fixed point, assuming coincidence between the center of mass and the fixed point, and derived sufficient stability conditions [38]. Similar problems were independently examined by Crespo da Silva [39] and by Kane and Fowler [40], with all studies arriving at consistent stability criteria. Additional cases and refinements were later explored in works by Rumiantsev [38], Anchev [41, 42], and Kolesnikov [43], among others.

More recently, the energy–Casimir method has become a powerful tool for assessing Lyapunov stability in systems possessing conserved quantities, eliminating the explicit construction of a Lyapunov function. Stability is ensured when the second variation of the combined energy–Casimir functional is positive definite [44]. This approach has been successfully applied to various rigid-body gyrostat configurations. Iñarrea et al. [21] employed the method to analyze the permanent rotations of an asymmetric gyrostat in a uniform Newtonian field, establishing both necessary and sufficient stability conditions. Elmandouh [24] studied a heavy charged gyrostat with an internal rotor aligned with an external magnetic field, deriving the equations of motion and identifying stability criteria for specific families of steady rotations. The influence of combined gravitational and magnetic fields on the stability of an axisymmetric gyrostat was later examined in [26]. Furthermore, the stability of stationary motions of a rigid-body gyrostat immersed in an ideal incompressible fluid, with distinct mass and buoyancy centers, was investigated in [27]. The dynamics of a heavy symmetric gyrostat with a rotor axis not coinciding with the mass-center axis was analyzed by Iñarrea et al. [45], where necessary and sufficient conditions for stability and instability were derived using the energy–Casimir framework. This work generalizes earlier results reported in [21]. Finally, Guirao and Vera [46] studied gyrostat motion in an ideal incompressible fluid by reformulating Kirchhoff’s equations within a Lie–Poisson structure, thereby reinforcing the applicability of the energy–Casimir method. Their analysis included both stability assessments and the identification of Hopf bifurcations at equilibrium configurations.

This study examines the stability of specific motions of a heavy, electrically charged gyrostat subjected to a central attractive field. The problem is of considerable interest because it encompasses a broad class of important mechanical systems, as will be discussed later. To the best of our knowledge, the stability of these particular motions has not been addressed in the existing literature. This gap motivates us to contribute to and extend the current body of research.

This study is arranged as follows: Motion equations and their first integrals are introduced in Section 2, which includes the Lie-Poisson Hamilton formulation to motion equation. Permanent rotations are discussed in Section 3, and their mechanical explanation is provided. We announce in Section 4 the necessary and sufficient conditions for Lyapunov stability of the permanent rotations stated in Section 3. Finally, Section 5 concludes the work.

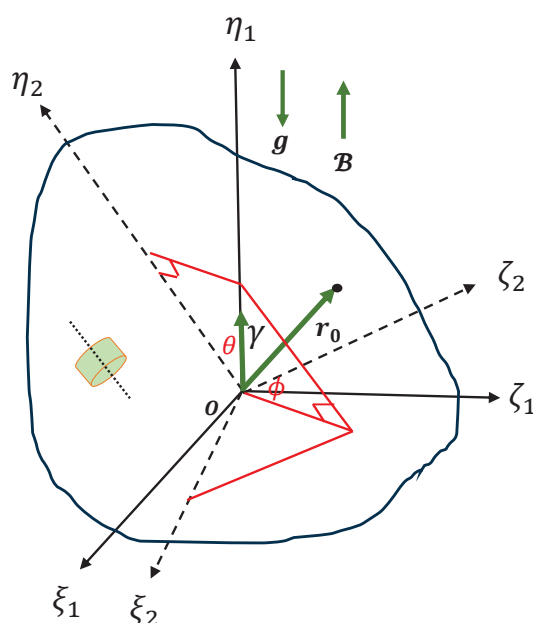
## 2. Motion's equations

This section strives to present the Euler–Poisson equations for a heavy charged gyrostat. This motion is assumed to result from the combined action of three fields: Gravitational, uniform magnetic, and attractive central fields.

The physical configuration is illustrated in Figure 1, where the body-fixed and space-fixed frames, rotor, and directions of the external fields are shown. Two reference frames are chosen suitably to describe the motion. The first one,  $O\xi_1\xi_2$ , is fixed in space, and the unit vector along the axis  $O\eta_1$  is denoted by  $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3)$ . The second,  $O\xi_1\xi_2$ , is fixed in the body and is chosen to be the inertia principal system at  $O$ ; hence, the gyrostat inertia tensor is  $\mathbb{I} = \text{diag}(A, B, C)$ . The body's angular velocity is denoted by  $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ . Both vectors  $\boldsymbol{\gamma}$  and  $\boldsymbol{\omega}$  refer to the body frame. To derive the governing equations, the theory of angular momentum is utilized [1]. It provides

$$\frac{d\mathbf{G}^*}{dt} = \mathbf{N}_O, \quad (2.1)$$

where  $\mathbf{N}_O$  and  $\mathbf{G}^*$  are the resultant external torque about  $O$  and the gyrostat total angular momentum, respectively.



**Figure 1.** Schematic of the heavy charged gyrostat: rigid body with an internal rotor (angular momentum  $\sigma$  along the third axis), fixed point  $O$ , body axes, vertical direction  $\boldsymbol{\gamma}$ , and external fields (gravity, magnetic, central attractive).

The gyrostat total angular momentum is

$$\mathbf{G}^* = \mathbf{G} + \mathbf{n}, \quad (2.2)$$

where  $\mathbf{G} = \mathbb{I}\boldsymbol{\omega}$  is the rigid-body angular momentum in the absence of rotor motion. The vector  $\mathbf{n} = (0, 0, \sigma)$ ,  $\sigma \in \mathbb{R}$  is the rotor relative angular momentum with respect to the body. In literature,  $\mathbf{n}$  is called the gyrostatic momentum [1].

The magnetic field  $\mathcal{B}$  is assumed to be constant and directed along the  $O\eta_1$ -axis. Thus,

$$\mathcal{B} = \mathcal{B}\boldsymbol{\gamma}, \quad (2.3)$$

where  $\mathcal{B}$  denotes its magnitude. Consider now a charged element  $de$  within the body at point  $q$ , moving with velocity  $\mathbf{v}(q)$  at position  $\mathbf{r}$ . The Lorentz force acting on this element is given by

$$d\mathbf{F} = de(\mathbf{v} \times \mathcal{B}) = \mathcal{B}\sigma d\tau[(\boldsymbol{\omega} \times \mathbf{r}) \times \boldsymbol{\gamma}], \quad (2.4)$$

where  $\sigma$  is the charge density\* and  $d\tau$  is the volume element of the gyrostat. Expanding the cross product gives

$$d\mathbf{F} = \mathcal{B}\sigma d\tau[(\boldsymbol{\omega} \cdot \boldsymbol{\gamma})\mathbf{r} - (\mathbf{r} \cdot \boldsymbol{\gamma})\boldsymbol{\omega}]. \quad (2.5)$$

The total magnetic torque is found by integrating over the entire charge distribution. Thus, we have

$$\begin{aligned} N_O^1 &= \int_{\tau} \mathbf{r} \times d\mathbf{F} = \boldsymbol{\omega} \times \sigma\mathcal{B} \int_{\tau} \mathbf{r}(\mathbf{r} \cdot \boldsymbol{\gamma}) d\tau \\ &= \boldsymbol{\omega} \times K\boldsymbol{\gamma}, \end{aligned} \quad (2.6)$$

where  $K$  is a constant  $3 \times 3$  matrix. For simplicity, we take  $K = \text{diag}(k_1, k_2, k_3)$ , where  $k_1$ ,  $k_2$ , and  $k_3$  are arbitrary parameters representing the magnetic susceptibility coefficients along the principal axes of the gyrostat. These coefficients quantify how the distributed electric charges and magnetic properties of the body respond to the external magnetic field  $\mathcal{B}$ . In essence, the term  $K\boldsymbol{\gamma}$  in the torque expression can be interpreted as an effective magnetic moment vector that is linearly related to the direction vector  $\boldsymbol{\gamma}$  through the matrix  $K$ . This linear relationship arises from the assumption that the body's electromagnetic properties are anisotropic but constant in the body-fixed frame. For more details, see [24, 48].

Following [9], the potential energy for the gravitational and attractive center fields is

$$V = \mathbf{a} \cdot \boldsymbol{\gamma} + \frac{1}{2} \boldsymbol{\gamma} \cdot \boldsymbol{\gamma} \mathbf{J}, \quad (2.7)$$

where  $\mathbf{a} = Mg\mathbf{r}_0$  ( $M$  is the gyrostat mass,  $g$  is the acceleration of the gravity, and  $\mathbf{r}_0$  is the mass center position), and  $\mathbf{J}$  is an arbitrary  $3 \times 3$  matrix. The first term in (2.7) represents the gravitational potential, whereas the second accounts for the attraction toward a central point and partially describes the electric interaction with a remote Coulomb charge positioned along the line parallel to  $\boldsymbol{\gamma}$  that passes through  $O$ . Under this interpretation, the two matrices  $\mathbf{J}$  and the inertia matrix associated with

\*Here MKS units are employed. In Gaussian units,  $de$  should be divided by the light speed  $c$  (see, e.g., [47]). The velocity and acceleration are assumed to be appropriately small to ignore both relativistic impacts and classical radiation damping.

the system's distributed electric charges are proportional. Consequently, the torque arising from these potential forces is expressed as

$$\mathbf{N}_O^2 = \boldsymbol{\gamma} \times \frac{\partial V}{\partial \boldsymbol{\gamma}} = \boldsymbol{\gamma} \times (\mathbf{a} + \boldsymbol{\gamma} \mathbf{J}). \quad (2.8)$$

Hence, the torque of combined external forces about  $O$  is

$$\begin{aligned} \mathbf{N}_O &= \mathbf{N}_O^1 + \mathbf{N}_O^2, \\ &= \boldsymbol{\gamma} \times (\mathbf{a} + \boldsymbol{\gamma} \mathbf{J}) + \boldsymbol{\omega} \times \mathbf{K} \boldsymbol{\gamma}. \end{aligned} \quad (2.9)$$

By inserting Eq (2.9) into Eq (2.1), we obtain

$$\dot{\mathbf{G}} + \boldsymbol{\omega} \times (\mathbf{G} + \mathbf{n} + \boldsymbol{\gamma} \mathbf{K}) = \boldsymbol{\gamma} \times (\boldsymbol{\gamma} \mathbf{J} + \mathbf{a}), \quad (2.10a)$$

$$\dot{\boldsymbol{\gamma}} + \boldsymbol{\omega} \times \boldsymbol{\gamma} = 0. \quad (2.10b)$$

The equations of the motion (2.10) have three classical integrals of motion, which are

$$\begin{aligned} C_0 : \frac{1}{2} \boldsymbol{\omega} \cdot \boldsymbol{\omega} \mathbb{I} + \mathbf{a} \cdot \boldsymbol{\gamma} + \frac{1}{2} \boldsymbol{\gamma} \cdot \mathbf{J} \boldsymbol{\gamma} &= h, && \text{Jacobi-integral,} \\ C_1 : \boldsymbol{\gamma} \cdot \boldsymbol{\gamma} &= 1, && \text{Geometric integral,} \\ C_2 : (\boldsymbol{\omega} \mathbb{I} + \mathbf{n} + \frac{1}{2} \boldsymbol{\gamma} \mathbf{K}) \cdot \boldsymbol{\gamma} &= f, && \text{Area's integral,} \end{aligned} \quad (2.11)$$

where  $h$  and  $f$  are two constants. The physical system governed by Eq (2.10) is of particular interest because it encompasses a wide range of classical mechanical models. This can be seen through the following special cases:

- (i) When  $\mathbf{n} = 0$  and  $\mathbf{K} = \mathbf{J} = \mathbf{0}$ , Eq (2.10) generalizes the rigid-body rotation problem about  $O$  under the influence of gravity [1].
- (ii) When  $\mathbf{K} = \mathbf{J} = \mathbf{0}$ , the current problem reduces to that of a heavy gyrostat [1].
- (iii) When  $\mathbf{J} = \mathbf{0}$ , the current problem represents the heavy charged gyrostat rotating about  $O$  [1].
- (iv) When both matrices  $\mathbf{J}$  and  $\mathbb{I}$  are proportional, Eq (2.10) describes a heavy charged gyrostat motion subjected to a Newtonian force field [1].
- (v) When the matrix  $\mathbf{J}$  is non-singular, the immersed rigid-body dynamics in an ideal, unbounded, and incompressible fluid are equivalently described by Eq (2.10) [49].

To simplify the calculations, we postulate

$$\mathbf{a} = (0, 0, a_3), \quad \mathbb{I} = \text{diag}(A, B, C), \quad \mathbf{J} = \text{diag}(j_1, j_2, j_3), \quad \mathbf{K} = \text{diag}(k_1, k_2, k_3), \quad (2.12)$$

where  $a_3, \sigma, j_i$ , and  $k_i$  ( $i = 1, 2, 3$ ) are constants. Under the assumptions in (2.12), the scalar form of

the equations of the motion (2.10) takes the form

$$\begin{aligned}\dot{G}_1 &= \frac{B-C}{BC}G_2G_3 + \frac{k_2G_3}{C}\gamma_2 - \frac{k_3\gamma_3 + \sigma}{B}G_2 + \gamma_2\gamma_3(j_3 - j_2) + a_3\gamma_2, \\ \dot{G}_2 &= \frac{C-A}{AC}G_1G_3 - \frac{k_1\gamma_1}{C}G_3 + \frac{k_3\gamma_3 + \sigma}{A}G_1 + \gamma_1\gamma_3(j_1 - j_3) - a_3\gamma_1, \\ \dot{G}_3 &= \frac{(A-B)}{AB}G_1G_2 + \frac{k_1\gamma_1G_2}{B} - \frac{k_2\gamma_2G_1}{A} + \gamma_1\gamma_2(j_2 - j_1), \\ \dot{\gamma}_1 &= \frac{\gamma_2G_3}{C} - \frac{\gamma_3G_2}{B}, \quad \dot{\gamma}_2 = \frac{\gamma_3G_1}{A} - \frac{\gamma_1G_3}{C}, \quad \dot{\gamma}_3 = \frac{\gamma_1G_2}{B} - \frac{\gamma_2G_1}{A}.\end{aligned}\tag{2.13}$$

Despite the fact that the variables used are not canonical, the problem can be characterized by considering a Hamilton function within the context of Lie-Poisson systems that have the form

$$H = \frac{1}{2}\left(\frac{G_1^2}{A} + \frac{G_2^2}{B} + \frac{G_3^2}{C}\right) + a_3\gamma_3 + \frac{j_1}{2}\gamma_1^2 + \frac{j_2}{2}\gamma_2^2 + \frac{j_3}{2}\gamma_3^2.\tag{2.14}$$

With the help of [50], the motion equation (2.13) can be written as a Hamiltonian-Poisson system spanning by the matrix  $\mathbb{T}_l$  that is given by

$$\mathbb{T}_l = \begin{pmatrix} 0 & -G_3 - l_3 & G_2 + l_2 & 0 & -\gamma_3 & \gamma_2 \\ G_3 + l_3 & 0 & -G_1 - l_1 & \gamma_3 & 0 & -\gamma_1 \\ -G_2 - l_2 & G_1 + l_1 & 0 & -\gamma_2 & \gamma_1 & 0 \\ 0 & -\gamma_3 & \gamma_2 & 0 & 0 & 0 \\ \gamma_3 & 0 & -\gamma_1 & 0 & 0 & 0 \\ -\gamma_2 & \gamma_1 & 0 & 0 & 0 & 0 \end{pmatrix},\tag{2.15}$$

where the vector  $l$  is the gyrostatic momentum given by

$$l = (l_1, l_2, l_3) = (k_1\gamma_1, k_2\gamma_2, \sigma + k_3\gamma_3),\tag{2.16}$$

satisfying the Jacobi identity

$$\mathbb{T}_l^{si}\partial_s\mathbb{T}_l^{\delta\alpha} + \mathbb{T}_l^{s\delta}\partial_s\mathbb{T}_l^{\alpha i} + \mathbb{T}_l^{s\alpha}\partial_s\mathbb{T}_l^{i\delta} = 0, \quad i, \delta, \alpha = 1, 2, \dots, 6.\tag{2.17}$$

The motion equation (2.13) reads as

$$\dot{Y} = \mathbb{T}_l\nabla H,\tag{2.18}$$

where  $\nabla H$  is the naive gradient of  $H$ . System (2.18) has two Casimirs in the form

$$C_1 : \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1,\tag{2.19a}$$

$$C_2 : G_1\gamma_1 + G_2\gamma_2 + (G_3 + \sigma)\gamma_3 + \frac{k_1}{2}\gamma_1^2 + \frac{k_2}{2}\gamma_2^2 + \frac{k_3}{2}\gamma_3^2 = f.\tag{2.19b}$$

### 3. Permanent rotations

Permanent rotations have attracted the interest of researchers for a variety of perspectives. In [51], and recently [22, 24, 26, 46], linear stability analysis was conducted for many instances. In [32], Rumiantsev depended on the extensive application of linearized equations in the variations and Lyapunov functions construction to discover his results.

**Theorem 1.** *The motion of a heavy charged gyrostat about a fixed point  $O$  under the action of a central attractive potential admits four families of permanent rotations. They are:*

(a) *The first family is given by the points*

$$Q_{1\pm} = (0, 0, \pm\omega_0 C, 0, 0, \pm 1), \quad (3.1)$$

where  $\omega_0 \in \mathbb{R}$ .

(b) *The second family is defined by the points*

$$Q_{2\pm} = (0, \pm B\omega_0 \sin \theta_0, C\omega_0 \cos \theta_0, 0, \pm \sin \theta_0, \cos \theta_0), \quad (3.2)$$

where  $\theta_0 \in (0, \pi)$  and  $\omega_0 \in \mathbb{R}$  satisfy

$$\cos \theta_0 [\omega_0^2 (B - C) + \omega_0 (k_2 - k_3) + j_3 - j_2] + a_3 - \omega_0 \sigma = 0. \quad (3.3)$$

(c) *The third family is determined by the points*

$$Q_{3\pm} = (\pm A\omega_0 \sin \theta_0, 0, C\omega_0 \cos \theta_0, \pm \sin \theta_0, 0, \cos \theta_0), \quad (3.4)$$

where  $\omega_0 \in \mathbb{R}$  and  $\theta_0 \in (0, \pi)$  satisfy

$$\gamma_3 [\omega_0^2 (A - C) + \omega_0 (k_1 - k_3) + j_3 - j_1] + a_3 - \omega_0 \sigma = 0. \quad (3.5)$$

(d) *The fourth family is represented by the points*

$$Q_{4\pm} = (A\omega_0 \sin \theta_0 \sin \phi, B\omega_0 \sin \theta_0 \cos \phi, C\omega_0 \cos \theta_0, \sin \theta_0 \sin \phi, \sin \theta_0 \cos \phi, \cos \theta_0), \quad (3.6)$$

where  $\omega_0 \in \mathbb{R}$ ,  $\theta_0 \in (0, \pi)$ , and  $\phi \in (0, 2\pi) \setminus \{\pi/2, \pi, 3\pi/2\}$ , provided that

$$j_1 = (A - B)\omega_0^2 + (k_1 - k_2)\omega_0 + j_2, \quad (3.7a)$$

$$\cos \theta_0 [(B - C)\omega_0^2 + (k_2 - k_3)\omega_0 + j_3 - j_2] - \sigma\omega_0 + a_3 = 0. \quad (3.7b)$$

*Proof.* Permanent rotations correspond to equilibrium solutions of the equations of motion (2.10), obtained by imposing  $\dot{\omega} = \dot{\gamma} = \mathbf{0}$ . The condition  $\dot{\omega} = \mathbf{0}$  defines permanent rotations—motions where the angular velocity vector remains fixed in both magnitude and direction relative to the body, and the axis of rotation remains fixed in space. From the kinematic equation (2.10b),  $\dot{\gamma} = 0$  directly implies  $\omega \parallel \gamma$ . Hence, we can write  $\omega = \omega_0 \gamma$  with  $\omega_0 \in \mathbb{R}$ . This form of the angular velocity is valid for all permanent rotations and does not depend on the specific form of the torques or system parameters [1]. Substituting  $\omega = \omega_0 \gamma$  into Eq (2.10a) gives

$$\gamma \times [\omega_0^2 \gamma \mathbb{I} + \omega_0 (\mathbf{n} + \gamma \mathbf{K}) - \mathbf{a} - \gamma \mathbf{J}] = \mathbf{0}. \quad (3.8)$$

Using the expressions in (2.12), the scalar form of Eq (3.8) becomes

$$\gamma_2 [\gamma_3 (\omega_0^2 (B - C) + \omega_0 (k_2 - k_3) + j_3 - j_2) + a_3 - \omega_0 \sigma] = 0, \quad (3.9a)$$

$$\gamma_1[\gamma_3(\omega_0^2(A - C) + \omega_0(k_1 - k_3) + j_3 - j_1) + a_3 - \omega_0\sigma] = 0, \quad (3.9b)$$

$$\gamma_1\gamma_2[\omega_0^2(A - B) + \omega_0(k_1 - k_2) - j_1 + j_2] = 0. \quad (3.9c)$$

The components of  $\boldsymbol{\gamma}$  are given by

$$\gamma_1 = \sin \theta \sin \phi, \quad \gamma_2 = \sin \theta \cos \phi, \quad \gamma_3 = \cos \theta, \quad (3.10)$$

with  $\theta \in [0, \pi]$  and  $\phi \in [0, 2\pi]$ . The angular momentum components are

$$\mathbf{G} = (G_1, G_2, G_3) = \omega_0 \boldsymbol{\gamma} \mathbb{I} = (A\omega_0 \sin \theta \sin \phi, B\omega_0 \sin \theta \cos \phi, C\omega_0 \cos \theta). \quad (3.11)$$

In the following, we express a permanent rotation as  $Q = (G_1, G_2, G_3, \gamma_1, \gamma_2, \gamma_3)$ . Equation (3.9c) is satisfied in four distinct cases:

- (a) If  $\gamma_1 = \gamma_2 = 0$ , then  $\sin \theta = 0$ , so  $\theta = 0$  or  $\theta = \pi$ , and  $\boldsymbol{\gamma} = (0, 0, \pm 1)$ . All equations in (3.9) hold identically, giving the permanent rotations  $Q_{1\pm}$  in (3.1). The rotation  $Q_{1+}$  corresponds to spinning with angular velocity  $\omega_0$  in the positive  $\boldsymbol{\gamma}$ -direction (center of mass above  $O$ ), while  $Q_{1-}$  corresponds to rotation in the opposite direction (center of mass below  $O$ ).
- (b) If  $\gamma_1 = 0$  and  $\gamma_2 \neq 0$ , then  $\boldsymbol{\gamma} = (0, \pm \sin \theta_0, \cos \theta_0)$ , with  $\theta_0 \in (0, \pi)$ . Equation (3.9b) is satisfied identically, and (3.9a) holds precisely when condition (3.3) is satisfied. Thus, the family  $Q_{2\pm}$  in (3.2) exists under condition (3.3). Mechanically, this corresponds to rotation with angular velocity  $\omega_0$  about an axis with direction cosines  $(0, \pm \sin \theta_0, \cos \theta_0)$ .
- (c) If  $\gamma_1 \neq 0$  and  $\gamma_2 = 0$ , then  $\boldsymbol{\gamma} = (\pm \sin \theta_0, 0, \cos \theta_0)$ , with  $\theta_0 \in (0, \pi)$ . Equation (3.9a) is satisfied automatically. The permanent rotations  $Q_{3\pm}$  in (3.4) exist when condition (3.5) holds. This corresponds to rotation with angular velocity  $\omega_0$  about an axis with direction cosines  $(\pm \sin \theta_0, 0, \cos \theta_0)$ .
- (d) If  $\gamma_1\gamma_2 \neq 0$ , then Eq (3.9c) requires condition (3.7a) to be satisfied. The equation resulting by substituting (3.7a) into (3.9b) to eliminate  $j_1$  holds only when (3.7b) is also satisfied, with  $\theta_0 \in (0, \pi)$ . Under condition (3.7b), Eq (3.9a) is also satisfied. Thus, the permanent rotations  $Q_{4\pm}$  in (3.6) exist exactly when both conditions in (3.7) hold.

□

#### 4. Stability analysis

To find the stability sufficient conditions to the attained permanent rotations in Theorem 1, the energy-Casimir method is applied [33, 44]. This is a generalized result provided by Ortega and Ratiu [52], which can be stated as follows:

**Theorem 2.** (*Generalized energy-Casimir method*) Let  $(\mathbb{M}, \{\cdot, \cdot\}, H)$  be a Poisson system and  $Q \in \mathbb{M}$  be an equilibrium point for the Hamiltonian vector field  $X_H$ . If  $C_1, C_2, \dots, C_n \in C^\infty(\mathbb{M})$  is a set of first integrals such that

$$d(H + C_1 + C_2 + \dots + C_n)(Q) = 0, \quad (4.1)$$

and

$$d^2(H + C_1 + C_2 + \cdots + C_n)(Q)|_{U \times U} \quad (4.2)$$

is definite for  $U = dC_1(Q) \cap dC_2(Q) \cap \cdots \cup dC_n(Q)$ , then  $Q$  is stable. If  $U = \{0\}$ ,  $Q$  is always stable.

In applying Theorem 2, it is essential to explicitly construct a basis for the vector space  $U = \ker dC_1 \cap \ker dC_2$  at each equilibrium point and to verify the linear independence of these basis vectors. This ensures that the subsequent computation of the reduced Hessian matrix and the application of Sylvester's criterion are performed on the correct subspace. In the following analysis, we provide these details for each family of permanent rotations.

Using the integrals of the motion in (2.19), we define the next augmented Hamiltonian

$$\begin{aligned} \mathcal{H} = & \frac{1}{2} \left( \frac{G_1^2}{A} + \frac{G_2^2}{B} + \frac{G_3^2}{C} \right) + a_3 \gamma_3 + \frac{j_1}{2} \gamma_1^2 + \frac{j_2}{2} \gamma_2^2 + \frac{j_3}{2} \gamma_3^2 + \rho_1 (\gamma_1^2 + \gamma_2^2 + \gamma_3^2) + \rho_2 (\gamma_1 G_1 \\ & + \gamma_2 G_2 + (G_3 + \sigma) \gamma_3 + \frac{k_1}{2} \gamma_1^2 + \frac{k_2}{2} \gamma_2^2 + \frac{k_3}{2} \gamma_3^2), \end{aligned} \quad (4.3)$$

where  $\rho_1, \rho_2$  are free parameters and they are selected in such a way to make the equilibrium point  $Q$  a critical point for the augmented Hamiltonian (4.3), i.e.,

$$\frac{\partial \mathcal{H}}{\partial G_i} \Big|_Q = 0, \quad \frac{\partial \mathcal{H}}{\partial \gamma_i} \Big|_Q = 0, \quad i = 1, 2, 3. \quad (4.4)$$

We start investigating the stability of the permanent rotations, which are presented by Theorem 1 by utilizing Theorem 2. We start by examining the stability of  $Q_{1\pm}$ .

**Theorem 3.** *Sufficient for the Lyapunov stability of the permanent rotation  $Q_{1\pm}$  is*

$$\pm a_3 < \max(\chi_1, \chi_2), \quad (4.5)$$

where

$$\begin{aligned} \chi_1 &= \pm \omega_0 \sigma + j_1 - j_3 - \omega_0 (k_1 - k_3) - (A - C) \omega_0^2, \\ \chi_2 &= \pm \omega_0 \sigma + j_2 - j_3 + \omega_0 (k_3 - k_2) - (B - C) \omega_0^2. \end{aligned} \quad (4.6)$$

*Proof.* The permanent rotations  $Q_{1\pm}$  are critical points for the augmented Hamiltonian (4.3) if  $d(\mathcal{H})(Q_{1\pm}) = 0$ . Hence, we get:

$$\rho_1 = \frac{1}{2} C \omega_0^2 + \frac{\omega_0}{2} (k_3 \pm \sigma) \mp \frac{1}{2} a_3 - \frac{1}{2} j_3, \quad \rho_2 = -\omega_0. \quad (4.7a)$$

We can use Theorem 2, where  $U$  is defined as follows:

$$U = \ker dC_1(Q_{1\pm}) \cap \ker dC_2(Q_{1\pm}), \quad (4.8)$$

where

$$\begin{aligned} dC_1(Q_{1\pm}) &= \pm 2d\gamma_3, \\ dC_2(Q_{1\pm}) &= \pm dG_3 + (\pm C \omega_0 \pm k_3 + \sigma) d\gamma_3. \end{aligned} \quad (4.9)$$

To construct a basis for the vector space (4.8), the kernel conditions  $dC_1(Q_{1\pm}) = 0$  and  $dC_2(Q_{1\pm}) = 0$  impose constraints on the variations. Solving these yields that the vectors  $e_i, i = 1, 2, 4, 5$ , where  $e_i, i = 1, 2, \dots, 6$ , are the canonical basis of  $\mathbb{R}^6$  satisfy both conditions and are linearly independent. Hence, they form a basis for  $U$ . The linear independence is verified by noting that the matrix formed by these four vectors has full rank. Consequently, the vector  $y = (y_1, y_2, y_3, y_4) \in U$  is written as

$$y = y_1 e_1 + y_2 e_2 + y_3 e_4 + y_4 e_5, \quad (4.10)$$

where  $y_i \in \mathbb{R}, i = 1, 2, 3, 4$ . The quadratic form (4.2) in the variables  $y_i$  can be found by  $y^T \cdot \text{Hess}(\mathcal{H}) \cdot y$ . Hence,

$$\text{Hess}(\mathcal{H})|_{U \times U} = \begin{pmatrix} \frac{2}{A} & 0 & 2\rho_2 & 0 \\ 0 & \frac{2}{B} & 0 & 2\rho_2 \\ 2\rho_2 & 0 & 4\rho_1 + 2k_1\rho_2 + 2j_1 & 0 \\ 0 & 2\rho_2 & 0 & 4\rho_1 + 2k_2\rho_2 + 2j_2 \end{pmatrix}. \quad (4.11)$$

The definiteness of the reduced Hessian matrix (4.11) is assessed using the Sylvester criterion. Equivalently, it is considered positive definite if all its principal minors are positive. Denoting the principal minors by  $\Delta_j, j = 1, 2, 3, 4$ , they are

$$\Delta_1 = \frac{2}{A}, \quad \Delta_2 = \frac{4}{AB}, \quad (4.12a)$$

$$\Delta_3 = \frac{-8}{AB} [\pm a_3 \mp \omega_0 \sigma - j_1 + j_3 + \omega_0(k_1 - k_3) + (A - C)\omega_0^2], \quad (4.12b)$$

$$\Delta_4 = -2\Delta_3 [\pm a_3 \mp \omega_0 \sigma - j_2 + j_3 - \omega_0(k_3 - k_2) + (B - C)\omega_0^2]. \quad (4.12c)$$

It is obvious that  $\Delta_1 > 0$  and  $\Delta_2 > 0$ , while  $\Delta_3 > 0$  if

$$\pm a_3 < \pm \omega_0 \sigma + j_1 - j_3 - \omega_0(k_1 - k_3) - (A - C)\omega_0^2 = \chi_1. \quad (4.13)$$

$\Delta_4$  is positive if

$$\pm a_3 < \pm \omega_0 \sigma + j_2 - j_3 + \omega_0(k_3 - k_2) - (B - C)\omega_0^2 = \chi_2. \quad (4.14)$$

The two conditions (4.13) and (4.14) are met together if

$$\pm a_3 < \min(\chi_1, \chi_2). \quad (4.15)$$

□

We complete these results by presenting the necessary conditions for stability. Thus, we introduce the following theorem:

**Theorem 4.** *If the permanent rotation  $Q_{1\pm}$  is stable, then it must be verified that*

$$\left[ \pm a_3 - j_2 + j_3 - (k_3 - k_2 \pm \sigma)\omega_0 + (B - C)\omega_0^2 \right] \left[ \pm a_3 - j_1 + j_3 - (k_3 - k_1 \pm \sigma)\omega_0 + (A - C)\omega_0^2 \right] \geq 0. \quad (4.16)$$

*Proof.* The proof follows readily by noting that spectral stability is a necessary condition for Lyapunov stability. Consequently, the coefficient matrix (or Jacobian) of the perturbed system, linearized about

the equilibrium points  $Q_{1\pm}$ , must possess eigenvalues whose real parts are non-positive. In this case, the Jacobian takes the form

$$J_{Q_{1\pm}} = \begin{pmatrix} 0 & \frac{\mp(B-C)\omega_0 \mp k_3 - \sigma}{B} & 0 & 0 & \pm\omega k_2 + a_3 \mp j_2 \pm j_3 & 0 \\ \frac{\pm(C-A)\omega_0 \pm k_3 + \sigma}{A} & 0 & 0 & \mp\omega_0 k_1 - a_3 \pm j_1 \mp j_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mp\frac{1}{B} & 0 & 0 & \pm\omega_0 & 0 \\ \pm\frac{1}{A} & 0 & 0 & \mp\omega_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (4.17)$$

The eigenvalues of the Jacobian matrix (4.17) are the roots of the characteristic equation  $\lambda^2(\lambda^4 + p_1\lambda^2 + q_1) = 0$ , where  $q_1 = \frac{1}{16}\Delta_4$  and where  $\Delta_4$  is given by Eq (4.12c). The characteristic equation is invariant under the transformation  $\lambda \rightarrow -\lambda$ . For this reason, if  $\lambda_1$  is a root, then  $-\lambda_1$  is also a root. Therefore, the necessary condition for the linear stability is that all the eigenvalues are pure imaginary, i.e., they must have zero real parts, which indicates  $q_1 \geq 0$ ; the case  $q_1 < 0$  would produce a real positive eigenvalue and thus the permanent rotation is Lyapunov unstable and no further analysis is required. This proves condition (4.16).  $\square$

**Remark 1.** When both bracketed expressions in Eq (4.16) are negative, the necessary and sufficient conditions coincide. In other words, condition (4.5) becomes a necessary and sufficient condition for stability.

**Remark 2.** When condition (4.16) is not satisfied, the permanent rotation  $Q_{1\pm}$  is Lyapunov unstable, and no further analysis is required.

**Remark 3.** Consider the stability condition for the permanent rotation  $Q_1^\pm$  in the case of an axisymmetric gyrostat ( $A = B$ ). Theorem 3 provides the sufficient condition for Lyapunov stability (4.5):

$$\pm a_3 < \min(\chi_1, \chi_2), \quad (4.18)$$

where

$$\begin{aligned} \chi_1 &= \pm\omega_0\sigma + j_1 - j_3 - \omega_0(k_1 - k_3) - (A - C)\omega_0^2, \\ \chi_2 &= \pm\omega_0\sigma + j_2 - j_3 + \omega_0(k_3 - k_2) - (A - C)\omega_0^2. \end{aligned} \quad (4.19)$$

The expressions  $\chi_1$  and  $\chi_2$  in (4.19) differ due to the distinct parameters  $j_1, j_2$  and  $k_1, k_2$ . When the external field parameters satisfy  $k_1 = k_2$  and  $j_1 = j_2$ , condition (4.18) simplifies to

$$\pm a_3 < \pm\omega_0\sigma + j_1 - j_3 - \omega_0(k_1 - k_3) - (A - C)\omega_0^2. \quad (4.20)$$

### Special cases:

(a) For a heavy gyrostat under gravitational influence alone ( $k_i = j_i = 0$ ), the stability condition (4.18) for  $Q_1^\pm$  reduces to

$$\pm a_3 < \pm\omega_0\sigma - (A - C)\omega_0^2, \quad (4.21)$$

which agrees with the result in [21].

(b) Neglecting the central attractive potential ( $j_1 = j_3 = 0$ ), the system becomes a heavy charged gyrostat, and the stability condition simplifies to

$$\pm a_3 < \pm\omega_0\sigma - \omega_0(k_1 - k_3) - (A - C)\omega_0^2, \quad (4.22)$$

consistent with [24].

(c) For a charged heavy gyrostat moving in a Newtonian gravitational field, we set  $j_1 = j_2 = nA$  and  $j_3 = nC$ , where  $n = 3g/R$  and  $R$  represents the distance from the attraction center to the fixed point  $O$  (with  $R$  much larger than the body's dimensions). Under these conditions, the stability criterion reduces to

$$\pm a_3 < (A - C)(n - \omega_0^2) + \omega_0(\pm\sigma + k_3 - k_1), \quad (4.23)$$

which matches exactly the result derived in [25].

Now, we are going to investigate the stability of the second family of permanent rotation  $Q_{2\pm}$ .

**Theorem 5.** The sufficient condition for the permanent rotation  $Q_{2\pm}$  describing the rotation of a heavy charged gyrostat in an attractive central field with a constant velocity  $\omega_0$  around an axis with direction cosines  $(0, \pm \sin \theta_0, \cos \theta_0)$  is

$$j_2 < \min(\tau_1, \tau_2), \quad (4.24)$$

where

$$\begin{aligned} \tau_1 &= j_1 - \omega_0^2(A - B) - \omega_0(k_1 - k_2), \\ \tau_2 &= j_3 + \frac{1}{B \sin^2 \theta + C \cos^2 \theta} \left[ \cos^2 \theta_0 [3(B - C)^2 \omega_0^2 + 3(k_2 - k_3)(B - C)\omega_0 + (k_2 - k_3)^2] \right. \\ &\quad \left. - 2\sigma \cos \theta_0 [2(B - C)\omega_0 + k_2 - k_3] + B(B - C)\omega_0^2 + B\omega_0(k_2 - k_3) + \sigma^2 \right]. \end{aligned} \quad (4.25)$$

*Proof.* We consider the augmented Hamiltonian (4.3). The permanent rotation  $Q_{2\pm}$  is a critical point for the augmented Hamiltonian (4.3) if

$$\rho_1 = \frac{B}{2}\omega_0^2 + \frac{k_2}{2}\omega_0 - \frac{j_2}{2}, \quad \rho_2 = -\omega_0. \quad (4.26)$$

Now, we are going to find the base of the vector space  $U$ ,

$$U = \ker dC_1(Q_{2\pm}) \cap \ker dC_2(Q_{2\pm}),$$

where

$$\begin{aligned} dC_1(Q_{2\pm}) &= 2[\pm \sin \theta_0 d\gamma_2 + \cos \theta_0 d\gamma_3], \\ dC_2(Q_{2\pm}) &= \pm \sin \theta_0 dG_2 + \cos \theta_0 dG_3 \pm \sin \theta_0 (B\omega_0 + k_2) d\gamma_2 + ((C\omega_0 + k_3) \cos \theta_0 + \sigma) d\gamma_3. \end{aligned} \quad (4.27)$$

Now, we consider the two permanent rotations  $Q_{2\pm}$ , individually.

**Case 1:** In this case, we investigate the Lyapunov stability for  $Q_{2+}$ . To determine a basis for  $U = \ker dC_1(Q_{2+}^+) \cap \ker dC_2(Q_{2+}^+)$ , we solve the linear system  $dC_1(Q_{2+}^+)(v) = 0$ ,  $dC_2(Q_{2+}^+)(v) = 0$ , where  $v = (v_1, v_2, v_3, v_4, v_5, v_6) \in \mathbb{R}^6$  represents the variation vector. Using the expressions in (4.27), we obtain the following independent solutions:

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{e}_1, \\
\mathbf{v}_2 &= -\cos \theta_0 \mathbf{e}_2 + \sin \theta_0 \mathbf{e}_3, \\
\mathbf{v}_3 &= \mathbf{e}_4, \\
\mathbf{v}_4 &= \{\cos \theta_0[(B - C)\omega_0 + k_2 - k_3] - \sigma\} \mathbf{e}_2 - \cos \theta_0 \mathbf{e}_5 + \sin \theta_0 \mathbf{e}_6,
\end{aligned} \tag{4.28}$$

where  $\mathbf{e}_i, i = 1, 2, \dots, 6$ , are the canonical basis of  $\mathbb{R}^6$ . These four vectors shown in (4.28) are linearly independent, as can be verified by proving that the matrix formed by these four vectors has full rank 4. Therefore, any vector  $\mathbf{z} = (z_1, z_2, z_3, z_4) \in U$  can be expressed as a linear combination of the vectors  $\mathbf{v}_i, i = 1, 2, 3, 4$ , i.e., we have

$$\begin{aligned}
\mathbf{z} &= z_1 \mathbf{v}_1 + z_2 \mathbf{v}_2 + z_3 \mathbf{v}_3 + z_4 \mathbf{v}_4, \\
&= z_1 \mathbf{e}_1 + [\cos \theta_0[-z_2 + z_4((B - C)\omega_0 + k_2 - k_3)] - z_4 \sigma] \mathbf{e}_2 \\
&\quad + z_2 \sin \theta_0 \mathbf{e}_3 + z_3 \mathbf{e}_4 - z_4 \cos \theta_0 \mathbf{e}_5 + z_4 \sin \theta_0 \mathbf{e}_6,
\end{aligned} \tag{4.29}$$

where  $z_i \in \mathbb{R}, i = 1, 2, 3, 4$ . The quadratic form (4.2) in the variables  $z_i$  is evaluated by  $\mathbf{z}^T \cdot \text{Hess}(\mathcal{H}) \cdot \mathbf{z}$ . The corresponding Hessian matrix takes the form

$$\text{Hess}(\mathcal{H})|_{U \times U} = \begin{pmatrix} \frac{2}{A} & 0 & 2\rho_2 & 0 \\ 0 & \frac{2}{BC}(B \sin^2 \theta_0 + C \cos^2 \theta_0) & 0 & \xi_1 \\ 2\rho_2 & 0 & 4\rho_1 + 2k_1\rho_2 + 2j_1 & 0 \\ 0 & \xi_1 & 0 & \xi_2 \end{pmatrix}, \tag{4.30}$$

where

$$\begin{aligned}
\xi_1 &= \frac{-2}{B} [\cos^2 \theta_0((B - C)\omega_0 + k_2 - k_3) - \sigma \cos \theta_0 + B\omega_0], \\
\xi_2 &= \frac{2}{B} \left\{ \cos^2 \theta_0[\omega_0(2B - C) + k_2 - k_3]^2 + 2\sigma \cos \theta_0[\omega_0(C - 2B) - k_2 + k_3] \right. \\
&\quad \left. + B \sin^2 \theta_0[B\omega_0^2 + \omega_0(k_2 - k_3) - j_2 + j_3] + \sigma^2 \right\}.
\end{aligned} \tag{4.31}$$

The definiteness of matrix (4.30) is inspected by computing its principal minors, denoted as  $D_j, j = 1, 2, 3, 4$ . These are given by

$$\begin{aligned}
D_1 &= \frac{2}{A}, \\
D_2 &= \frac{4}{ABC} [B \sin^2 \theta_0 + C \cos^2 \theta_0], \\
D_3 &= 2D_2[j_1 - j_2 - (A - B)\omega_0^2 - (k_1 - k_2)\omega_0], \\
D_4 &= 2D_3 \sin^2 \theta_0 \left\{ j_3 - j_2 + \frac{1}{B \sin^2 \theta_0 + C \cos^2 \theta_0} \right. \\
&\quad \times \left[ \cos^2 \theta_0(3(B - C)^2 \omega_0^2 + 3(k_2 - k_3)(B - C)\omega_0 + (k_2 - k_3)^2) \right. \\
&\quad \left. - 2\sigma \cos \theta_0(2(B - C)\omega_0 + k_2 - k_3) \right. \\
&\quad \left. \left. + B(B - C)\omega_0^2 + B\omega_0(k_2 - k_3) + \sigma^2 \right] \right\}.
\end{aligned} \tag{4.32}$$

It is evident that  $D_1$  and  $D_2$  remain positive at all times, while  $D_3$  is positive if

$$j_2 < j_1 - \omega_0^2(A - B) - \omega_0(k_1 - k_2) = \tau_1 \quad (4.33)$$

is met.  $D_4$  is always positive if the following condition,

$$j_2 < j_3 + \frac{1}{B \sin^2 \theta + C \cos^2 \theta} \left[ \cos^2 \theta_0 [3(B - C)^2 \omega_0^2 + 3(k_2 - k_3)(B - C)\omega_0 + (k_2 - k_3)^2] \right. \\ \left. - 2\sigma \cos \theta_0 [2(B - C)\omega_0 + k_2 - k_3] + B(B - C)\omega_0^2 + B\omega_0(k_2 - k_3) + \sigma^2 \right] = \tau_2, \quad (4.34)$$

is verified. The two inequalities (4.33) and (4.34) are satisfied together if

$$j_2 < \min(\tau_1, \tau_2). \quad (4.35)$$

**Case 2:** In this case, we consider the permanent rotation  $Q_{2-}$ . For the equilibrium  $Q_{2-}$ , solving the kernel conditions yields the basis vectors

$$\begin{aligned} \mathbf{v}_1 &= \mathbf{e}_1, \\ \mathbf{v}_2 &= \cos \theta_0 \mathbf{e}_2 + \sin \theta_0 \mathbf{e}_3, \\ \mathbf{v}_3 &= \mathbf{e}_4, \\ \mathbf{v}_4 &= \sin \theta_0 \{ \cos \theta_0 [(B - C)\omega_0 + k_2 - k_3] - \sigma \} \mathbf{e}_3 + \cos^2 \theta_0 \mathbf{e}_5 + \sin \theta_0 \cos \theta_0 \mathbf{e}_6. \end{aligned} \quad (4.36)$$

It is obvious that these vectors are linearly independent, as their coordinate matrix has rank 4. Hence any vector  $\mathbf{z} = (z_1, z_2, z_3, z_4) \in U$  can be expressed as

$$\begin{aligned} \mathbf{z} &= z_1 \mathbf{e}_1 + z_2 \cos \theta_0 \mathbf{e}_2 + \sin \theta_0 [z_2 + z_4 \{ \cos \theta_0 [(B - C)\omega_0 + k_2 - k_3] - \sigma \}] \mathbf{e}_3 + z_3 \mathbf{e}_4 \\ &+ z_4 \cos^2 \theta_0 \mathbf{e}_5 + z_4 \sin \theta_0 \cos \theta_0 \mathbf{e}_6, \end{aligned} \quad (4.37)$$

where  $z_i \in \mathbb{R}$ . The Hessian matrix corresponding to the quadratic  $\mathbf{z}^T \cdot \text{Hess}(\mathcal{H}) \cdot \mathbf{z}$  in the variables  $z_i$  admits the form

$$\text{Hess}(\mathcal{H})|_{U \times U} = \begin{pmatrix} \frac{2}{A} & 0 & 2\rho_2 & 0 \\ 0 & \frac{2}{BC}(B \sin^2 \theta_0 + C \cos^2 \theta_0) & 0 & \xi_3 \\ 2\rho_2 & 0 & 4\rho_1 + 2k_1\rho_2 + 2j_1 & 0 \\ 0 & \xi_3 & 0 & \xi_4 \end{pmatrix}, \quad (4.38)$$

where

$$\begin{aligned} \xi_3 &= \frac{2}{C} [\cos^3 \theta_0 ((C - B)\omega_0 - k_2 + k_3) + \sigma \cos^2 \theta_0 + ((B - 2C)\omega_0 + k_2 - k_3) \cos \theta_0 - \sigma], \\ \xi_4 &= \frac{2 \cos^2 \theta_0}{C} \left\{ \sin^2 \theta_0 [(B^2 + 3C^2)\omega_0^2 + (k_2 - k_3)(2B - 3C)\omega_0 + C(j_3 - j_2) + (k_2 - k_3)^2] \right. \\ &\quad \left. - 2\sigma \tan \theta_0 \sin \theta_0 [(B - 2C)\omega_0 + k_2 - k_3] + \sigma^2 \tan^2 \theta_0 + BC\omega_0^2(4 \cos^2 \theta_0 - 3) \right\}. \end{aligned} \quad (4.39)$$

The definiteness of matrix (4.38) is inspected by computing its principal minors, denoted as  $D_j$ ,  $j = 1, 2, 3, 4$ . These are given by

$$\begin{aligned}
 D_1 &= \frac{2}{A}, \\
 D_2 &= \frac{4}{ABC}[B \sin^2 \theta_0 + C \cos^2 \theta_0], \\
 D_3 &= -2D_2[(A - B)\omega_0^2 + \omega_0(k_1 - k_2) - j_1 + j_2], \\
 D_4 &= 2D_3 \sin^2 \theta_0 \cos^2 \theta_0 \left\{ j_3 - j_2 + \frac{1}{C \cos^2 \theta_0 + B \sin^2 \theta_0} \right. \\
 &\quad \times \left[ (3(B - C)^2 \omega_0^2 + 3(k_2 - k_3)(B - C)\omega_0 + (k_2 - k_3)^2) \cos^2 \theta_0 \right. \\
 &\quad \left. - 2\sigma \cos \theta_0 (2(B - C)\omega_0 + k_2 - k_3) \right. \\
 &\quad \left. \left. + B(B - C)\omega_0^2 + B(k_2 - k_3)\omega_0 + \sigma^2 \right] \right\}. \tag{4.40}
 \end{aligned}$$

It is clear that  $D_1 > 0$  and  $D_2 > 0$  while  $D_3 > 0$  if

$$j_2 < j_1 - (A - B)\omega_0^2 + \omega_0(k_2 - k_3) = \tau_1. \tag{4.41}$$

$D_4$  is positive if

$$\begin{aligned}
 j_2 &< j_3 + \frac{1}{C \cos^2 \theta_0 + B \sin^2 \theta_0} \left[ (3(B - C)^2 \omega_0^2 + 3(k_2 - k_3)(B - C)\omega_0 + (k_2 - k_3)^2) \right. \\
 &\quad \times \cos^2 \theta_0 - 2\sigma \cos \theta_0 [2(B - C)\omega_0 + k_2 - k_3] + B(B - C)\omega_0^2 + B(k_2 - k_3)\omega_0 + \sigma^2 \left. \right] \\
 &= \tau_2. \tag{4.42}
 \end{aligned}$$

Hence, the two inequalities (4.41) and (4.42) are held together if condition (4.35) is met.

□

For completeness, we provide the necessary condition for the stability of the permanent rotation. Thus, we introduce the next theorem.

**Theorem 6.** *If the permanent rotation  $Q_{2\pm}$  is stable, then the condition*

$$\begin{aligned}
 [j_1 - j_2 - (A - B)\omega_0^2 - (k_1 - k_2)\omega_0] &\left\{ j_3 - j_2 + \frac{1}{B \sin^2 \theta + C \cos^2 \theta_0} \left[ \cos^2 \theta_0 [3(B - C)^2 \omega_0^2 \right. \right. \\
 &\quad \left. \left. + 3(k_2 - k_3)(B - C)\omega_0 + (k_2 - k_3)^2] - 2\sigma \cos \theta_0 [2(B - C)\omega_0 + k_2 - k_3] + B(B - C)\omega_0^2 \right. \right. \\
 &\quad \left. \left. + B\omega_0(k_2 - k_3) + \sigma^2 \right] \right\} \geq 0 \tag{4.43}
 \end{aligned}$$

is verified.

*Proof.* Spectral stability is a necessary condition for Lyapunov stability. The Jacobian matrix linearized about the equilibrium  $Q_{2+}$  is given by

$$J_{Q_{2+}} = \begin{pmatrix} 0 & a_{12} & \frac{\sin \theta_0 [\omega_0 (B-C) + k_2]}{C} & 0 & a_3 + \cos \theta_0 [k_2 \omega_0 - j_2 + j_3] & a_{16} \\ a_{21} & 0 & 0 & a_{24} & 0 & 0 \\ a_{31} & 0 & 0 & a_{34} & 0 & 0 \\ 0 & -\frac{\cos \theta_0}{B} & \frac{\sin \theta_0}{C} & 0 & \omega_0 \cos \theta_0 & -\omega_0 \sin \theta_0 \\ \frac{\cos \theta_0}{A} & 0 & 0 & -\omega_0 \cos \theta_0 & 0 & 0 \\ -\frac{\sin \theta_0}{A} & 0 & 0 & \omega_0 \sin \theta_0 & 0 & 0 \end{pmatrix}, \quad (4.44)$$

where

$$\begin{aligned} a_{12} &= \frac{\cos \theta_0 [\omega_0 (B-C) - k_3] - \sigma}{B}, & a_{16} &= \sin \theta_0 [j_3 - j_2 - k_3 \omega_0], \\ a_{21} &= \frac{\cos \theta_0 [k_3 + (C-A)\omega_0] + \sigma}{A}, & a_{24} &= [j_1 - j_3 - k_1 \omega_0] \cos \theta_0 - a_3, \\ a_{31} &= \frac{\sin \theta_0}{A} [\omega_0 (A-B) - k_2], & a_{34} &= \sin \theta_0 [k_1 \omega_0 + j_2 - j_1]. \end{aligned} \quad (4.45)$$

Its characteristic polynomial is  $\det(J_{Q_{2+}} - \lambda I) = \lambda^2(\lambda^4 + p_2 \lambda^2 + q_2) = 0$ , where  $q_2 = \frac{1}{16} D_4$  and  $D_4$  is given by Eq (4.32). The factor  $\lambda^2$  accounts for two zero eigenvalues inherent to the Poisson structure.

The condition of linear stability is that all eigenvalues are pure imaginary. Hence, the condition  $q_2 \geq 0$  must be verified. The case  $q_2 < 0$  would produce a real positive eigenvalue and thus the permanent rotation is Lyapunov unstable and no further analysis is required. This proves directly condition (4.43).  $\square$

**Remark 4.** When both bracketed expressions in Eq (4.43) are positive, the necessary and sufficient conditions are the same. Or, equivalently, condition (4.24) is sufficient.

**Remark 5.** When condition (4.43) is not satisfied, the permanent rotation  $Q_{2\pm}$  is Lyapunov unstable, and no further analysis is required.

With similar procedures, we can prove the next theorems.

**Theorem 7.** The sufficient condition for the permanent rotation  $Q_{3\pm}$  describing the rotation of a heavy charged gyrostat in an attractive central field with a constant angular velocity  $\omega_0$  about an axis with direction cosines  $(\pm \sin \theta_0, 0, \cos \theta_0)$  is

$$j_1 < \min(\tau_3, \tau_4), \quad (4.46)$$

where

$$\begin{aligned} \tau_3 &= j_2 + (A-B)\omega_0^2 + (k_1 - k_2)\omega_0, \\ \tau_4 &= j_3 + \frac{1}{A \sin^2 \theta_0 + C \cos^2 \theta_0} \left[ [3(A-C)^2 \omega_0^2 + 3(k_1 - k_3)(A-C)\omega_0 + (k_1 - k_3)^2] \cos^2 \theta_0 \right. \\ &\quad \left. - 2\sigma \cos \theta_0 [2(A-C)\omega_0 + k_1 - k_3] + A(A-C)\omega_0^2 + A(k_1 - k_3)\omega_0 + \sigma^2 \right]. \end{aligned} \quad (4.47)$$

**Theorem 8.** *If the permanent rotation  $Q_{3\pm}$  is stable, then the condition*

$$\begin{aligned} & \left[ j_2 + (A - B)\omega_0^2 + (k_1 - k_2)\omega_0 \right] \\ & \times \left\{ j_3 + \frac{1}{A \sin^2 \theta_0 + C \cos^2 \theta_0} \left[ (3(A - C)^2 \omega_0^2 + 3(k_1 - k_3)(A - C)\omega_0 + (k_1 - k_3)^2) \cos^2 \theta_0 \right. \right. \\ & \left. \left. - 2\sigma \cos \theta_0 (2(A - C)\omega_0 + k_1 - k_3) + A(A - C)\omega_0^2 + A(k_1 - k_3)\omega_0 + \sigma^2 \right] \right\} > 0 \end{aligned} \quad (4.48)$$

is satisfied.

Now, we introduce a diagram of the stability zones in the plane of any two parameters, say  $\theta_0$  and  $\sigma$ , while the other parameters characterizing the gyrostat geometry, the center of mass, the uniform magnetic effect, and the attractive center effects are kept fixed. It is evident that the equilibrium  $Q_{2+}$  exists if condition (3.3) is satisfied. This condition is quadratic in the angular velocity  $\omega_0$  and, consequently, it has real solutions if

$$L_0 : f(\theta_0, \sigma; k_i, j_i, a_3) = \left[ \sigma + \cos \theta_0 (k_3 - k_2) \right]^2 + 4(B - C) \cos \theta_0 \left[ \cos \theta_0 (j_2 - j_3) - a_3 \right] \geq 0. \quad (4.49)$$

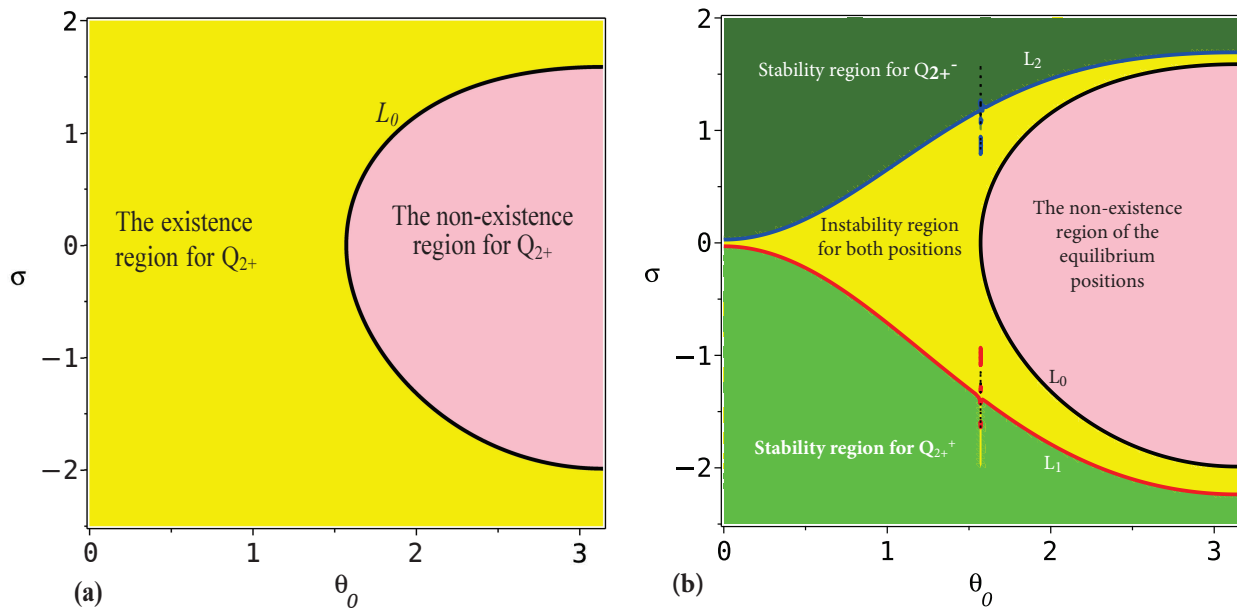
The existence and stability of the equilibrium  $Q_{2+}$  depend critically on the nutation angle  $\theta_0$  and the gyrostatic momentum  $\sigma$ , as illustrated in Figure 2. Panel (a) shows that the pink region corresponds to parameter values where the quadratic equation (3.3) has no real solutions for  $\omega_0$ ; hence, the permanent rotation does not exist. In the yellow region where the condition (4.49) is satisfied, real solutions exist, giving rise to two distinct angular velocities  $\omega_+$  and  $\omega_-$  given by

$$\begin{aligned} \omega_{\pm} = & \frac{1}{2(B - C) \cos \theta_0} \left\{ \sigma + (k_3 - k_2) \right. \\ & \left. \pm \left[ \cos^2 \theta_0 (4(j_2 - j_3)(B - C) + (k_2 - k_3)^2) + 2 \cos \theta_0 [\sigma(k_3 - k_2) - 2a_3(B - C)] + \sigma^2 \right]^{1/2} \right\}. \end{aligned} \quad (4.50)$$

Panel (b) displays the stability zones for these two branches. By substituting  $\omega_{\pm}$  into condition (4.24), we obtain two limiting curves:

$$L_1 : J_2 = \min(\tau_1, \tau_2) \Big|_{\omega_+}, \quad L_2 : J_2 = \min(\tau_1, \tau_2) \Big|_{\omega_-}.$$

The green and dark green regions satisfy the sufficient stability condition (4.24) derived in Theorem 5, indicating that small perturbations will not destabilize the rotation. The yellow region represents unstable configurations where even infinitesimal disturbances lead to growing deviations.



**Figure 2.** Existence and stability of  $Q_{2+}$  in the  $(\theta_0, \sigma)$  plane. (a) Existence (yellow) vs. non-existence (pink) zones. (b) Stability (green) vs. instability (yellow) for  $\omega_+$  and  $\omega_-$ , with boundaries  $L_1, L_2$ .

It is noteworthy that when the equilibrium point  $Q_{2+}^+$  is stable, the equilibrium point  $Q_{2+}^-$  is unstable, and vice versa. Furthermore, the stability and instability regions of  $Q_{2+}^\pm$  do not intersect with the pink region, where the equilibrium positions do not exist. A similar analysis can be performed for  $Q_{2-}$ .

Physically, these diagrams illustrate how the internal rotor momentum and the orientation of the rotation axis affect the stability of the gyrostat motion under the influence of external fields.

Finally, we investigate the stability analysis for the permanent rotation  $Q_4$ . This analysis is presented in the following theorems:

**Theorem 9.** *The sufficient condition for the permanent rotation  $Q_4$  to be Lyapunov stable is*

$$j_2 < \omega_0^2(B - C) + \omega_0(k_2 - k_3) + j_3. \tag{4.51}$$

*Proof.* The equilibrium point  $Q_4$  is a critical point to the augmented Hamiltonian if

$$\rho_1 = \frac{1}{2}[\omega_0^2 B + \omega_0 k_2 - j_2], \quad \rho_2 = -\omega_0. \tag{4.52}$$

We can utilize Theorem 2, where  $U$  is defined as follows:

$$U = \ker dC_1(Q_4) \cap \ker dC_2(Q_4), \tag{4.53}$$

where

$$\begin{aligned} dC_1(Q_4) &= 2 \sin \theta_0 \sin \phi d\gamma_1 + 2 \sin \theta_0 \cos \phi d\gamma_2 + 2 \cos \theta_0 d\gamma_3, \\ dC_2(Q_4) &= \sin \theta_0 \sin \phi dG_1 + \sin \theta_0 \cos \phi dG_2 + \cos \theta_0 dG_3 + \sin \theta_0 \sin \phi(k_1 + A\omega_0) d\gamma_1 \\ &\quad + \sin \theta_0 \cos \phi(k_2 + B\omega_0) d\gamma_2 + \cos \theta_0(j_3 + \sigma + C\omega_0) d\gamma_3. \end{aligned} \tag{4.54}$$

We now construct a basis for  $U = \ker dC_1(Q_4) \cap \ker dC_2(Q_4)$ . Solving the homogeneous linear system defined by (4.54) yields four linearly independent solutions:

$$\begin{aligned}
\mathbf{v}_1 &= \cos \theta_0 \mathbf{e}_1 - \sin \theta_0 \sin \phi \mathbf{e}_3, \\
\mathbf{v}_2 &= \cos \theta_0 \mathbf{e}_2 - \sin \theta_0 \sin \phi \mathbf{e}_3, \\
\mathbf{v}_3 &= -\sin \theta_0 \mathbf{e}_2 + \sin \phi [\cos \theta_0 ((A - C)\omega_0 + k_1 - k_3) - \sigma] \mathbf{e}_3 + \cos^2 \theta_0 \mathbf{e}_4 - \sin \theta_0 \cos \theta_0 \sin \phi \mathbf{e}_6, \\
\mathbf{v}_4 &= -\sin \theta_0 \mathbf{e}_2 + \cos \phi [\cos \theta_0 ((B - C)\omega_0 + k_2 - k_3) - \sigma] \mathbf{e}_3 + \cos^2 \theta_0 \mathbf{e}_5 - \sin \theta_0 \cos \theta_0 \cos \phi \mathbf{e}_6,
\end{aligned} \tag{4.55}$$

where  $\mathbf{e}_i, i = 1, 2, \dots, 6$ , are the canonical basis for  $\mathbb{R}^6$ . Linear independence is verified by noting that the matrix formed by these four vectors has full rank 4. Hence any vector  $\mathbf{x} = (x_1, x_2, x_3, x_4) \in U$  can be expressed as a linear combination of  $\mathbf{v}_i, i = 1, 2, 3, 4$ .

$$\begin{aligned}
\mathbf{x} &= x_1 \mathbf{v}_1 + x_2 \mathbf{v}_2 + x_3 \mathbf{v}_4 + x_4 \mathbf{v}_5, \\
&= x_1 \cos \theta_0 \mathbf{e}_1 + x_2 \cos \theta_0 \mathbf{e}_2 - \sin \theta_0 \left\{ \cos \theta_0 [x_3 \sin \phi (\omega_0 (A - C) + k_1 - k_3) + x_4 \cos \phi ((B - C) \right. \\
&\quad \times \omega_0 + k_2 - k_3) + (x_2 + x_2 - \sigma x_3) \sin \phi - x_4 \sigma \cos \phi] \mathbf{e}_3 + x_3 \cos^2 \theta_0 \mathbf{e}_4 + x_4 \cos^2 \theta_0 \mathbf{e}_5 \\
&\quad \left. - \sin \theta_0 \cos \theta_0 [x_3 \sin \phi + x_4 \cos \phi] \mathbf{e}_6, \right\}
\end{aligned} \tag{4.56}$$

where  $\cos \theta_0 \neq 0$  and  $x_i \in \mathbb{R}, i = 1, 2, 3, 4$ . The quadratic form (4.2) in the variables  $x_i$  can be found by  $\mathbf{x}^T \cdot \text{Hess}(\mathcal{H}) \cdot \mathbf{x}$ . Hence,

$$\text{Hess}(\mathcal{H})|_{U \times U} = \begin{pmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ r_{12} & r_{22} & r_{23} & r_{24} \\ r_{13} & r_{23} & r_{33} & r_{34} \\ r_{14} & r_{24} & r_{34} & r_{44} \end{pmatrix}, \tag{4.57}$$

where  $r_{ij}$  are given by

$$\begin{aligned}
r_{11} &= \frac{2}{A} \cos^2 \theta_0 + \frac{2}{C} \sin^2 \theta_0 \sin^2 \phi, \\
r_{12} &= \frac{2}{C} \sin^2 \theta_0 \sin^2 \phi, \\
r_{13} &= -2\omega_0 \cos^3 \theta_0 + \frac{2}{C} \sin^2 \theta_0 \sin^2 \phi \left( \cos \theta_0 [(A - 2C)\omega_0 + k_1 - k_3] - \sigma \right), \\
r_{14} &= \frac{2}{C} \sin^2 \theta_0 \sin \phi \cos \phi \left( \cos \theta_0 [(B - 2C)\omega_0 + k_2 - k_3] - \sigma \right), \\
r_{22} &= \frac{2}{B} \cos^2 \theta_0 + \frac{2}{C} \sin^2 \theta_0 \sin^2 \phi, \\
r_{23} &= \frac{2}{C} \sin^2 \theta_0 \sin^2 \phi \left( \cos \theta_0 [(A - 2C)\omega_0 + k_1 - k_3] - \sigma \right), \\
r_{24} &= -2\omega_0 \cos^3 \theta_0 + \frac{2}{C} \sin^2 \theta_0 \sin \phi \cos \phi \left( \cos \theta_0 [(B - 2C)\omega_0 + k_2 - k_3] - \sigma \right), \\
r_{33} &= 2 \cos^4 \theta_0 [B\omega_0^2 + \omega_0(k_2 - k_1) + j_1 - j_2] \\
&\quad + \frac{2}{C} \sin^2 \theta_0 \sin^2 \phi \left\{ \cos^2 \theta_0 [3C\omega_0^2 + C((B - 4A)\omega_0^2 \right. \\
&\quad \left. + \omega_0(k_2 + 3k_3 - 4k_1) - j_2 + j_3) + (A\omega_0 + k_1 - k_3)^2] \right. \\
&\quad \left. - 4\sigma \cos \theta_0 [\omega_0(A - 2C) + k_1 - k_3] + 2\sigma^2 \right\},
\end{aligned}$$

$$\begin{aligned}
r_{34} &= \frac{2}{C} \sin^2 \theta_0 \sin \phi \cos \phi \left\{ \cos^2 \theta_0 [\omega_0^2 (AB - (2A + B)C + 3C^2) \right. \\
&\quad + \omega_0 (Bk_1 + Ak_2 - k_3(A + B)) + C(3k_3 - k_2 - 2k_1) \\
&\quad + C(j_3 - j_2) + (k_2 - k_3)(k_1 - k_3)] \\
&\quad \left. - \sigma \cos \theta_0 [k_1 + k_2 - 2k_3 + \omega_0(A + B - 4C)] + \sigma^2 \right\}, \\
r_{44} &= 2B\omega_0^2 \cos^4 \theta_0 + \frac{2}{C} \sin^2 \theta_0 \cos^2 \phi \left\{ \cos^2 \theta_0 [\omega_0^2 (B^2 - 3BC + 3C^2) \right. \\
&\quad + (2B - 3C)(k_2 - k_3)\omega_0 + C(j_3 - j_2) + (k_2 - k_3)^2] \\
&\quad \left. - 2\sigma \cos \theta_0 [(B - 2C)\omega_0 + k_2 - k_3] + \sigma^2 \right\}.
\end{aligned}$$

The definiteness of the reduced Hessian matrix (4.57) is tested by using the Sylvester criterion. Equivalently, it is considered positive definite if all its principal minors are positive. Denoting the principal minors by  $D_j$ ,  $j = 1, 2, 3, 4$ , they are

$$\begin{aligned}
D_1 &= \frac{2}{AC} [A \sin^2 \theta_0 \sin^2 \phi + C \cos^2 \theta_0], \\
D_2 &= \frac{4 \cos^2 \theta_0}{ABC} [(A + B) \sin^2 \theta_0 \sin^2 \phi + C \cos^2 \theta_0], \\
D_3 &= \frac{8 \cos^4 \theta_0 \sin^2 \theta_0 \sin^2 \phi}{ABC} [(A + B) \sin^2 \theta_0 \sin^2 \phi + C \cos^2 \theta_0] \\
&\quad \times \left\{ j_3 - j_2 + \frac{1}{(A + B) \sin^2 \theta_0 \sin^2 \phi + C \cos^2 \theta_0} \right. \\
&\quad \times \left[ \cos^2 \theta_0 ((k_1 - k_3)^2 + \omega_0^2 (C(3C + B) + 4A(A - 2C))) \right. \\
&\quad \left. + \omega_0 (4A(k_1 - k_3) - C(4k_1 - 3k_3 - 2k_2)) \right. \\
&\quad \left. - 2\sigma \cos \theta_0 (2(A - C)\omega_0 + k_1 - k_3) + \sigma^2 \right. \\
&\quad \left. \left. + \omega_0 (A + B) ((B - C)\omega_0 + k_2 - k_3) \sin^2 \phi \sin^2 \theta_0 \right] \right\}, \\
D_4 &= \frac{16 \cos^8 \theta_0 \sin^4 \theta_0 \sin^2 \phi}{ABC} [\cos \phi ((2A - B)\omega_0 + k_1 - k_2) - B\omega_0 \sin \phi]^2 \\
&\quad \times [j_3 - j_2 + \omega_0^2 (B - C) + \omega_0 (k_2 - k_3)].
\end{aligned} \tag{4.58}$$

It is obvious that  $D_1 > 0$  and  $D_2 > 0$ , while  $D_3 > 0$  if

$$\begin{aligned}
j_2 < j_3 + \frac{1}{(A + B) \sin^2 \theta_0 \sin^2 \phi + C \cos^2 \theta_0} \left\{ \omega_0^2 [4A(A - 2C) + C(3C + B)] \omega_0^2 + \cos^2 \theta_0 [4A(k_1 \right. \\
&\quad \left. - k_3) - C(4k_1 - k_2 - 3k_3)] - \sigma [4(A - C)\omega_0 + 2(k_1 - k_3)] \cos \theta_0 + \omega_0^2 (A + B) [(B - C)\omega_0 \right. \\
&\quad \left. + k_2 - k_3] \sin^2 \theta_0 \times \sin^2 \phi + \sigma^2 \right\} = \tau_6.
\end{aligned} \tag{4.59}$$

$D_4$  is positive if

$$j_2 < \omega_0^2(B - C) + \omega_0(k_2 - k_3) + j_3 = \tau_5. \quad (4.60)$$

The two conditions (4.59) and (4.60) are met together if the following condition holds:

$$j_2 < \min(\tau_5, \tau_6). \quad (4.61)$$

Direct calculations show that

$$\tau_6 - \tau_5 = \frac{[\cos \theta_0(2(A - C)\omega_0 + k_1 - k_2) - \sigma]^2}{(A + B) \sin^2 \theta_0 \sin^2 \phi + C \cos^2 \theta_0} > 0. \quad (4.62)$$

This shows  $\tau_6 > \tau_5$  and, consequently, condition (4.61) holds if condition (4.51) is satisfied.  $\square$

The stability study of the permanent rotation  $Q_4$  is completed by establishing the next theorem.

**Theorem 10.** *If the permanent rotation  $Q_4$  is stable, then the condition*

$$j_2 < \omega_0^2(B - C) + \omega_0(k_2 - k_3) + j_3 \quad (4.63)$$

*holds.*

*Proof.* The proof can be obtained directly by calculating the Jacobian matrix around the equilibrium point  $Q_4$ . After similar calculations to the above proofs, the Jacobian matrix eigenvalues are found to be the roots of the characteristic equation  $\lambda^2(\lambda^4 + p_3\lambda^2 + q_3) = 0$ , where  $q_3$  is a real number given by

$$q_3 = \frac{\sec^8 \theta_0 \cos^2 \phi [2(A - B)\omega_0 + k_1 - k_2]^2}{16[\cos \phi [(2A - B)\omega_0 + k_1 - k_2] - B\omega_0 \sin \phi]^2} D_4. \quad (4.64)$$

The condition for linear stability is that all the eigenvalues are pure imaginary, i.e., their real part must be zero. This happens when  $q_3 > 0$ , which is equivalent to  $D_4 > 0$ . This proves condition (4.63).  $\square$

**Remark 6.** *It is obvious that condition (4.51) is sufficient and necessary for the permanent rotations  $Q_4$  to be stable.*

We present the regions of existence and non-existence of the equilibrium position  $Q_4$ , in addition to the stability and instability regions, in the plane of the two parameters  $\theta_0$  and  $\sigma$ , which characterize the gyrostatic momentum.

The existence and stability of the fourth family of permanent rotations  $Q_4$  are illustrated in Figure 3 as functions of the nutation angle  $\theta_0$  and the gyrostatic momentum  $\sigma$ .

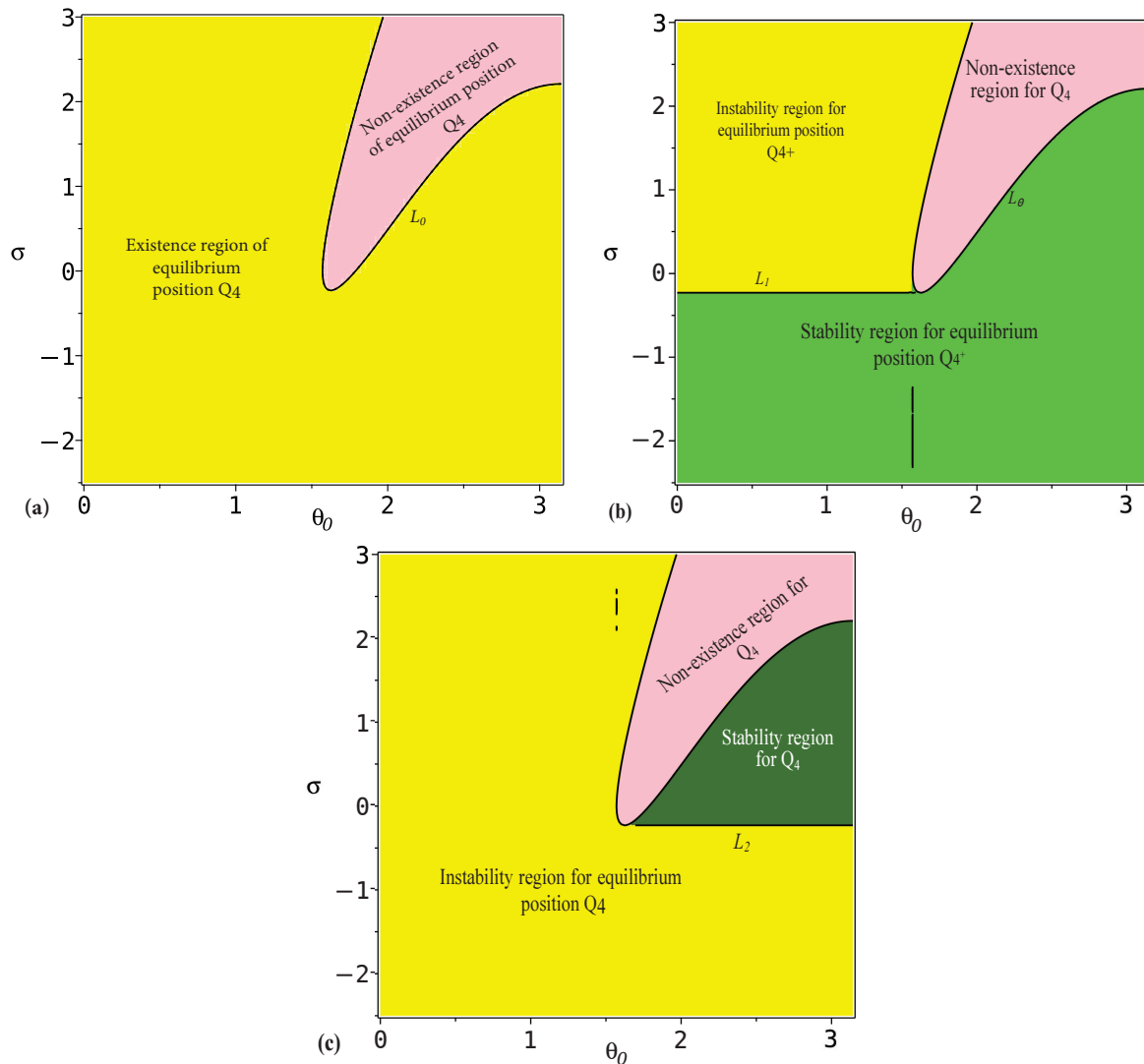
The equilibrium position  $Q_4$  exists if the quadratic polynomial in  $\omega_0$  shown in Eq (3.7b) has real solutions. This is satisfied if

$$L_0 : g(\theta_0, \sigma; k_i, j_i, a_3) = [\sigma - (k_2 - k_3)]^2 - 4(B - C) \cos \theta_0 [a_3 + (j_3 - j_2)] \geq 0, \quad (4.65)$$

in addition to  $j_1$  being strictly chosen to satisfy condition (3.7a). The curve  $L_0 : g(\theta_0, \sigma; k_i, j_i, a_3) = 0$  splits the  $\theta_0\sigma$ -plane into two regions. One of them, in pink, characterizes the zone of non-existence of the equilibrium position  $Q_4$ , while this position exists in the yellow zone, as shown in Figure 3(a).

Now, we outline the regions of instability and stability. If condition (4.65) is satisfied, then there exist two possible values of the angular velocity  $\omega_0$ . They are

$$\omega_{0\pm} = \frac{1}{2(B - C) \cos \theta_0} \left\{ \sigma + (k_3 - k_2) \pm \left[ \cos^2 \theta_0 (4(j_2 - j_3)(B - C) + (k_2 - k_3)^2) + 2 \cos \theta_0 [\sigma(k_3 - k_2) - 2a_3(B - C)] + \sigma^2 \right]^{1/2} \right\}. \tag{4.66}$$



**Figure 3.** Existence and stability of  $Q_4$  in the  $(\theta_0, \sigma)$  plane. (a) Existence (yellow) vs. non-existence (pink). (b)-(c) Stability (green) vs. instability (yellow) for  $\omega_+$  and  $\omega_-$ , with boundaries  $L_1, L_2$ .

By substituting (4.66) into Eq (3.7a),  $j_1$  is computed as

$$j_1^\pm = (A - B)\omega_{0\pm}^2 + (k_1 - k_2)\omega_{0\pm} + j_2. \tag{4.67}$$

The necessary and sufficient condition for the stability of the permanent rotation  $Q_{4\pm}$ , computed at  $\omega_{0\pm}$ , is

$$j_2^\pm < \omega_{0\pm}^2 (B - C) + \omega_{0\pm} (k_2 - k_3) + j_3. \quad (4.68)$$

Thus, we have two limiting curves

$$L_i : j_2^\pm - \left[ \omega_{0\pm}^2 (B - C) + \omega_{0\pm} (k_2 - k_3) + j_3 \right], \quad i = 1, 2.$$

Curve  $L_1$  splits the region of stability for the equilibrium positions  $Q_{4+}$  in bright green and the region of instability in yellow, as displayed in Figure 3(b). Curve  $L_2$  separates the zones of stability for the equilibrium positions  $Q_{4-}$  in dark green and the instability region in yellow, as shown in Figure 3(c). It is obvious that the regions of instability or stability do not intersect the pink region, which characterizes the non-existence of the equilibrium point. Moreover, in the dark green region, both equilibrium positions  $Q_{4\pm}$  are stable.

Physically, these diagrams reveal how the internal rotor momentum and the orientation of the rotation axis influence the robustness of the gyrostat's motion under the combined external fields.

#### 4.1. Numerical results

This section intends to present a numerical confirmation of the necessary stability and sufficient conditions that are formed analytically in the prior section, as well as clarify the impact of the key parameters on the stability boundary. Hence, we should carefully elect the values of the physical parameters guaranteeing the existence conditions appearing in Section 3 are fulfilled.

Now, we offer proper physical values of the parameters allowing us to conduct a comprehensive examination of the stability conditions for some permanent rotations. Let us consider a gyrostat model having principal inertia moments  $A = 1$ ,  $B = 2$ ,  $C = 3$ , and the angular momentum for the rotor along the third axis is  $\sigma = 0.5$ . The magnetic field is determined by  $k_1 = 0.1$ ,  $k_2 = 0.2$ ,  $k_3 = 0.3$ , while the parameters of the central field are  $j_1 = 0.5$ ,  $j_2 = 0.4$ ,  $j_3 = 0.6$  in addition to the gravitational field parameter, which is  $a_3 = 0.2$ .

We first consider the permanent rotations ( $Q_{1\pm}$ ). It is obvious that the angular velocity  $\omega_0$  can be picked out arbitrary and, therefore, we adopt  $\omega_0 = 0.8$ . Theorem 3 provides the stability sufficient condition as

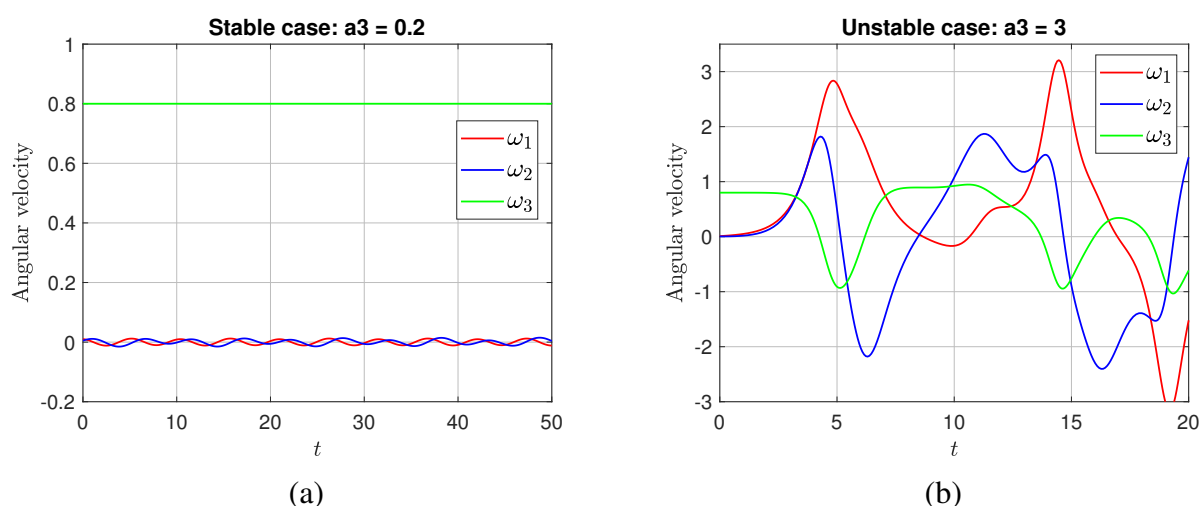
$$\pm a_3 < \min(\chi_1, \chi_2),$$

where  $\chi_1, \chi_2$  are defined by (4.6).

For  $Q_{1+}$ , we calculate  $\chi_1 \approx 2.56$  and  $\chi_2 \approx 2.02$ , thus  $\min(\chi_1, \chi_2) = 2.02$ . Because  $a_3 = 0.2 < 2.02$ , the stability condition holds. Then,  $Q_{1+}$  is stable.

For  $Q_{1-}$ , direct computation yields  $\chi_1 \approx 2.40$ ,  $\chi_2 \approx 1.86$ , so  $\min(\chi_1, \chi_2) = 1.86$ . The stability condition  $-a_3 = -0.2 < 1.86$  is verified. Hence,  $Q_{1-}$  is stable as well.

The Euler-Poisson equations (2.13) are solved numerically utilizing a fourth-order Runge–Kutta method, applying small initial perturbations ( $\delta G_i = 0.01$ ,  $\delta \gamma_i = 0.01$ ). Figure 4(a) displays the time evolution of angular velocity components  $\omega_1(t)$ ,  $\omega_2(t)$ ,  $\omega_3(t)$  for the stable  $Q_{1+}$  case. The perturbations stay bound and have shown oscillatory behavior, confirming the Lyapunov stability. On the contrary, when parameters are adjusted to make the stability condition not met (for example,  $a_3 = 3$  while the other parameters are still unaltered), Figure 4(b) shows that the trajectory diverges significantly. This confirms the Lyapunov instability.



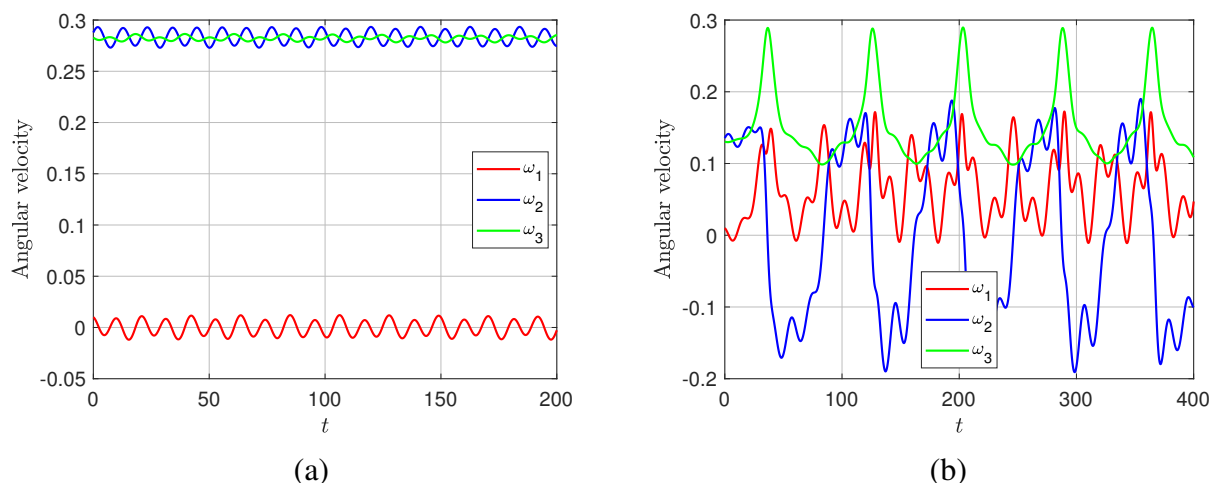
**Figure 4.** Time evolution of  $\omega_1, \omega_2,$  and  $\omega_3$  for  $Q_{1+}$ : (a) stable case ( $a_3 = 0.2$ ); (b) unstable case ( $a_3 = 3$ ).

Second, we inspect the permanent rotations ( $Q_{2\pm}$ ). We kept all the parameters unchanged and assumed  $\theta_0 = \pi/4$ . Solving the quadratic equation (3.3), which is the existence condition for  $Q_{2\pm}$ , results in two real zeros. They are  $\omega_0^{(1)} \approx 0.346$  and  $\omega_0^{(2)} \approx 0.754$ , where each of them characterizes diverse permanent rotation. The stability condition (4.24) is then checked for both values. This is summed up in Table 1.

**Table 1.** Stability analysis for  $Q_{2\pm}$  with  $\theta_0 = \pi/4$ .

$\omega_0$	$\tau_1$	$\tau_2$	$j_2$	$\min(\tau_1, \tau_2)$	Condition $j_2 < \min$	Stable?
0.346	0.712	0.503	0.4	0.503	$0.4 < 0.503$ (true)	Yes
0.754	0.712	0.278	0.7	0.278	$0.7 < 0.278$ (false)	No

Figure 5(a) points out the two components  $\omega_2$  and  $\omega_3$  are still approximately fixed near 0.28 while the component  $\omega_1$  is close to zero. Hence, there is no systematic growth or divergence, i.e., all quantities are still bound over time. This confirms the Lyapunov stability for  $Q_{2+}$ . On the other side, the unstable case which is characterized by the second row in Table 1, is described by Figure 5(b). It is noticeable that the trajectory diverges significantly. This confirms the Lyapunov instability.



**Figure 5.** Time evolution of  $\omega_1, \omega_2,$  and  $\omega_3$  for  $Q_{2+}$ : (a) stable case ( $j_2 = 0.4, \omega_0 = 0.346$ ); (b) unstable case ( $j_2 = 0.7, \omega_0 = 0.754$ ).

#### 4.2. Key parameter influences

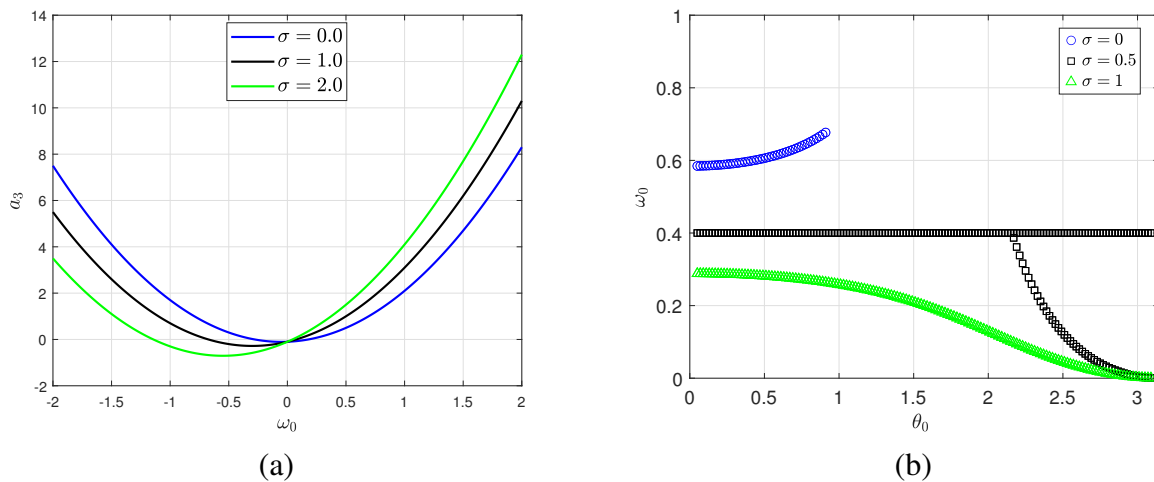
This subsection examines how variations in the rotor momentum parameter  $\sigma$  influence the system's stability while keeping all other parameters fixed. The analysis concentrates on the two permanent motions  $Q_{1+}$  and  $Q_{2+}$ .

The effect of the rotor momentum  $\sigma$  on the stability domain associated with the permanent rotation  $Q_{1+}$  is investigated through numerical simulations. When the angular velocity is positive ( $\omega_0 > 0$ ), larger values of  $\sigma$  shift the stability threshold upward in the  $(\omega_0, a_3)$  plane, which enlarges the region where the motion remains stable. This behavior reflects the stabilizing gyroscopic contribution of a rotor spinning in the same direction as the main body. For negative angular velocities ( $\omega_0 < 0$ ), however, the product  $\sigma\omega_0$  becomes negative, pushing the stability boundary downward and reducing the stable set. In physical terms, the rotor then acts against the body's rotation, which can promote instability. As a result, the overall stability landscape is not symmetric with respect to  $\omega_0$ , a feature clearly illustrated in Figure 6(a).

These observations highlight that the rotor momentum plays a decisive role in shaping the attitude stability and can either reinforce or weaken it depending on the relative direction of rotation. These results reveal that increasing the rotor momentum  $\sigma$  does not universally enhance stability; in this parameter configuration, it actually reduces the stability region and eventually leads to complete instability as shown in Figure 6(b). This counterintuitive behavior arises because the rotor's effect enters both the existence condition and the stability criteria, and for the chosen moments of inertia ( $C > B > A$ ) and magnetic coefficients, the sign of  $\omega_0$  (which may become negative for some  $\theta_0$ ) can cause the term  $\sigma\omega_0$  to lower the stability threshold. Thus, the rotor's influence is highly parameter-dependent, and the numerical exploration provides essential insight into the practical stability boundaries.

A similar study can be performed for the remaining permanent rotations. All numerical simulations fully corroborate our theoretical stability criteria, confirming the validity of the analytical derivations through the energy-Casimir approach. These computational results not only reinforce the robustness of our stability conditions but also offer practical guidance for engineers developing gyrostat-based

satellite attitude control systems.



**Figure 6.** The impact of the rotor momentum  $\sigma$  on the stability and instability zones for (a) the permanent rotation  $Q_{1+}$  and (b) the permanent rotation  $Q_{2+}$ .

## 5. Conclusions

The stability of a particular heavy charged rigid body carrying a rotor along the third axis of inertia, subjected to a combination of three fields—gravitational, uniform magnetic, and attractive central—has been investigated. The equations of motion have been derived and rewritten within a Lie–Poisson Hamiltonian system. It has been shown that the problem under consideration represents a generalization of a large class of rigid-body problems. The permanent rotations have been explored and mechanically explained.

The first permanent rotation is the rotation of the body about a fixed axis in the direction of the vector fixed in space,  $\gamma$ , with constant angular velocity. This is sometimes known as a standing gyroscope, since the fixed point lies above the center of mass. If the axis of rotation is opposite to the direction of the fixed vector, the body is called a suspended gyroscope, since the center of mass lies below the fixed point. The second permanent rotation is the rotation of the body with constant angular velocity about an axis oblique with respect to the vertical and having direction cosines  $(0, \pm \sin \theta_0, \cos \theta_0)$ . The third permanent rotation has a similar mechanical interpretation to the second. The fourth permanent rotation is the rotation of the body with constant angular velocity about an axis with direction cosines  $(\sin \theta_0 \sin \phi, \sin \theta_0 \cos \phi, \cos \theta_0)$ . Sufficient and necessary conditions for stability have been obtained.

## Author contributions

Conceptualization, A.E. and M.A.; methodology, A.E.; software, M.A. and A.E.; validation, M.A. and A.E.; formal analysis, M.A. and A.E.; investigation, M.A.; resources, A.E. and M.A.; data curation, M.A.; writing original draft preparation, A.E. and M.A.; writing—review and editing, M.A. and A.E.; visualization, A.E.; supervision, A.E. and M.A.; project administration, A.E.; funding acquisition, M.A. All authors have read and agreed to the published version of the manuscript.

## Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

## Data availability

The data underlying the results presented in the study are available from the author upon request.

## Funding

This work was supported by the Deanship of Scientific Research, Vice Presidency for Graduate Studies and Scientific Research, King Faisal University, Saudi Arabia [Grant No. KFU251198].

## Acknowledgments

The authors acknowledge the Deanship of Scientific Research at King Faisal University for the financial support.

## Conflict of interest

The authors declare no conflict of interest.

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