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Research article

Existence of solutions to fractional elliptic problem with nonlocal gradient term and lower order term

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Abstract: Let

$$\Omega \subset \mathbb{R}^N \left(3 \leqslant N < \frac{2s(2-\theta)-4t}{\theta}\right)$$

be a smooth bounded domain. For

$$0 < t < \frac{2s(2-\theta)-3\theta}{4} < s < 1,$$

we consider the following fractional partial differential equation:

$$\begin{cases} (-\Delta)^s u(x) = B\left|\mathbb{D}_t\left(u^{\frac{2-\theta}{2}}(x)\right)\right|^2 + \lambda f(x), & x \in \Omega, \\ \qquad u(x) > 0, & x \in \Omega, \\ \qquad u(x) = 0, & x \in \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $B > 0$ is a constant, $\theta \in \left(0, \frac{4s}{2s+3}\right)$, $0 < f \in L^m(\Omega)$, and $\lambda > 0$ is a real parameter. In addition, $\mathbb{D}_t\left(u^{\frac{2-\theta}{2}}(x)\right)$ denotes a nonlocal gradient term. Problems involving local gradient terms and lower-order terms have been extensively investigated in the existing literature. Motivated by this, we focus on the corresponding problem with nonlocal gradient terms and lower-order terms in the present paper. Specifically, our aim is to analyze the influence of the lower-order term on the existence of solutions to the fractional Laplace problem. For $0 < \lambda \leqslant \lambda^*$, we prove the existence of solutions to this problem when $m > \frac{N}{s}$.

Keywords: fractional Laplace problem; nonlocal gradient term; lower-order term
Mathematics Subject Classification: 35A01, 35R11

1. Introduction

This paper deals with the existence of solutions to the following nonlocal elliptic problem:

$$\begin{cases}(-\Delta)^s u(x) = B\left|\mathbb{D}_t\left(u^{\frac{2-\theta}{2}}(x)\right)\right|^2 + \lambda f(x), & x \in \Omega,\\ \quad u(x) > 0, & x \in \Omega,\\ \quad u(x) = 0, & x \in \mathbb{R}^N \setminus \Omega,\end{cases} \tag{1.1}$$

under the assumption

$$\begin{cases}\Omega \subset \mathbb{R}^N \left(3 \leqslant N < \frac{2s(2-\theta)-4t}{\theta}\right) \text{ is a bounded domain with } \partial\Omega \text{ of class } C^2,\\ 0 < t < \frac{2s(2-\theta)-3\theta}{4} < s < 1 \text{ and } \theta \in \left(0, \frac{4s}{2s+3}\right),\\ B > 0 \text{ is a constant},\\ 0 < f \in L^m(\Omega) \text{ with } m > \frac{N}{s},\end{cases} \tag{1.2}$$

where the fractional Laplacian operator $(-\Delta)^s$ is defined as

$$(-\Delta)^s u(x) = a_{N,s}\ \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}} dy, \quad s \in (0,1),$$

where

$$a_{N,s} = -\frac{2^{2s}\Gamma\left(\frac{N}{2}+s\right)}{\pi^{\frac{N}{2}}\Gamma(-s)} \tag{1.3}$$

is a normalization constant and P.V. stands for Cauchy principal value. The nonlocal gradient term is defined as

$$\mathbb{D}_t\left(u(x)\right) = \left(\frac{a_{N,s}}{2}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2t}} dy\right)^{\frac{1}{2}},$$

where $a_{N,s}$ comes from (1.3). The initial motivation for studying problem (1.1) stems from the local case, i.e.,

$$\begin{cases}-\Delta u(x) = B\dfrac{|Du(x)|^2}{u(x)} + f(x), & x \in \Omega,\\ \quad u(x) = 0, & x \in \partial\Omega,\end{cases} \tag{1.4}$$

where $B > 0$ is a constant, and $f(x) > 0$ is a Lebesgue integrable function. In this case, problem (1.4) is strongly related to the porous medium equation. The local problem involving a singular lower-order term has been extensively studied by Arcoya, Carmona, and others. In particular, Arcoya et al. [1, Theorems 1.1–1.3 and Theorem 3.8] proved the existence of solutions to problem (1.4) with $f(x) \in L^m(\Omega)(1 < m < \infty)$ by the approximation method. This approach is a standard method to address the nonlinear problem with a singular lower-order term.

Problem (1.4) is a particular case of the following general model

$$\begin{cases}-\Delta u(x) = \eta\dfrac{|Du(x)|^q}{|u(x)|^\theta} + f(x,u), & x \in \Omega,\\ \quad u(x) = 0, & x \in \partial\Omega,\end{cases} \tag{1.5}$$

where $\eta \geqslant 0$. Giachetti and Murat [2, Theorems 2.2 and 2.3] have examined the existence of solutions to the elliptic problem with $\eta < 0$ and singular dependence of the lower-order term. For $f(x) \in L^m(\Omega)\left(m \geqslant \frac{N}{2}\right)$, Giachetti et al. [3, Theorem 2.1] demonstrated the existence of weak solutions to problem (1.5) with $\eta = 1$, $q = 2$, $\theta \in (0,1)$, and $f(x,u) \equiv f(x)$. This result holds without imposing any sign restrictions on $f(x)$. They defined the meaning of solutions to ensure that the gradient term remains meaningful even when $u(x) = 0$. Furthermore, they established the boundedness of solutions for $f(x) \in L^m(\Omega)\left(m > \frac{N}{2}\right)$. Arcoya et al. [4, Theorem 1.1] studied the existence of solutions to problem (1.5) with $q = 2$, $\theta = 1$, $\eta = \mu(x) \in L^\infty(\Omega)$, and $f(x,u) = \lambda u(x) + f(x)$. For $0 < f(x) \in L^m(\Omega)\left(1 < m < \frac{N}{2}\right)$, they demonstrated that the problem admits a positive solution provided that $\lambda < \frac{\lambda_1}{1+\|\mu\|_{L^\infty(\Omega)}}$, where λ_1 denotes the first eigenvalue of the Laplacian operator. Carmona et al. [5, Theorem 1.2] investigated the existence of solutions to an elliptic problem with a singular term of the form $\frac{|\nabla u|^q}{u^{q-1}}$. Problem (1.5) reduces to their model with $\theta = q - 1 \in (0,1]$, $\eta = \mu(x) \in L^\infty(\Omega)$, and $f(x,u) = \lambda u(x) + f(x)$. For $f(x) \in L^m(\Omega)\left(m > \frac{N}{2}\right)$, they extended the traditional case of $q = 2$ (see [4]) to a broader range of q and proved that problem (1.5) admits a unique solution for $\lambda \leqslant 0$, at least one solution for $\lambda < \lambda^*$ and no solution for $\lambda > \lambda^*$. López-Martínez [6, Theorem 1.1] investigated the existence, multiplicity, and uniqueness of solutions to an elliptic problem with a singular term of the form $\frac{|\nabla u|^q}{u^\alpha}$. Problem (1.5) reduces to their model with $\theta = \alpha \in [0,1]$, $\eta = \mu(x) \in L^\infty(\Omega)$, and $f(x,u) = \lambda u(x) + f(x)$. The study revealed that $\alpha = q - 1$ plays the role of a break point for the multiplicity or uniqueness of solutions. Kumar and Tyagi [7, Theorem 2.1] introduced the Hardy potential based on problem (1.4) and proved the existence of solutions. For related works on the p-Laplacian, see [8, 9].

The fractional Laplacian operator $(-\Delta)^s (0 < s < 1)$ is pivotal in modeling long-range interactions. It has applications in obstacle problems [10], crystal dislocations [11], and phase transitions [12]. It is widely used in modern scientific analysis and modeling.

This leads us to the nonlocal boundary value problem

$$\begin{cases} (-\Delta)^s u(x) = |\mathcal{D}_t u(x)|^q + \rho f(x), & x \in \Omega, \\ \qquad u(x) = 0, & x \in \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.6}$$

where $\mathcal{D}_t(\cdot)$ denotes the nonlocal gradient term, which has three distinct forms. Abdellaoui et al. [13, Theorems 1.1–1.4] established the existence of solutions with $t = s \in \left(\frac{1}{2}, 1\right)$, where one of the forms of the nonlocal gradient term $\mathcal{D}_t(\cdot)$ was considered. Subsequently, Abdellaoui et al. [14, Theorem 1.4] extended these results to the full parameter range with $s \in (0,1)$ and $t \in \left(0, \min\left\{1, s\left(1 + \frac{1}{N}\right)\right\}\right)$. Abdellaoui et al. [15, Theorem 4.7] demonstrated the existence of solutions to problem (1.6) with Hardy potentials. Recently, Younes et al. [16, Theorem 1.2] made significant progress by generalizing this framework to multi-operator fractional elliptic systems.

Problem (1.1) can be seen as a nonlocal generalization of the singular problem

$$\begin{cases} -\Delta u(x) = \mu_\theta(x)\dfrac{|Du(x)|^2}{u^\theta(x)} + \lambda f(x), & x \in \Omega, \\ u(x) > 0, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{1.7}$$

where

$$\mu_\theta(x) = \frac{(2-\theta)^2}{4} B$$

and

$$\frac{|Du(x)|^2}{u^\theta(x)} = \frac{4}{(2-\theta)^2}\left|Du^{\frac{2-\theta}{2}}(x)\right|^2.$$

The existence of solutions to problem (1.7) have been analyzed in [3]. On the other hand, problem (1.1) can be seen as a singular perturbation of the nonlocal problem that has been studied in [13, 14].

Next, we introduce the definition of the solution to problem (1.1).

Definition 1.1. We say that $u(x)$ is a weak solution to problem (1.1) if $0 < u(x) \in L^1(\Omega)$, $\left|\mathbb{D}_t\left(u^{\frac{2-\theta}{2}}(x)\right)\right|^2 \in L^1(\Omega)$, and $u(x) = 0$ for all $x \in \mathbb{R}^N \setminus \Omega$, and

$$\iint_{\mathcal{D}_\Omega} \frac{(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+2s}}dxdy = B\left|\mathbb{D}_t\left(u^{\frac{2-\theta}{2}}(x)\right)\right|^2\phi(x)dx + \lambda\int_\Omega f(x)\phi(x)dx, \tag{1.8}$$

for all ϕ belonging to

$$\mathcal{X}^s(\Omega) := \left\{\phi \in C^{0,s}\left(\mathbb{R}^N\right) : \phi(x) = 0 \text{ for all } x \in \mathbb{R}^N \setminus \Omega \text{ and } (-\Delta)^s\phi \in L^\infty(\Omega)\right\},$$

where $C^{0,s}\left(\mathbb{R}^N\right)$ denotes the Hölder continuous space, and $s \in (t, 1)$ is the Hölder continuity exponent.

The paper is organized as follows. Section 2 presents the functional framework and relevant spaces. Section 3 establishes the existence of solutions to the problem (1.1).

2. Preliminaries

First, we introduce some notations used in the paper:

- For a bounded open set $\Omega \subset \mathbb{R}^N$, we denote its complementary as $\mathcal{C}_\Omega$, that is $\mathcal{C}_\Omega = \mathbb{R}^N \setminus \Omega$.
- For $p \in (1, \infty)$, we denote by p' the conjugate exponent of p, namely $p' = \frac{p}{p-1}$ and by p_s^*, the fractional Sobolev critical exponent, i.e., $p_s^* = \frac{Np}{N-sp}$ if $sp < N$ and $p_s^* = +\infty$ if $sp \geqslant N$.
- C represents a generic positive constant that may be different even in the same formula.

We now collect some definitions of the main function spaces involved in our results and some additional tools here. First of all, for all $s \in (0, 1)$ and $1 \leqslant p < +\infty$, the fractional Sobolev space $\mathcal{W}^{s,p}\left(\mathbb{R}^N\right)$ is defined as

$$\mathcal{W}^{s,p}\left(\mathbb{R}^N\right) := \left\{u \in L^p\left(\mathbb{R}^N\right) : \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}dxdy < +\infty\right\},$$

which is a Banach space endowed with the norm

$$\|u\|_{\mathcal{W}^{s,p}(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} |u(x)|^p dx + \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}dxdy\right)^{\frac{1}{p}}.$$

Define the space $\mathcal{W}_0^{s,p}(\Omega)$ as

$$\mathcal{W}_0^{s,p}(\Omega) := \left\{u \in \mathcal{W}^{s,p}\left(\mathbb{R}^N\right) : u(x) \equiv 0, x \in \mathbb{R}^N \setminus \Omega\right\}.$$

Thanks to the Sobolev inequality, $\mathcal{W}_0^{s,p}(\Omega)$ is a Banach space endowed with norm

$$\|u\|_{\mathcal{W}_0^{s,p}(\Omega)} = \left(\iint_{\mathcal{D}_\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy\right)^{\frac{1}{p}},$$

where

$$\mathcal{D}_\Omega := \left(\mathbb{R}^N \times \mathbb{R}^N\right) \setminus (C_\Omega \times C_\Omega).$$

In this case, the spaces $\mathcal{W}^{s,2}(\mathbb{R}^N)$ and $\mathcal{W}_0^{s,2}(\Omega)$ are commonly denoted by $\mathcal{H}^s(\mathbb{R}^N)$ and $\mathcal{H}_0^s(\Omega)$, respectively.

Next, we recall that for all $s \in (0,1)$ and $1 \leqslant p < +\infty$, the Bessel potential space is denoted as

$$\mathcal{L}^{s,p}\left(\mathbb{R}^N\right) = \overline{\{u \in C_c^\infty(\mathbb{R}^N)\}}^{|\cdot|_{\mathcal{L}^{s,p}(\mathbb{R}^N)}},$$

where

$$|u|_{\mathcal{L}^{s,p}(\mathbb{R}^N)} = \left\|(1-\Delta)^{\frac{s}{2}} u\right\|_{L^p(\mathbb{R}^N)}$$

and

$$(1-\Delta)^{\frac{s}{2}} u = \mathcal{F}^{-1}\left(\left(1+|\cdot|^2\right)^{\frac{s}{2}} \mathcal{F} u\right), \text{ for all } u \in C_c^\infty\left(\mathbb{R}^N\right).$$

We note that for $s \in (0,1)$ and $1 < p < +\infty$, the norm

$$\|u\|_{\mathcal{L}^{s,p}(\mathbb{R}^N)} = \|u\|_{L^p(\mathbb{R}^N)} + \left\|(-\Delta)^{\frac{s}{2}} u\right\|_{L^p(\mathbb{R}^N)}$$

is equivalent to that of the space $\mathcal{L}^{s,p}\left(\mathbb{R}^N\right)$ (see [17, Theorem 2]). Moreover, we have the additional result that if $\frac{2N}{N+2s} < p < +\infty$, then an equivalent norm

$$\|u\|_{\mathcal{L}^{s,p}(\mathbb{R}^N)} = \|u\|_{L^p(\mathbb{R}^N)} + \|\mathbb{D}_s u\|_{L^p(\mathbb{R}^N)}$$

can be defined on $\mathcal{L}^{s,p}\left(\mathbb{R}^N\right)$ [17, Theorem 1.1]. By analogy with the space $\mathcal{W}_0^{s,p}(\Omega)$, we define the space

$$\mathcal{L}_0^{s,p}(\Omega) = \left\{u \in \mathcal{L}^{s,p}\left(\mathbb{R}^N\right) : u \equiv 0 \text{ in } \mathbb{R}^N \setminus \Omega\right\}.$$

We emphasize that for $0 < s < 1$ and $1 < p < +\infty$, this space is a Banach space with respect to the norm

$$\|u\|_{\mathcal{L}_0^{s,p}(\Omega)} = \left\|(-\Delta)^{\frac{s}{2}} u\right\|_{L^p(\mathbb{R}^N)}.$$

Furthermore, if the additional condition $\frac{2N}{N+2s} < p < +\infty$ holds, then an equivalent norm can be imposed on $\mathcal{L}_0^{s,p}(\Omega)$, given by

$$\|u\|_{\mathcal{L}_0^{s,p}(\Omega)} = \|\mathbb{D}_s(u)\|_{L^p(\mathbb{R}^N)}.$$

We also recall the embeddings that hold for all $0 < \epsilon < \sigma < 1$ and all $1 < p < +\infty$ (see [18, Theorem 7.63(g)]):

$$\mathcal{L}^{\sigma+\epsilon,p}\left(\mathbb{R}^N\right) \subset \mathcal{W}^{\sigma,p}\left(\mathbb{R}^N\right) \subset \mathcal{L}^{\sigma-\epsilon,p}\left(\mathbb{R}^N\right).$$

It is a well-known result that for all $1 \leqslant p < +\infty$ and all $0 < \sigma \leqslant \sigma' < 1$, the embedding

$$\mathcal{W}^{\sigma',p}\left(\mathbb{R}^N\right) \subset \mathcal{W}^{\sigma,p}\left(\mathbb{R}^N\right)$$

holds [18, Theorem 7.63(c)], and that for $1 < p < \infty$, we have

$$\mathcal{L}^{\sigma',p}\left(\mathbb{R}^N\right) \subset \mathcal{L}^{\sigma,p}\left(\mathbb{R}^N\right).$$

As the constant below will be of use in subsequent arguments, we point out that there exists a constant $k = k(\sigma, \sigma', p) \geqslant 1$ such that

$$||u||_{\mathcal{L}^{\sigma,p}\left(\mathbb{R}^N\right)} \leqslant k||u||_{\mathcal{L}^{\sigma',p}\left(\mathbb{R}^N\right)} \tag{2.1}$$

for all $u \in \mathcal{L}^{\sigma',p}\left(\mathbb{R}^N\right)$, and

$$||u||_{\mathcal{W}^{\sigma,p}\left(\mathbb{R}^N\right)} \leqslant k||u||_{\mathcal{W}^{\sigma',p}\left(\mathbb{R}^N\right)}$$

for all $u \in \mathcal{W}^{\sigma',p}\left(\mathbb{R}^N\right)$. Having introduced the basic notions of the relevant function spaces, we next present several lemmas that will be applied in the subsequent analysis.

Lemma 2.1. *[19, Lemma 2.2] Let* $0 < \alpha \leqslant 1$*; then for every* $x,\ y \geqslant 0$*, we have*

$$|x^\alpha - y^\alpha| \leqslant |x - y|^\alpha.$$

Lemma 2.2. *[14, Proposition 2.4] Let* $0 < t < \min\left\{1, s\left(1 + \frac{1}{N}\right)\right\}$ *and* $1 < p < \frac{N}{N(1+t-s)-s}$*. The solution map*

$$\mathbb{G}_s : L^1(\Omega) \to \mathcal{L}_0^{t,p}(\Omega), \quad h \mapsto \mathbb{G}_s[h] = \int_\Omega G_s(x, y)h(y)dy \tag{2.2}$$

is well defined, continuous, and compact, where $G_s : \mathbb{R}^{2N}_* \to \mathbb{R}$ *denotes the Green function associated with the fractional Laplacian operator* $(-\Delta)^s$ *in the domain* Ω *with homogeneous Dirichlet boundary conditions. Note that*

$$\mathbb{R}^{2N}_* := \left\{(x, y) \in \mathbb{R}^{2N} : x \neq y\right\}.$$

Lemma 2.3. *[13, Lemma 4.1] Let* $a, b > 0$*,* $p > 1$*, and*

$$c^* := \frac{p-1}{p}(\frac{1}{pa^p b})^{\frac{1}{p-1}}.$$

Then, the function g: $[0, \infty) \to \mathbb{R}$ *is given by*

$$g(t) = a^p(bt + c^*)^p - t.$$

Therefore, $g(t) = 0$ *has exactly one root* $t^* \in (0, \infty)$*.*

Lemma 2.4. *[14, Proposition 2.1] Let* $0 < s \leqslant t < \min\left\{1, s\left(1 + \frac{1}{N}\right)\right\}$*, and let* u *be the unique weak solution to*

$$\begin{cases} (-\Delta)^s u(x) = h(x), & x \in \Omega, \\ \qquad u(x) = 0, & x \in \mathbb{R}^N \setminus \Omega, \end{cases} \tag{2.3}$$

with $h \in L^m(\Omega)$ *for some* $m \geqslant 1$*, where*

$$\tilde{p}(m, s, t) = \begin{cases} \frac{1}{(t-s)^+}, & \text{if } m > \frac{N}{2s-t}, \\ \min\left\{\frac{mN}{N-ms+mN(t-s)}, \frac{1}{(t-s)^+}\right\}, & \text{if } 1 \leqslant m < \frac{N}{2s-t}. \end{cases} \tag{2.4}$$

Then, for all $1 < p < \tilde{p}$*, there exists* $C(N, s, p, m, \Omega) > 0$ *such that*

$$\|u\|_{\mathcal{L}^{t,p}(\mathbb{R}^N)} \leqslant C\|h\|_{L^m(\Omega)} \ \text{ and } \ \|u\|_{W^{t,p}(\mathbb{R}^N)} \leqslant C\|h\|_{L^m(\Omega)}. \tag{2.5}$$

Lemma 2.5. *Let* a, b *be real numbers and* $p > 0$*. Then,*

$$(|a| + |b|)^p \leqslant C(|a|^p + |b|^p),$$

where

$$C = \begin{cases} 1, & 0 < p \leqslant 1, \\ 2^{p-1}, & p > 1. \end{cases}$$

Lemma 2.6. *Assume that* (1.2) *holds, and that* α *and* ε *are defined in* (3.3) *and* (3.5)*, respectively. Then,*

$$\left(\int_\Omega \left(\int_{\mathbb{R}^N} \frac{\left|u^{\frac{2-\theta}{2}}(x) - u^{\frac{2-\theta}{2}}(y)\right|^2}{|x-y|^{N+2t}} dy \right)^{\frac{1+\alpha}{2}} dx \right)^{\frac{1}{1+\alpha}} \leqslant \|u\|^{\frac{2-\theta}{2}}_{\mathcal{L}_0^{s+\varepsilon,1+\alpha}(\Omega)}.$$

Proof. Using Lemma 2.1, convexity, the Hölder inequality, and (2.1), we have

$$\begin{aligned} \left(\int_\Omega \left(\int_{\mathbb{R}^N} \frac{\left|u^{\frac{2-\theta}{2}}(x) - u^{\frac{2-\theta}{2}}(y)\right|^2}{|x-y|^{N+2t}} dy \right)^{\frac{1+\alpha}{2}} dx \right)^{\frac{1}{1+\alpha}} &\leqslant \left(\int_\Omega \left(\int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{2-\theta}}{|x-y|^{N+2t}} dy \right)^{\frac{1+\alpha}{2}} dx \right)^{\frac{1}{1+\alpha}} \\ &\leqslant C \left(\int_\Omega \left(\int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x-y|^{(N+2t)\frac{2}{2-\theta}}} dy \right)^{\frac{(2-\theta)(1+\alpha)}{4}} dx \right)^{\frac{1}{1+\alpha}} \\ &\leqslant C \left(\int_\Omega \left(\int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2\frac{4t+N\theta}{2(2-\theta)}}} dy \right)^{\frac{1+\alpha}{2}} dx \right)^{\frac{2-\theta}{2(1+\alpha)}} \\ &= C \left\| \mathbb{D}_{\frac{4t+N\theta}{2(2-\theta)}}(u) \right\|^{\frac{2-\theta}{2}}_{L^{1+\alpha}(\Omega)} \\ &= C\|u\|^{\frac{2-\theta}{2}}_{\mathcal{L}_0^{\frac{4t+N\theta}{2(2-\theta)},1+\alpha}(\Omega)} \\ &\leqslant C\|u\|^{\frac{2-\theta}{2}}_{\mathcal{L}_0^{s+\varepsilon,1+\alpha}(\Omega)}. \end{aligned}$$

This completes the proof.

□

3. Existence result

We now prove the existence of solutions to problem (1.1). Let (1.2) hold, and define $\mathcal{T}\colon E \to \mathcal{L}_0^{s+\varepsilon,1+\alpha}(\Omega)$ as

$$\mathcal{T}(v) = u,$$

where u is the weak solution to

$$\begin{cases} (-\Delta)^s u(x) = B\left|\mathbb{D}_t\left(v^{\frac{2-\theta}{2}}(x)\right)\right|^2 + \lambda f(x), & x \in \Omega, \\ \quad u(x) > 0, & x \in \Omega, \\ \quad u(x) = 0, & x \in \mathbb{R}^N \setminus \Omega, \end{cases} \tag{3.1}$$

and $v(x) > 0$ in Ω, while $v(x) = 0$ in $\mathbb{R}^N \setminus \Omega$, where

$$E = \left\{0 < u \in \mathcal{L}_0^{s+\varepsilon,1+\alpha}(\Omega) :\ \|u\|_{\mathcal{L}_0^{s+\varepsilon,\tau}(\Omega)} \leqslant \ell^{\frac{1}{2-\theta}}\right\}, \tag{3.2}$$

and ℓ satisfies (3.10),

$$1 < 1 + \alpha < \tau < \min\left\{2, \frac{N}{N(1+\varepsilon) - s}\right\}, \tag{3.3}$$

where

$$s + \varepsilon > \frac{4t + N\theta}{2(2-\theta)}, \quad \tau = \tau(r, \theta, m, s) = \frac{r(2-\theta)}{2},$$

and

$$r > \max\left\{2m, \frac{1+\alpha}{\alpha}\right\}, \tag{3.4}$$

where

$$\varepsilon < \min\left\{1 - s, \frac{s}{N}\right\} \tag{3.5}$$

is a small enough positive constant.

In order to prove the existence of a weak solution to problem (1.1), we show that the function $\mathcal{T}$ has a fixed point in E, that is,

$$u = \mathcal{T}(u).$$

Lemma 3.1. *$\mathcal{T}$ is well defined if* (1.2) *holds.*

Proof. By the Hölder inequality and (3.4), we have

$$\begin{aligned} \int_\Omega \left|\mathbb{D}_t\left(v^{\frac{2-\theta}{2}}(x)\right)\right|^2 dx &\leqslant |\Omega|^{\frac{r-2}{r}} \left(\int_\Omega \left|\mathbb{D}_t\left(v^{\frac{2-\theta}{2}}(x)\right)\right|^r dx\right)^{\frac{2}{r}} \\ &\leqslant C\left(\int_\Omega \left(\int_{\mathbb{R}^N} \frac{\left|v^{\frac{2-\theta}{2}}(x) - v^{\frac{2-\theta}{2}}(y)\right|^2}{|x-y|^{N+2t}} dy\right)^{\frac{r}{2}} dx\right)^{\frac{2}{r}}, \end{aligned}$$

and by Lemma 2.1 and convexity, we obtain that

$$\begin{aligned}
\int_\Omega \left|\mathbb{D}_t\left(v^{\frac{2-\theta}{2}}(x)\right)\right|^2 dx &\leqslant C\left(\int_\Omega\left(\int_{\mathbb{R}^N}\frac{|v(x)-v(y)|^{2-\theta}}{|x-y|^{N+2t}}dy\right)^{\frac{r}{2}}dx\right)^{\frac{2}{r}}\\
&\leqslant C\left(\int_\Omega\left(\int_{\mathbb{R}^N}\frac{|v(x)-v(y)|^{2}}{|x-y|^{(N+2t)\frac{2}{2-\theta}}}dy\right)^{\frac{r(2-\theta)}{4}}dx\right)^{\frac{2}{r}}\\
&= C\left(\int_\Omega\left(\int_{\mathbb{R}^N}\frac{|v(x)-v(y)|^{2}}{|x-y|^{N+2\cdot\frac{4t+N\theta}{2(2-\theta)}}}dy\right)^{\frac{r(2-\theta)}{4}}dx\right)^{\frac{2}{r}}\\
&= C\left\|\mathbb{D}_{\frac{4t+N\theta}{2(2-\theta)}}\left(v(x)\right)\right\|^{2-\theta}_{L^{\frac{r(2-\theta)}{2}}(\Omega)}\\
&= C\|v\|^{2-\theta}_{\mathcal{L}_0^{\frac{4t+N\theta}{2(2-\theta)},\frac{r(2-\theta)}{2}}(\Omega)}\\
&\leqslant C\|v\|^{2-\theta}_{\mathcal{L}_0^{s+\varepsilon,\tau}(\Omega)}\\
&\leqslant C\ell.
\end{aligned}\tag{3.6}$$

By [20, Theorem 23] and Lemma 2.2, $\mathcal{T}$ is well defined, using (3.3).

This completes the proof. □

Lemma 3.2. *$\mathcal{T}(E)\subset E$ if (1.2) holds.*

Proof. For any $v\in E$, let $u=\mathcal{T}(v)$. Now, using Lemma 2.4, we obtain that

$$\begin{aligned}
\|u\|_{\mathcal{L}_0^{s+\varepsilon,\tau}(\Omega)} &\leqslant C\left\|B\left|\mathbb{D}_t\left(v^{\frac{2-\theta}{2}}(x)\right)\right|^2+\lambda f(x)\right\|_{L^m(\Omega)}\\
&\leqslant C\left\|\left|\mathbb{D}_t\left(v^{\frac{2-\theta}{2}}(x)\right)\right|^2\right\|_{L^m(\Omega)}+C\lambda^*\|f\|_{L^m(\Omega)},
\end{aligned}\tag{3.7}$$

where λ^* appears in (3.11), $m>\frac{N}{s}$, and we have used (3.3).

By the Hölder inequality, (3.4), and Lemma 2.1 again, we find

$$\begin{aligned}
\left\|\left|\mathbb{D}_t\left(v^{\frac{2-\theta}{2}}(x)\right)\right|^2\right\|_{L^m(\Omega)} &=\left(\int_\Omega\left(\int_{\mathbb{R}^N}\frac{\left|v^{\frac{2-\theta}{2}}(x)-v^{\frac{2-\theta}{2}}(y)\right|^2}{|x-y|^{N+2t}}dy\right)^{m}dx\right)^{\frac{1}{m}}\\
&\leqslant|\Omega|^{\frac{r-2m}{rm}}\left(\int_\Omega\left(\int_{\mathbb{R}^N}\frac{\left|v^{\frac{2-\theta}{2}}(x)-v^{\frac{2-\theta}{2}}(y)\right|^2}{|x-y|^{N+2t}}dy\right)^{\frac{r}{2}}dx\right)^{\frac{2}{r}}\\
&\leqslant C\left(\int_\Omega\left(\int_{\mathbb{R}^N}\frac{|v(x)-v(y)|^{2-\theta}}{|x-y|^{N+2t}}dy\right)^{\frac{r}{2}}dx\right)^{\frac{2}{r}},
\end{aligned}$$

and by (3.6), we have

$$\left(\int_{\Omega}\left(\int_{\mathbb{R}^N}\frac{|v(x)-v(y)|^{2-\theta}}{|x-y|^{N+2t}}dy\right)^{\frac{r}{2}}dx\right)^{\frac{2}{r}}\leqslant C\ell. \tag{3.8}$$

Combining (3.7) and (3.8), we have

$$\|u\|_{\mathcal{L}_0^{s+\varepsilon,\tau}(\Omega)}\leqslant C\left(\ell+\lambda^*\|f\|_{L^m(\Omega)}\right). \tag{3.9}$$

Let

$$C\left(\ell+\lambda^*\|f\|_{L^m(\Omega)}\right)=\ell^{\frac{1}{2-\theta}}, \tag{3.10}$$

where

$$\lambda^*=C\frac{1}{\|f\|_{L^m(\Omega)}}. \tag{3.11}$$

By Lemma 2.3, we know that (3.10) has a unique solution ℓ. Thus, by (3.9) and (3.10), we have that

$$\|u\|_{\mathcal{L}_0^{s+\varepsilon,\tau}(\Omega)}\leqslant \ell^{\frac{1}{2-\theta}}. \tag{3.12}$$

By Lemma 2.4, we also know that $u\in\mathcal{L}_0^{s+\varepsilon,1+\alpha}(\Omega)$. This, together with (3.12), leads to $u\in E$. Subsequently, we deduce that $\mathcal{T}(E)\subset E$.

This completes the proof. □

Lemma 3.3. *$\mathcal{T}$ is continuous if* (1.2) *holds.*

Proof. Let $\{v_n\}\subset E$ be a sequence such that $v_n\to v$ in $\mathcal{L}_0^{s+\varepsilon,1+\alpha}(\Omega)$. Define $u_n=\mathcal{T}(v_n)$ and $u=\mathcal{T}(v)$. Now, we show that $u_n\to u$ in $\mathcal{L}_0^{s+\varepsilon,1+\alpha}(\Omega)$. In order to show this fact, by Lemma 2.2, we only need to show that

$$g_n(x)=B\left|\mathbb{D}_t\left(v_n^{\frac{2-\theta}{2}}(x)\right)\right|^2+\lambda f(x)\to g(x)=B\left|\mathbb{D}_t\left(v^{\frac{2-\theta}{2}}(x)\right)\right|^2+\lambda f(x)\ \text{ in } L^1(\Omega). \tag{3.13}$$

That is,

$$\begin{aligned}\left\|\left|\mathbb{D}_t\left(v_n^{\frac{2-\theta}{2}}(x)\right)\right|^2-\left|\mathbb{D}_t\left(v^{\frac{2-\theta}{2}}(x)\right)\right|^2\right\|_{L^1(\Omega)}&=\iint_{\mathcal{D}_\Omega}\frac{\left|\left|v_n^{\frac{2-\theta}{2}}(x)-v_n^{\frac{2-\theta}{2}}(y)\right|^2-\left|v^{\frac{2-\theta}{2}}(x)-v^{\frac{2-\theta}{2}}(y)\right|^2\right|}{|x-y|^{N+2t}}dxdy\\&=\iint_{\mathcal{D}_\Omega}\frac{\left|v_n^{\frac{2-\theta}{2}}(x)-v_n^{\frac{2-\theta}{2}}(y)\right|+\left|v^{\frac{2-\theta}{2}}(x)-v^{\frac{2-\theta}{2}}(y)\right|}{|x-y|^{\frac{N}{2}+t}}\\&\quad\times\frac{\left|\left|v_n^{\frac{2-\theta}{2}}(x)-v_n^{\frac{2-\theta}{2}}(y)\right|-\left|v^{\frac{2-\theta}{2}}(x)-v^{\frac{2-\theta}{2}}(y)\right|\right|}{|x-y|^{\frac{N}{2}+t}}dxdy.\end{aligned}$$

Using the Hölder inequality, we have

$$\begin{aligned}\left\|\left|\mathbb{D}_t\left(v_n^{\frac{2-\theta}{2}}(x)\right)\right|^2-\left|\mathbb{D}_t\left(v^{\frac{2-\theta}{2}}(x)\right)\right|^2\right\|_{L^1(\Omega)} &\leqslant \int_\Omega\left(\left(\int_{\mathbb{R}^N}\frac{\left(\left|v_n^{\frac{2-\theta}{2}}(x)-v_n^{\frac{2-\theta}{2}}(y)\right|+\left|v^{\frac{2-\theta}{2}}(x)-v^{\frac{2-\theta}{2}}(y)\right|\right)^2}{|x-y|^{N+2t}}dy\right)^{\frac{1}{2}}\right.\\ &\quad\left.\times\left(\int_{\mathbb{R}^N}\frac{\left(\left|v_n^{\frac{2-\theta}{2}}(x)-v_n^{\frac{2-\theta}{2}}(y)\right|-\left|v^{\frac{2-\theta}{2}}(x)-v^{\frac{2-\theta}{2}}(y)\right|\right)^2}{|x-y|^{N+2t}}dy\right)^{\frac{1}{2}}\right)dx\\ &\leqslant\left(\iint_{\mathcal{D}_\Omega}\frac{\left(\left|v_n^{\frac{2-\theta}{2}}(x)-v_n^{\frac{2-\theta}{2}}(y)\right|+\left|v^{\frac{2-\theta}{2}}(x)-v^{\frac{2-\theta}{2}}(y)\right|\right)^2}{|x-y|^{N+2t}}dxdy\right)^{\frac{1}{2}}\\ &\quad\times\left(\iint_{\mathcal{D}_\Omega}\frac{\left(\left|v_n^{\frac{2-\theta}{2}}(x)-v_n^{\frac{2-\theta}{2}}(y)\right|-\left|v^{\frac{2-\theta}{2}}(x)-v^{\frac{2-\theta}{2}}(y)\right|\right)^2}{|x-y|^{N+2t}}dxdy\right)^{\frac{1}{2}}\\ &=M_1^{\frac{1}{2}}\cdot M_2^{\frac{1}{2}}. \end{aligned}\tag{3.14}$$

Now, we prove that M_1 is bounded and $M_2\to 0$ as $n\to\infty$. Then,

$$\left\|\left|\mathbb{D}_t\left(v_n^{\frac{2-\theta}{2}}(x)\right)\right|^2-\left|\mathbb{D}_t\left(v^{\frac{2-\theta}{2}}(x)\right)\right|^2\right\|_{L^1(\Omega)}\to 0.$$

First, we claim that M_1 is bounded. By (3.6), we have

$$\begin{aligned}M_1&\leqslant 2\left(\iint_{\mathcal{D}_\Omega}\frac{\left|v_n^{\frac{2-\theta}{2}}(x)-v_n^{\frac{2-\theta}{2}}(y)\right|^2}{|x-y|^{N+2t}}dxdy+\iint_{\mathcal{D}_\Omega}\frac{\left|v^{\frac{2-\theta}{2}}(x)-v^{\frac{2-\theta}{2}}(y)\right|^2}{|x-y|^{N+2t}}dxdy\right)\\ &\leqslant 2|\Omega|^{\frac{r-2}{r}}\left(\left(\int_\Omega\left(\int_{\mathbb{R}^N}\frac{\left|v_n^{\frac{2-\theta}{2}}(x)-v_n^{\frac{2-\theta}{2}}(y)\right|^2}{|x-y|^{N+2t}}dy\right)^{\frac{r}{2}}dx\right)^{\frac{2}{r}}+\left(\int_\Omega\left(\int_{\mathbb{R}^N}\frac{\left|v^{\frac{2-\theta}{2}}(x)-v^{\frac{2-\theta}{2}}(y)\right|^2}{|x-y|^{N+2t}}dy\right)^{\frac{r}{2}}dx\right)^{\frac{2}{r}}\right)\\ &\leqslant C\left(\|v_n\|^{2-\theta}_{\mathcal{L}_0^{s+\varepsilon,\tau}(\Omega)}+\|v\|^{2-\theta}_{\mathcal{L}_0^{s+\varepsilon,\tau}(\Omega)}\right).\end{aligned}$$

By $v_n\in E$ and $v\in E$, we have

$$M_1\leqslant C\left(\|v_n\|^{2-\theta}_{\mathcal{L}_0^{s+\varepsilon,\tau}(\Omega)}+\|v\|^{2-\theta}_{\mathcal{L}_0^{s+\varepsilon,\tau}(\Omega)}\right)\leqslant C\ell,$$

which implies that M_1 is bounded.

Next, we claim $M_2\to 0$. Denote

$$\phi_n=v_n^{\frac{2-\theta}{2}}-v^{\frac{2-\theta}{2}}.$$

Combining with (3.14) and the Hölder inequality, we obtain that

$$\begin{aligned}
M_2 &\leqslant \int_\Omega\left(\int_{\mathbb{R}^N}\frac{|\phi_n(x)-\phi_n(y)|^2}{|x-y|^{N+2t}}dy\right)dx\\
&=\int_\Omega\left(\int_{\mathbb{R}^N}\frac{|\phi_n(x)-\phi_n(y)|}{|x-y|^{\frac{N}{2}+t}}\cdot\frac{|\phi_n(x)-\phi_n(y)|}{|x-y|^{\frac{N}{2}+t}}dy\right)dx\\
&\leqslant\int_\Omega\left(\int_{\mathbb{R}^N}\frac{|\phi_n(x)-\phi_n(y)|^2}{|x-y|^{N+2t}}dy\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^N}\frac{|\phi_n(x)-\phi_n(y)|^2}{|x-y|^{N+2t}}dy\right)^{\frac{1}{2}}dx\\
&\leqslant\left(\int_\Omega\left(\int_{\mathbb{R}^N}\frac{|\phi_n(x)-\phi_n(y)|^2}{|x-y|^{N+2t}}dy\right)^{\frac{1+\alpha}{2}}dx\right)^{\frac{1}{1+\alpha}}\left(\int_\Omega\left(\int_{\mathbb{R}^N}\frac{|\phi_n(x)-\phi_n(y)|^2}{|x-y|^{N+2t}}dy\right)^{\frac{1+\alpha}{2\alpha}}dx\right)^{\frac{\alpha}{1+\alpha}}.
\end{aligned}$$

Hence, by $v_n \to v$ in $\mathcal{L}_0^{s+\varepsilon,1+\alpha}(\Omega)$ and Lemma 2.6, we have

$$\left(\int_\Omega\left(\int_{\mathbb{R}^N}\frac{|\phi_n(x)-\phi_n(y)|^2}{|x-y|^{N+2t}}dy\right)^{\frac{1+\alpha}{2}}dx\right)^{\frac{1}{1+\alpha}}\to 0.$$

If we show that

$$\int_\Omega\left(\int_{\mathbb{R}^N}\frac{|\phi_n(x)-\phi_n(y)|^2}{|x-y|^{N+2t}}dy\right)^{\frac{1+\alpha}{2\alpha}}dx \tag{3.15}$$

is bounded, we can get $M_2 \to 0$. Due to Lemma 2.5 and Lemma 2.1, we get

$$\begin{aligned}
\int_\Omega\left(\int_{\mathbb{R}^N}\frac{|\phi_n(x)-\phi_n(y)|^2}{|x-y|^{N+2t}}dy\right)^{\frac{1+\alpha}{2\alpha}}dx &= \int_\Omega\left(\int_{\mathbb{R}^N}\frac{\left|v_n^{\frac{2-\theta}{2}}(x)-v^{\frac{2-\theta}{2}}(x)-v_n^{\frac{2-\theta}{2}}(y)+v^{\frac{2-\theta}{2}}(y)\right|^2}{|x-y|^{N+2t}}dy\right)^{\frac{1+\alpha}{2\alpha}}dx\\
&\leqslant\int_\Omega\left(\int_{\mathbb{R}^N}\frac{\left|\left|v_n^{\frac{2-\theta}{2}}(x)-v_n^{\frac{2-\theta}{2}}(y)\right|+\left|v^{\frac{2-\theta}{2}}(x)-v^{\frac{2-\theta}{2}}(y)\right|\right|^2}{|x-y|^{N+2t}}dy\right)^{\frac{1+\alpha}{2\alpha}}dx\\
&\leqslant C\int_\Omega\left(\int_{\mathbb{R}^N}\frac{\left|v_n^{\frac{2-\theta}{2}}(x)-v_n^{\frac{2-\theta}{2}}(y)\right|^2+\left|v^{\frac{2-\theta}{2}}(x)-v^{\frac{2-\theta}{2}}(y)\right|^2}{|x-y|^{N+2t}}dy\right)^{\frac{1+\alpha}{2\alpha}}dx\\
&\leqslant C\int_\Omega\left(\int_{\mathbb{R}^N}\frac{|v_n(x)-v_n(y)|^{2-\theta}+|v(x)-v(y)|^{2-\theta}}{|x-y|^{N+2t}}dy\right)^{\frac{1+\alpha}{2\alpha}}dx\\
&\leqslant C\int_\Omega\left(\int_{\mathbb{R}^N}\frac{|v_n(x)-v_n(y)|^{2-\theta}}{|x-y|^{N+2t}}dy\right)^{\frac{1+\alpha}{2\alpha}}dx\\
&+C\int_\Omega\left(\int_{\mathbb{R}^N}\frac{|v(x)-v(y)|^{2-\theta}}{|x-y|^{N+2t}}dy\right)^{\frac{1+\alpha}{2\alpha}}dx\\
&=C(P_1+P_2).
\end{aligned}$$

Using convexity, the Hölder inequality, and (3.4), we have

$$\begin{aligned}
P_1 =& \int_\Omega \left(\int_{\mathbb{R}^N} \frac{|v_n(x) - v_n(y)|^{2-\theta}}{|x-y|^{N+2t}} dy \right)^{\frac{1+\alpha}{2\alpha}} dx \\
\leqslant& C \int_\Omega \left(\int_{\mathbb{R}^N} \frac{|v_n(x) - v_n(y)|^{2}}{|x-y|^{(N+2t)\frac{2}{2-\theta}}} dy \right)^{\frac{(2-\theta)(1+\alpha)}{4\alpha}} dx \\
\leqslant& C \left(\int_\Omega \left(\int_{\mathbb{R}^N} \frac{|v_n(x) - v_n(y)|^{2}}{|x-y|^{N+2\frac{4t+N\theta}{2(2-\theta)}}} dy \right)^{\frac{r(2-\theta)}{4}} dx \right)^{\frac{1+\alpha}{\alpha r}} \\
=& C \left\| \mathbb{D}_{\frac{4t+N\theta}{2(2-\theta)}} (v_n(x)) \right\|_{L^{\frac{r(2-\theta)}{2}}(\Omega)}^{\frac{(2-\theta)(1+\alpha)}{2\alpha}},
\end{aligned}$$

and by $v_n \in E$, we have

$$P_1 \leqslant C \|v_n\|_{\mathcal{L}_0^{s+\varepsilon,\tau}(\Omega)}^{\frac{(2-\theta)(1+\alpha)}{2\alpha}} \leqslant C \ell^{\frac{1+\alpha}{2\alpha}}.$$

Similarly, we can obtain that P_2 is bounded.

Combining claims 1 and 2, we find that

$$\left\| \left| \mathbb{D}_t \left(v_n^{\frac{2-\theta}{2}} \right) \right|^2 - \left| \mathbb{D}_t \left(v^{\frac{2-\theta}{2}} \right) \right|^2 \right\|_{L^1(\Omega)} \to 0,$$

which implies that $g_n \to g$ in $L^1(\Omega)$, leading to the following result. □

Lemma 3.4. *$\mathcal{T}$ is compact if* (1.2) *holds.*

Proof. Let $\{v_n\} \subset E$ be a bounded sequence in $\mathcal{L}_0^{s+\varepsilon,1+\alpha}(\Omega)$ and denote $u_n = \mathcal{T}(v_n)$. We need to prove that $u_n \to u$ in $\mathcal{L}_0^{s+\varepsilon,1+\alpha}(\Omega)$ for some $u \in \mathcal{L}_0^{s+\varepsilon,1+\alpha}(\Omega)$. By Lemma 3.1, we obtain that $\left\{ \left| \mathbb{D}_t \left(v_n^{\frac{2-\theta}{2}} \right) \right|^2 \right\}$ is a bounded sequence in $L^1(\Omega)$. By (3.13), we get that $\{g_n\}$ is a bounded sequence in $L^1(\Omega)$. The result follows from Lemma 2.2. □

Proof of Theorem 3.5. Since E is a closed convex set of $\mathcal{L}_0^{s+\varepsilon,1+\alpha}(\Omega)$, by Lemmas 3.1–3.4, we know that $\mathcal{T}$ is continuous, compact, and satisfies $\mathcal{T}(E) \subset E$, and we can apply the Schauder fixed point Theorem to obtain $u \in E$ such that $\mathcal{T}(u) = u$. Therefore, we deduce that problem (1.1) has a weak solution u for any $0 < \lambda \leqslant \lambda^*$. Thanks to Lemma 2.4 and [21, Proposition 1.4(iii)], we have $u \in \mathcal{H}^s(\mathbb{R}^N) \cap C^{0,s}(\mathbb{R}^N)$. □

4. Conclusions

In this paper, we have studied the fractional elliptic problem (1.1) involving a nonlocal gradient term and a lower-order term. Under suitable assumptions on the parameters $N, s, t, \theta, B, \lambda$ and the source term $f \in L^m(\Omega)$ with $m > N/s$, we established the existence of weak solutions. More precisely, for any $0 < \lambda \leq \lambda^*$, there exists a positive weak solution $u \in \mathcal{L}_0^{s+\epsilon,1+\alpha}(\Omega) \cap H^s(\mathbb{R}^N) \cap C^{0,s}(\mathbb{R}^N)$. The proof relies on a fixed-point argument combined with compactness and continuity properties of the solution operator $\mathcal{T}$. Our results extend classical local problems with singular quadratic gradient terms to the nonlocal fractional setting, and highlight the influence of the lower-order term on the existence of solutions.

Author contributions

Le Zhang: wrote the initial draft of the paper; Xiao Long: assisted in revising the paper and provided the range of N; Shuibo Huang: provided the main research ideas of the paper and further improved the paper quality; Yonglin Xu: revised the paper. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

References

1. D. Arcoya, L. Boccardo, T. Leonori, A. Porretta, Some elliptic problem with singular natural growth lower order terms, *J. Differ. Equations*, **249** (2010), 2771–2795. https://doi.org/10.1016/j.jde.2010.05.009
2. D. Giachetti, F. Murat, An elliptic problem with a lower order term having singular behaviour, *Boll. Unione Mat. Ital.*, **2** (2009), 349–370.
3. D. Giachetti, F. Petitta, S. S. de León, Elliptic equations having a singular quadratic gradient term and a changing sign datum, *Commun. Pure Appl. Anal.*, **11** (2012), 1875–1895. https://doi.org/10.3934/cpaa.2012.11.1875
4. D. Arcoya, L. Moreno-Mérida, The effect of a singular term in a quadratic quasi-linear problem, *J. Fixed Point Theory Appl.*, **19** (2017), 815–831. https://doi.org/10.1007/s11784-016-0374-0
5. J. Carmona, T. Leonori, S. López-Martínez, P. J. Martínez-Aparicio, Quasilinear elliptic problem with singular and homogeneous lower order terms, *Nonlinear Anal.*, **179** (2019), 105–130. https://doi.org/10.1016/j.na.2018.08.002
6. S. López-Martínez, A singularity as a break point for the multiplicity of solutions to quasilinear elliptic problem, *Adv. Nonlinear Anal.*, **9** (2020), 1351–1382. https://doi.org/10.1515/anona-2020-0056

7. D. Kumar, J. Tyagi, Singular elliptic equations with quadratic gradient term, *J. Math. Anal. Appl.*, **502** (2021), 125245. https://doi.org/10.1016/j.jmaa.2021.125245
8. F. Oliva, Anosov flows with stable and unstable differentiable distributions, *Ann. Mat. Pura Appl.*, **200** (2021), 287–314. https://doi.org/10.1007/s10231-020-00996-1
9. W. Zhou, X. Wei, *Multiplicity of solutions for a singular p-laplacian elliptic equation*, Annales Polonici Mathematici, 2010, 157–180.
10. B. Barrios, A. Figalli, X. Ros-Oton, Free boundary regularity in the parabolic fractional obstacle problem, *Commun. Pure Appl. Math.*, **71** (2018), 2129–2159. https://doi.org/10.1002/cpa.21745
11. S. Dipierro, A. Figalli, E. Valdinoci, Strongly nonlocal dislocation dynamics in crystals, *Commun. Partial Differ. Equations*, **39** (2014), 2351–2387. https://doi.org/10.34657/3292
12. Y. Sire, E. Valdinoci, Fractional Laplacian phase transitions and boundary reactions: a geometric inequality and a symmetry result, *J. Funct. Anal.*, **256** (2009), 1842–1864. https://doi.org/10.1016/j.jfa.2009.01.020
13. B. Abdellaoui, A. J. Fernández, Nonlinear fractional Laplacian problem with nonlocal 'gradient terms', *Proc. R. Soc. Edinburgh Sect. A Math.*, **150** (2020), 2682–2718. https://doi.org/10.1017/prm.2019.60
14. B. Abdellaoui, A. J. Fernández, T. Leonori, A. Younes, Deterministic KPZ-type equations with nonlocal "gradient terms", *Ann. Mat. Pura Appl.*, **202** (2023), 1451–1468. https://doi.org/10.1007/s10231-022-01288-6
15. B. Abdellaoui, K. Biroud, A. Primo, F. Soria, A. Younes, Fractional KPZ equations with fractional gradient term and Hardy potential, *Math. Eng.*, **5** (2023), 1–36. https://doi.org/10.3934/mine.2023042
16. A. Younes, K. Biroud, F. Mahmoudi, B. Abdellaoui, Fractional KPZ system with nonlocal "gradient terms", *Discrete Contin. Dyn. Syst.*, **44** (2024), 342–364. https://doi.org/10.3934/dcds.2023106
17. E. M. Stein, The characterization of functions arising as potentials, *Bull. Amer. Math. Soc.*, **67** (1961), 102–104. https://doi.org/10.1090/s0002-9904-1961-10517-x
18. R. A. Adams, *Sobolev spaces*, Academic Press, 1975.
19. A. Youssfi, G. O. M. Mahmoud, Nonlocal semilinear elliptic problem with singular nonlinearity, *Calc. Var. Partial Differ. Equations*, **60** (2021), 153. https://doi.org/10.1007/s00526-021-02034-1
20. T. Leonori, I. Peral, A. Primo, F. Soria, Basic estimates for solutions of a class of nonlocal elliptic and parabolic equations, *Discrete Contin. Dyn. Syst.*, **35** (2015), 6031–6068. https://doi.org/10.3934/dcds.2015.35.6031
21. X. Ros-Oton, J. Serra, The extremal solution for the fractional Laplacian, *Calc. Var. Partial Differ. Equations*, **50** (2014), 723–750. https://doi.org/10.1007/s00526-013-0653-1

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