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Research article

On strong solutions of Navier-Stokes equations with two velocity components

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Abstract: In this paper, we established a continuation criterion for local strong solutions to the three-dimensional incompressible Navier-Stokes equations based on partial velocity components. More precisely, we showed that a unique local strong solution u does not blow up at time T provided that the two horizontal velocity components u_h belong to the Banach spaces $\dot{V}^s_{p,q,\theta}$ and $\dot{U}^s_{p,\beta,\sigma}$. These functional spaces strictly contained the homogeneous Besov space $\dot{B}^s_{p,q}$, and thus allowed a wider admissible class than those considered in earlier works. Our results can therefore be viewed as an extension and refinement of previously known criteria.

Keywords: Navier-Stokes equations; extension criteria; two velocity components
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1. Introduction

We consider the three-dimensional incompressible Navier-Stokes equations

$$\begin{cases} \partial_t u - \Delta u + (u \cdot \nabla)u + \nabla P = 0, \\ \nabla \cdot u = 0, \\ u(x,0) = u_0(x), \end{cases} \tag{1.1}$$

posed on $\mathbb{R}^3 \times (0,T)$. Here $u = u(x,t) \in \mathbb{R}^3$ denotes the velocity field and $P = P(x,t) \in \mathbb{R}$ is the associated scalar pressure. The initial data u_0 is assumed to be divergence-free in the sense of distributions.

The foundational work of Leray [16] guarantees the existence of at least one global weak solution to (1.1) for any divergence-free initial data $u_0 \in L^2(\mathbb{R}^3)$. Such solutions satisfy the classical energy inequality

$$\frac{1}{2}\|u(t)\|_{L^2}^2 + \int_0^t \|\nabla u(\tau)\|_{L^2}^2 \, d\tau \le \frac{1}{2}\|u_0\|_{L^2}^2, \quad \text{for every } t \in [0,\infty). \tag{1.2}$$

On the other hand, the local well-posedness and uniqueness of strong solutions were established by Fujita and Kato [8]. In particular, if $u_0 \in H^s(\mathbb{R}^3)$ with $s > 1/2$, then there exists a maximal time $T^* > 0$ such that the corresponding strong solution satisfies

$$u \in C([0,T^*); H^s(\mathbb{R}^3)) \cap C^1((0,T^*); H^s(\mathbb{R}^3)) \cap C((0,T^*); H^{s+2}(\mathbb{R}^3)). \tag{1.3}$$

A fundamental open problem is whether such a local strong solution can develop singularities at the maximal time T^* or whether it can be extended smoothly for all time. Although this question remains unresolved, extensive research has been devoted to identifying sufficient conditions that rule out finite-time blow-up.

Among the classical results, the Prodi-Serrin criterion [20, 21] provides a sufficient condition based on space-time integrability of the velocity field, namely

$$u \in L^q\left(0,T; L^p\left(\mathbb{R}^3\right)\right) \quad \text{with} \quad \frac{2}{q} + \frac{3}{p} = 1 \quad \text{and} \quad 3 < p \le \infty. \tag{1.4}$$

The endpoint case $(p,q) = (\infty,3)$ was later resolved by Escauriaza, Seregin, and Šverák [6]. In a related direction, Beirão da Veiga [2] derived continuation criteria formulated in terms of the velocity gradient,

$$\nabla u \in L^q\left(0,T; L^p\left(\mathbb{R}^3\right)\right) \quad \text{with} \quad \frac{2}{q} + \frac{3}{p} = 2 \quad \text{and} \quad \frac{3}{2} < p < \infty. \tag{1.5}$$

Subsequent developments have focused on weakening these assumptions by exploiting partial information of the velocity field, its gradient, or the vorticity; see, for example, [3, 4, 10, 11, 17, 19] and the references therein. Using Littlewood-Paley decomposition techniques, Dong and Zhang [5] obtained a regularity criterion involving only the horizontal derivatives of the velocity,

$$\int_0^T \|\nabla_h \tilde{u}(\tau)\|_{\dot{B}^0_{\infty,\infty}}^2 \, \mathrm{d}\tau < \infty, \tag{1.6}$$

where $\nabla_h \tilde{u} = (\partial_1 \tilde{u}, \partial_2 \tilde{u}, 0)$. This result was later refined by Gala and Ragusa [9], who showed that a condition imposed directly on the two horizontal velocity components,

$$\int_0^T \|\tilde{u}(\tau)\|_{\dot{B}^0_{\infty,\infty}}^2 \, \mathrm{d}\tau < \infty, \tag{1.7}$$

with $\tilde{u} = (u_1, u_2, 0)$, suffices to ensure regularity. More recently, Kanamaru [12] introduced Vishik-type spaces and established logarithmic interpolation inequalities that lead to Beale-Kato-Majda type criteria. In particular, if a weak solution satisfies

$$\int_0^T \|\operatorname{rot} u(\tau)\|_{\dot{V}^0_{p,\infty,\theta}}^\theta \, \mathrm{d}\tau < \infty, \quad \frac{2}{\theta} + \frac{3}{p} = 2, \quad p \in (3,\infty], \tag{1.8}$$

then the solution remains smooth on $(0, T]$. Further extensions of this approach were obtained using Littlewood-Paley methods in [23], and additional results in Vishik spaces can be found in [7,14,15,24]. Very recently, Farwig and Kanamaru [7] proved continuation criteria formulated in terms of the spaces $\dot{U}^s_{p,\beta,\sigma}$ and $\dot{V}^s_{p,q,\theta}$, which may be strictly larger than the homogeneous Besov space $\dot{B}^s_{p,q}$.

Motivated by these developments, the aim of the present paper is to derive extension criteria for (1.1) within a broader functional framework, relying only on reduced velocity information. In particular, we focus on conditions expressed in terms of the two horizontal velocity components in larger Banach spaces. We will prove the followin

Theorem 1.1. *Let $u_0 \in H^2\left(\mathbb{R}^3\right)$ be a divergence-free initial data, and let $u \in C\left(0, T; H^2\left(\mathbb{R}^3\right)\right)$ denote the corresponding strong solution to the three-dimensional Navier-Stokes Eq* (1.1). *Assume that the two horizontal velocity components $\tilde{u} = (u_1, u_2, 0)$ satisfy*

$$\int_0^T \|\tilde{u}(\tau)\|^2_{\dot{V}^0_{\infty,\infty,2}} \, d\tau < \infty. \tag{1.9}$$

Then the solution u admits a smooth continuation beyond the time $t = T$.

Theorem 1.2. *Let $u_0 \in H^1\left(\mathbb{R}^3\right)$ be divergence-free, and let $u \in C\left(0, T; H^1\left(\mathbb{R}^3\right)\right)$ denote the associated strong solution to the three-dimensional Navier-Stokes Eq* (1.1). *Suppose that the horizontal gradient of the reduced velocity field, $\nabla_h \tilde{u} = (\partial_1 \tilde{u}, \partial_2 \tilde{u}, 0)$, satisfies*

$$\int_0^T \|\nabla_h \tilde{u}(\tau)\|_{\dot{U}^{\frac{3}{p}}_{p,\frac{1}{\theta},\infty}} \, d\tau < \infty, \quad 1 \le p \le \infty, \ 1 \le \theta < \infty. \tag{1.10}$$

Then, the solution u admits a continuation beyond the time $t = T$.

We emphasize that this enlargement is not merely formal. Indeed, the inclusion

$$\dot{B}^s_{p,q}(\mathbb{R}^3) \subset \dot{V}^s_{p,q,\theta}(\mathbb{R}^3)$$

is strict when $\theta < q$. In particular, one can construct functions f such that $f \in \dot{V}^s_{p,q,\theta}$ but $f \notin \dot{B}^s_{p,q}$, showing that the $\dot{V}$-framework admits velocity fields with weaker summability properties than those allowed in the classical Besov setting(see Remark 1.2). This observation highlights that our continuation criteria apply to a strictly larger class of solutions that are not covered by earlier results.

From a methodological viewpoint, our approach combines Littlewood-Paley decomposition, logarithmic interpolation inequalities in Vishik-type spaces, and energy estimates for the Navier-Stokes equations. A key feature of our analysis is that the logarithmic interpolation mechanism allows us to transfer partial information on the horizontal velocity components into global control of the solution, thereby ruling out blow-up under weaker assumptions.

Remark 1.1. *Notice the following continuous embeddings (see Lemma 2.2):*

$$\dot{B}^s_{p,\infty}\left(\mathbb{R}^3\right) \hookrightarrow \dot{V}^s_{p,\infty,\theta}\left(\mathbb{R}^3\right) \hookrightarrow \dot{U}^s_{p,\frac{1}{\theta},\infty}\left(\mathbb{R}^3\right), \quad s \in \mathbb{R}, \ 1 \le \theta < \infty;$$

$$\dot{B}^s_{p,q}\left(\mathbb{R}^3\right) = \dot{V}^s_{p,q,q}\left(\mathbb{R}^3\right) \hookrightarrow \dot{V}^s_{p,q,\theta_2}\left(\mathbb{R}^3\right) \hookrightarrow \dot{V}^s_{p,q,\theta_1}\left(\mathbb{R}^3\right), \quad s \in \mathbb{R}, 1 \le p, q \le \infty, 1 \le \theta_1 \le \theta_2 \le q.$$

Naturally, we see that Theorem 1.1 and 1.2 are the optimal extension criteria for the Navier-Stokes equation (NSE) (1.1). *Our results represent an improvement over the corresponding findings presented in [5, 7, 9]*

Remark 1.2. *To see that $\dot{B}^s_{p,q} \subsetneq \dot{V}^s_{p,q,\theta}$ for any $\theta < q$, it suffices to construct a sequence $\{a_j\}$ that diverges in l^q but remains controlled under the l^θ-average weight. Consider f such that $\|\dot{\Delta}_j f\|_{L^p} = 2^{-js} a_j$, where the coefficients are supported on a lacunary set:*

$$a_j = \begin{cases} k^{-1/q}, & j = 2^k, k \in \mathbb{N}, \\ 0, & \textit{otherwise}. \end{cases}$$

A direct calculation yields $\sum_{j=1}^\infty a_j^q = \sum_{k=1}^\infty k^{-1} = \infty$, hence, $f \notin \dot{B}^s_{p,q}$. However, for any $N \geq 1$, the number of active terms in the sum $\sum_{j=1}^N a_j^\theta$ is at most $\lfloor \log_2 N \rfloor$. Thus,

$$\sum_{j=1}^{N} a_j^\theta = \sum_{k=1}^{\lfloor \log_2 N \rfloor} k^{-\theta/q} \lesssim (\log N)^{1-\theta/q}.$$

It follows that

$$\sup_{N\geq 1} \frac{(\sum_{j=1}^N a_j^\theta)^{1/\theta}}{N^{1/q}} \lesssim \sup_{N \geq 1} \frac{(\log N)^{\frac{1}{\theta} - \frac{1}{q}}}{N^{1/q}} < \infty,$$

where we used the fact that any power of $\log N$ is dominated by $N^{1/q}$ as $N \to \infty$. This confirms $f \in \dot{V}^s_{p,q,\theta}$, completing the proof.

2. Preliminaries

In this section, we will recall some definitions and present several lemmas that will be used in the proof of Theorems 1.1 and 1.2.

We begin by fixing some notation and recalling basic facts from Littlewood-Paley theory. Let $\mathcal{S}(\mathbb{R}^3)$ denote the Schwartz space of rapidly decreasing functions. For any $f \in \mathcal{S}(\mathbb{R}^3)$, its Fourier transform $\mathcal{F}f = \hat{f}$ is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^3} f(x) e^{-ix\cdot\xi} \, \mathrm{d}x.$$

Choose a pair of smooth functions (χ, φ) taking values in $[0, 1]$ such that χ is supported in $B = \{\xi \in \mathbb{R}^3 : |\xi| \leq \frac{4}{3}\}$ and φ is supported in $C = \{\xi \in \mathbb{R}^3 : \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$. These functions are required to satisfy

$$\chi(\xi) + \sum_{j\geq 0} \varphi(2^{-j}\xi) = 1, \quad \forall\, \xi \in \mathbb{R}^3,$$

and

$$\sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1, \quad \forall\, \xi \in \mathbb{R}^3 \setminus \{0\}.$$

Setting $\varphi_j = \varphi(2^{-j}\xi)$, and denoting by $h = \mathcal{F}^{-1}\varphi$ and $\widetilde{h} = \mathcal{F}^{-1}\chi$, we define the homogeneous dyadic blocks by

$$\dot{\Delta}_j f = \varphi(2^{-j}D) f = 2^{3j} \int_{\mathbb{R}^3} h(2^j y) f(x-y) \, \mathrm{d}y, \qquad j \in \mathbb{Z},$$

and the corresponding low-frequency cut-off operators by

$$\dot{S}_j f = \sum_{k \leq j-1} \dot{\Delta}_k f = \chi(2^{-j}D) f = 2^{3j} \int_{\mathbb{R}^3} \widetilde{h}(2^j y) f(x-y) \, \mathrm{d}y, \qquad j \in \mathbb{Z}.$$

This dyadic decomposition enjoys the standard quasi-orthogonality property, namely,

$$\dot{\Delta}_j\dot{\Delta}_q f \equiv 0 \qquad \text{whenever } |j-q| \geq 2.$$

As a consequence, any tempered distribution $f \in \mathcal{S}'(\mathbb{R}^3) \setminus \mathcal{P}(\mathbb{R}^3)$ admits the formal decomposition

$$f = \sum_{j=-\infty}^{\infty} \dot{\Delta}_j f,$$

where $\mathcal{P}(\mathbb{R}^3)$ denotes the space of polynomials. We refer to [1] for a detailed exposition of the Littlewood-Paley theory.

Definition 2.1. *Let $s \in \mathbb{R}$ and $(p,\sigma) \in [1,\infty]^2$. The homogeneous Besov space $\dot{B}^s_{p,\sigma}$ consists of all distributions $f \in \mathcal{Z}'(\mathbb{R}^3)$ such that*

$$\|f\|_{\dot{B}^s_{p,\sigma}} < \infty,$$

where

$$\|f\|_{\dot{B}^s_{p,\sigma}} = \begin{cases} \Big(\sum_{j\in\mathbb{Z}} 2^{js\sigma}\|\dot{\Delta}_j f\|^{\sigma}_{L^p}\Big)^{1/\sigma}, & \sigma < \infty, \\ \sup_{j\in\mathbb{Z}} 2^{js}\|\dot{\Delta}_j f\|_{L^p}, & \sigma = \infty. \end{cases}$$

Where $\mathcal{Z}'(\mathbb{R}^3)$ is the dual space of

$$\mathcal{Z}(\mathbb{R}^3) = \{f \in \mathcal{S}(\mathbb{R}^3) : D^\alpha \widehat{f}(0) = 0 \, \textit{for all } \alpha \in \mathbb{N}^3\}.$$

The following Bernstein inequalities [1] will be used repeatedly throughout this work.

Lemma 2.1. *Let $k \in \mathbb{N}$. Then, for any $1 \leq p \leq q \leq \infty$, the following estimate holds:*

$$\sup_{|\alpha|=k} \|\partial^\alpha \dot{\Delta}_j f\|_{L^q} \leq C 2^{jk+3j\left(\frac{1}{p}-\frac{1}{q}\right)}\|\dot{\Delta}_j f\|_{L^p},$$

where C denotes an absolute constant independent of both f and j.

Remark 2.1. *A direct consequence of the above Bernstein inequality is*

$$\|\dot{\Delta}_j f\|_{L^q} \leq C 2^{3j\left(\frac{1}{p}-\frac{1}{q}\right)}\|\dot{\Delta}_j f\|_{L^p}.$$

Next, we define the Banach spaces $\dot{V}^s_{p,q,\theta}$ and $\dot{U}^s_{p,\beta,\sigma}$, which are strictly larger than the homogeneous Besov spaces $\dot{B}^s_{p,q}$. These spaces can be interpreted as modified versions of those introduced by Nakao-Taniuchi [18] and Vishik [22].

Definition 2.2. *Let $s \in \mathbb{R}, 1 \leq p,q,\theta \leq \infty$. Then, $\dot{V}^s_{p,q,\theta}(\mathbb{R}^n) := \left\{f \in Z'; \|f\|_{\dot{V}^s_{p,q,\theta}} < \infty\right\}$ is introduced by the norm*

$$\|f\|_{\dot{V}^s_{p,q,\theta}} := \begin{cases} \sup_{N=1,2,\cdots} \dfrac{\left(\sum_{|j|\leq N} 2^{js\theta}\|\dot{\Delta}_j f\|^{\theta}_{L^p}\right)^{\frac{1}{\theta}}}{N^{\frac{1}{\theta}-\frac{1}{q}}}, & \theta \neq \infty, \\ \sup_{N=1,2,\cdots} N^{\frac{1}{q}} \max_{|j|\leq N} 2^{js}\left\|\dot{\Delta}_j f\right\|_{L^p}, & \theta = \infty. \end{cases}$$

Definition 2.3. *Let $s,\beta \in \mathbb{R}, 1 \leq p,\sigma \leq \infty$. Then, $\dot{U}^s_{p,\beta,\sigma}(\mathbb{R}^n) := \left\{f \in Z'; \|f\|_{\dot{U}^s_{p,\beta,\sigma}} < \infty\right\}$ is equipped with the norm*

$$\|f\|_{\dot{U}^s_{p,\beta,\sigma}} := \begin{cases} \sup_{N=1,2,\cdots} \frac{\left(\sum_{|j|\leq N} 2^{js\sigma}\|\dot{\Delta}_j f\|^{\sigma}_{L^p}\right)^{\frac{1}{\sigma}}}{N^\beta}, & \sigma \neq \infty, \\ \sup_{N=1,2,\cdots} \frac{\max_{|j|\leq N} 2^{js}\|\dot{\Delta}_j f\|_{L^p}}{N^\beta}, & \sigma = \infty. \end{cases}$$

The following lemma indicates that $\dot{V}^s_{p,q,\theta}$ extends $\dot{B}^s_{p,q}$, while $\dot{U}^s_{p,\beta,\sigma}$ is an extension of $\dot{V}^s_{p,q,\theta}$.

Lemma 2.2. *[7] (i) Let $s \in \mathbb{R}, 1 \leq p,q \leq \infty$, and $1 \leq \theta_1 \leq \theta_2 \leq q < \theta_3$. Then, it holds that*

$$\{0\} = \dot{V}^s_{p,q,\theta_3} \subset \dot{B}^s_{p,q} = \dot{V}^s_{p,q,q} \subset \dot{V}^s_{p,q,\theta_2} \subset \dot{V}^s_{p,q,\theta_1}.$$

(ii) Let $s \in \mathbb{R}, 1 \leq p,\sigma \leq \infty$, and $\beta_1 < 0 \leq \beta_2 \leq \beta_3$. Then, it holds that

$$\{0\} = \dot{U}^s_{p,\beta_1,\sigma} \subset \dot{B}^s_{p,\sigma} = \dot{U}^s_{p,0,\sigma} \subset \dot{U}^s_{p,\beta_2,\sigma} \subset \dot{U}^s_{p,\beta_3,\sigma}.$$

(iii) Let $s,\beta \in \mathbb{R}, 1 \leq p,q,\theta \leq \infty, \tilde{\beta} = \frac{1}{\theta} - \frac{1}{q}$, and $1 \leq \sigma_1 \leq \sigma_2 \leq \infty$. Then, it holds that

$$\dot{V}^s_{p,q,\theta} = \dot{U}^s_{p,\tilde{\beta},\theta} \quad \text{and} \quad \dot{U}^s_{p,\beta,\sigma_1} \subset \dot{U}^s_{p,\beta,\sigma_2}.$$

Lemma 2.3. *(Logarithmic interpolation inequality [7]) Let $s_0, s_1, s_2 \in \mathbb{R}$ satisfy $s_1 < s_0 < s_2$, and let $0 \leq \beta < \infty$ and $1 \leq p,\sigma \leq \infty$. Then, there exists a positive constant C, which depends solely on s_0, s_1, s_2 but is independent of p,β,σ, such that*

$$\|f\|_{\dot{B}^{s_0}_{p,\sigma}} \leq C\Big(1 + \|f\|_{\dot{U}^{s_0}_{p,\beta,\sigma}} \log^\beta\left(e + \|f\|_{\dot{B}^{s_1}_{p,\infty}\cap\dot{B}^{s_2}_{p,\infty}}\right)\Big)$$

for every $f \in \dot{B}^{s_1}_{p,\infty} \cap \dot{B}^{s_2}_{p,\infty}$. More specifically, by taking $\beta = \frac{1}{\theta} - \frac{1}{q}$ and $\sigma = \theta$ (with $1 \leq q \leq \infty$ and $1 \leq \theta \leq q$), it follows that

$$\|f\|_{\dot{B}^{s_0}_{p,\theta}} \leq C\Big(1 + \|f\|_{\dot{V}^{s_0}_{p,q,\theta}} \log^{\frac{1}{\theta}-\frac{1}{q}}\left(e + \|f\|_{\dot{B}^{s_1}_{p,\infty}\cap\dot{B}^{s_2}_{p,\infty}}\right)\Big).$$

Lemma 2.4. *[5,9] Suppose $u = (u_1, u_2, u_3)$ is a smooth, divergence-free vector field ($\nabla \cdot u = 0$). Define $\tilde{u} = (u_1, u_2, 0)$ and $\nabla_h\tilde{u} = (\partial_1\tilde{u}, \partial_2\tilde{u}, 0)$. Then, for a generic constant C, the following estimates hold:*

$$\left|\int_{\mathbb{R}^3} (u \cdot \nabla u) \cdot \Delta u \,\mathrm{d}x\right| \leq C \int_{\mathbb{R}^3} |\tilde{u}|\,|\nabla u|\,|\Delta u| \,\mathrm{d}x, \tag{2.1}$$

$$\left|\int_{\mathbb{R}^3} (u \cdot \nabla u) \cdot \Delta u \,\mathrm{d}x\right| \leq C \int_{\mathbb{R}^3} |\nabla_h\tilde{u}|\,|\nabla u|^2 \,\mathrm{d}x. \tag{2.2}$$

3. Proof of Theorem 1.1

In this section, we focus on proving Theorem 1.1. The argument relies on deriving a priori estimates under the condition (1.9).

By multiplying the first equation of (1.1) by u, integrating by parts, and exploiting the divergence-free property, it follows that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2_{L^2} + \|\nabla u(t)\|^2_{L^2} = 0.$$

This identity allows us to get

$$\|u(t)\|_{L^2}^2 + \int_0^t \|\nabla u(s)\|_{L^2}^2 \, \mathrm{d}s \le \|u_0\|_{L^2}^2.$$

Multiplying the first equation in (1.1) by Δu and integrating throughout $\mathbb{R}^3$, one can use Lemma 2.4 to get

$$\frac{1}{2}\frac{\mathrm{d}}{dt}\|\nabla u(t)\|_{L^2}^2 + \|\Delta u(t)\|_{L^2}^2 = -\int_{\mathbb{R}^3} (u\cdot\nabla u)\cdot\Delta u \, \mathrm{d}x \le C\int_{\mathbb{R}^3} |\tilde{u}||\nabla u||\Delta u| \, \mathrm{d}x. \tag{3.1}$$

We recall the following property of Hardy spaces and Bounded Mean Oscillation (BMO) spaces:

$$\int_{\mathbb{R}^3} fgh \, \mathrm{d}x \le \|fg\|_{\mathcal{H}^1}\|h\|_{\mathrm{BMO}} \le \|f\|_{L^2}\|g\|_{L^2}\|h\|_{\mathrm{BMO}}, \tag{3.2}$$

for any vector fields satisfying $\nabla\cdot f = 0$ and $\nabla\times g = 0$. Applying the inequality (3.2) above together with Young's inequality gives

$$\begin{aligned}\int_{\mathbb{R}^3} |\tilde{u}||\nabla u||\Delta u| \, \mathrm{d}x \le& \|\nabla u\|_{L^2}\|\Delta u\|_{L^2}\|\tilde{u}\|_{\mathrm{BMO}} \\ \le& \epsilon\|\Delta u\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2\|\tilde{u}\|_{\mathrm{BMO}}^2 \\ \le& \epsilon\|\Delta u\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2\|\tilde{u}\|_{\dot{B}^0_{\infty,2}}^2.\end{aligned} \tag{3.3}$$

where we used the continuous embedding $\dot{B}^0_{\infty,2} \subset BMO$. Inserting (3.3) to the righthand side of (3.1),

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u(t)\|_{L^2}^2 + \|\Delta u(t)\|_{L^2}^2 \le C\|\tilde{u}\|_{\dot{B}^0_{\infty,2}}^2\|\nabla u(t)\|_{L^2}^2. \tag{3.4}$$

Next, by applying Lemmas 2.2 and 2.3, it follows that

$$\|\tilde{u}(\tau)\|_{\dot{B}^0_{\infty,2}} \le C\left(1 + \|\tilde{u}(\tau)\|_{\dot{V}^0_{\infty,\infty,2}} \log^{\frac{1}{2}}\left(e + \|\tilde{u}(\tau)\|_{\dot{B}^{-\frac{3}{2}}_{\infty,\infty}\cap\dot{B}^{\frac{1}{2}}_{\infty,\infty}}\right)\right).$$

By the following embeddings:

$$\dot{B}^0_{2,\infty} \subset \dot{B}^{-n/2}_{\infty,\infty}, \dot{B}^s_{2,\infty} \subset \dot{B}^{-n/2}_{\infty,\infty} \cap \dot{B}^{s-n/2}_{\infty,\infty}$$

and

$$H^s \subset B^s_{2,\infty} = L^2 \cap \dot{B}^s_{2,\infty} \subset \dot{B}^0_{2,\infty} \cap \dot{B}^s_{2,\infty},$$

hence, we have

$$\|\tilde{u}(\tau)\|_{\dot{B}^{-\frac{3}{2}}_{\infty,\infty}\cap\dot{B}^{\frac{1}{2}}_{\infty,\infty}} \le C\|\tilde{u}(\tau)\|_{\dot{B}^0_{2,\infty}\cap\dot{B}^2_{2,\infty}} \le C\|\tilde{u}(\tau)\|_{\dot{B}^2_{2,\infty}} \le C\|\tilde{u}(\tau)\|_{H^2}.$$

Moreover, for any small constant $\varepsilon > 0$, one can find a $T_0 = T_0(\varepsilon) \in (0, T)$ such that

$$\int_{T_0}^T \|\tilde{u}(\tau)\|_{\dot{V}^0_{\infty,\infty,2}}^2 \, \mathrm{d}\tau \le \varepsilon << 1.$$

Inserting the previously derived estimates into (3.4) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 \le C\Big(1 + \|\tilde{u}(\tau)\|_{\dot{V}^0_{\infty,\infty,2}}^2 \log(e + \|u(\tau)\|_{H^2})\Big)\|\nabla u\|_{L^2}^2. \tag{3.5}$$

For any $t \in (T_0, T)$, we define

$$Y(t) = \max_{\tau\in[T_0,t]} \|u(\tau)\|_{H^2}^2,$$

and it follows from (3.5) that

$$\begin{aligned}
&\|\nabla u(t)\|_{L^2}^2 + \int_{T_0}^t \|\triangle u(\tau)\|_{L^2}^2 \mathrm{d}\tau \\
\le&\|\nabla u(T_0)\|_{L^2}^2 \exp\Big\{C\int_{T_0}^t \Big[1 + \|\tilde{u}(\tau)\|_{\dot{V}^0_{\infty,\infty,2}}^2 \log(e + \|u(\tau)\|_{H^2})\Big]\mathrm{d}\tau\Big\} \\
\le&C(T_0)\exp\Big\{C\int_{T_0}^t \Big[\|\tilde{u}(\tau)\|_{\dot{V}^0_{\infty,\infty,2}}^2 \log(e + \|u(\tau)\|_{H^2})\Big]\mathrm{d}\tau\Big\} \\
\le&C(T_0)(e + Y(t))^{C\int_{T_0}^t \|\tilde{u}(\tau)\|_{\dot{V}^0_{\infty,\infty,2}}^2 \mathrm{d}\tau} \\
\le&C(T_0)(e + Y(t))^{C\varepsilon}.
\end{aligned} \tag{3.6}$$

Taking the Λ^2 on NSE (1.1), then multiplying by $\Lambda^2 u$, and integrating by parts, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda^2 u\|_{L^2}^2 + \|\Lambda^3 u\|_{L^2}^2 = -\int_{\mathbb{R}^3} \Lambda^2[(u\cdot\nabla)u]\cdot\Lambda^2 u\,\mathrm{d}x \tag{3.7}$$

Using the Kato-Ponce commutator bounds [13], along with the Gagliardo-Nirenberg and Young inequalities, one finds

$$\begin{aligned}
-\int_{\mathbb{R}^3} \Lambda^2[(u\cdot\nabla)u]\cdot\Lambda^2 u\,\mathrm{d}x &= -\int_{\mathbb{R}^3} [\Lambda^2(u\cdot\nabla u) - (u\cdot\Lambda^2\nabla u)]\cdot\Lambda^2 u\,\mathrm{d}x \\
&\le \|\Lambda^2(u\cdot\nabla u) - (u\cdot\Lambda^2\nabla u)\|_{L^{\frac{4}{3}}}\|\Lambda^2 u\|_{L^4} \\
&\le C\|\nabla u\|_{L^2}\|\Lambda^2 u\|_{L^4}^2 \\
&\le C\|\nabla u\|_{L^2}\|u\|_{L^2}^{\frac{1}{6}}\|\Lambda^3 u\|_{L^2}^{\frac{11}{6}} \\
&\le \frac{1}{4}\|\Lambda^3 u\|_{L^2}^2 + C\|\nabla u\|_{L^2}^{12}.
\end{aligned} \tag{3.8}$$

Hence, by combining (3.7) and (3.8), we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta u\|_{L^2}^2 + \|\Lambda^3 u\|_{L^2}^2 \le C\|\nabla u\|_{L^2}^{12}.$$

Integrating the preceding inequality on (T_0, t) yields

$$\begin{aligned}
\|u(t)\|_{\dot{H}^2}^2 + \int_{T_0}^t \|\Lambda^3 u\|_{L^2}^2 \mathrm{d}\tau &\le \|u(T_0)\|_{\dot{H}^2}^2 + C\int_{T_0}^t \|\nabla u\|_{L^2}^{12}\,\mathrm{d}\tau \\
&\le \|u(T_0)\|_{\dot{H}^2}^2 + C(T_0)\int_{T_0}^t (e + Y(\tau))^{6C\varepsilon}\,\mathrm{d}\tau.
\end{aligned} \tag{3.9}$$

Let $0 < \varepsilon \leq \frac{1}{6C}$. From (3.9), we obtain

$$Y(t) \leq Y(T_0) + C(T_0) \int_{T_0}^{t} (e + Y(\tau))\, \mathrm{d}\tau,$$

Gronwall's inequality ensures the boundedness of the H^2-norm of u, thus concluding the proof of Theorem 1.1.

4. Proof of Theorem 1.2

We now turn to the proof of Theorem 1.2. The strategy relies on employing Littlewood-Paley decomposition within a newly defined, larger Vishik-type space.

By taking the inner product of the first equation in (1.1) with Δu and integrating over $\mathbb{R}^3$, one obtains, via Lemma 2.4,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u(t)\|_{L^2}^2 + \|\Delta u(t)\|_{L^2}^2 = -\int_{\mathbb{R}^3} (u \cdot \nabla u) \cdot \Delta u\, dx \leq C \int_{\mathbb{R}^3} |\nabla_h \tilde{u}||\nabla u|^2 dx. \tag{4.1}$$

Employing the Littlewood-Paley decomposition for the term $\nabla_h \tilde{u}$ yields

$$\begin{aligned}
&\int_{\mathbb{R}^3} |\nabla_h \tilde{u}||\nabla u|^2 \mathrm{d}x \\
\leq &\int_{\mathbb{R}^3} |\sum_{j<-N} \dot{\Delta}_j \nabla_h \tilde{u}|\, |\nabla u|^2\, \mathrm{d}x + \int_{\mathbb{R}^3} |\sum_{j=-N}^{j=N} \dot{\Delta}_j \nabla_h \tilde{u}||\nabla u|^2 \mathrm{d}x + \int_{\mathbb{R}^3} |\sum_{j>-N} \dot{\Delta}_j \nabla_h \tilde{u}||\nabla u|^2 \mathrm{d}x \\
:=& I_{11} + I_{12} + I_{13}.
\end{aligned} \tag{4.2}$$

Regarding I_{11}, one can use the Hölder and Bernstein inequalities to obtain

$$\begin{aligned}
|I_{11}| &\leq \sum_{j<-N} \left\|\dot{\Delta}_j \nabla_h \tilde{u}\right\|_{L^\infty} \|\nabla u\|_{L^2}^2 \\
&\leq C \sum_{j<-N} 2^{\frac{3}{2}j} \left\|\dot{\Delta}_j \nabla_h \tilde{u}\right\|_{L^2} \|\nabla u\|_{L^2}^2 \\
&\leq C \left(\sum_{j<-N} 2^{3j}\right)^{\frac{1}{2}} \left(\sum_{-\infty}^{-N} \left\|\dot{\Delta}_j \nabla_h \tilde{u}\right\|_{L^2}^2\right)^{\frac{1}{2}} \|\nabla u\|_{L^2}^2 \\
&\leq C 2^{-\frac{3}{2}N} \|\nabla_h \tilde{u}\|_{\dot{B}^0_{2,2}} \|\nabla u\|_{L^2}^2 \\
&\leq C 2^{-\frac{3}{2}N} \|\nabla u\|_{L^2}^3.
\end{aligned} \tag{4.3}$$

Due to Definition 2.3 and Bernstein's inequality, the term I_{12} can be estimated as

$$
\begin{aligned}
|I_{12}| &\leq \sum_{j=-N}^{N} \left\|\dot{\Delta}_j \nabla_h \tilde{u}\right\|_{L^\infty} \|\nabla u\|_{L^2}^2 \\
&\leq C \sum_{j=-N}^{N} 2^{\frac{3j}{p}} \left\|\dot{\Delta}_j \nabla_h \tilde{u}\right\|_{L^p} \|\nabla u\|_{L^2}^2 \\
&\leq C(2N+1) \max_{|j|\leq N} \left\{ 2^{\frac{3j}{p}} \|\dot{\Delta}_j \nabla_h \tilde{u}\|_{L^p} \right\} \|\nabla u\|_{L^2}^2 \\
&\leq CN \cdot N^{\frac{1}{\theta}} \frac{\max_{|j|\leq N} \left\{ 2^{\frac{3j}{p}} \|\dot{\Delta}_j \nabla_h \tilde{u}\|_{L^p} \right\}}{N^{\frac{1}{\theta}}} \|\nabla u\|_{L^2}^2 \\
&\leq CN^{1+\frac{1}{\theta}} \|\nabla_h \tilde{u}\|_{\dot{U}^{\frac{3}{p}}_{p,\frac{1}{\theta},\infty}} \|\nabla u\|_{L^2}^2,
\end{aligned} \tag{4.4}
$$

where $1 \leq \theta < \infty$. For I_{13}, using both Hölder's inequality and Bernstein's inequality, the following conclusion follows,

$$
\begin{aligned}
|I_{13}| &\leq C \sum_{j>N} \|\dot{\Delta}_j \nabla_h \tilde{u}\|_{L^3} \|\nabla u\|_{L^2} \|\nabla u\|_{L^6} \\
&\leq C \sum_{j>N} 2^{\frac{j}{2}} \|\dot{\Delta}_j \nabla_h \tilde{u}\|_{L^2} \|\nabla u\|_{L^2} \|\Delta u\|_{L^2} \\
&\leq C (\sum_{j>N} 2^{-j})^{\frac{1}{2}} (\sum_{j>N} 2^{2j} \|\dot{\Delta}_j \nabla_h \tilde{u}\|_{L^2}^2)^{\frac{1}{2}} \|\nabla u\|_{L^2} \|\Delta u\|_{L^2} \\
&\leq C 2^{-\frac{N}{2}} \|\nabla u\|_{L^2} \|\Delta u\|_{L^2}^2.
\end{aligned} \tag{4.5}
$$

Substitute (4.3)–(4.5) into (4.1) to obtain

$$
\begin{aligned}
&\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\nabla u(t)\|_{L^2}^2 + \|\Delta u(t)\|_{L^2}^2 \\
\leq & C 2^{-\frac{3}{2}N} \|\nabla u\|_{L^2}^3 + CN^{1+\frac{1}{\theta}} \|\nabla_h \tilde{u}\|_{\dot{U}^{\frac{3}{p}}_{p,\frac{1}{\theta},\infty}} \|\nabla u\|_{L^2}^2 + C 2^{-\frac{N}{2}} \|\nabla u\|_{L^2} \|\Delta u\|_{L^2}^2.
\end{aligned} \tag{4.6}
$$

Next, let us choose N large enough such that

$$
C 2^{-\frac{N}{2}} \|\nabla u\|_{L^2} \leq \frac{1}{2}
$$

i.e., choosing

$$
N \geq 2 \log_2 (2C \|\nabla u\|_{L^2})
$$

it follows that

$$
\frac{\mathrm{d}}{\mathrm{d}t} \|\nabla u(t)\|_{L^2}^2 + \|\Delta u(t)\|_{L^2}^2 \leq C \|\nabla u\|_{L^2}^2 + C \left[\ln(e + \|\nabla u\|_{L^2}^2) \right]^{1+\frac{1}{\theta}} \|\nabla_h \tilde{u}\|_{\dot{U}^{\frac{3}{p}}_{p,\frac{1}{\theta},\infty}} \|\nabla u\|_{L^2}^2. \tag{4.7}
$$

Let $X(t) = e + \|\nabla u(t)\|_{L^2}^2$. The inequality simplifies to

$$\frac{\mathrm{d}X(t)}{\mathrm{d}t} \leq C\left(1 + \|\nabla_h \tilde{u}(t)\|_{\dot{U}^{\frac{3}{p}}_{p,\frac{1}{\theta},\infty}}\right) X(t)[\ln X(t)]^{1+\frac{1}{\theta}}. \tag{4.8}$$

Applying the logarithmic Gronwall inequality, and given the assumption (1.10) that

$$\int_0^T \|\nabla_h \tilde{u}(\tau)\|_{\dot{U}^{\frac{3}{p}}_{p,\frac{1}{\theta},\infty}} \mathrm{d}\tau < \infty,$$

we conclude that

$$\sup_{0<t\leq T} \|\nabla u(t)\|_{L^2}^2 + \int_0^T \|\Delta u(t)\|_{L^2}^2 \mathrm{d}t \leq C < \infty. \tag{4.9}$$

This ensures that the strong solution u does not blow up at $t = T$ and can be continued beyond T. This completes the proof of Theorem 1.2.

5. Conclusions

The authors establish a continuation criterion for local strong solutions to the three-dimensional incompressible Navier-Stokes equations based on partial velocity components. They show that a unique local strong solution u does not blow up at time T provided that the two horizontal velocity components u_h belong to the Banach spaces $\dot{V}^s_{p,q,\theta}$ and $\dot{U}^s_{p,\beta,\sigma}$.

Author contributions

All authors contributions are equally to work all the parts of the paper.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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