Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(4): 9319–9333.
DOI: 10.3934/math.2026385
Received: 15 December 2025
Revised: 27 February 2026
Accepted: 10 March 2026
Published: 07 April 2026

Research article

Rigidity of extrinsic spheres in Einstein spacetimes and the Goddard conjecture

Mohammed Guediri*

Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia

* **Correspondence:** Email: mguediri@ksu.edu.sa.

Abstract: Goddard's conjecture asserts that within de Sitter space, a spacelike hypersurface that is complete and has constant mean curvature must exhibit total umbilicity. Although counterexamples show that the conjecture does not hold in full generality, it remains valid under additional geometric assumptions, notably in the compact case. In this paper, we establish a broad extension of Goddard's conjecture to compact, spacelike hypersurfaces in Lorentzian manifolds that admit a timelike conformal vector field. Under a natural integral condition involving the ambient Ricci tensor and a mild assumption of the behavior of the mean curvature along the induced tangential flow, we prove that the hypersurface must be totally umbilical. As an application of our approach, we establish rigidity phenomena in Einstein Lorentzian manifolds and retrieve, as a special instance, the well-known classification of compact, spacelike hypersurfaces with constant mean curvature in de Sitter space as round spheres.

Keywords: Lorentzian manifolds; Einstein manifolds; spacelike hypersurfaces; Ricci and scalar curvatures; extrinsic spheres; Goddard conjecture
Mathematics Subject Classification: 53C24, 53C25, 53C42, 53E20, 53Z05

---

1. Introduction

Spacelike hypersurfaces of constant mean curvature in Lorentzian manifolds, and in particular maximal hypersurfaces (i.e., those with $\mathrm{H} = 0$), play a central role in Lorentzian geometry and general relativity. They arise naturally in the study of Einstein field equations and the Cauchy problem, where such hypersurfaces provide distinguished foliations and canonical spacetime slices; see, for example, [1–4].

Among the various ambient Lorentzian spaces, the geometry of spacelike hypersurfaces in the de Sitter space $\mathbb{S}_1^{n+1}$ has attracted considerable interest. This is largely due to the strong rigidity and

Bernstein-type phenomena exhibited by such hypersurfaces, which make de Sitter space a natural Lorentzian counterpart of space forms in Riemannian geometry.

In this context, Goddard [5] conjectured in 1977 that every complete spacelike hypersurface in $\mathbb{S}_1^{n+1}$ with constant mean curvature must be totally umbilical. Although this conjecture was later shown to fail in full generality, it has motivated a substantial amount of research aimed at identifying geometric hypotheses under which such a rigidity statement does hold.

More precisely, the conjecture is known to be valid under suitable bounds on the mean curvature. For $n = 2$, it holds when $0 \leq \mathrm{H}^2 \leq 1$ and fails when $\mathrm{H}^2 > 1$. For $n \geq 3$, Akutagawa [6] proved that the conjecture remains true provided

$$0 \leq \mathrm{H}^2 < \frac{4(n-1)}{n^2}.$$

In addition, it has been established that Goddard's conjecture holds when the hypersurface is compact [7], and an alternative proof in the two-dimensional case was provided by Ramanathan [8].

Beyond the constant mean curvature assumption, several rigidity results have been obtained under the hypothesis of constant scalar curvature. In particular, Cheng and Ishikawa [9] showed that totally umbilical round spheres are the only compact spacelike hypersurfaces in de Sitter space whose scalar curvature satisfies $\mathrm{Scal} < n(n-1)$. Further characterizations under constant scalar curvature assumptions can be found in [10–12]. Moreover, Aledo et al. [13] introduced new criteria involving higher-order mean curvatures, characterizing totally umbilical round spheres as the only compact spacelike hypersurfaces in $\mathbb{S}_1^{n+1}$ which satisfy suitable geometric conditions.

From a different viewpoint, Oliker [14] studied the stability of the Bernstein-type property in de Sitter space. He proved that if a complete spacelike hypersurface has mean curvature H satisfying

$$\mathrm{H}^2 < \frac{4(n-1)}{n^2},$$

and H is sufficiently close to being constant, then the hypersurface is nearly totally umbilical.

Many of the affirmative results mentioned above rely on the position vector field of the ambient Minkowski space. This vector field is concurrent and therefore a particular instance of a conformal vector field. This observation suggests that conformal vector fields play a fundamental role in extending Goddard-type rigidity phenomena beyond the classical de Sitter setting.

Motivated by this perspective, in the present paper, we study compact spacelike hypersurfaces (M, g) immersed in Lorentzian manifolds $(\overline{M}, \overline{g})$ which admit a timelike conformal vector field $\overline{\Xi}$. The main objective is to establish rigidity results that generalize previously known affirmative cases of Goddard's conjecture to this broader geometric framework.

More precisely, our main result shows that if (M, g) is a compact spacelike hypersurface in a Lorentzian manifold which admits a timelike conformal vector field, and if a natural integral condition involving the ambient Ricci tensor is satisfied together with the assumption that the mean curvature remains constant along the integral curves of the tangential component of the conformal vector field, then (M, g) must be totally umbilical (Theorem 4.1). This condition is natural and consistent with earlier rigidity results in related contexts, having appeared in various forms in the literature.

As consequences of the main theorem, we derive rigidity results for compact spacelike hypersurfaces in Einstein Lorentzian manifolds admitting a timelike conformal vector field. In particular, we prove that any compact spacelike hypersurface of constant mean curvature in such an ambient space is an extrinsic sphere, that is, a totally umbilical hypersurface of constant mean curvature (Theorem 4.3). When the ambient manifold is a Lorentzian space form, this result reduces to the de Sitter case, yielding a complete classification: compact spacelike hypersurfaces of constant mean curvature in $\mathbb{S}_1^{n+1}(c)$ are necessarily round spheres.

We also mention several references closely related to the present work, including [15–17], where various characterizations of round spheres are established. See also [18, 19], where round spheres are characterized in terms of Ricci solitons.

The paper is organized as follows. In Section 3, we derive an identity involving the divergence of the tangential component $\Xi^\top$ of the restriction of $\overline{\Xi}$ to M, the mean curvature of M, and the Ricci curvature of the ambient manifold $(\overline{M}, \overline{g})$. In Section 4, which contains the main results, we integrate this identity over M to obtain a global integral formula leading to our principal rigidity theorems.

For comparison, we recall that in the Riemannian setting, when $\overline{\Xi}$ is a Killing or conformal vector field, formulas for the Laplacian $\Delta\Theta$ of the support function $\Theta = \overline{g}(\Xi, N)$ were obtained in [20] and [21], respectively, in terms of the Ricci curvature of the ambient manifold and the squared norm of the shape operator. Corresponding formulas in the Lorentzian setting were derived in [22] and, following a different approach, in [23]. In contrast to these works, our approach does not rely on the Laplacian of the support function. Instead, we exploit identities involving the divergence of $\Xi^\top$ and of its image $\mathrm{A}(\Xi^\top)$ under the shape operator A, which provides a natural and effective framework for establishing the rigidity results presented here.

2. Preliminary notions

Let (M, g) be a connected semi-Riemannian manifold of dimension $n \geq 2$, where the metric g is not necessarily positive definite. Throughout this paper, we follow the curvature sign convention of [24], which is opposite to that adopted in [25]. Under this convention, the Riemann curvature tensor is the $(1, 3)$-tensor field defined by

$$\mathrm{R}(X, Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z,$$

for any vector fields $X, Y, Z \in \mathfrak{X}(M)$.

Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of the tangent space T_pM, $p \in M$. The Ricci curvature is the tensor field on M obtained by contracting the Riemann curvature tensor, and it can be written as

$$\mathrm{Ric}(X, Y) = \sum_{i=1}^{n} \epsilon_i \, g(\mathrm{R}(X, e_i)e_i, Y),$$

where $\epsilon_i = g(e_i, e_i)$.

The scalar curvature is the function on M defined as the trace of the Ricci tensor. With respect to the above orthonormal basis, it is given by

$$\mathrm{Scal}(p) = \sum_{i=1}^{n} \epsilon_i \ \mathrm{Ric}(e_i, e_i).$$

The covariant derivative tensor field T of type $(1, 1)$ is given by

$$(\nabla_X T)(Y) = \nabla_X(T(Y)) - T(\nabla_X Y).$$

If T is self-adjoint (that is, symmetric with respect to the metric g), then its covariant derivative $\nabla_X T$ is also self-adjoint. Equivalently,

$$g((\nabla_X T)(Y), Z) = g((\nabla_X T)(Z), Y), \tag{2.1}$$

for all $X, Y, Z \in \mathfrak{X}(M)$. This can be shown by a direct computation or follows from [24, Exercise 2.5.9, part (2)].

For a tensor field T of type $(1, 1)$, its divergence is introduced as the trace of its covariant derivative. More precisely,

$$\operatorname{div} T = \operatorname{tr}(\nabla T) = \sum_{i=1}^{n} (\nabla_{e_i} T)(e_i),$$

where tr stands for the trace operator with respect to a local orthonormal frame $\{e_i\}_{i=1}^{n}$.

If $\{e_i\}$ is a parallel frame, the above formula reduces to

$$\operatorname{div} T = \sum_{i=1}^{n} \nabla_{e_i}(T(e_i)).$$

For a vector field $X \in \mathfrak{X}(M)$, its divergence is the function

$$\operatorname{div}(X) = \sum_{i=1}^{n} \epsilon_i \, g(\nabla_{e_i} X, e_i),$$

with respect to any local orthonormal frame $\{e_i\}$.

From now on, assume that the metric g is Riemannian (i.e., of index 0) and that (M, g) is a hypersurface in a Lorentzian manifold $(\overline{M}, \overline{g})$ of dimension $n + 1$. The Levi-Civita connections of g and $\overline{g}$ will be denoted by ∇ and $\overline{\nabla}$, respectively.

Let $\mathfrak{X}(M)$ and $\mathfrak{X}(\overline{M})$ be the spaces of smooth vector fields on M and $\overline{M}$, and let $\overline{\mathfrak{X}}(M)$ denote the restrictions to M of vector fields on $\overline{M}$. For a vector field X and a smooth function f, we write $X(f)$ or $X \cdot f$.

Let N be a unit timelike normal vector field along M. Then, for any $X, Y \in \mathfrak{X}(M)$, the Gauss and Weingarten formulas can be expressed as

$$\overline{\nabla}_X Y = \nabla_X Y - g(Y, \mathrm{A}\, X)\, N, \tag{2.2}$$

$$\overline{\nabla}_X N = -\, \mathrm{A}\, X, \tag{2.3}$$

where A is the shape operator of M with respect to N.

The curvature tensors of M and $\overline{M}$ are connected through the Gauss equation, which reads

$$\mathrm{R}(X, Y)Z = (\overline{\mathrm{R}}(X, Y)Z)^{\top} - g(\mathrm{A}(Y), Z)\, \mathrm{A}(X) + g(\mathrm{A}(X), Z)\, \mathrm{A}(Y), \tag{2.4}$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

Expressed through the shape operator, the Codazzi equation reads

$$(\nabla_Y \mathrm{A})(X) - (\nabla_X \mathrm{A})(Y) = \overline{\mathrm{R}}(X, Y)N, \tag{2.5}$$

for every $X, Y \in \mathfrak{X}(M)$.

The mean curvature of the hypersurface (M, g) is given by

$$\mathrm{H} = -\frac{1}{n}\ \mathrm{tr}(\mathrm{A}).$$

The hypersurface (M, g) is called an extrinsic sphere whenever it is totally umbilical, and its mean curvature is constant.

Taking the trace of the Gauss equation (2.4), one obtains the expression of the Ricci tensor of (M, g):

$$\mathrm{Ric}(X, Y) = \overline{\mathrm{Ric}}(X, Y) + \overline{g}(\overline{\mathrm{R}}(N, X)Y,\ N) + g(\mathrm{A}(X),\ \mathrm{A}(Y) + n\ \mathrm{H}\,Y), \tag{2.6}$$

for every $X, Y \in \mathfrak{X}(M)$.

Further taking the trace of (2.6) yields the scalar curvature of (M, g):

$$\mathrm{Scal} = \overline{\mathrm{Scal}} + 2\,\overline{\mathrm{Ric}}(N, N) - (n^2\,\mathrm{H}^2 - |\,\mathrm{A}\,|^2), \tag{2.7}$$

with $|\,\mathrm{A}\,|^2$ being the square of the norm of the shape operator,

$$|\,\mathrm{A}\,|^2 = \mathrm{tr}(\mathrm{A}^2).$$

Let $\overline{\Xi} \in \mathfrak{X}(\overline{M})$ be a timelike vector field, and let

$$\overline{\omega}(X) = \overline{g}(\overline{\Xi}, X)$$

be its associated 1-form.

We know that

$$\mathrm{L}_{\overline{\Xi}}\,\overline{g}(X, Y) + d\overline{\omega}(X, Y) = 2\,\overline{g}(\overline{\nabla}_X\overline{\Xi}, Y),$$

for all $X, Y \in \mathfrak{X}(M)$, where $\mathrm{L}_{\overline{\Xi}}\,\overline{g}$ represents the Lie derivative of $\overline{g}$ taken in the direction of $\overline{\Xi}$.

We introduce the $(1, 1)$-tensor $\overline{\nabla}\,\overline{\Xi}$ by

$$\overline{\nabla}\,\overline{\Xi}(X) := \overline{\nabla}_X\overline{\Xi}.$$

Decomposing $\overline{\nabla}, \overline{\Xi}$ into its symmetric (self-adjoint) part S and its skew-symmetric (skew self-adjoint) part W, we obtain

$$\mathrm{L}_{\overline{\Xi}}\,\overline{g}(X, Y) = 2\,\overline{g}(\mathrm{S}(X), Y), \tag{2.8}$$

$$d\overline{\omega}(X, Y) = 2\,\overline{g}(\mathrm{W}(X), Y). \tag{2.9}$$

Let Ξ be the restriction of $\overline{\Xi}$ to M, so that $\Xi \in \overline{\mathfrak{X}}(M)$. We set

$$\overline{\nabla}\,\Xi(X) = \overline{\nabla}_X\Xi, \qquad X \in \mathfrak{X}(M),$$

which is well-defined (cf. [25], p. 99). Then,

$$\overline{\nabla}\,\Xi(X) = (\overline{\nabla}_X\Xi)^\top - \overline{g}(\overline{\nabla}\,\Xi(X), N)\,N. \tag{2.10}$$

Define $\zeta \in \mathfrak{X}(M)$ by

$$g(\zeta, X) = \overline{g}(\overline{\nabla}\,\Xi(X), N), \qquad X \in \mathfrak{X}(M). \tag{2.11}$$

Then,

$$g(\zeta, X) = \overline{g}(\mathrm{S}(X) + \mathrm{W}(X), N) = \overline{g}(X, \mathrm{S}(N)) - \overline{g}(X, \mathrm{W}(N)) = g(X, \mathrm{S}(N)^{\top}) - \overline{g}(X, \mathrm{W}(N)),$$

and because $\mathrm{W}(N)$ lies in the tangent bundle of M (indeed, $\overline{g}(N, \mathrm{W}(N)) = 0$ due to the skew-symmetry of W), it follows that

$$\zeta = \mathrm{S}(N)^{\top} - \mathrm{W}(N). \tag{2.12}$$

Because $\overline{M}$ is Lorentzian, and M is spacelike, we may choose a globally defined unit timelike normal N with $\overline{g}(\Xi, N) < 0$. Define the support function $\Theta = \overline{g}(\Xi, N)$. The reverse Cauchy-Schwarz inequality implies $|\Theta| = |\overline{g}(\Xi, N)| \geq |\Xi|\,|N|$, where $|\Xi| = \sqrt{-\overline{g}(\Xi, \Xi)}$, and $|N| = 1$ (see [25, Propsition 30]).

Because N and Ξ lie in the same time cone,

$$\Theta \leq -\sqrt{-\overline{g}(\Xi, \Xi)} < 0. \tag{2.13}$$

The decomposition of Ξ into tangential and normal components reads

$$\Xi = \Xi^{\top} - \Theta N. \tag{2.14}$$

By means of (2.2) and (2.3), we calculate

$$\begin{aligned} \overline{\nabla}_X \Xi &= \overline{\nabla}_X (\Xi^{\top} - \Theta N) \\ &= \overline{\nabla}_X \Xi^{\top} - (X \cdot \Theta) N - \Theta\, \overline{\nabla}_X N \\ &= \nabla_X \Xi^{\top} - g(\mathrm{A}(X), \Xi^{\top}) N - g(\nabla\Theta, X) N + \Theta\, \mathrm{A}(X) \\ &= \nabla_X \Xi^{\top} + \Theta\, \mathrm{A}(X) - g(\mathrm{A}(\Xi^{\top}) + \nabla\Theta, X) N. \end{aligned} \tag{2.15}$$

By comparing (2.10) with (2.15) and using the definition of ζ given in (2.11), we obtain

$$\nabla_X \Xi^{\top} = (\overline{\nabla}_X \Xi)^{\top} - \Theta\, \mathrm{A}(X), \tag{2.16}$$

$$\nabla\Theta = \zeta - \mathrm{A}(\Xi^{\top}). \tag{2.17}$$

3. An explicit formula for the Laplacian of the support function

Note that the identities (2.16) and (2.17) are valid for any timelike vector field $\Xi \in \mathfrak{X}(\overline{M})$. In what follows, we take Ξ to be a conformal vector field on $(\overline{M}, \overline{g})$. By definition, this means there exists a smooth function Ψ (called the conformal function, or conformal factor) on $\overline{M}$ such that the Lie derivative of the metric with respect to Ξ satisfies

$$\mathrm{L}_{\Xi}\, \overline{g}(X, Y) = 2\Psi\, \overline{g}(X, Y) \tag{3.1}$$

for all $X, Y \in \mathfrak{X}(\overline{M})$.

By comparing (2.8) and (3.1), we see that $\mathrm{S} = \Psi \mathrm{I}$, with I being the identity operator acting on the tangent bundle $T\overline{M}$.

It follows that $\mathrm{S}(N)^{\top} = 0$, and consequently,

$$\zeta = -\mathrm{W}(N).$$

On the other hand, because (3.1) is equivalent to

$$\overline{g}(\overline{\nabla}_X\Xi, Y) + \overline{g}(X, \overline{\nabla}_Y\Xi) = 2\Psi\,\overline{g}(X, Y), \qquad X, Y \in \mathfrak{X}(\overline{M}), \tag{3.2}$$

it follows that for any $X \in \mathfrak{X}(M)$,

$$\overline{g}(\overline{\nabla}_X\Xi, N) + \overline{g}(X, \overline{\nabla}_N\Xi) = 0,$$

and therefore, by (2.11),

$$\zeta = -(\overline{\nabla}_N\Xi)^\top.$$

Thus, Eq (2.17) may be rewritten in the form

$$\nabla\Theta = -(\overline{\nabla}_N\Xi)^\top - \mathrm{A}(\Xi^\top). \tag{3.3}$$

Remark 1. *If Ξ is a closed conformal vector field, that is,*

$$\overline{\nabla}_X\Xi = \Psi\,X, \qquad \textit{for all } X \in \mathfrak{X}(\overline{M}),$$

one obtains, besides $\mathrm{S} = \Psi\mathrm{I}$*, the additional relation* $\mathrm{W} = 0$.

Consequently,

$$(\overline{\nabla}_N\Xi)^\top = 0.$$

Hence, (3.3) *reduces to*

$$\nabla\Theta = -\,\mathrm{A}(\Xi^\top). \tag{3.4}$$

As indicated in the introduction, the main aim of this section is to derive an identity that will play a key role in proving the main theorem of the paper. To achieve this, we compute explicit expressions for the divergences of $\Xi^\top$ and $\mathrm{A}(\Xi^\top)$, respectively.

Lemma 1. *Consider a spacelike hypersurface (M, g) in a Lorentzian manifold $(\overline{M}, \overline{g})$ of dimension $n + 1$ equipped with a timelike conformal vector field Ξ with conformal factor Ψ. Under the notation and assumptions of this section, the following identities hold:*

$$\mathrm{div}(\Xi^\top) = n(\Psi + \Theta\,\mathrm{H}), \tag{3.5}$$

$$\mathrm{div}(\mathrm{A}(\Xi^\top)) = g(\mathrm{div}(\mathrm{A}), \Xi^\top) - n\Psi\;\mathrm{H} - \Theta\,|\mathrm{A}|^2. \tag{3.6}$$

Proof. Let $\{e_1, \ldots, e_n\}$ be a local orthonormal frame in M. To compute $\mathrm{div}(\Xi^\top)$, we employ Eqs (2.16) and (3.2), which yield

$$\begin{aligned}\mathrm{div}(\Xi^\top) &= \sum_{i=1}^{n} g(\nabla_{e_i}\Xi^\top, e_i)\\ &= \sum_{i=1}^{n} g((\overline{\nabla}_{e_i}\Xi)^\top, e_i) - \sum_{i=1}^{n} g(\Theta\;\mathrm{A}(e_i), e_i)\\ &= \sum_{i=1}^{n} \overline{g}(\overline{\nabla}_{e_i}\Xi, e_i) - \Theta\sum_{i=1}^{n} g(\mathrm{A}(e_i), e_i)\\ &= n\Psi - \Theta\,\mathrm{tr}(\mathrm{A})\end{aligned}$$

$$= n(\Psi + \Theta\,\mathrm{H}).$$

To compute $\mathrm{div}(\mathrm{A}(\Xi^\top))$, we use (2.16) together with the fact that if T is a self-adjoint operator, then its covariant derivative $\nabla_X T$ is also self-adjoint (see (2.1)).

$$\begin{aligned}
\mathrm{div}(\mathrm{A}(\Xi^\top)) &= \sum_{i=1}^{n} g(\nabla_{e_i}\mathrm{A}(\Xi^\top),\, e_i)\\
&= \sum_{i=1}^{n} g\big((\nabla_{e_i}\mathrm{A})(\Xi^\top),\, e_i\big) + \sum_{i=1}^{n} g(\mathrm{A}(\nabla_{e_i}\Xi^\top),\, e_i)\\
&= \sum_{i=1}^{n} g\big((\nabla_{e_i}\mathrm{A})(e_i),\, \Xi^\top\big) + \sum_{i=1}^{n} g(\nabla_{e_i}\Xi^\top,\, \mathrm{A}(e_i))\\
&= g\big(\mathrm{div}(\mathrm{A}),\, \Xi^\top\big) + \sum_{i=1}^{n} g\Big((\overline{\nabla}_{e_i}\Xi)^\top,\, \mathrm{A}(e_i)\Big) - \sum_{i=1}^{n} g(\Theta\,\mathrm{A}(e_i),\, \mathrm{A}(e_i))\\
&= g\big(\mathrm{div}(\mathrm{A}),\, \Xi^\top\big) + \sum_{i=1}^{n} g(\Psi\, e_i,\, \mathrm{A}(e_i)) - \Theta\sum_{i=1}^{n} g\Big(\mathrm{A}^2(e_i),\, e_i\Big)\\
&= g\big(\mathrm{div}(\mathrm{A}),\, \Xi^\top\big) - n\Psi\,\mathrm{H} - \Theta|\mathrm{A}|^2.
\end{aligned}$$

□

Lemma 2. *Consider a spacelike hypersurface (M, g) in a Lorentzian manifold $(\overline{M}, \overline{g})$ of dimension $n+1$ equipped with a timelike conformal vector field Ξ with conformal factor Ψ. With the notation and hypotheses fixed throughout this section, one has the identity*

$$\mathrm{div}(\mathrm{A}) = -(\overline{\mathrm{Q}}(N))^\top - n\,\nabla\mathrm{H}, \tag{3.7}$$

where $\overline{\mathrm{Q}}$ is the Ricci operator of $(\overline{M}, \overline{g})$, characterized by $\overline{\mathrm{Ric}}(X, Y) = \overline{g}\Big(\overline{\mathrm{Q}}(X),\, Y\Big)$ for every $X, Y \in \mathfrak{X}(\overline{M})$. In particular, if $(\overline{M}, \overline{g})$ is Einstein, then

$$\mathrm{div}(\mathrm{A}) = -n\,\nabla\mathrm{H}\,. \tag{3.8}$$

Proof. Let $\{e_1, \ldots, e_n\}$ be a local orthonormal frame on M, chosen so that the fields are parallel at the point under consideration. As previously mentioned, extending $\{N, e_1, \ldots, e_n\}$ arbitrarily to vector fields on $\overline{M}$ allows the symmetries of $\overline{R}$ to yield, for any $X \in \mathfrak{X}(M)$,

$$\begin{aligned}
\overline{\mathrm{Ric}}(N, X) &= \sum_{i=1}^{n} \overline{g}\Big(\overline{\mathrm{R}}(N, e_i)e_i, X\Big) - \overline{g}\Big(\overline{\mathrm{R}}(N, N)N, X\Big)\\
&= \sum_{i=1}^{n} \overline{g}\Big(\overline{\mathrm{R}}(e_i, X)N, e_i\Big)\\
&= \sum_{i=1}^{n} g(\nabla_X(\mathrm{A}(e_i))\,, e_i) - \sum_{i=1}^{n} g\big((\nabla_{e_i}\mathrm{A})(X), e_i\big)\\
&= \sum_{i=1}^{n} X(g(\mathrm{A}(e_i), e_i)) - g\left(X, \sum_{i=1}^{n}(\nabla_{e_i}\mathrm{A})(e_i)\right)
\end{aligned}$$

$$\begin{aligned}&= X(\mathrm{tr}(\mathrm{A})) - g(X, \operatorname{div} \mathrm{A})\\ &= -\,n\,X(\mathrm{H}) - g(\operatorname{div} \mathrm{A}, X)\\ &= -\,n\,g(X, \nabla \mathrm{H}) - g(\operatorname{div} \mathrm{A}, X).\end{aligned}$$

Hence, for all $X \in \mathfrak{X}(M)$,

$$\overline{g}\Big(\overline{\mathrm{Q}}(N), X\Big) = -\,n\,g(\nabla \mathrm{H}, X) - g(\operatorname{div} \mathrm{A}, X),$$

which gives (3.7).

If $(\overline{M}, \overline{g})$ is Einstein, then

$$\overline{\mathrm{Q}} = \left(\tfrac{\mathrm{Scal}}{n+1}\right)\mathrm{I},$$

and hence,

$$(\overline{\mathrm{Q}}(N))^{\top} = 0,$$

which gives (3.8). □

In the next lemma, we derive a concrete and convenient expression for $\operatorname{div}(\mathrm{A}(\Xi^{\top}))$, written in terms of the Ricci curvature (see also formula (14) in [23]).

Lemma 3. *Let (M, g) be a spacelike hypersurface in a Lorentzian manifold $(\overline{M}, \overline{g})$ of dimension $n + 1$ equipped with a timelike conformal vector field $\overline{\Xi}$ with conformal factor Ψ, and let the notation and assumptions be as above. Then,*

$$\operatorname{div}(\mathrm{A}(\Xi^{\top})) = -\,\overline{\mathrm{Ric}}(N, \Xi^{\top}) - n\,\Xi^{\top}(\mathrm{H}) - n\Psi\,\mathrm{H} - \Theta\,|\mathrm{A}|^2. \tag{3.9}$$

Proof. The identity (3.9) is obtained immediately by inserting formula (3.7) into (3.6). □

4. Integral formulas, main result, and the Goddard conjecture

As stated in the introduction, Goddard's conjecture claims that the complete spacelike hypersurfaces of $\mathbb{S}_1^{n+1}$ with constant mean curvature must be totally umbilical [5]. This formulation was later shown not to hold in full generality, yet it continues to motivate efforts to determine the precise geometric hypotheses that ensure such a characterization. In particular, it is known that the conjecture becomes true when the hypersurface is compact [7]. Proofs of these affirmative cases commonly employ the position vector field in Minkowski space, which is a concurrent vector field (and thus a special example of a conformal vector field).

The aim of this paper, and of this section in particular, is to establish an extension of Goddard's conjecture to the setting in which (M, g) is a compact, spacelike hypersurface of a Lorentzian manifold $(\overline{M}, \overline{g})$ of dimension $n + 1$ that carries a timelike conformal vector field $\overline{\Xi}$. To start, we demonstrate the following theorem in the context described above.

Theorem 4.1. *Let (M, g) be a compact, spacelike hypersurface in an $n + 1$-dimensional Lorentzian manifold $(\overline{M}, \overline{g})$ which admits a timelike conformal vector field $\overline{\Xi}$ with conformal factor Ψ. Let $\Xi^{\top}$ denote the tangential component of the vector field induced on M by $\overline{\Xi}$, and denote by N a unit normal vector field along M.*

If the Ricci tensor $\overline{\mathrm{Ric}}$ *of* $(\overline{M}, \overline{g})$ *satisfies*

$$\int_M \overline{\mathrm{Ric}}(\Xi^\top, N)\, dV \leq 0, \tag{4.1}$$

and the mean curvature H *of* (M, g) *remains constant along the integral curves of* $\Xi^\top$*, then* (M, g) *is totally umbilical.*

Proof. From (3.5), it follows that

$$n\Psi = \mathrm{div}(\Xi^\top) - n\Theta\, \mathrm{H}.$$

Substituting this expression into (3.9), we obtain

$$-\mathrm{H}\,\mathrm{div}(\Xi^\top) - \mathrm{div}(\mathrm{A}(\Xi^\top)) = \overline{\mathrm{Ric}}(N, \Xi^\top) + n\, \Xi^\top(\mathrm{H}) + \Theta(|\mathrm{A}|^2 - n\,\mathrm{H}^2).$$

Moreover, using the identity

$$\mathrm{div}(\mathrm{H}\, \Xi^\top) = \mathrm{H}\,\mathrm{div}(\Xi^\top) + \Xi^\top(\mathrm{H}),$$

we can rewrite the previous equation as

$$-\mathrm{div}(\mathrm{H}\, \Xi^\top) - \mathrm{div}(\mathrm{A}(\Xi^\top)) = \overline{\mathrm{Ric}}(N, \Xi^\top) + (n-1)\, \Xi^\top(\mathrm{H}) + \Theta(|\mathrm{A}|^2 - n\,\mathrm{H}^2).$$

Integrating both sides over M and applying the divergence theorem, we obtain

$$\int_M \Big(\overline{\mathrm{Ric}}(\Xi^\top, N) + (n-1)\, \Xi^\top(\mathrm{H}) + \Theta(|\mathrm{A}|^2 - n\,\mathrm{H}^2)\Big)\, dV = 0. \tag{4.2}$$

Because the mean curvature H of (M, g) is assumed to be constant along the integral curves of $\Xi^\top$, we have $\Xi^\top(\mathrm{H}) = 0$. Furthermore, under assumption (4.1), together with $\Theta < 0$ and $|\mathrm{A}|^2 - n\,\mathrm{H}^2 \geq 0$, Eq (4.2) implies $|\mathrm{A}|^2 = n\,\mathrm{H}^2$. Consequently, $\mathrm{A} = \mathrm{H}\,\mathrm{I}$, and hence, (M, g) is totally umbilical. □

When $(\overline{M}, \overline{g})$ is an Einstein manifold admitting a timelike conformal vector field, the following theorems follow as consequences of Theorem 4.1. The second one extends previously known affirmative cases of the Goddard conjecture.

Theorem 4.2. *Let* (M, g) *be a compact, spacelike hypersurface in an Einstein Lorentzian manifold* $(\overline{M}, \overline{g})$ *that admits a timelike conformal vector field* $\overline{\Xi}$*. Denote by* $\Xi^\top$ *the tangential projection of the restriction of* $\overline{\Xi}$ *to* M*. If the mean curvature* H *remains constant along the flow lines of* $\Xi^\top$*, then* (M, g) *is necessarily totally umbilical.*

Theorem 4.3. *Let* (M, g) *be a compact, spacelike hypersurface of constant mean curvature in an Einstein Lorentzian manifold* $(\overline{M}, \overline{g})$ *that admits a timelike conformal vector field. Then,* (M, g) *must be an extrinsic sphere.*

We note that a Riemannian analogue of Theorem 4.3 was obtained in [26] under the additional assumption that the support function Θ is nonzero and does not change sign.

To obtain Theorem 4.3 when $(\overline{M}, \overline{g})$ is a Lorentzian space form, that is, a connected Lorentzian manifold with constant sectional curvature, because totally umbilical hypersurfaces in Lorentzian space forms are well-known (see [27, 28]), it suffices to restrict our attention to the case where $(\overline{M}, \overline{g})$ is the de Sitter space $\mathbb{S}_1^{n+1}(c)$ with $c > 0$. This is the only Lorentzian space form which admits compact totally umbilical hypersurfaces.

In this setting, the timelike conformal vector field may be chosen as any constant (parallel) vector field of the ambient Minkowski space $\mathbb{R}_1^{n+2}$. Conformal vector fields on Minkowski space, and more generally on semi-Riemannian manifolds, are well understood (see [29]). Consequently, we obtain the following result, which may be viewed as a solution to Goddard's conjecture in the compact case (cf. [7, Theorem 4]).

Theorem 4.4. *A compact, spacelike hypersurface with constant mean curvature in the de Sitter space* $\mathbb{S}_1^{n+1}(c)$ *must be a round sphere.*

5. Geometric meaning and examples of the integral Ricci condition (4.1)

Although in Theorem 4.1, the integral Ricci condition (4.1) and the assumption that the mean curvature remains constant along the flow generated by the tangential component of the conformal vector field may appear somewhat technical, it is important to emphasize that condition (4.1) is in fact natural and has precedents in earlier works. Similar hypotheses occur in the literature; see, for example, [30, Theorem 1.1].

We now explain these hypotheses' geometric meaning and provide examples of situations where they hold naturally, beyond the Einstein and space form settings.

More precisely, for a spacelike hypersurface in a Lorentzian manifold which admits a timelike conformal vector field, the inequality (4.1) in Theorem 4.1, or the stronger condition

$$\overline{\mathrm{Ric}}(\Xi^{\top}, N) \leq 0, \tag{5.1}$$

expresses that the ambient Ricci curvature in the normal (timelike) direction does not interact positively with the conformal flow.

Beyond Einstein manifolds and space forms, condition (5.1) naturally appears in several geometric and physical frameworks, such as cosmological spacetimes, generalized Robertson–Walker (GRW) spacetimes, and manifolds with nonpositive sectional curvature. In particular, if the ambient Lorentzian manifold has nonpositive sectional curvature on every plane containing the vectors $\Xi^{\top}$ and N, then averaging in the definition of the Ricci tensor yields (5.1). Such situations arise, for example, in certain noncompact symmetric Lorentzian spaces.

The case of GRW spacetimes is especially explicit. The following example illustrates that condition (4.1), although it may appear technical at first, is in fact natural.

Let (M, g) be a Riemannian manifold of dimension $n \geq 2$, (a, b) an open interval of $\mathbb{R}$, and f a positive smooth function on (a, b). Endowing (a, b) with the metric $-dt^2$, we obtain a Lorentzian warped product manifold $(\overline{M}, \overline{g})$ of dimension $(n + 1)$, where

$$\overline{M} = (a, b) \times M, \qquad \overline{g} = -dt^2 + f^2 g.$$

The Lorentzian manifold $(\overline{M}, \overline{g})$ is then labeled a GRW spacetime. This extends the classical Robertson–Walker spacetime, obtained when (M, g) is a Riemannian manifold of dimension 3 with constant sectional curvature.

The coordinate vector field ∂_t is a globally defined unit timelike vector field on $\overline{M}$, providing a time orientation (see, for example, [2, 31]).

Let (Σ, g_Σ) be a spacelike hypersurface in $(\overline{M}, \overline{g})$, and let N be a globally defined unit vector field normal to Σ consistent with the chosen orientation. Let θ denote the support function of ∂_t along Σ, defined by $\theta = \overline{g}(\partial_t, N)$.

Using the above notation, it follows that $\Theta = f\theta$, and from (2.13), it follows that on Σ,

$$\Theta \leq -f. \tag{5.2}$$

Let $\partial_t^\top$ be the tangential component of ∂_t along Σ. Then,

$$\partial_t^\top = \partial_t + \theta N. \tag{5.3}$$

For the following lemma, see [25, 32].

Lemma 4. *On $(\overline{M}, \overline{g})$, one has*

$$\overline{\nabla}_{\partial_t} X = \frac{f'}{f} X$$

for every $X \in \mathfrak{X}(M)$.

This immediately implies that the vector field $\overline{\Xi} = f\partial_t$ satisfies

$$\overline{\nabla}_X \overline{\Xi} = f' X$$

for all $X \in \mathfrak{X}(\overline{M})$; hence, $\overline{\Xi}$ is a closed conformal vector field on $(\overline{M}, \overline{g})$.

From (5.3), its restriction Ξ to Σ decomposes as

$$\Xi = \Xi^\top - \Theta N. \tag{5.4}$$

The following result [31, Proposition 16] shows that Condition (4.1) is naturally fulfilled in this setting.

Proposition 1. *Let (Σ, g_Σ) be a compact, spacelike hypersurface in the Lorentzian warped product manifold $(\overline{M}, \overline{g}) = \mathbb{R} \times_f M$ with $\overline{g} = -dt^2 + f^2(t)g$. With the above notation, one has*

$$\int_\Sigma \overline{\mathrm{Ric}}(\Xi^\top, N)\, dV = -\int_\Sigma f\Big(\theta(\| \mathrm{A} \|^2 - n\,\mathrm{H}^2) + (n-1)\partial_t^\top(\mathrm{H})\Big)\, dV. \tag{5.5}$$

From (5.5), we deduce that if (Σ, g_Σ) is totally umbilical with constant mean curvature, then

$$\int_\Sigma \overline{\mathrm{Ric}}(\Xi^\top, N)\, dV = 0,$$

meaning that Condition (4.1) holds. This applies in particular to an important class of spacelike hypersurfaces known as slices of $(\overline{M}, \overline{g})$, which are fundamental in physics and general relativity as reference frames associated with particular observers.

Let h denote the height function of Σ, defined by $h = \pi \circ \psi$, where π is the projection onto the $\mathbb{R}$-factor, and ψ is the immersion of Σ into $\overline{M}$. One then has the following (cf. [31]):

Lemma 5. *The gradient of* h *on* Σ *is* $\nabla h = -\partial_t^{\top}$, *and its norm satisfies*

$$\|\nabla h\|^2 = \theta^2 - 1. \tag{5.6}$$

A slice of $(\overline{M}, \overline{g})$ is a spacelike hypersurface where the height function h remains constant. Equivalently, by (5.6), this occurs precisely when $\theta \equiv -1$, that is, $\Theta = -f$. In this situation, the shape operator of the slice $\Sigma_0 = \{t_0\} \times M$ is given by $\mathrm{A} = -\mathrm{H}\, I$, where

$$\mathrm{H} = -\frac{f'(t_0)}{f(t_0)}$$

denotes the (constant) mean curvature of Σ_0. Consequently, the slices are totally umbilical, spacelike hypersurfaces of $(\overline{M}, \overline{g})$ with constant mean curvature, and therefore they satisfy condition (5.1) and, a fortiori, condition (4.1).

6. Conclusions and future directions

In this paper, a rigidity result was established for compact, spacelike hypersurfaces in Lorentzian manifolds which admit a timelike conformal vector field. Under a natural integral condition on the ambient Ricci tensor and a mild assumption on the behavior of the mean curvature along the induced tangential flow, we showed that these hypersurfaces must be totally umbilical. In the Einstein case, this provides a characterization of compact spacelike hypersurfaces with constant mean curvature as extrinsic spheres and, in particular, recovers the classical result in the de Sitter space, thereby confirming the well-known Goddard conjecture.

Several problems remain open for future study. One direction is to weaken or replace the integral Ricci condition with pointwise or variational assumptions, possibly allowing noncompact hypersurfaces. Another direction is to investigate similar rigidity results for hypersurfaces with constant higher order mean curvatures or prescribed scalar curvature. It would also be interesting to see whether the same methods work for more general vector fields and to explore applications to problems in mathematical relativity.

We hope that the ideas developed in this paper will help improve the understanding of rigidity phenomena for spacelike hypersurfaces in Lorentzian geometry.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This research was supported by the Ongoing Research Funding Program (ORF-2026-1053), King Saud University, Riyadh, Saudi Arabia.

Conflict of interest

The author declares that he has no conflict of interest.

References

1. R. Bartnik, L. Simon, Spacelike hypersurfaces with prescribed boundary values and mean curvature, *Commun. Math. Phys.*, **87** (1982), 131–152. https://doi.org/10.1007/BF01211061
2. L. J. Alías, A. Romero, M. Sánchez, Uniqueness of complete spacelike hypersurfaces of constant mean curvature in generalized Robertson-Walker spacetimes, *Gen. Relat. Gravit.*, **27** (1995), 71–84. https://doi.org/10.1007/BF02105675
3. J. E. Marsden, F. J. Tipler, Maximal hypersurfaces and foliations of constant mean curvature in general relativity, *Phys. Rep.*, **66** (1980), 109–139. https://doi.org/10.1016/0370-1573(80)90154-4
4. M. Caballero, A. Romero, R. Rubio, Constant mean curvature spacelike hypersurfaces in Lorentzian manifolds with a timelike gradient conformal vector field, *Class. Quantum Grav.*, **28** (2011), 145009. https://doi.org/10.1088/0264-9381/28/14/145009
5. A. J. Goddard, Some remarks on the existence of spacelike hypersurfaces of constant mean curvature, *Math. Proc. Cambridge Philos. Soc.*, **82** (1977), 489–495. https://doi.org/10.1017/S0305004100054153
6. K. Akutagawa, On spacelike hypersurfaces with constant mean curvature in the de Sitter space, *Math. Z.*, **196** (1987), 13–19. https://doi.org/10.1007/BF01179263
7. S. Montiel, An integral inequality for compact spacelike hypersurfaces in de Sitter space and applications to the case of constant mean curvature, *Indiana Univ. Math. J.*, **37** (1988), 909–917. https://doi.org/10.1512/iumj.1988.37.37045
8. J. Ramanathan, Complete spacelike hypersurfaces of constant mean curvature in de Sitter space, *Indiana Univ. Math. J.*, **36** (1987), 349–359.
9. Q. M. Cheng, S. Ishikawa, Spacelike hypersurfaces with constant scalar curvature, *Manuscripta Math.*, **95** (1998), 499–505. https://doi.org/10.1007/BF02678045
10. H. Li, Global rigidity theorems of hypersurfaces, *Ark. Mat.*, **35** (1997), 327–351. https://doi.org/10.1007/BF02559973
11. Y. Zheng, On space-like hypersurfaces in the de Sitter space, *Ann. Glob. Anal. Geom.*, **13** (1995), 317–321. https://doi.org/10.1007/BF00773403
12. Y. Zheng, Space-like hypersurfaces with constant scalar curvature in the de Sitter spaces, *Differ. Geom. Appl.*, **6** (1996), 51–54.
13. J. A. Aledo, L. J. Alías, A. Romero, Integral formulas for compact space-like hypersurfaces in de Sitter space: applications to the case of constant higher order mean curvature, *J. Geom. Phys.*, **31** (1999), 195–208. https://doi.org/10.1016/S0393-0440(99)00008-X
14. V. Oliker, A priori estimates of the principal curvatures of spacelike hypersurfaces in de Sitter space with applications to hypersurfaces in hyperbolic space, *Amer. J. Math.*, **114** (1989), 605–626. https://doi.org/10.2307/2374771
15. M. Guediri, S. Deshmukh, Hypersurfaces in a Euclidean space with a Killing vector field, *AIMS Math.*, **9** (2024), 1899–1910. https://doi.org/10.3934/math.2024093
16. Y. Li, M. Bin-Asfour, K. S. Albalawi, M. Guediri, Spacelike hypersurfaces in de Sitter space, *Axioms*, **14** (2025), 155. https://doi.org/10.3390/axioms14030155
17. M. Bin-Asfour, K. S. Albalawi, M. Guediri, Integral formulae and applications for compact Riemannian hypersurfaces in Riemannian and Lorentzian manifolds admitting concircular vector fields, *Mathematics*, **13** (2025), 1672. https://doi.org/10.3390/math13101672

18. M. Guediri, N. Alshehri, Rigidity of almost Ricci solitons on compact Riemannian manifolds, *AIMS Math.*, **10** (2025), 13524–13539. https://doi.org/10.3934/math.2025608
19. M. Guediri, Rigidity of non-steady gradient Ricci solitons, *Axioms*, **14** (2025), 842. https://doi.org/10.3390/axioms14110842
20. S. Furnari, J. Ripoll, Killing fields, mean curvature, translation maps, *Illinois J. Math.*, **48** (2004), 1385–1403. https://doi.org/10.1215/ijm/1258138517
21. L. J. Alías, M. Dajczer, J. Ripoll, A Bernstein-type theorem for Riemannian manifolds with a Killing field, *Ann. Glob. Anal. Geom.*, **31** (2007), 363–373. https://doi.org/10.1007/s10455-006-9045-5
22. A. Barros, A. Brasil, A. Caminha, Stability of spacelike hypersurfaces in foliated spacetimes, *Differ. Geom. Appl.*, **26** (2008), 357–365. https://doi.org/10.1016/j.difgeo.2007.11.028
23. A. L. Albujer, J. A. Aledo, L. J. Alías, On the scalar curvature of hypersurfaces in spaces with a Killing field, *Adv. Geom.*, **10** (2010), 487–503. https://doi.org/10.1515/advgeom.2010.017
24. P. Petersen, *Riemannian geometry*, 3 Eds., Vol. 171, Springer, 2016. https://doi.org/10.1007/978-3-319-26654-1
25. B. O'Neill, *Semi-Riemannian geometry with applications to relativity*, Academic Press, 1983.
26. Y. Katsurada, On a certain property of closed hypersurfaces in an Einstein space, *Comment. Math. Helv.*, **38** (1964), 165–171. https://doi.org/10.1007/BF02566914
27. N. Abe, N. Koike, S. Yamaguchi, Congruence theorems for proper semi-Riemannian hypersurfaces in a real space form, *Yokohama Math. J.*, **35** (1987), 123–136.
28. N. Alshehri, M. Guediri, Ricci solitons on Riemannian hypersurfaces arising from closed conformal vector fields in Riemannian and Lorentzian manifolds, *J. Nonlinear Math. Phys.*, **31** (2024), 25. https://doi.org/10.1007/s44198-024-00190-4
29. D. S. Kim, S. B. Kim, Y. H. Kim, S. H. Park, Conformal vector fields and totally umbilic hypersurfaces, *Bull. Korean Math. Soc.*, **39** (2002), 671–680. https://doi.org/10.4134/BKMS.2002.39.4.671
30. P. Petersen, W. Wylie, Rigidity of gradient Ricci solitons, *Pacific J. Math.*, **241** (2009), 329–345. https://doi.org/10.2140/pjm.2009.241.329
31. N. Alessa, M. Guediri, On spacelike hypersurfaces in generalized Robertson–Walker spacetimes, *Axioms*, **13** (2024), 636. https://doi.org/10.3390/axioms13090636
32. A. L. Besse, *Einstein manifolds*, Springer Berlin, Heidelberg, 1987. https://doi.org/10.1007/978-3-540-74311-8

AIMS Press