AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(4): 9303–9318.
DOI: 10.3934/math.2026384
Received: 26 July 2025
Revised: 10 March 2026
Accepted: 27 March 2026
Published: 07 April 2026

Research article

Maximum entropy principle for uncertain sets with expectation and variance constraints

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Abstract: Up to now, the entropy of uncertain sets has found many applications in areas including uncertain finance, uncertain inference, and learning curves. Providing a maximum entropy principle for uncertain sets is helpful for selecting appropriate membership functions in these areas. This paper proposes the maximum entropy principle for uncertain sets and obtains the membership function which achieves the maximum entropy by using the Euler equation in the calculus of variations. Additionally, the entropy of some commonly used uncertain sets is computed in the manuscript.

Keywords: entropy; maximum entropy principle; uncertainty theory; uncertain set; membership function; Euler equation
Mathematics Subject Classification: 60A40, 94A17

1. Introduction

To handle human uncertainty, in 2007, Liu [1] proposed uncertainty theory, which is an axiomatic system, grounded in normality, duality, sub-additivity, and product axioms. Uncertainty theory has been widely applied in various fields, including uncertain statistics [2,3], uncertain programming [4,5], reliability analysis [6–8], and uncertain finance [9–11].

Uncertain sets were initially introduced by Liu [12] in 2010 to model imprecise notions like “young”, “tall”, and “most” within the framework of uncertainty theory. Uncertain sets are completely different from fuzzy sets, which was introduced by Zadeh [13] in 1965 as an extension of the classical notion of sets. The essential difference between them is that different measures are used, i.e., fuzzy sets use possibility measures, and uncertain sets use uncertain measures. So far, uncertain sets have become a valuable tool in diverse fields, including the learning curve [14, 15], uncertain inference [16, 17], and multivalued logic [18].

In the mid-19th century, Clausius proposed the entropy concept to quantify thermodynamic disorder. Up to now, entropy has become a fundamental tool across diverse fields. Shannon's information entropy [19] measures the unpredictability of random variables, becoming crucial in information theory (see [20]). In 1958, Kolmogorov [21] extended entropy to dynamical systems for analyzing Bernoulli shifts. In 2009, the logarithm entropy was introduced into uncertainty theory by Liu [22] to measure the complexity for the prediction of uncertain variables. Today, entropy has founded extensive applications across multiple disciplines including machine learning, data analysis, dynamical systems, ergodic theory, probability theory, and uncertainty theory. For a detailed treatment, the readers can refer to Einsiedler and Thomas [23], Martin and England [24], Walters [25], and Liu [26].

The maximum entropy principle (for simplicity, we call it the MaxEnt principle hereafter), originated by Jaynes [27, 28], offers a criterion for probability distribution selection. It states that, among all distributions matching a set of constraints, the one with maximum Shannon entropy should be chosen. Choosing a lower-entropy distribution assumes unknown information. The maximum entropy distribution is therefore the only justified choice. This principle has found widespread applications in fields such as decision-making [29, 30], time-series analysis [31], communication theory [32], statistical inference [33], reliability theory [34, 35], cellular automata analysis [36], and moment equations in the kinetic theory of gases and wave velocities [37]. Here, we will investigate the MaxEnt principle for the uncertain set. It is worth emphasizing that our result differs fundamentally from the MaxEnt principles proposed by Chen and Dai [38], Yao et al. [39], and Dai [40]. The key distinction lies in the fact that their conclusions are derived in the framework of uncertain variables, whereas our result is established in the context of uncertain sets. This distinction remains despite the fact that their papers and ours utilize the same primary tool: the Euler equation.

In 2011, Liu [22] first proposed logarithm entropy to quantify the difficulty of predicting the outcome of an uncertain set. Following closely, Yao and Ke [41] provided a formula of logarithm entropy by using of the inverse membership function. Based on the entropy formula, Yao and Ke [41] also proved the linearity of the logarithm entropy. Afterward, Yao [42] introduced sine entropy for uncertain set and also provided several characteristics of the sine entropy. Wang and Ha [43] proposed the quadratic entropy for uncertain sets and obtained several characteristics of the quadratic entropy. Gao and Ralescu [44] suggested elliptic entropy of uncertain sets and provided an entropy formula and applications to portfolio selection and clustering. Tan and Yu [14] proposed hyperbolic entropy and provided an entropy formula and application to uncertain learning curves. Zhao et al. [45] introduced the Tsallis entropy and presented applications in portfolio optimization of uncertain returns. Wang et al. [15] introduced exponential entropy of uncertain sets and provided applications in learning curves and portfolio optimization.

We also point out that, in 2006, Li and Liu [46] investigated the maximum entropy principle for fuzzy variables. Like fuzzy sets, fuzzy variables can be characterized by their membership functions. Using the Euler equation, they proved that normal fuzzy variables achieve maximum entropy under the constraints of a given expected value and variance. The distinction between their results and ours lies in the nature of the functions involved: they investigated the entropy of fuzzy variables, which are real-valued functions mapping a credibility space to the set of real numbers; in contrast, we focus on the entropy of uncertain sets, which are set-valued functions mapping an uncertainty space to a collection of sets of real numbers.

As mentioned above, the entropy of uncertain sets has been applied in various fields, such as uncertain finance [42, 45], uncertain inference [14, 15], and learning curves [14]. However, up to now, no results regarding the MaxEnt principle for uncertain sets have been reached. Therefore, providing a MaxEnt principle on the entropy for uncertain sets is helpful both theoretically and practically. In this paper, we will derive the membership function that achieves maximum entropy for uncertain set based on Jaynes' principle of maximum entropy. Our main tool is the Euler equation in the calculus of variations.

The remainder of the article is organized in the following order. In Section 2, we will collect some necessary facts on uncertain set. In Section 3, the entropy and entropy formulas on uncertain sets will be reviewed. In Section 4, we will establish the MaxEnt principle of logarithm entropy for uncertain sets. Section 5 provides a brief discussion and recommendations for future investigation. For the reader's convenience, some essential facts about the Euler equation are provided in the Appendix.

2. Preliminaries

In this section, some requisite facts on uncertain sets will be collected. First, we recall the uncertainty space.

Let Γ be the universal set, and let $\mathcal{B}$ be a σ-algebra of subsets on Γ. Members in $\mathcal{B}$ are referred to as events. Liu [1] proposed the uncertain measure to deal with human's uncertainty rationally.

Definition 2.1. *A set-valued map $\mathcal{M}$: $\mathcal{B} \to [0, 1]$ is defined as an uncertain measure if it has the next three properties:*

(1) Normality: $\mathcal{M}\{\Gamma\} = 1$;
(2) Duality: $\mathcal{M}\{B\} + \mathcal{M}\{B^c\} = 1$, for any $B \in \mathcal{B}$;
(3) Subadditivity: For any given series of events $B_1, B_2, \cdots$, one gets

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} B_i\right\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{B_i\}.$$

The triplet $(\Gamma, \mathcal{B}, \mathcal{M})$ is called an uncertainty space. In addition, Liu [47] suggested the following product axiom:

(4) Suppose $(\Gamma_i, \mathcal{B}_i, \mathcal{M}_i)$, $i = 1, 2, \cdots$ is a sequence of uncertainty space. $\mathcal{M}$ is called a product uncertain measure if for each sequence of events $B_i \in \mathcal{B}_i, i = 1, 2, \cdots$, one has

$$\mathcal{M}\left\{\prod_{i=1}^{\infty} B_i\right\} = \bigwedge_{i=1}^{\infty} \mathcal{M}_i\{B_i\},$$

where $\bigwedge_{i=1}^{\infty} b_i = \min\{b_i : b_i \geq 0, i = 1, 2, \cdots\}$.

Liu [12] introduced uncertain sets in 2010 to model imprecise concepts within the framework of uncertainty theory. In essence, an uncertain set is a set-valued function on an uncertainty space designed to model imprecise concepts that are essentially sets whose borders are not clearly expressed due to the inherent uncertainty in natural language. Some representative instances are "cold", "old", and "short". The precise definition is provided below.

Definition 2.2. *[12] Let* $(\Gamma, \mathcal{B}, \mathcal{M})$ *be an uncertainty space. A map X*

$$X\colon (\Gamma, \mathcal{B}, \mathcal{M}) \to \{B \,|\, B \text{ is the Borel subsets of real numbers}\} \tag{2.1}$$

is called an uncertain set, provided that $\{B \subset X\}$ *and* $\{X \subset B\}$ *are events for each Borel set B.*

We point out that $\{B \subset X\}$ and $\{X \subset B\}$ are subsets of Γ in this paper, i.e.,

$$\{B \subset X\} = \{\gamma \in \Gamma \mid B \subset X(\gamma)\},$$
$$\{X \subset B\} = \{\gamma \in \Gamma \mid X(\gamma) \subset B\}.$$

The complement X^c of the uncertain set X is defined by

$$X^c(\gamma) = X(\gamma)^c, \gamma \in \Gamma, \tag{2.2}$$

where for a subset of real numbers A, A^c is the complement of A.

Just as a subset of the real numbers can be characterized by its indicator function, an uncertain set also can be represented by its membership function.

Definition 2.3. *[12] A function* $\mu : \mathbb{R} \to [0, 1]$ *is called the membership function of an uncertain set X, provided that for each Borel set B, it is true that*

$$\mathcal{M}\{B \subset X\} = \inf_{x \in B} \mu(x),$$
$$\mathcal{M}\{X \subset B\} = 1 - \sup_{x \in B^C} \mu(x).$$

The equations above are referred to as measure inversion formulas.

Liu [12] also provided the following intuitive result.

Proposition 2.1. *[12] Assume X is an uncertain set having membership function* μ*. For each number x, it follows that*

$$\mu(x) = \mathcal{M}\{x \in X\}. \tag{2.3}$$

Remark 2.1. $\mu(x)$ *quantifies the membership grade of x' in the uncertain set X.* $\mu(x)$*=1 signifies that x is entirely within X, whereas* $\mu(x)$*=0 indicates that x is not included in X completely. Generally, a higher* $\mu(x)$ *signifies a stronger association of x with X.*

Liu [22] provided an equivalent condition for a function μ to be the membership function of an uncertain set.

Proposition 2.2. *[22] Let* μ *be a real-valued function. Then,* μ *qualifies as a membership function precisely when*

$$0 \le \mu(x) \le 1, \; x \in \mathbb{R}.$$

Below are some commonly used uncertain sets in uncertainty theory.

Example 2.1. An uncertain set characterized by the following membership function is termed as triangular uncertain set:

$$\mu(x) = \begin{cases} \dfrac{x-a}{b-a}, & a \le x \le b, \\ \dfrac{x-c}{b-c}, & b \le x \le c, \\ 0, & otherwise. \end{cases}$$

We use (a, b, c) to denote a triangular uncertain set.

Example 2.2. An uncertain set characterized by the following membership function is termed a Gaussian uncertain set:

$$\mu(x) = \exp\left(-\frac{(x-e)^2}{2\sigma^2}\right),\ x \in \mathbb{R}.$$

We use $\mathcal{G}(e, \sigma)$ to denote a Gaussian uncertain set.

Liu [48] also suggested the notions of regular membership functions and inverse membership functions. In what follows are their formal definitions.

Definition 2.4. *[48] A membership function μ is said to be regular if there is a point x_0 for which $\mu(x_0) = 1$, and μ is increasing on $(-\infty, x_0)$ and decreasing on $(x_0, +\infty)$.*

Specially, the membership function $\mu \equiv 1$ is regular, but $\mu \equiv 0$ is not. Obviously, all triangular and Gaussian membership functions are regular.

Definition 2.5. *[48] For an uncertain set X having membership function μ, we define the inverse membership function for X by*

$$\mu^{-1}(\alpha) = \{x \in \mathbb{R} | \mu(x) \ge \alpha\},\ \forall \alpha \in (0, 1]. \tag{2.4}$$

The set $\mu^{-1}(\alpha)$ is as well termed as the α-cut of μ.

Example 2.3. For a triangular uncertain set (a, b, c), its α-cut is

$$\mu^{-1}(\alpha) = [(1-\alpha)a + \alpha b, \alpha b + (1-\alpha)c],\ \alpha \in [0, 1]. \tag{2.5}$$

As an extension of the independence of uncertain variables, Liu [49] introduced the independence of uncertain sets.

Definition 2.6. *[49] Uncertain sets $X_1, X_2, \cdots, X_n$ are independent when the following two equations hold for any Borel sets $B_1, B_2, \cdots, B_n$:*

$$\mathcal{M}\{\bigcap_{i=1}^{n}(X_i^* \subset B_i)\} = \bigwedge_{i=1}^{n} \mathcal{M}(X_i^* \subset B_i), \tag{2.6}$$

$$\mathcal{M}\{\bigcup_{i=1}^{n}(X_i^* \subset B_i)\} = \bigvee_{i=1}^{n} \mathcal{M}(X_i^* \subset B_i). \tag{2.7}$$

Here, $X_i^* \in \{X_i, X_i^C\}$, $i = 1, 2, \cdots, n$, are chosen arbitrarily, and $\bigvee_{i=1}^{n} \mathcal{M}(X_i)$ means the maximum of $\mathcal{M}(X_i)$, $i = 1, 2, \cdots, n$.

Liu [22] proposed the expectation of an uncertain set as the center of gravity in the sense of uncertain measure. A rigorous definition is provided below.

Definition 2.7. *[22] Let X be a nonempty uncertain set. The expectation of X is defined as follows:*

$$\mathrm{E}[X]=\frac{1}{2}\int_{0}^{+\infty}(\mathcal{M}\{X\geq x\}+1-\mathcal{M}\{X<x\})\,dx-\frac{1}{2}\int_{-\infty}^{0}(\mathcal{M}\{X\leq x\}+1-\mathcal{M}\{X>x\})\,dx \tag{2.8}$$

if both integrals exist.

For uncertain sets having regular membership function, Liu [49] provided a simple calculation formula.

Proposition 2.3. *[50] Given an uncertain set X having regular membership function μ, we have*

$$\mathrm{E}[X]=x_0+\frac{1}{2}\int_{x_0}^{+\infty}\mu(x)dx-\frac{1}{2}\int_{-\infty}^{x_0}\mu(x)dx. \tag{2.9}$$

Here, $\mu(x_0)$=1 for some point x_0.

To measure the degree of the spread of the uncertain set around its expectation, Liu [22] proposed the variance of uncertain sets.

Definition 2.8. *[22] Assume X is a nonempty uncertain set having finite expectation e. We define its variance as*

$$\mathrm{V}[X]=\mathrm{E}[(X-e)^2]. \tag{2.10}$$

For an uncertain set X having regular membership function, a formula for calculating the variance of X is provided in Example 4.1 by Yang and Gao [51].

Proposition 2.4. *[51] Assume X is an uncertain set having regular and symmetric membership function μ. If the expectation e of X is finite, it has variance*

$$\mathrm{V}[X]=\int_{e}^{+\infty}(x-e)\mu(x)\,dx. \tag{2.11}$$

3. Entropy of the uncertain set

This section focuses on recalling some necessary facts on the logarithm entropy of uncertain sets.

Definition 3.1. *[22] Given an uncertain set X having membership function μ, we define its logarithm entropy by*

$$\mathrm{H}[X]=\int_{-\infty}^{+\infty}S(\mu(x))\,dx. \tag{3.1}$$

Here, $S(t)=-t\ln t-(1-t)\ln(1-t),\ t\in[0,1]$.

Below are two illustrative examples of entropy for uncertain sets.

Example 3.1. For a crisp set A of the real number, the uncertain set $X \equiv A$ has the following membership function:

$$\mu(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

By direct computation, we have

$$\mathrm{H}[X] = \int_{-\infty}^{+\infty} S(\mu(x))dx = \int_{-\infty}^{+\infty} 0dx = 0.$$

Example 3.2. Let X be an uncertain set with discrete membership function such as Lerch-type, for example, the "Good" membership function [36], i.e.,

$$\mu(x) = \begin{cases} \dfrac{x^{-\alpha} r^x}{\Phi_m(r, \alpha)}, & x = 1, 2, \ldots, m, \\ 0, & x \notin \{1, 2, \ldots, m\}, \end{cases} \tag{3.2}$$

where

$$\Phi_m(r, \alpha) = \sum_{i=1}^{m} \frac{r^i}{i^\alpha}. \tag{3.3}$$

Note that uncertainty theory adopts the Riemann-Stieltjes integral, and by direct computation, we have

$$\mathrm{H}[X] = 0.$$

In the following, we present the symmetry and positive linearity of the entropy of the uncertainty set.

Proposition 3.1. *[22] Let X be an uncertain set, and let X^c be its complement. Then,*

$$\mathrm{H}[X^c] = \mathrm{H}[X].$$

Proposition 3.2. *(Positive linearity [41]) Let X and Y be independent uncertain sets with regular membership functions. Then, for any real numbers a and b, we have*

$$\mathrm{H}[aX + bY] = |a|\mathrm{H}[X] + |b|\mathrm{H}[Y].$$

Yao and Ke [41] summarized several properties regarding logarithm entropy on uncertain sets, and interested readers can refer to the article by Yao and Ke [41] and the textbook by Liu [26].

4. MaxEnt principle for logarithm entropy

In this section, following Jaynes' principle of maximum entropy, we derive the membership function that maximizes the logarithm entropy.

Theorem 4.1. *Let X be an uncertain set with a regular membership function μ whose expectation is e and variance is σ^2. Further, assume $\mu(x_0) = 1$ and μ is symmetric about x_0. Then,*

$$\mathrm{H}[X] \leq \frac{2\sqrt{3}\pi\sigma}{3},$$

and the maximum entropy attains letting

$$\mu(x) = \begin{cases} \left(1 + \exp\left(\frac{\sqrt{3}\pi|x-e|}{6\sigma}\right)\right)^{-1}, & x \neq e, \\ 1, & x = e. \end{cases}$$

Proof of Theorem 4.1. By Theorem 2.3 [50] and noting that μ is symmetric about x_0, we have

$$\mathrm{E}[X] = x_0.$$

Then, $e = x_0$. According to Theorem 2.4 [51], it follows that

$$\mathrm{V}[X] = \int_e^{+\infty} (x - e)\mu(x)\, dx.$$

From the definition of logarithm entropy, it can be shown that

$$\begin{aligned} \mathrm{H}[X] = -\bigg(&\int_{-\infty}^{e} (\mu(x) \ln \mu(x) + (1 - \mu(x)) \ln(1 - \mu(x)))\, dx \\ &+ \int_e^{+\infty} (\mu(x) \ln \mu(x) + (1 - \mu(x)) \ln(1 - \mu(x)))\, dx\bigg). \end{aligned}$$

Noting that μ is symmetric about e, we have

$$\begin{aligned} \mathrm{H}[X] = -\bigg(&\int_{-\infty}^{e} (\mu(2e - x) \ln \mu(2e - x) + (1 - \mu(2e - x)) \ln(1 - \mu(2e - x)))\, dx \\ &+ \int_e^{+\infty} (\mu(x) \ln \mu(x) + (1 - \mu(x)) \ln(1 - \mu(x)))\, dx\bigg). \end{aligned}$$

Letting $2e - x = y$, we have

$$\begin{aligned} \mathrm{H}[X] &= -\bigg(\int_e^{+\infty} (\mu(y) \ln \mu(y) + (1 - \mu(y)) \ln(1 - \mu(y)))\, dx \\ &\qquad + \int_e^{+\infty} (\mu(x) \ln \mu(x) + (1 - \mu(x)) \ln(1 - \mu(x)))\, dx\bigg) \\ &= -2 \int_e^{+\infty} (\mu(x) \ln \mu(x) + (1 - \mu(x)) \ln(1 - \mu(x)))\, dx. \end{aligned}$$

According to Theorem B in the Appendix, noting that H[X] does not depend on the derivative of the membership function μ and letting μ be the function y in Theorem B, we obtain

$$J[\mu] = -2 \int_e^{+\infty} (\mu(x) \ln \mu(x) + (1 - \mu(x)) \ln(1 - \mu(x)))\, dx, \tag{4.1}$$

and the constraint of the variance is

$$K[\mu] = \int_e^{+\infty} (x - e)\mu(x)\, dx = \sigma^2. \tag{4.2}$$

Then, there exists a constant λ such that $\mu = \mu(x)$ is an extremal of the following functional:

$$\int_e^{+\infty} (-2(\mu(x)\ln\mu(x) + (1-\mu(x))\ln(1-\mu(x))) + \lambda(x-e)\mu(x))\,dx.$$

In order to satisfy the regularity condition for the membership function μ, we introduce a natural constraint $\mu(e) = 1$. Thus, the maximum entropy membership function μ^* should meet

$$-2\ln\mu^*(x) + 2\ln(1-\mu^*(x)) + \lambda(x-e) = 0.$$

Then,

$$\mu^*(x) = (1+\exp(-\lambda(x-e)/2))^{-1},\ x \geq e.$$

Substituting μ^* into the variance formula, we get

$$\int_e^{+\infty} \frac{x-e}{1+\exp(-\lambda(x-e)/2)}dx = \sigma^2.$$

Letting $x - e = y$, we have

$$\int_0^{+\infty} \frac{y}{1+\exp(-\lambda y/2)}dy = \sigma^2.$$

Note that the above equation implies the convergence of the improper integral. Thus, it follows that $\lambda < 0$. An equivalent transformation yields

$$\frac{4}{\lambda^2}\int_0^{+\infty} \frac{\frac{-\lambda y}{2}}{1+\exp(-\lambda y/2)}d\frac{-\lambda y}{2} = \sigma^2.$$

Letting $\frac{-\lambda}{2}y = z$, we have

$$\frac{4}{\lambda^2}\int_0^{+\infty} \frac{z}{1+\exp(z)}dz = \sigma^2.$$

By the following integration formula (see [52, Page 1043]):

$$\int_0^{+\infty} \frac{z}{1+\exp(z)}dy = \frac{\pi^2}{12},$$

we have

$$\lambda = -\frac{\pi}{\sigma\sqrt{3}}.$$

Noting that μ is symmetric about e and $\mu(e) = 1$, then, the maximum entropy membership function μ^* is

$$\mu^*(x) = \begin{cases} (1+\exp(\frac{\sqrt{3}\pi|x-e|}{6\sigma}))^{-1}, & x \neq e, \\ 1, & x = e. \end{cases}$$

Substituting μ^* into the quadratic entropy formula, we obtain

$$\mathrm{H}[X] = -2\int_e^{+\infty} (\mu^*(x)\ln\mu^*(x) + (1-\mu^*(x))\ln(1-\mu^*(x)))\,dx.$$

Writing

$$y = \frac{\sqrt{3}\pi}{6\sigma}(x - e),$$

we have

$$dx = \frac{2\sqrt{3}\sigma}{\pi}dy.$$

Then,

$$\mathrm{H}[X] = -\frac{4\sqrt{3}\sigma}{\pi}\int_0^{+\infty}\left(\frac{1}{1+\exp(y)}\ln\frac{1}{1+\exp(y)} + (1 - \frac{1}{1+\exp(y)})\ln(1 - \frac{1}{1+\exp(y)})\right)dy.$$

Writing

$$z = \frac{1}{1+\exp(y)},$$

we have

$$dy = \frac{1}{z(1-z)}dz.$$

Thus,

$$\mathrm{H}[X] = -\frac{4\sqrt{3}\sigma}{\pi}\int_0^{0.5}\left(\frac{\ln z}{1-z} + \frac{\ln(1-z)}{z}\right)dz.$$

By writing $1 - z == u$, we obtain

$$\int_0^{0.5}\frac{\ln z}{1-z}dz = \int_{0.5}^{1}\frac{\ln(1-u)}{u}du.$$

Then,

$$\mathrm{H}[X] = -\frac{4\sqrt{3}\sigma}{\pi}\int_0^{1}\frac{\ln(1-z)}{z}dz.$$

According to Eq (4.291.2) in Section 4.29 of reference [52], we have

$$\int_0^1\frac{\ln(1-z)}{z}dz = -\frac{\pi^2}{6}.$$

Then, the maximum entropy is

$$\mathrm{H}[X] = 2\sqrt{3}\pi\sigma/3.$$

The proof is complete.

Remark 4.1. *Although the current derivation relies heavily on a balanced configuration, this property is not an essential prerequisite for establishing the optimal entropy model. We introduced this constraint solely to simplify the proof. Nevertheless, the proposed analytical framework remains fully applicable for determining the optimal membership function, even under asymmetric conditions.*

5. Conclusions

Up to now, the entropy of uncertain sets has found numerous applications in areas such as uncertain finance, uncertain inference, and multivalued logic. It is meaningful to provide a MaxEnt principle for uncertain set in these areas. In this paper, by using the Euler equation in the calculus of variations, we obtained the membership function which achieves the maximum entropy for uncertain set. As far as we know, this is the first result on MaxEnt principle for the entropy of uncertain sets in uncertain theory. Our finding will provide considerable convenience for selecting suitable membership functions in practice.

Three promising directions for future research are:

(1) Investigate the MaxEnt principle of the sine entropy, quadratic entropy, and elliptic entropy for uncertain sets.
(2) Introduce the entropy into chance theory and propose the MaxEnt principle of uncertain random variables.
(3) To investigate the maximum entropy principle for the logarithmic entropy of uncertain sets without the symmetry assumption of the membership function.

Author contributions

Chenyan Liu: Conceptualization, Methodology, Validation, Writing–original draft preparation; Guanzhong Ma: Conceptualization, Writing–review and editing, Supervision. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this paper.

Acknowledgments

The work was supported by Natural Science Foundation of Henan (Grant No. 252300420312) and Postgraduate Education Reform and Quality Improvement Project of Henan Province (Grant No. YJS2024JC36).

Conflict of interest

The authors declare no conflicts of interest in this paper.

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Appendix

For the convenience of readers, we provide some essential facts on the Euler equation in the calculus of variations. For detailed content, readers may refer to the monograph by Gelfand et al. [53].

(1) Euler equation

For a function $F(x, y, z)$ having continuous first and second partial derivatives, the basic variational task seeks to extremize the functional

$$J[y] = \int_a^b F(x, y, y')\, dx$$

among all continuously differentiable functions $y(x)$ on [a, b] that satisfy

$$y(a) = A, \qquad y(b) = B.$$

The following theorem provides a necessary condition for the extremum in the above problem.

Theorem A. *[53] Let $J[y]$ be the following functional:*

$$J[y] = \int_a^b F(x, y, y')\, dx,$$

defined on the set of functions $y(x)$ that are continuously differentiable on $[a, b]$ and satisfy the boundary conditions $y(a) = A$, $y(b) = B$. A necessary condition for $J[y]$ to have an extremum at a given function $y(x)$ is that $y(x)$ satisfies Euler's equation

$$F_y - \frac{d}{dx} F_{y'} = 0. \tag{A.1}$$

In the calculus of variations, the Euler equation (A.1), which plays a fundamental role, is generally a second-order differential equation. However, under specific conditions, it can be reduced to a first-order equation. For example, when F is independent of y', Euler's equation (A.1) simplifies to

$$F_y(x, y) = 0.$$

(2) Constrained minima/maxima

Many variational problems involve constraints beyond boundary conditions. A typical goal is to determine $y = y(x)$ that extremizes the functional

$$J[y] = \int_a^b F(x, y, y')\, dx$$

subject to boundary conditions

$$y(a) = A, \qquad y(b) = B,$$

and the constraint that

$$K[y] = \int_a^b G(x, y, y')\, dx$$

takes a fixed value l, where the functions F and G have continuous first and second derivatives. Gelfand et al. [53] provides a solution to this type of problem.

Theorem B. *[53] Given the functional*

$$J[y] = \int_a^b F(x, y, y')\, dx,$$

consider suitable curves satisfying

$$y(a) = A, \qquad y(b) = B,$$

and the constraint

$$K[y] = \int_a^b G(x, y, y')\, dx = l.$$

Assume $J[y]$ *attains an extremum at* $y = y(x)$. *If* $y = y(x)$ *is not an extremal of* $K[y]$, *then there is a constant* λ *for which* $y = y(x)$ *extremizes the functional*

$$\int_a^b (F + \lambda G)\, dx.$$

Then, $y = y(x)$ *fulfills the differential equation*

$$F_y - \frac{d}{dx}F_{y'} + \lambda(G_y - \frac{d}{dx}G_{y'}) = 0. \tag{A.2}$$

The above method is called the Lagrange multiplier method, and the constant λ is called the Lagrangian multiplier. As in the previous section, if F does not depend on y', Euler's equation (A.2) becomes a first-order differential equation. In this case, Euler's equation (A.2) is given by

$$F_y(x, y) + \lambda G_y(x, y) = 0.$$

Although the Euler equation is necessary but insufficient for an extremum. However, the existence of an extremum can frequently be inferred from the problem itself. If the Euler equation yields a unique solution under these circumstances, that solution can be confidently identified as the curve yielding the extremum.

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