AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(4): 9260–9283.
DOI: 10.3934/math.2026382
Received: 13 January 2026
Revised: 06 March 2026
Accepted: 19 March 2026
Published: 03 April 2026

Research article

Maximum strong diameter of the strong product of complete multipartite graph and path

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Abstract: Strong product graphs are well-suited for modeling interconnected networks in parallel computing systems. In such networks, the strong diameter is defined as the maximum strong distance between any two vertices, which serves as a key measure of transmission efficiency. A smaller strong diameter corresponds to higher efficiency and lower latency. Optimizing this parameter can significantly enhance information transmission speed. In this paper, we form the strong product network $K_{m_1,m_2,\dots,m_k} \otimes P_n$ by taking the complete multipartite graph $K_{m_1,m_2,\dots,m_k}|\{m_i \geq 1, i = 1,2,\dots,k\}$ and the path P_n as the subgraphs. On this basis, we summarize and apply different strong orientation methods to investigate its maximum strong diameter. Specifically, we investigate the maximum strong diameter of $K_{m_1,m_2,\dots,m_k} \otimes P_n$, establishing its exact value and bounds for the cases in which $K_{m_1,m_2,\dots,m_k}|\{m_i \geq 1, i = 1,2,\dots,k\}$ does or does not admit a Hamiltonian cycle. In addition, a new algorithm is proposed to find the maximum strong diameter of $K_{m_1,m_2,\dots,m_k} \otimes P_n$. Through simulation experiments, we find that the high-dimensional strong product network $K_{m_1,m_2,\dots,m_k} \otimes P_n$ demonstrates superior information transfer efficiency when the underlying graph $K_{m_1,m_2,\dots,m_k}|\{m_i \geq 1, i = 1,2,\dots,k\}$ lacks a Hamiltonian cycle.

Keywords: strong product graph; maximum strong diameter; complete multipartite graph; path
Mathematics Subject Classification: 05C12, 05C69, 05C76

1. Introduction

The design of efficient interconnection networks relies on both unoriented and oriented graph models. While unoriented graphs capture topological connectivity, oriented graphs, particularly strongly oriented ones, are essential for modeling directional information flow. In such networks, the strong diameter measures the worst-case directed distance between vertices and directly impacts transmission latency and efficiency. Thus, understanding and optimizing the strong diameter is key to improving real-world network performance, motivating its theoretical and applied study.

The graph G can be seen as a binary group $(V(G), E(G))$, where $V(G)$ is the set of vertices, and $E(G)$ is the set of edges. To convert the graph G into a simple graph, we eliminate all loops and ensure that each pair of vertices is connected by only one edge. A trail of different vertices is called a path, the path of length $n-1$ is called P_n.

A complete multipartite graph is denoted by $K_{m_1,m_2,\ldots,m_k}|\{m_i \geq 1, i = 1, 2, \ldots, k\}$, such that every two vertices from distinct classes are adjacent.

Given a simple unoriented graph G, the eccentricity of a vertex u, denoted by $e(u)$, is the maximum distance from u to any other vertex in G. The diameter of G, denoted by $diam(G)$, is the maximum eccentricity among all the vertices.

After orienting all edges in G, a oriented graph D is obtained. An orientation D of G is called a strong orientation if D is strongly connected; that is, for every pair of vertices u and v in D, there is an oriented path from u to v and an oriented path from v to u. Meanwhile, we need to understand some basic definitions.

Chartrand et al. [1] introduced the definition of strong distance. While in a strongly connected oriented graph D, the strong distance between vertices u and v is defined as the sum of the lengths of the minimum oriented path from u and v and the minimum oriented path from v to u, the formula is

$$sd(u, v) = d(u, v) + d(v, u), \tag{1.1}$$

where $d(u, v)$ denotes the value of arcs in the minimum oriented path from u to v, and $d(v, u)$ denotes the value of arcs in the minimum oriented path from v to u. As shown in Figure 1, the strong distance from u to the other six vertices is as follows:

$$\begin{aligned} &sd(u, x) = 3, sd(u, \psi) = 3, sd(u, \theta) = 3; \\ &sd(u, w) = 3, sd(u, y) = 4, sd(u, v) = 4. \end{aligned} \tag{1.2}$$

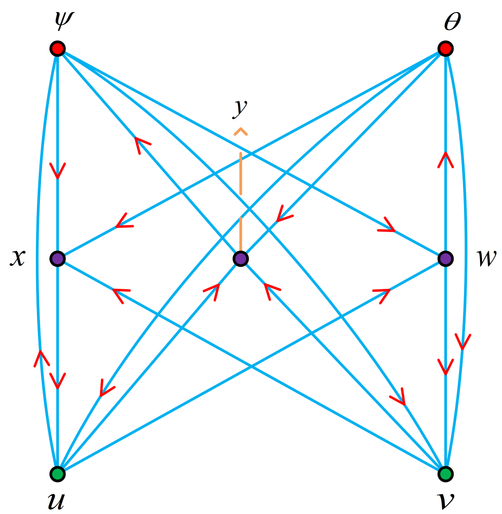

Figure 1. Strong product oriented graph.

The strong eccentricity of a vertex u is the maximum strong distance from u to all other vertices in D, and the formula is

$$se(u) = max\{sd(u, v)|v \in V(D)\}. \tag{1.3}$$

The minimum strong diameter of D is the maximum strong eccentricity over all vertices in D, and the formula is

$$sdiam(D) = max\{se(u)|u \in V(D)\}. \tag{1.4}$$

Following their definitions of strong eccentricity and the strong diameter of an oriented graph, Balakrishnan [2] introduced, for a given unoriented graph G, its minimum strong diameter and maximum strong diameter. These are, respectively, the smallest and largest possible strong diameter achievable by a strong orientation of G. The formulas are

$$\begin{aligned} sdiam(G) &= min\{sdiam(D)|D \in \mathbf{D}(G)\}; \\ SDIAM(G) &= max\{sdiam(D)|D \in \mathbf{D}(G)\}, \end{aligned} \tag{1.5}$$

where $\mathbf{D}(G)$ denotes the set of all strong orientations of G.

The strong product of simple graphs makes it easy to build large-scale graph networks and facilitates the intuitive discovery of their information transfer ability. Here is a formal definition of the strong product: Assume two unoriented simple graph networks $G_1 = (V(G_1), E(G_1))$, and $G_2 = (V(G_2), E(G_2))$. The network vertex set of $G_1 \otimes G_2$ is denoted by $V(G_1 \otimes G_2) = V(G_1) \times V(G_2)$, and two vertices (u_1, v_1), (u_2, v_2) are adjacent in the strong network of $G_1 \otimes G_2$ only if (1) $u_1 = u_2$ and $(v_1, v_2) \in E(G_2)$; (2) $v_1 = v_2$ and $(u_1, u_2) \in E(G_1)$; (3) $(u_1, u_2) \in E(G_1)$ and $(v_1, v_2) \in E(G_2)$. In order to better understand the method of constructing a strong product network, as shown in Figure 2, the strong product is performed using $K_{2,1,2}$ and P_3 as an example. More properties of other graph products can be found in references [3, 4].

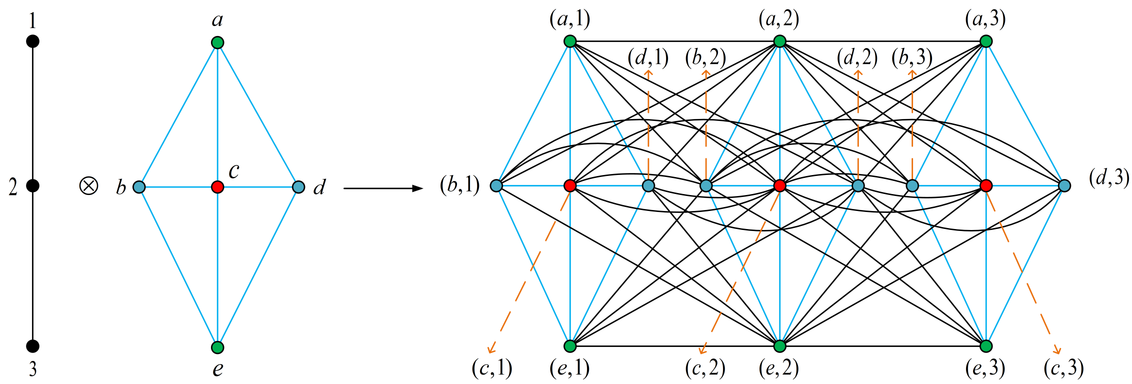

Figure 2. Strong product graph $K_{2,1,2} \otimes P_3$.

The problem of determining the strong diameter of a graph is known to be computationally hard. Hence, bounds on the strong diameter in terms of other graph parameters have been studied in the literature, for example, the fault diameter [5], the oriented diameter [6–9], and the diameter [10–12]. In order to better use strong diameter to evaluate the superiority of network information transmission. Many related superior results on strong diameter have been explored and obtained. Zhou [13] constructed a strong product graph network by two different cycles, and proposed various strong orientation methods under different situations, and obtained the upper and lower bounds on the maximum and minimum strong diameter of the strong product network of two cycles. Haller et al. [14] investigated the properties of Banach space for a given finite value of unit spheres and proved symmetric strong diameter under weak conditions under the weak star version. López-Pérez et al. [15] evaluated the case when the convex combinations of slices of the unit ball are relatively open or have non-empty relative interior for different topologies, obtaining the properties of L_∞-space and L_1-space.

Before obtaining the strong diameter, it is of utmost importance to strongly orient all the edges in the strong product network. Dankelmann et al. [16] proposed the minimum oriented diameter among all strong orientations of G and obtained bounds on the oriented diameter of the complement of a graph G. Špacapan in [17] considered the orientation of the cartesian product of graphs with bridges and gave an upper bound on the minimum diameter of such a orientation. Aksoy et al. [18] established a mild condition under which a possibly irregular sparse graph G has many strong orientations and proved that if G satisfies the minimality condition and the value of Cheeger's constant is to be minimal, the resulting randomly oriented graph is strongly connected with high probability. More researches on orientation can been found in [19–22].

More researchers are now applying the strong product of graphs to model real-world networks, particularly in evaluating information transfer efficiency. In this context, network models based on the strong product of two specific graph classes have attracted considerable attention. The complete multipartite graph is a well-known interconnection network model due to its structural generality and widespread use. The path graph, on the other hand, represents simple and unidirectional information transmission, making it easy to implement and analyze. Determining the strong diameter of general graphs is computationally difficult; in fact, computing this parameter is known to be NP-hard. To obtain exact results, we therefore focus on these two special classes of graphs.

In the first part of this paper, we investigate the relationships among several graph parameters: the minimum and maximum strong diameters, the diameter, the number of vertices, and the strong distance between arbitrary vertex pairs in the strong product graph $G = K_{m_1,m_2,\ldots,m_k} \otimes P_n$. Based on these relationships, we propose many distinct strong orientation rules. Utilizing these rules, we derive exact values and tight upper and lower bounds for the maximum strong diameter of $G = K_{m_1,m_2,\ldots,m_k} \otimes P_n$. All the bounds that we establish are sharp.

In the second part, we present a new algorithm and compare the maximum strong diameter of $G = K_{m_1,m_2,\ldots,m_k} \otimes P_n$ in two cases: when the base complete multipartite graph $K_{m_1,m_2,\ldots,m_k}|\{m_i \geq 1, i = 1, 2, \ldots, k\}$ contains a Hamiltonian cycle, and when it does not. Our analysis demonstrates that when $K_{m_1,m_2,\ldots,m_k}|\{m_i \geq 1, i = 1, 2, \ldots, k\}$ is non-Hamiltonian, the resulting product network achieves a smaller maximum strong diameter. This implies that such networks possess superior information transmission efficiency in terms of the worst-case oriented distances.

The organizational structure of this paper is as follows: Section 2 provides preliminaries that lay the foundation for investigating the maximum strong diameter of the strong product of a complete multipartite graph and a path. Section 3 delves into the maximum strong diameter of the strong product of a complete multipartite graph and a path. In Section 4, we present a new algorithm, application examples, and an analysis of the maximum strong diameter of this product under two different conditions. Section 5 summarizes the full paper.

2. Preliminaries

Before proving the main theorem of this paper, we first present several lemmas that will be useful for analyzing the maximum strong diameter of the strong product network $K_{m_1,m_2,\ldots,m_k} \otimes P_n$.

Lemma 2.1. *[23] For any strongly connected oriented graph D and any of its vertices u, the following inequality holds:*

$$srad(D) \leq se(u) \leq sdiam(D) \leq SDIAM(D) \leq 2srad(D). \tag{2.1}$$

$srad(D)$ denotes the strong radius of a strong orientation D. This inequality reveals a direct relationship between the maximum and minimum strong diameters, the strong eccentricity, and the strong radius.

Lemma 2.2. *[24] For any unoriented graph G, the following inequality holds:*

$$2diam(G) \leq sdiam(G) \leq SDIAM(G). \tag{2.2}$$

The work of Boesch et al. [24] establishes relationships between different diameters for mixed multigraphs. Combining their results with Lemma 2.1, which gives bounds on the maximum strong diameter, we can effectively analyze the maximum strong diameter of the specific strong product network $K_{m_1,m_2,\ldots,m_k} \otimes P_n$.

Lemma 2.3. *[25] Let G be a graph that contains a Hamiltonian path, and the following inequality holds:*

$$|V(G)| \leq SDIAM(G). \tag{2.3}$$

Jaradat [25] investigated properties of product graphs, including results that relate the number of vertices and the maximum strong diameter, particularly when a Hamiltonian cycle is present. This foundational work is essential for our analysis of the maximum strong diameter of $K_{m_1,m_2,\ldots,m_k} \otimes P_n$, in cases where $K_{m_1,m_2,\ldots,m_k}$ may or may not contain a Hamiltonian cycle.

Lemma 2.4. *[26] Let $K_{m_1,m_2,\ldots,m_k}$ be a complete multipartite graph with $k \geq 3$ and $1 \leq m_1 \leq \ldots \leq m_k$. Define $m = \sum_{i=1}^{k-1} m_i$, then the maximum strong diameter of $K_{m_1,m_2,\ldots,m_k}$ is*

$$\text{SDIAM}\left(K_{m_1,m_2,\ldots,m_k}\right) = \begin{cases} 2m+2, & \text{if } m < m_k; \\ m+m_k+1, & \text{if } m \geq m_k. \end{cases} \tag{2.4}$$

Miao [26] discussed the maximum strong diameter of $K_{m_1,m_2,\ldots,m_k}$ under two different cases, conducting in-depth research on the diameter-related results of complete multipartite graphs.

Lemma 2.5. *[27] If G_1 and G_2 are any two unoriented simple graphs, and (u_1, v_1) and (u_2, v_2) are any two vertices in $G_1 \otimes G_2$, then the distance between (u_1, v_1) and (u_2, v_2) is*

$$d_{G_1 \otimes G_2}((u_1, v_1), (u_2, v_2)) = max\,\{d_{G_1}(u_1, u_2), d_{G_2}(v_1, v_2)\}. \tag{2.5}$$

Based on the above lemmas, a key question arises: Under what conditions does a graph G admit a strong orientation? The answer is provided by the following theorem.

Theorem 2.1. *Let F_1 and F_2 be any non-trivial connected simple graphs of order m and n, respectively. Then, the strong product $F_1 \otimes F_2$ admits a strong orientation.*

Proof. Let $G = F_1 \otimes F_2$. Since F_1 and F_2 are connected, according to the definition of strong product, G is also connected. We show that G has no bridge, and every edge lies on a cycle. Then, by Robbins' theorem, G admits a strong orientation.

Consider any edge $e = \{(u, v), (u', v')\} \in G$, and three cases exist:

Case 1: $u = u'$ (horizontal edge). Then, v and v' are adjacent in F_2. Because F_1 is connected and non-trivial, u has w in F_1 (different from u). The following path connects (u, v) and (u, v') without using e:

$$(u, v) - (w, v) - (w, v') - (u, v'), \tag{2.6}$$

where the first and last edges are vertical (since $u \sim w$), and the middle edge is horizontal (since $v \sim v'$). Thus, e is not a bridge.

Case 2: $v = v'$ (vertical edge). Symmetrically, let z be a neighbor of v in F_2. Then we have the path:

$$(u, v) - (u, z) - (u', z) - (u', v), \tag{2.7}$$

where the first and last edges are horizontal and the middle edge is vertical. Thus, e is not a bridge.

Case 3: $u \neq u'$ and $v \neq v'$ (diagonal edge). Then $u \sim u'$ in F_1 and $v \sim v'$ in F_2. Consider the path:

$$(u, v) - (u, v') - (u', v'), \tag{2.8}$$

which uses the horizontal edge $(u, v) - (u, v')$ and the vertical edge $(u, v') - (u', v')$. Both exist under adjacency conditions and are distinct from e. Thus, e is not a bridge.

In all cases, e is not a bridge, so G is 2-edge-connected. According to Robbins' theorem, G has a strong orientation, and the proof is complete.

The following section is dedicated to an investigation of the maximum strong diameter of a new strong orientation on the strong product graph $G = K_{m_1,m_2,\ldots,m_k} \otimes P_n$.

When $m_1 = m_2 = \ldots = m_k = 1$, a complete multipartite graph $K_{m_1,m2,\ldots,m_k}$ is a complete graph K_k, and then the maximum strong diameter of $Km_1, m_2 \ldots, m_k \otimes P_n$ is equivalent to the maximum strong diameter of a complete graph K_k and a path P_n. When $m_2 = m_3 = \ldots = m_k = 0$, the maximum strong diameter of $K_{m_1,m_2\ldots,m_k} \otimes P_n$ is equivalent to investigating the maximum strong diameter of the path P_n, and we can obtain $SDIAM(P_n) = n - 1$. The following corollary will be given to investigate the maximum strong diameter of $SDIAM(K_k \otimes P_n)$.

Corollary 2.1. *Let K_k be a complete graph, P_n be a path, k and n are both any real numbers; then, the maximum strong diameter of $G = K_k \otimes P_n$ is*

$$SDIAM(K_k \otimes P_n) = 2(n - 1). \tag{2.9}$$

Proof. Using the relationship between k and n, we can easily construct the strong product graph $G = K_k \otimes P_n$.

According to Lemma 2.2, for $G = K_k \otimes P_n$, there exists

$$\begin{aligned} &diam(K_k \otimes P_n) = max\{diam(K_k), diam(P_n)\};\\ &2diam(K_k \otimes P_n) \leq sdiam(K_k \otimes P_n) \leq SDIAM(K_k \otimes P_n). \end{aligned} \tag{2.10}$$

We can easily obtain that $diam(K_k) = 1, diam(P_n) = n - 1, diam(K_k \otimes P_n) = n - 1$.

Let D_1 be the minimum strong oriented graph of $G = K_k \otimes P_n$. Since the set of vertices of $G = K_k \otimes P_n$ are symmetric, the set of vertices of D_1 is

$$\begin{aligned} DM_1 &= \{(u, v) | 1 \leq u \leq k, 1 \leq v \leq \lfloor n/2 \rfloor\};\\ DM_2 &= \{(u, v) | 1 \leq u \leq k, \lfloor n/2 \rfloor \leq v \leq n\}. \end{aligned} \tag{2.11}$$

The edges of DM_1 and DM_2 are oriented according to the following rules for an arbitrary vertex (u, v):

- For $1 < u \leq k$, and $1 \leq v \leq n$, orient $(u, v) \to (u-1, v)$.
- For $1 \leq u \leq k$, and $1 \leq v \leq \lfloor n/2 \rfloor$, orient $(u, v) \to (u, v+1)$.
- For $1 \leq u \leq k$, and $\lfloor n/2 \rfloor \leq v \leq n$, orient $(u, v) \to (u, v-1)$.
- For $1 \leq u \leq \lfloor k/2 \rfloor$, and $1 \leq v \leq n$, orient $(u, v) \to (\lfloor k/2 \rfloor, \lfloor n/2 \rfloor)$.
- For $\lfloor k/2 \rfloor \leq u \leq k$, and $1 \leq v \leq n$, orient $(\lfloor k/2 \rfloor, \lfloor n/2 \rfloor) \to (u, v)$.
- For $u \equiv 0 (mod\ 2)$, and $v \equiv 1 (mod\ 2)$, orient $(u, v) \to (u-1, v+2)$.
- For $u \equiv 1 (mod\ 2)$, and $v \equiv 0 (mod\ 2), v \neq 2$, orient $(u, v) \to (u+1, v-2)$.

After completing the strong orientation of D_1, it can be clearly observed that the maximum strong distance between any two vertices in D_1 is $2(n-1)$. To better understand the strong orientation of D_1, as shown in Figure 3, we use $K_3 \otimes P_3$ as an example, and the maximum strong distance for any two vertices is 4. Meanwhile, through the example diagram, it can be clearly observed that any two vertices are strongly connected. The proof is complete.

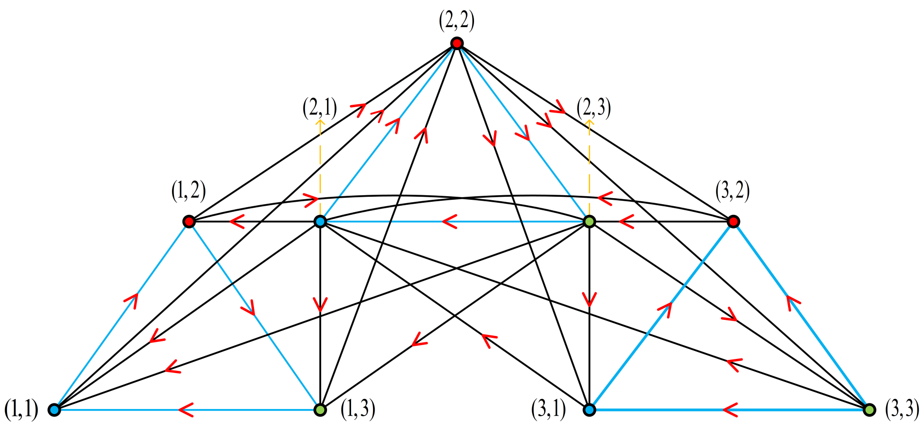

Figure 3. Strong product oriented graph of $K_3 \otimes P_3$.

In summary, this corollary considers a strong product graph $G = K_{m_1,m_2,\ldots,m_k} \otimes P_n$ when the complete multipartite graph is in a special case. In the following sections, we will provide some general cases.

3. Main results of maximum strong diameter

In Section 2, we introduced the relevant preliminary knowledge about exploring the maximum strong diameter, which helps us to actively investigate the advantages of strong product network in information transmission ability. In this section, we will specifically research the maximum strong diameter of strong product of $K_{m_1,m_2,\ldots,m_k}$ and P_n in two cases.

Theorem 3.1. *Let $K_{m_1,m_2,\ldots,m_k}$ be a complete multipartite graph with $\{m_i \geq 1 | i = 1, 2, \ldots, k\}$ and $k \geq 3$, and P_n be a path with $n \geq 2$. Suppose there exists Hamiltonian cycle in $K_{m_1,m_2,\ldots,m_k}$ and $m = \sum_{i=1}^{k} m_i$, then, the maximum strong diameter of $G = K_{m_1,m_1,\ldots,m_k} \otimes P_n$ is*

$$SDIAM(K_{m_1,m_2,\ldots,m_k} \otimes P_n) = mn. \tag{3.1}$$

Proof. According to the parity of the path, since $K_{m_1,m_2,\ldots,m_k} \otimes P_n \cong P_n \otimes K_{m_1,m_2,\ldots,m_k}$, without loss of generality, we will only discuss the cases of when $n \equiv 1\ (mod\ 2)$ and $n \equiv 0\ (mod\ 2)$.

Case 1. $n \equiv 1\ (mod\ 2)$. Let D_2 be the minimum strong oriented graph of $G = K_{m_1,m_2,\ldots,m_k} \otimes P_n$. So, the vertex set of D_2 can be represented as

$$V(D_2) = \{(u, v) | 1 \le u \le m, 1 \le v \le n\}. \tag{3.2}$$

Next, we will demonstrate Case 1 in 5 steps.

Step 1.1. Preliminary results.

When $K_{m_1,m_2,\ldots,m_k}$ contains Hamiltonian cycle, then for each part V_i of value m_i $(i = 1, 2, \ldots, k)$, the following condition satisfies:

$$|m_i| \le \sum_{j \ne i} |m_j|\ (i = 1, 2, \ldots, k). \tag{3.3}$$

According to Lemma 2.3, the maximum diameter of $G = K_{m_1,m_2,\ldots,m_k} \otimes P_n$ satisfies

$$mn = V|(K_{m_1,m_2,\ldots,m_k})| \times V|(P_n)| \le SDIAM(K_{m_1,m_2,\ldots,m_k} \otimes P_n). \tag{3.4}$$

Furthermore, the diameter of the product graph relates to its factors as

$$diam(K_{m_1,m_2,\ldots,m_k} \otimes P_n) = max\ \{diam(K_{m_1,m_2,\ldots,m_k}), diam(P_n)\}. \tag{3.5}$$

According to Lemma 2.4, the strong eccentricity of any vertex $(u, v) \in D_2$ satisfies

$$V|(K_{m_1,m_2,\ldots,m_k})| \times V|(P_n)| - 1 \le se(u, v) \le mn. \tag{3.6}$$

Step 1.2. Strong orientation rules.

The edges of D_2 are oriented according to the following rules for an arbitrary vertex (u, v):

- For $u \equiv 1(mod\ 2)$, and $1 \le v \le \lfloor n/2 \rfloor$, orient $(u, v) \to (u + 2, v)$.
- For $u \equiv 1(mod\ 2)$, and $\lfloor n/2 \rfloor \le v \le n$, orient $(u, v) \to (u - 2, v)$.
- For $u \equiv 1(mod\ 2)$, and $1 < v \le n - 2$, orient $(u, v) \to (u + \lfloor m/2 \rfloor, v - 1)$.
- For $u \equiv 0(mod\ 2)$, and $1 < v \le n - 2$, orient $(u, v) \to (u + \lfloor 3m/2 \rfloor, v - 1)$.
- For $u \equiv 0(mod\ 2)$, $u \ne 2$, and $1 < v \le n$, orient $(u, v) \to (u + \lfloor 3m/2 \rfloor - 1, v)$.
- For $u \equiv 1(mod\ 2), u \ne \lfloor m/2 \rfloor$, and $1 \le v \le n$, orient $(u, v) \to (u + \lfloor m/2 \rfloor - 1, v)$.
- For $\lfloor m/2 \rfloor \le u \le m$, and $1 \le v \le n$, orient $(u, v) \to (u - 1, v) \to (u, v - 1) \to (u - 1, v - 1)$.
- For $1 \le u \le \lfloor m/2 \rfloor$, and $1 \le v \le n$, orient $(u, v) \to (u - 1, v + 1) \to (\lfloor 3m/2 \rfloor + u, v) \to (u, \lfloor n/2 \rfloor)$.
- For $\lfloor m/2 \rfloor < u \le m$, and $1 \le v \le n$, orient $(u, v) \to (u + 1, v - 1) \to (\lfloor m/2 \rfloor + u, v) \to (u, \lfloor 3n/2 \rfloor)$.
- For $1 \le u \le \lfloor m/2 \rfloor$, and $1 \le v \le n$, orient $(u, v) \to (u+1, v), (u, v) \to (u, v+1), (u, v) \to (u+1, v+1)$.

Step 1.3. Upper bound on strong eccentricity.

After performing strong orientation on the edges that meet the conditions according to the strong orientation rule in Step 1.2, for any vertex $(u, v) \in V(D_2)$, the strong eccentricity $se(u, v)$ is defined as

$$se(u, v) = \max\ \{sd((u, v), (u', v')) | (u', v') \in V(D_2)\}, \tag{3.7}$$

where $sd(u, v)$ denotes the length of a minimum oriented path from (u, v) to (u', v') in D_2.

Since any minimum oriented path between two distinct vertices contains no repeated vertices, its length cannot exceed the total number of vertices, and we can obtain

$$se(u, v) \leq mn - 1 < mn. \tag{3.8}$$

In particular, the inequality $se(u, v) \leq mn$ holds trivially for every vertex $(u, v)|(u, v) \in V(D_2)$.

Step 1.4. Lower bound and tightness on strong distance.

We now show that the bound of the strong distance in $K_{m_1,m_2,...,m_k} \otimes P_n$ is tight. Consider two vertices (u_1, v_1) and (u_2, v_2) chosen as follows:

(a) u_1 and u_2 are two distinct vertices of $K_{m_1,m_2,...,m_k}$ lying in the same partite set. This means that in the original graph, there is no edge between u_1 and u_2.
(b) v_1 and v_2 are the two endpoints of the path P_n. Hence, $d_{P_n}(v_1, v_2) = n - 1$.

According to Lemma 2.5, in the underlying unoriented graph of $K_{m_1,m_2,...,m_k} \otimes P_n$, the distance between the vertices (u_1, v_1) and (u_2, v_2) is

$$d_{K_{m_1,m_2,...,m_k} \otimes P_n}((u_1, v_1), (u_2, v_2)) = max\{d_{K_{m_1,m_2,...,m_k}}(u_1, u_2), d_{P_n}(v_1, v_2)\} = max\{2, n - 1\}. \tag{3.9}$$

Now, consider the digraph D_2 obtained after applying the strong orientation rules in Step 1.2. Our orientation ensures that there is no directed path longer than $mn - 1$. To prove that $mn - 1$ is tight, we need to argue that any directed path from (u_1, v_1) to (u_2, v_2) has length at least $mn - 1$.

The key observation is: To go from (u_1, v_1) to (u_2, v_2), any oriented path must pass through a certain set of "intermediate" states. Specifically, because u_1 and u_2 belong to the same partite set and are nonadjacent, the path must go through an intermediate vertex (u', v'), where u' is adjacent to both u_1 and u_2. Moreover, since v_1 and v_2 are the endpoints of P_n, the path must traverse all intermediate vertices of P_n along the way.

Combining these with the orientation rules, one can show that any strong directed path from (u_1, v_1) to (u_2, v_2) must visit all vertices of the form (u, v), where u ranges over all vertices not in the same partite set as u_1 (or equivalently, over all vertices that are adjacent to both u_1 and u_2), and v ranges over all vertices of P_n. Consequently, the length of such a path is at least the total number of vertices, and we have

$$sd_{D_2}((u_1, v_1), (u_2, v_2)) \geq mn - 1. \tag{3.10}$$

Combined with the upper bound $se(u, v) \leq mn - 1$ proved in Step 1.3, we conclude that for every vertex (u, v), the strong eccentricity attains this bound

$$se(u, v) = mn - 1| \,\forall (u, v) \in V(D_2). \tag{3.11}$$

Therefore, based on the above analysis, for any vertex $(u, v)|(u, v) \in V(D_2)$, the strong distance from (u, v) back to itself satisfies

$$sd_{D_2}((u, v), (u, v)) = mn. \tag{3.12}$$

Since D_2 is the minimum strong orientation of $K_{m_1,m_2,...,m_k} \otimes P_n$, the minimum strong diameter over all strong orientations is at most mn. But since we have exhibited an orientation with strong diameter exactly mn, it follows that

$$SDIAM(K_{m_1,m_2,...,m_k} \otimes P_n) = mn. \tag{3.13}$$

Step 1.5. Example.

As shown in Figure 4, the strong product graph $K_{m_1,m_2,\ldots,m_k} \otimes P_n$ illustrates this result under the assumption that $K_{m_1,m_2,\ldots,m_k}$ contains a Hamiltonian cycle. The red arrow represents the edge orientation that satisfies the orientation rules in Step 1.2 for all unoriented edges. Here, m_1 to m_k, and n are real numbers. Applying the orientation rules above yields a strong orientation where the maximum strong distance between any two vertices is mn, and every vertex has a strong eccentricity $mn-1$. Meanwhile, through the example diagram, it can be clearly observed that any two vertices are strongly connected.

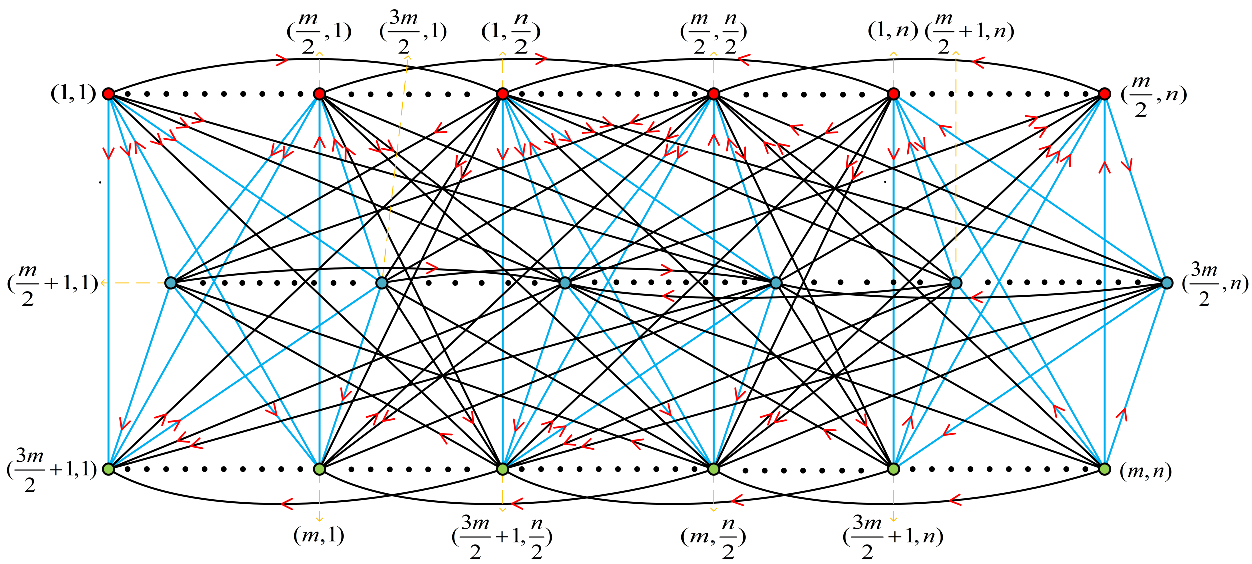

Figure 4. Strong product oriented graph $K_{m_1,m_2,\ldots,m_k} \otimes P_n$.

Case 2. $n \equiv 0$ $(mod\ 2)$. Let D_3 be the minimum strong oriented graph of $G = K_{m_1,m_2,\ldots,m_k} \otimes P_n$, and the vertex set classification of D_3 is similar to that of Case 1. Demonstrate Case 2 in 2 steps.

Step 2.1. Preliminary results and strong orientation rules.

The edges of D_3 are oriented according to the following rules for an arbitrary vertex (u, v):

- For $u \equiv 0(mod\ 2)$, and $1 \le v \le n/2$, orient $(u, v) \to (u + 2, v)$.
- For $u \equiv 0(mod\ 2), u \ne 2$, and $1 \le v \le n/2$, orient $(u, v) \to (u - 2, v)$.
- For $u \equiv 1(mod\ 2)$, and $1 < v \le n - 1$, orient $(u, v) \to (u + \lfloor m/2 \rfloor, v + 1)$.
- For $u \equiv 0(mod\ 2)$, and $1 < v \le n - 1$, orient $(u, v) \to (u + \lfloor 3m/2 \rfloor, v + 1)$.
- For $u \equiv 0(mod\ 2), u \ne 2$, and $1 < v \le n$, orient $(u, v) \to (u + \lfloor m/2 \rfloor + 1, v)$.
- For $u \equiv 1(mod\ 2), u \ne 1$, and $1 < v \le n$, orient $(u, v) \to (u + \lfloor 3m/2 \rfloor + 1, v)$.
- For $1 \le u \le \lfloor m/2 \rfloor$, and $1 \le v \le n$, orient $(u, v) \to (u - 1, v) \to (u, v - 1) \to (u - 1, v - 1)$.
- For $\lfloor m/2 \rfloor \le u \le m$, and $1 \le v \le n$, orient $(u, v) \to (u + 1, v) \to (u, v + 1) \to (u + 1, v + 1)$.
- For $1 \le u \le \lfloor m/2 \rfloor$, and $1 \le v \le n$, orient $(u, v) \to (u + 1, v - 1) \to (\lfloor m/2 \rfloor + u, v) \to (u, \lfloor 3n/2 \rfloor)$.
- For $\lfloor m/2 \rfloor < u \le m$, and $1 \le v \le n$, orient $(u, v) \to (u - 1, v + 1) \to (\lfloor 3m/2 \rfloor + u, v) \to (u, \lfloor n/2 \rfloor)$.

Step 2.2. Upper bound on strong eccentricity.

The proof process of the strong eccentricity of any vertex $(u, v) \in E(D_3)$ is similar to that of Case 1.

Step 2.3. Lower bound and tightness on strong distance.

The proof process of the strong distance of any two vertices is similar to that of Case 1.

Therefore, according to above analysis, we can easily obtain that the maximum strong diameter of $G = K_{m_1,m_2,\ldots,m_k} \otimes P_n$ is mn.

In summary, from Case 1 to Case 2, and formula (3.1) to formula (3.13) in Theorem 3.8, we have analyzed and discussed two different conditions of the maximum strong diameter of $K_{m_1,m_2,\ldots,m_k} \otimes P_n$ when complete multipartite graphs have a Hamiltonian cycle. In the following chapter content, we will continue to investigate the maximum strong diameter when there is no Hamiltonian cycle in $K_{m_1,m_2,\ldots,m_k}$. The proof is complete.

Theorem 3.2. *Let $K_{m_1,m_2,\ldots,m_k}$ be a complete multipartite graph with $\{m_i \geq 1 | i = 1, 2, \ldots, k\}$ and $k \geq 3$, and P_n be a path with $1 < n \leq 5$. Suppose there is no Hamiltonian cycle in $K_{m_1,m_2,\ldots,m_k}$ and $m = \sum_{i=1}^{k} m_i$, then, the maximum strong diameter of $G = K_{m_1,m_1,\ldots,m_k} \otimes P_n$ is*

$$SDIAM(K_{m_1,m_2,\ldots,m_k} \otimes P_n) = 2(n-1). \tag{3.14}$$

Proof. According to the parity of the path, since $K_{m_1,m_2,\ldots,m_k} \otimes P_n \cong P_n \otimes K_{m_1,m_2,\ldots,m_k}$, without loss of generality, it is evident from Theorem 3.1 and the definition of the maximum strong diameter that the conclusion holds. Below, we will discuss when $n \equiv 1\ (mod\ 2)$ and $n \equiv 0\ (mod\ 2)$.

Case 1. $n \equiv 1\ (mod\ 2)$. Let D_4 be the minimum strong oriented graph of $G = K_{m_1,m_2,\ldots,m_k} \otimes P_n$. So, the vertex set of D_4 can be represented as

$$\begin{aligned} V(DE_1) &= \{(u,v) | 1 \leq u \leq \lfloor m/2 \rfloor, 1 \leq v \leq \lfloor n/2 \rfloor\}; \\ V(DE_2) &= \{(u,v) | \lfloor m/2 \rfloor \leq u \leq m, \lfloor n/2 \rfloor \leq v \leq n\}. \end{aligned} \tag{3.15}$$

Next, we will demonstrate Case 1 in 4 steps.

Step 1.1. Strong orientation rules and upper bound on strong eccentricity.

The edges of D_4 between DE_1 and DE_2 are oriented according to the following rules for an arbitrary vertex (u, v):

- For $1 \leq u \leq m$, and $v \equiv 0(mod\ 2)$, orient $(u,v) \to (u-1, v-1)$.
- For $1 \leq u \leq m$, and $v \equiv 1(mod\ 2)$, orient $(u,v) \to (u+1, v-1)$.
- For $1 \leq u \leq \lfloor m/2 \rfloor$, and $1 \leq v \leq \lfloor n/2 \rfloor$, orient $(u,v) \to (u+1, v)$.
- For $\lfloor m/2 \rfloor \leq u \leq m$, and $\lfloor n/2 \rfloor \leq v \leq n$, orient $(u,v) \to (u-1, v)$.
- For $u \equiv 1(mod\ 2), u \neq 1$, and $v \equiv 1(mod\ 2)$, orient $(u,v) \to (u-2, v)$.
- For $u \equiv 0(mod\ 2), u \neq 2$, and $v \equiv 1(mod\ 2)$, orient $(u,v) \to (u+2, v)$.
- For $1 \leq u \leq \lfloor m/2 \rfloor, u \equiv 1(mod\ 2)$, and $1 \leq v \leq n$, orient $(u,v) \to (u, v+1)$.
- For $\lfloor m/2 \rfloor \leq u \leq m, u \equiv 1(mod\ 2)$, and $1 \leq v \leq n$, orient $(u,v) \to (u, v-1)$.
- For $1 \leq u \leq \lfloor m/2 \rfloor, u \equiv 1(mod\ 2)$, and $1 \leq v \leq n$, orient $(m,v) \to (m, v-1), (m,v) \to (u,v)$.
- For $\lfloor m/2 \rfloor \leq u \leq m, u \equiv 1(mod\ 2)$, and $1 \leq v \leq n$, orient $(u,v) \to (m,v), (u,v) \to (m, v+1)$.

After completing the strong orientation for D_4, let J and $\overline{J}$ be two different sets of vertex $V(D_4)$, and suppose there are three different vertices $\{(u,v), (u_3,v_3), (u_4,v_4) | u \neq u_3 \neq u_4, v \neq v_3 \neq v_4\}$, expressed as

$$J = \{(u,v), (u_3,v_3)\}; \overline{J} = \{V(D_4) \backslash T\}. \tag{3.16}$$

For the set J, the maximum strong oriented distance from (u, v) to (u_4, v_4) satisfies

$$sd_{D_4}((u,v),(u_4,v_4)) > diam(K_{m_1,m_2,\ldots,m_k}) \times diam(P_n) - 1. \tag{3.17}$$

The maximum strong oriented loop distance from (u, v) back to itself satisfies

$$sd_{D_4}((u, v), (u, v)) \leq diam(K_{m_1,m_2,\ldots,m_k}) \times diam(P_n) = 2(n-1). \quad (3.18)$$

From the construction of D_4 and the strong distance between any of the vertices mentioned above, the strong eccentricity of any vertex (u, v) is at most $2(n-1)$:

$$sd(u, v) \leq 2(n-1)|(u, v) \in V(D_4). \quad (3.19)$$

Step 1.2. Lower bound and tightness on strong distance.

The proof process on strong distance is similar to Case 1 in Theorem 3.1.

Therefore, according to the above analysis, we can easily obtain that the maximum strong diameter of $K_{m_1,m_2,\ldots,m_k} \otimes P_n$ is $2(n-1)$.

Step 1.3. Example.

As shown in Figure 5, the graph $K_{1,1,3} \otimes P_3$ illustrates that the result under $K_{1,1,3}$ has a Hamiltonian cycle. The red arrow represents the edge orientation that satisfies the orientation rules in Step 1.2 for all unoriented edges in $K_{1,1,3} \otimes P_3$. Applying the orientation rules above yields a strong orientation, where the maximum strong distance between any two vertices is 4, and every vertex has a strong eccentricity 4. It can be clearly observed that any two vertices are strongly connected.

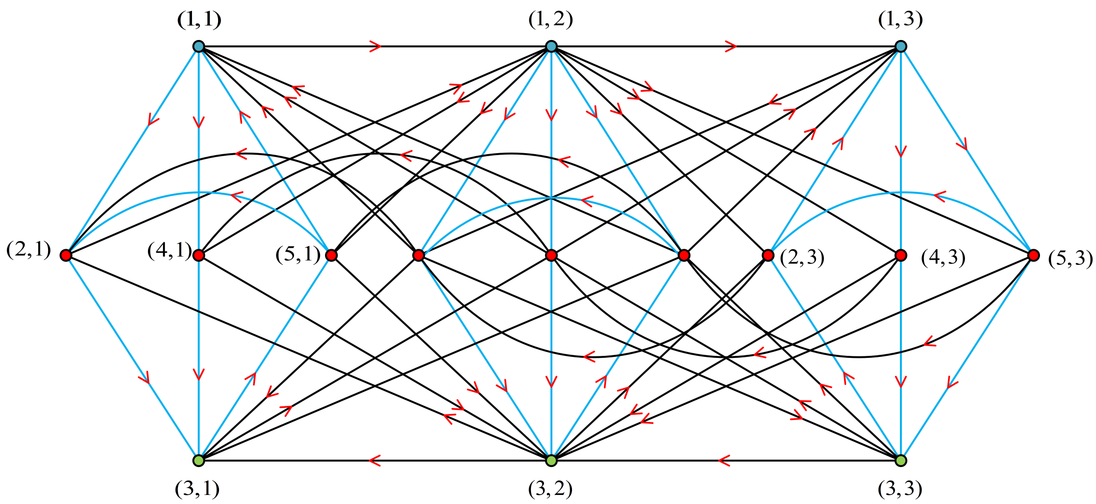

Figure 5. Strong product oriented graph $K_{1,1,3} \otimes P_3$.

Case 2. $n \equiv 0$ $(mod\ 2)$. Let D_5 be the minimum strong oriented graph of $G = K_{m_1,m_2,\ldots,m_k} \otimes P_n$. The vertex set of D_5 is similar to Case 1.

The edges of D_5 are oriented according to the following rules for an arbitrary vertex (u, v):

- For $1 < u \leq \lfloor m/2 \rfloor$, and $1 \leq v \leq n/2$, orient $(u, v) \rightarrow (u-1, v)$.
- For $\lfloor m/2 \rfloor \leq u \leq m$, and $n/2 \leq v \leq n$, orient $(u, v) \rightarrow (u+1, v)$.
- For $1 \leq u \leq m$, and $v \equiv 1(mod\ 2)$, orient $(u, v) \rightarrow (u-1, v-1)$.
- For $1 \leq u \leq m$, and $v \equiv 0(mod\ 2)$, orient $(u, v) \rightarrow (u+1, v-1)$.
- For $1 \leq u \leq \lfloor m/2 \rfloor, u \equiv 0(mod\ 2)$, and $1 \leq v \leq n$, orient $(u, v) \rightarrow (u, v+1)$.
- For $\lfloor m/2 \rfloor \leq u \leq m, u \equiv 0(mod\ 2)$, and $1 \leq v \leq n$, orient $(u, v) \rightarrow (u, v-1)$.

- For $u \equiv 0(mod\ 2), u \neq 2$, and $v \equiv 1(mod\ 2), v \neq 1$, orient $(u, v) \to (u + 2, v)$.
- For $u \equiv 1(mod\ 2), u \neq 1$, and $v \equiv 0(mod\ 2), v \neq 2$, orient $(u, v) \to (u - 2, v)$.
- For $\lfloor m/2 \rfloor \leq u \leq m, u \equiv 0(mod\ 2)$, and $1 \leq v \leq n$, orient $(u, v) \to (m, v), (u, v) \to (m, v + 1)$.
- For $1 \leq u \leq \lfloor m/2 \rfloor, u \equiv 0(mod\ 2)$, and $1 \leq v \leq n$, orient $(m, v) \to (m, v - 1), (m, v) \to (u, v)$.

The proof processes on strong eccentricity and strong distance between any two vertices in D_5 are similar to Case 1.

In summary, from Case 1 to Case 2, and formula (3.14) to formula (3.19) in Theorem 3.9, we have analyzed and discussed two different conditions of the maximum strong diameter of $K_{m_1,m_2,\ldots,m_k} \otimes P_n$ when complete multipartite graphs have Hamiltonian cycle under $1 < n \leq 5$. In the following, we will continue to investigate the maximum strong diameter when there is no Hamiltonian cycle in $K_{m_1,m_2,\ldots,m_k}$ under the condition of $n > 5$. The proof is complete.

Theorem 3.3. *Let $K_{m_1,m_2,\ldots,m_k}$ be a complete multipartite graph with $\{m_i \geq 1 | i = 1, 2, \ldots, k\}$ and $k \geq 3$, and P_n be a path with $n > 5$. Suppose there is no Hamiltonian cycle in $K_{m_1,m_2,\ldots,m_k}$ and $m = \sum_{i=1}^{k} m_i$, then, the upper and lower maximum strong diameter of $G = K_{m_1,m_2,\ldots,m_k} \otimes P_n$ is*

$$2n \leq SDIAM(K_{m_1,m_2,\ldots,m_k} \otimes P_n) \leq mn. \tag{3.20}$$

Proof. According to the parity of the path, since $K_{m_1,m_2,\ldots,m_k} \otimes P_n \cong P_n \otimes K_{m_1,m_2,\ldots,m_k}$, without loss of generality, we will discuss the general condition.

Let D_6 be the minimum strong oriented graph of $K_{m_1,m_2,\ldots,m_k} \otimes P_n$. Demonstrate Theorem 3.3 in 4 steps.

Step 1. Strong orientation rules.

Since the set of vertices of $K_{m_1,m_2,\ldots,m_k} \otimes P_n$ are symmetric, the set of vertices of D_6 can be divided into two parts, respectively:

$$\begin{aligned} V(DO_1) &= \{(u, v) | 1 \leq u \leq m, 1 \leq v \leq \lfloor n/2 \rfloor\}; \\ V(DO_2) &= \{(u, v) | 1 \leq u \leq m, \lfloor n/2 \rfloor \leq v \leq n\}. \end{aligned} \tag{3.21}$$

The edges of D_6 are oriented according to the following rules for an arbitrary vertex (u, v) in $V(DO_1)$ and $V(DO_2)$:

- For $u \equiv 1(mod\ 2)$, and $v \equiv 0(mod\ 2)$, orient $(u, v) \to (u + \lfloor m/2 \rfloor + 1, v)$.
- For $u \equiv 0(mod\ 2)$, and $v \equiv 1(mod\ 2)$, orient $(u, v) \to (u + \lfloor 3m/2 \rfloor - 1, v)$.
- For $1 \leq u \leq \lfloor m/2 \rfloor$, and $1 \leq v \leq n$, orient $(u, v) \to (u + \lfloor 3m/2 \rfloor, v + \lfloor n/2 \rfloor)$.
- For $\lfloor m/2 \rfloor \leq u \leq m$, and $1 \leq v \leq n$, orient $(u, v) \to (u + \lfloor m/2 \rfloor, v + \lfloor 3n/2 \rfloor)$.
- For $m - 3 \leq u \leq m, u \equiv 1(mod\ 2)$, and $1 \leq v \leq n$, orient $(\lfloor u/2 \rfloor, \lfloor v/2 \rfloor) \to (u, v)$.
- For $m - 4 \leq u \leq m, u \equiv 0(mod\ 2)$, and $1 \leq v \leq n$, orient $(\lfloor 3u/2 \rfloor, \lfloor 3v/2 \rfloor) \to (u, v)$.
- For $1 < u \leq m$, and $1 < v \leq \lfloor n/2 \rfloor$, orient $(u + 1, v - 1) \to (u, v), (u, v) \to (u + 1, v)$.
- For $1 < u \leq m$, and $\lfloor n/2 \rfloor \leq v \leq n$, orient $(u, v) \to (u - 1, v), (u - 1, v + 1) \to (u, v)$.
- For $m - 4 \leq u \leq m, u \equiv 0(mod\ 2)$, and $1 \leq v \leq n$, orient $(u, v) \to (\lfloor m/2 \rfloor + 1, v - 1)$.
- For $m - 3 \leq u \leq m, u \equiv 1(mod\ 2)$, and $1 \leq v \leq n$, orient $(u, v) \to (\lfloor 3m/2 \rfloor - 1, v + 1)$.
- For $u \equiv 1(mod\ 2)$, and $v \equiv 1(mod\ 2)$, orient $(u, v) \to (u + 1, v + 1), (u, v) \to (u, v + 1)$.
- For $u \equiv 0(mod\ 2)$, and $v \equiv 0(mod\ 2)$, orient $(u, v) \to (u - 1, v - 1), (u, v) \to (u, v - 1)$.

Step 2. Prove that the lower bound is strict.

According to Lemma 2.2 and the definition of diameter, when $n > 5$, there exists

$$diam(K_{m_1,m_2,\dots,m_k} \otimes P_n) > diam(K_{m_1,m_2,\dots,m_k}) \times diam(P_n). \tag{3.22}$$

We can obtain $diam(K_{m_1,m_2,\dots,m_k}) \times diam(P_n) = 2(n-1)$ and the diameter of $K_{m_1,m_2,\dots,m_k} \otimes P_n$ depends on the length of the P_n and $K_{m_1,m_2,\dots,m_k}$.

The strong eccentricity for any vertex $(u, v) \in V(D_6)$ satisfies

$$se(u, v) > diam(K_{m_1,m_2,\dots,m_k}) \times diam(P_n). \tag{3.23}$$

Therefore, according to the above analysis and the definition of maximum diameter, it can be concluded that $SDIAM(K_{m_1,m_2,\dots,m_k} \otimes P_n) \geq 2n$ is strict.

Step 3. Prove that the upper bound is sharp.

After completing the strong orientation for D_6, let Q be a proper subset of $V(D_6)$, and denote $\overline{Q} = V(D_6)\backslash Q$. Let $P_3 = \{(u, v), \dots, (u_5, v_5), (u_6, v_6)|[u_5, u_6] \neq u, [v_6, v_6] \neq v\}$ be a maximum oriented path from (u, v) to $\overline{Q}$. It is clear that $(u_5, v_5) \in Q$ and $(u_6, v_6) \in \overline{Q}$, and the segment $\{(u, v), (u_6, v_6)\}$ of P_3 must be a maximum path.

According to the results proven in Step 2 and the definition of diameter, for any vertex $(u, v) \in V(D_6)$, the strong eccentricity satisfies

$$2n \leq se(u, v) \leq mn. \tag{3.24}$$

From the construction of D_6, the maximum strong orientation distance P_3 is satisfied:

$$sd(P_3) \leq se(u, v) \leq V|(K_{m_1,m_2,\dots,m_k})| \times V|P(n)| = mn. \tag{3.25}$$

Thus, through the above analysis, it can be obtained that in D_6, for any vertex $(u, v) \in V(D_6)$, the relationship between $se(u, v), diam(D_6), SDIAM(D_6)$ can be summarized as follows:

$$2n \leq se(u, v) \leq SDIAM(D_6) \leq mn. \tag{3.26}$$

Therefore, according to the above analysis, we can obtain

$$2n \leq SDIAM(K_{m_1,m_2,\dots,m_k} \otimes P_n) \leq mn. \tag{3.27}$$

We can concluded that $SDIAM(K_{m_1,m_2,\dots,m_k} \otimes P_n)$ is sharp.

Step 4. Example.

As shown in Figure 6, the graph $K_{m_1,m_2,\dots,m_k} \otimes P_n$ illustrates this result under the assumption that $K_{m_1,m_2,\dots,m_k}$ does not contain a Hamiltonian cycle. The red arrow represents the edge orientation that satisfies the orientation rules in Step 1.2 for all unoriented edges. Here, m_1 to m_k, and n are real numbers. Applying the orientation rules above yields a strong orientation, where the maximum strong distance between any two vertices is $2n$ to mn, and every vertex has a strong eccentricity $2n$ to mn. It can be clearly observed that any two vertices are strongly connected. The proof is complete.

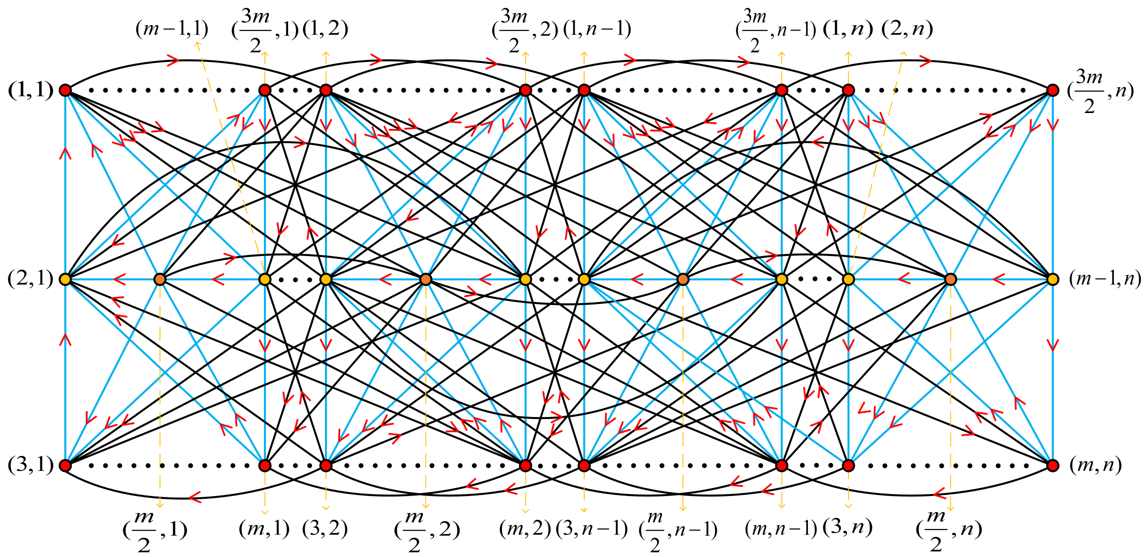

Figure 6. Strong product oriented graph $K_{m_1,m_2,\ldots,m_k} \otimes P_n$.

Therefore, according the above theorems, the strong product network is obtained according to different orders of $K_{m_1,m_2,\ldots,m_k}$ and P_n. The strong orientation of $K_{m_1,m_2,\ldots,m_k} \otimes P_n$ has been oriented. The specific value and the upper and lower bounds of maximum strong diameter of the strong product $K_{m_1,m_2,\ldots,m_k} \otimes P_n$ are investigated.

4. Applications and numerical simulations

Having established the theoretical value and bounds and the method for obtaining the maximum strong diameter of the strong product graph $K_{m_1,m_2,\ldots,m_k} \otimes P_n$ through the above theorems and corollaries, we now formalize this procedure into Algorithm 1.

Algorithm 1 Algorithm to calculate the maximum strong diameter

Input: Orders of complete multipartite graph $K_{m_1,m_2,\ldots,m_k}$: $m_1, m_2, \ldots, m_k$; Order of path P_n: n
Output: Maximum strong diameter of the strongly oriented product graph: $SDIAM(K_{m_1,m_2,\ldots,m_k} \otimes P_n)$

function Calculate maximum strong diameter $(m_1, m_2, \ldots, m_k, n)$
  Step 1: Construct the unoriented graph $G = K_{m_1,\ldots,m_k} \otimes P_n$
  $X \leftarrow \sum_{i=1}^{k} m_i$ ▹ Total vertices in complete multipartite graph
  $N \leftarrow X \times n$ ▹ Total vertices in strong product graph
  $G \leftarrow$ Create empty graph(N)
  for $u = 1$ to X **do** ▹ Build vertex labeling: vertex (u, v) where $u \in [1, X], v \in [1, n]$
    for $v = 1$ to n **do**
      current_vertex $\leftarrow (u, v)$ ▹ Add edges within same path position
      for $k = 1$ to X **do**
        if $u \neq k$ and vertices u, k are in different partite sets **then**
          Add Edge$(G, (u, v), (k, v))$
        end if
      end for ▹ Add edges along the P_n (path adjacency)

```
            if v < n then
                Add Edge(G, (u, v), (u, v + 1))                          ▷ Horizontal edge
            end if
            if v > 1 then
                Add Edge(G, (u, v), (u, v − 1))                          ▷ Horizontal edge
            end if
            if j < n then                          ▷ Add diagonal edges (strong product rule)
                for k = 1 to X do
                    if u ≠ k and vertices u, k are in different partite sets then
                        Add Edge(G, (u, v), (k, v + 1))                   ▷ Diagonal edge
                    end if
                end for
            end if
        end for
    end for

    Step 2: Generate strong orientation
    D ← Create empty digraph (N)
    for each undirected edge (u, v) in G do    ▷ Orientation rule based on both Theorems
    end for

    Step 3: Precalculate strong distance matrix
    Initialize sd(N)(N) ← ∞                 ▷ Find all strongly connected components in D
    SCCs ← Kosaraju Or Tarjan (D)                                ▷ Standard SCC algorithm
    for each SCC S in SCCs do                ▷ Update strong distances for all vertex pairs
        size_S ← |S|
        for each vertex (u, v) in S do
            sd(u, v) ← max(sd(u, v), size_S)
        end for
    end for
    for each SCC S in SCCs_containing_(u, v) do
        SCCs_containing_(u, v) ← Find All SCCs Containing ((u, v), D)
        for each vertex (u, v) in S do
            current_sd ← max(sd(u, v), |S|)
            sd(u, v) ← current_sd
        end for
    end for

    Step 4: Calculate strong eccentricity
    Initialize se(N) ← 0
    for u = 1 to N do
        max_distance ← 0
        for v = 1 to N do
```

if $u \neq v$ and $\mathrm{sd}(u, v) \neq \infty$ **then**
  max_distance ← max(max_distance, sd(u, v))
end if
end for
se(u, v) ← max_distance
end for

Step 5: Find maximum strong d iameter
$SDIAM(K_{m_1,m_2,\ldots,m_k} \otimes P_n) \leftarrow \infty$
for $(u = 1, v = 1)$ to (N, N) **do**
  $SDIAM(K_{m_1,m_2,\ldots,m_k} \otimes P_n) \leftarrow \max(SDIAM(K_{m_1,m_2,\ldots,m_k} \otimes P_n), \mathrm{se}(u, v))$
end for

Step 6: Return result
return $SDIAM(K_{m_1,m_2,\ldots,m_k} \otimes P_n)$
end function

This algorithm calculates the minimum strong diameter of a strong product graph through six main steps.

- Graph construction: The algorithm builds the strong product graph $G = K_{m_1,m_2,\ldots,m_k} \otimes P_n$ of a complete multipartite graph $K_{m_1,m_2,\ldots,m_k}$ and a path P_n, creating a structure with three types of edges: multipartite edges within layers, horizontal path edges, and diagonal product edges.
- Strong orientation: Applying a systematic orientation rule from Theorem 3.1 to Theorem 3.3 to the unoriented graph G yields a strongly connected oriented graph D.
- Strong distance calculation: This phase computes the strong distance matrix, defined as the value of the smallest strongly connected subgraph containing any two vertices. The algorithm first finds all strongly connected components (SCCs), then refines distance estimates by examining all SCCs that contain each vertex pair.
- Strong eccentricity calculation: For each vertex, the algorithm calculates its strong eccentricity as the maximum strong distance to any other vertex in the graph.
- Maximum strong diameter determination: The maximum strong diameter is found by identifying the smallest strong eccentricity among all vertices.
- Result return: The calculated maximum strong diameter value is returned as the final result.

According to the analysis of each step of the algorithm above, we can conclude that the time complexity of the entire algorithm is $O(X^2 n)$, where $X = \sum_{i=1}^{k} m_i$. We have verified the accuracy of the algorithm.

In this paper, we construct networks by taking the strong product of a complete multipartite graph $K_{m_1,m_2,\ldots,m_k}$ and a path P_n. Then, we determine the exact value and lower and upper bounds of the maximum diameter in the strong product sense of $K_{m_1,m_2,\ldots,m_k} \otimes P_n$ under two conditions: When $K_{m_1,m_2,\ldots,m_k}$ contains a Hamiltonian cycle, and when it does not.

To explore the applicability of such strong product networks in network design, we perform numerical simulations. Our aim is to compare the diameter, thus we can find the worst-case information transmission delay between the two cases. As illustrative examples, Figure 7 shows $K_{1,1,4} \otimes P_3$ (where the base complete multipartite graph has no Hamiltonian cycle) and Figure 8 shows $K_{2,2,3} \otimes P_3$ (with a Hamiltonian cycle in the base graph). Visually, the latter exhibits denser edge connections, suggesting a smaller diameter and higher transmission efficiency. We therefore hypothesize that if $K_{m_1,m_2,\ldots,m_k}$ has Hamiltonian, then $K_{m_1,m_2,\ldots,m_k} \otimes P_n$ has a smaller strong diameter, implying better network performance. To test this hypothesis, we use the maximum strong diameter as the evaluation metric in the following simulations.

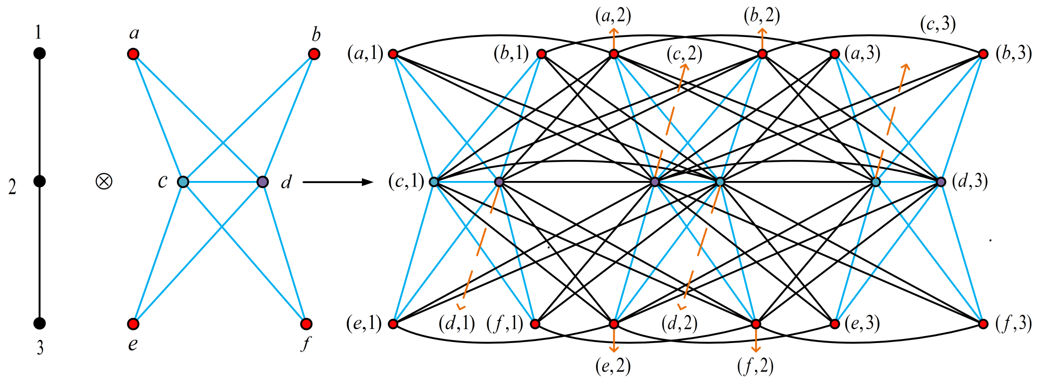

Figure 7. Strong product graph of $K_{1,1,4} \otimes P_3$.

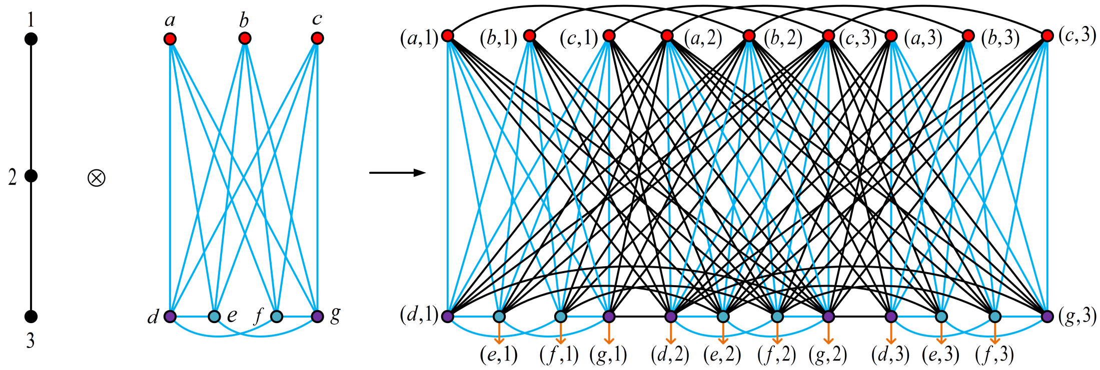

Figure 8. Strong product graph of $K_{2,2,4} \otimes P_3$.

In addition, this section explores the maximum strong diameter boundaries of strong product networks with different factor graphs, drawing on existing research results. We compare the results to reveal the influence of different product operations and factor graph combinations on crucial network parameters. This comparison not only highlights the impact of these factors on network properties in information transmission but also lays a crucial theoretical foundation and provides valuable design guidance for constructing complex network structures with specific strong diameter properties.

To facilitate a comparative analysis of the data, we first present the strong diameter of a strong product based on the findings of previous studies. In the following research, we will conduct two sets of comparative experiments, with a focus on the maximum strong diameter of a strong product network consisting of complete multipartite graphs and paths when $K_{m_1,m_2,\ldots,m_k}$ with and without Hamiltonian cycles.

The first set of experiments take the dimension of a complete multipartite graph containing Hamiltonian cycles as a specific invariant parameter, and compares the maximum strong diameter of the strong product network formed by the strong product of different paths with different dimensions. The second set of experiments considers a complete multipartite graph without Hamiltonian cycles as the invariant parameter and compares the maximum strong diameter of a strong product network composed of a complete multipartite graph without Hamiltonian cycles and paths of different dimensions. Through these series of experimental design comparisons, our goal is to gain a more comprehensive understanding of the information transmission behavior characteristics of networks under different conditions, leading to more accurate conclusions.

In the first comparative experiment, we completely fixed the multipartite graph $K_{m_1,m_1,\ldots,m_k}$ under the condition of containing a Hamiltonian cycle, met the conditions for any set of vertices of $K_{m_1,m_1,\ldots,m_k}$ to satisfy $|m_i| \leq \sum_{j\neq i} |m_j|$ $(i = 1, 2, \ldots, k)$, and calculated the maximum strong diameter of the strong product network of $K_{m_1,m_2,\ldots,m_k} \otimes P_n$ under different dimensional paths. We calculated the maximum oriented diameters of the $K_{m_1,m_2,\ldots,m_k} \otimes P_n$. As shown in Table 1, $m = m_1 + m_2 + \ldots + m_k$ represents the dimensionality of a complete multipartite graph, and n represents the path length. By analyzing the data changes in Table 1, we found that the observed changes were a sharp increase in the maximum diameter value as the value of m and n continued to expand.

Table 1. The maximum diameter with the Hamiltonian cycle in $K_{m_1,m_2,\ldots,m_k}$.

Sample value	Maximum strong diameter		
	Order ($K_{m_1,m_2,\ldots,m_k}$)	Order (P_n)	$SDIAM$ ($K_{m_1,m_2,\ldots,m_k} \otimes P_n$)
1	$m = 4$	$n = 5$	20
2	$m = 5$	$n = 6$	30
3	$m = 6$	$n = 7$	42
4	$m = 8$	$n = 8$	64
5	$m = 9$	$n = 9$	81
6	$m = 10$	$n = 11$	110
7	$m = 12$	$n = 13$	156
8	$m = 15$	$n = 14$	210
9	$m = 20$	$n = 15$	300
10	$m = 22$	$n = 20$	440

In the second comparative experiment, we completely fixed the multipartite graph $K_{m_1,m_1,\ldots,m_k}$ under the condition without Hamiltonian cycles, met the conditions for any set of vertices of $K_{m_1,m_1,\ldots,m_k}$ to satisfy $|m_i| > \sum_{j\neq i} |m_j|$ $(i = 1, 2, \ldots, k)$, and calculated the maximum strong diameter of the strong product network under different dimensional paths. We calculated the maximum oriented diameters of the $K_{m_1,m_2,\ldots,m_k} \otimes P_n$. As shown in Table 2, $m = m_1 + m_2 + \ldots + m_k$ represents the dimensionality of a completely multipartite graph, and n represents the path length. By analyzing the data changes in Table 2, we found that the observed changes occur as the value of m and n continues to increase, with the maximum diameter value being significantly smaller than the maximum diameter value in Table 1.

Table 2. The maximum diameter without the Hamiltonian cycle in $K_{m_1,m_2,\ldots,m_k}$.

Sample value	Maximum strong diameter		
	Order ($K_{m_1,m_2,\ldots,m_k}$)	Order (P_n)	$SDIAM$ ($K_{m_1,m_2,\ldots,m_k} \otimes P_n$)
1	$m = 4$	$n = 5$	15
2	$m = 5$	$n = 6$	22
3	$m = 6$	$n = 7$	30
4	$m = 8$	$n = 8$	45
5	$m = 9$	$n = 9$	56
6	$m = 10$	$n = 11$	66
8	$m = 15$	$n = 14$	100
9	$m = 20$	$n = 15$	126
10	$m = 22$	$n = 20$	200

According to the data of Tables 1 and 2, the difference in the data of the maximum diameter can be visually observed. In order to better simulate the different diameters of the two situations, the data curve of the maximum strong diameter of the strong product of $K_{m_1,m_2,\ldots,m_k}$ and P_n are derived. As shown in Figure 9, as the path length increases, the growth rate of the maximum strong diameter for $K_{m_1,m_2,\ldots,m_k} \otimes P_n$ without Hamiltonian cycle in $K_{m_1,m_1,\ldots,m_k}$ is obviously slower than that for $K_{m_1,m_2,\ldots,m_k} \otimes P_n$ with Hamiltonian cycle.

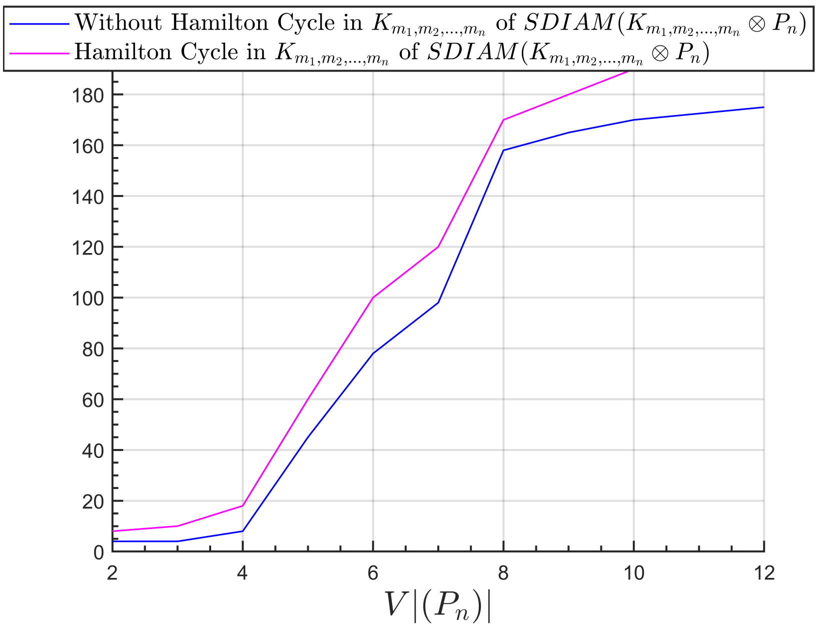

Figure 9. Two data change curves of maximum strong diameter.

In order to intuitively discover the difference of impaction on the information transmission efficiency of strong product networks between $K_{m_1,m_2,\ldots,m_k}$ with and without Hamiltonian cycles, as shown in Figure 10(a,b), the maximum strong diameter of two different three-dimensional graphs are constructed. First, comparing the maximum strong diameter of the three-dimensional graph, the strong product of $K_{m_1,m_2,\ldots,m_k} \otimes P_n$ for $K_{m_1,m_2,\ldots,m_k}$ without Hamiltonian cycle is closer to a simple graph. Thus, when the network transmits information in a strong product graph network, the transmission delay can be accurately judged. Second, after comparing the maximum strong diameter of the three-dimensional graph, it can be seen that the strong product of $K_{m_1,m_2,\ldots,m_k} \otimes P_n$ for $K_{m_1,m_2,\ldots,m_k}$ without Hamiltonian cycle has more concentrated data. The above comparison further confirms that our preliminary hypothesis is incorrect. Thus, a judgment can be made that strong product graphs usually have higher connectivity. This means that in a strong product graph, the path length between any two vertices quickly reaches the target vertex. Therefore, the maximum strong diameter is one of the most important metrics for oriented networks information transmission performance.

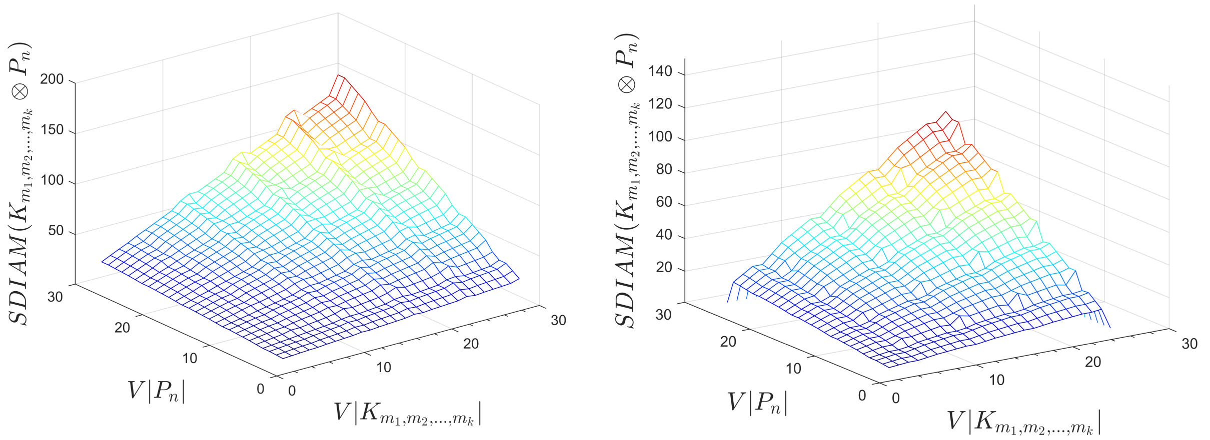

Figure 10. Three-dimension of the maximum strong diameter of $K_{m_1,m_2,\ldots,m_k} \otimes P_n$; (a) with Hamiltonian cycle in $K_{m_1,m_2,\ldots,m_k}$; (b) without Hamiltonian cycle in $K_{m_1,m_2,\ldots,m_k}$.

In summary, constructing the strong product graphs from multidimensional complete bipartite graphs without Hamiltonian cycles and paths offers certain advantages in network design and information communication, particularly reducing data transmission delays. Moreover, strong product graphs have higher connectivity, achieving lower latency and higher throughput. The strong product is especially important for real-time systems and large-scale load balancing, where resource sharing is critical. Therefore, based on the above experimental comparative analyses, the optimization of the network is realized.

5. Conclusions

In this paper, we successfully investigated the specific value and upper and lower bounds of the maximum strong diameter of $K_{m_1,m_2,\ldots,m_k} \otimes P_n$ in two different cases. At the same time, we compared the maximum strong diameter of $K_{m_1,m_2,\ldots,m_k} \otimes P_n$ depending on whether $K_{m_1,m_2,\ldots,m_k}$ contains a Hamiltonian cycle, across different dimensions of the graph. The comparison shows that without a Hamiltonian cycle, increasing the dimensionality leads to a smaller maximum strong diameter of the strong product of the complete multipartite graph and path. Therefore, constructing complex computer systems using higher-dimensional complete multipartite graphs without Hamiltonian cycles and paths can result in lower information transmission efficiency, network delay, and energy consumption. Due to the higher complexity of the strong product of complete multipartite graphs and paths, we will further investigate the relevant results in other products.

Author contributions

Ce Zhang: Conceptualization, Methodology, Formal analysis, Investigation, Writing–original draft preparation, Writing–review and editing; Feng Li: Resources, Supervision and project administration, Funding acquisition. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this paper.

Acknowledgments

This work was supported in part by the National Natural Science Foundation of China under Grant 11551002, in part by the Natural Science Foundation of Qinghai Province under Grant 2019-ZJ-7093.

Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work in this paper.

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