Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(4): 9210–9227.
DOI: 10.3934/math.2026380
Received: 30 December 2025
Revised: 11 March 2026
Accepted: 23 March 2026
Published: 03 April 2026

Research article

Minimum-norm least-squares solutions of generalized Sylvester-type quaternion matrix equation with bi-Hermitian and skew bi-Hermitian constraints

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Abstract: This paper investigated minimum-norm least-squares solutions to the quaternion matrix equation $KXL+MYN = O$, where the unknown matrices X and Y were subject to bi-Hermitian or skew bi-Hermitian constraints. Building on the derived theoretical results, an explicit numerical algorithm was proposed. Moreover, several numerical examples were presented to validate the accuracy of these results.

Keywords: quaternion matrix equations; real matrix representation; minimum norm least-squares solutions; bi-Hermitian solutions; skew bi-Hermitian solutions
Mathematics Subject Classification: 15A06, 15A09, 15A24, 15B33, 15B57

1. Introduction

Quaternions, first introduced by W. R. Hamilton in 1843 as a generalization of complex numbers [1], have been widely used in various disciplines, including signal processing [2], image processing [3], and quantum mechanics [4]. Consequently, matrices with quaternion entries and their related equations have received considerable attention in the literature [5].

In general, quaternions satisfy all the axioms of a field except the commutativity of multiplication; therefore, they are referred to as a skew field. For this reason, solving a quaternion matrix equation entails additional challenges compared to solving a real or complex matrix equation. Equations of this type can be solved using an appropriate inner product over the right or left vector space [6, 7], or by converting them into real or complex matrix equations through their respective matrix representations (see [8, 9]).

In this work, we consider the quaternion matrix equation,

$$KXL + MYN = O, \tag{1.1}$$

a generalized form encompassing both the Sylvester matrix equation [10], and the Lyapunov matrix equation [11]. The matrix Eq (1.1) and its variants have been extensively studied in the literature. Wei et al. addressed different constrained solutions of Eq (1.1) through the real representations of quaternion matrices [12], while Şimşek derived the general least-squares solutions, including the per-Hermitian and skew per-Hermitian least-squares solutions of the same equation [13]. Kyrchei derived explicit determinant-based expressions for the solutions of a two-sided quaternion generalized Sylvester matrix equation [14]. The η-Hermitian and η-anti-Hermitian solutions of the quaternion matrix Eq (1.1) were obtained using complex matrix representations of quaternion matrices [9], while other some types of solutions of (1.1) were derived using real representations of quaternion matrices [5, 8]. Another notable approach was proposed by Beik and Ahmadi-Asl [6], who developed the conjugate gradient least-squares (CGLS) method to compute the η-Hermitian and η-anti-Hermitian solutions of Eq (1.1). For large-scale instances of Eq (1.1), Krylov subspace methods such as the block generalized minimal residual (GMRES) and global GMRES have been employed, as reported in [15, 16], respectively. Additionally, Rehman et al. established theoretical foundations to solve Sylvester-type quaternion matrix equations and proposed a novel algorithm for the general solution [17]. The least-squares problem considered in this work is defined over quaternion bi-Hermitian and/or skew bi-Hermitian matrices. Before presenting the definitions of these specialized quaternion matrices, the relevant notations and terminology are introduced.

Throughout this paper, $\mathbb{R}^{l\times s}$, $\mathbb{Q}^{l\times s}$, $\mathbb{BR}^{s\times s}$, $\mathbb{SBR}^{s\times s}$, $\mathbb{BQ}^{s\times s}$, and $\mathbb{SBQ}^{s\times s}$ represent the sets of all $l\times s$ real, $l\times s$ quaternion, $s\times s$ real bi-symmetric, $s\times s$ real skew bi-symmetric, $s\times s$ quaternion bi-Hermitian, and $s\times s$ quaternion skew bi-Hermitian matrices, respectively. For $M\in\mathbb{Q}^{l\times s}$, the symbols $\overline{M}$, M^T, M^H, and $M^\dagger$ denote the conjugate, transpose, conjugate transpose, and Moore–Penrose inverse of M, respectively. The identity matrix of order l is denoted by I_l. The notation

$$\langle M, N\rangle = \mathrm{tr}\left(N^H M\right)$$

represents the inner product of matrices M, $N\in\mathbb{Q}^{l\times s}$. The set $\mathbb{Q}^{l\times s}$, together with this inner product, forms a Hilbert inner product space, and the induced matrix norm is the Frobenius norm, which is denoted by $\|\,.\,\|$. For $M=(m_1,\ m_2, ..., m_s)\in\mathbb{R}^{l\times s}$, where $m_i\in\mathbb{R}^l$ denotes the i. column of M, the vec operator is defined as $\mathrm{vec}(M)\in\mathbb{R}^{ls}$, which is obtained by placing the column vectors of M vertically. The Kronecker product of matrices M and N is denoted by $M\otimes N$.

Now, let us introduce the definition of quaternion bi-Hermitian and skew bi-Hermitian matrices.

Definition 1.1. *([18]) Let $M\in\mathbb{Q}^{l\times l}$. M is called quaternion bi-Hermitian if $M^H=M$ and $(JM)^H=(JM)$, and quaternion skew bi-Hermitian if $M^H=-M$ and $(JM)^H=-JM$, where $J=(e_l,\ e_{l-1},\ ...,\ e_1)$, and e_i denotes the i. column of the $l\times l$ identity matrix.*

Moreover, throughout the work, Ω_l refers to either $\mathbb{BQ}^{l\times l}$ or $\mathbb{SBQ}^{l\times l}$.

In this work, we consider the following problem:

Problem 1.1. *Let $K\in\mathbb{Q}^{mxp}$, $L\in\mathbb{Q}^{pxn}$, $M\in\mathbb{Q}^{mxq}$, $N\in\mathbb{Q}^{qxn}$, $O\in\mathbb{Q}^{mxn}$, and*

$$S_\Omega=\left\{\left(\widehat{X},\widehat{Y}\right):\widehat{X}\in\Omega_p,\widehat{Y}\in\Omega_q,\ \left\|K\widehat{X}L+M\widehat{Y}N-O\right\|=\min_{X\in\Omega_p,\ Y\in\Omega_q}\|KXL+MYN-O\|\right\}.$$

Determine $\left(\widehat{X}, \widehat{Y}\right) \in S_\Omega$ *for which*

$$\left\|\left(\widehat{X}, \widehat{Y}\right)\right\| = \min_{\left(\widehat{X}, \widehat{Y}\right) \in S_\Omega} \|X\| + \|Y\|.$$

Equivalently, seek least-squares solutions $\widehat{X}$ *and* $\widehat{Y}$ *with minimal norm that exhibit bi-Hermitian and/or skew bi-Hermitian properties for the quaternion matrix Eq* (1.1).

2. Preliminaries

A quaternion q is a four-dimensional hypercomplex number consisting of one real part and three imaginary components. It can be written in the following form:

$$q = q_1 + q_2 i + q_3 j + q_4 k,$$

where the coefficients q_1, q_2, q_3, q_4 are real numbers. The imaginary units i, j, and k obey the classical Hamiltonian multiplication rules: each squares to -1, their pairwise products generate the remaining unit, and the multiplication is non-commutative. This algebraic structure was first introduced by Hamilton [1].

Therefore, a quaternion matrix $M \in \mathbb{Q}^{l \times s}$ can be written as follows:

$$M = M_1 + M_2 i + M_3 j + M_4 k,$$

where $M_1, M_2, M_3, M_4 \in \mathbb{R}^{l \times s}$.

The real representation of any quaternion matrix

$$M = M_1 + M_2 i + M_3 j + M_4 k \in \mathbb{Q}^{l \times s},$$

as defined by Wei in [19], is given by the following:

$$M^R = \begin{bmatrix} M_1 & -M_2 & -M_3 & -M_4 \\ M_2 & M_1 & -M_4 & M_3 \\ M_3 & M_4 & M_1 & -M_2 \\ M_4 & -M_3 & M_2 & M_1 \end{bmatrix} \in \mathbb{R}^{4l \times 4s}. \tag{2.1}$$

Let the first column block of M^R be denoted by the following:

$$M_c^R = \begin{bmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \end{bmatrix}. \tag{2.2}$$

Some useful properties of matrices M^R and M_c^R are given below.

Lemma 2.1. *([20]) Let $K, L \in \mathbb{Q}^{l\times s}$, $M \in \mathbb{Q}^{s\times m}$, and $\lambda \in \mathbb{R}$. Then, the following properties hold:*

$$\begin{aligned}
&(i) \quad K = L \Leftrightarrow K^R = L^R;\\
&(ii) \quad (K+L)^R = K^R + L^R, \quad (\lambda K)^R = \lambda K^R, \quad (KM)^R = K^R M^R;\\
&(iii) \quad (K^\dagger)^R = (K^R)^\dagger;\\
&(iv) \quad (K+L)_c^R = K_c^R + L_c^R, \quad (\lambda K)_c^R = \lambda K_c^R, \quad (KM)_c^R = K^R M_c^R;\\
&(v) \quad \|K\| = \tfrac{1}{2}\|K^R\| = \|K_c^R\|.
\end{aligned}$$

Prior to presenting the lemmas used to solve Problem 1.1, we recall the following relation:

$$\mathrm{vec}(KLM) = (M^T \otimes K)\mathrm{vec}L, \tag{2.3}$$

for suitable real matrices K, L, and M [21].

Now, we give a relation between $\mathrm{vec}(M^R)$ and $\mathrm{vec}(M_c^R)$.

Lemma 2.2. *([22]) Let $M \in \mathbb{Q}^{l\times s}$. Then,*

$$\mathrm{vec}(M^R) = \mathcal{P}\,\mathrm{vec}(M_c^R),$$

where

$$\mathcal{P} = \begin{bmatrix} diag\,(I_{4l}, \ldots, I_{4l}) \\ diag\,(Q_l, \ldots, Q_l) \\ diag\,(R_l, \ldots, R_l) \\ diag\,(S_l, \ldots, S_l) \end{bmatrix} \in \mathbb{R}^{16ls\times 4ls}, \quad Q_l = \begin{bmatrix} \mathbf{0} & -I_l & \mathbf{0} & \mathbf{0} \\ I_l & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & I_l \\ \mathbf{0} & \mathbf{0} & -I_l & \mathbf{0} \end{bmatrix} \in \mathbb{R}^{4l\times 4l},$$

$$R_l = \begin{bmatrix} \mathbf{0} & \mathbf{0} & -I_l & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -I_l \\ I_l & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I_l & \mathbf{0} & \mathbf{0} \end{bmatrix} \in \mathbb{R}^{4l\times 4l}, \qquad S_l = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & -I_l \\ \mathbf{0} & \mathbf{0} & I_l & \mathbf{0} \\ \mathbf{0} & -I_l & \mathbf{0} & \mathbf{0} \\ I_l & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \in \mathbb{R}^{4l\times 4l}.$$

Lemma 2.3. *([23]) Let*

$$M = M_1 + M_2 i + M_3 j + M_4 k \in \mathbb{Q}^{l\times l}.$$

Then,

$$\mathrm{vec}\left(M_c^R\right) = \mathcal{S}\begin{bmatrix} \mathrm{vec}\,(M_1) \\ \mathrm{vec}\,(M_2) \\ \mathrm{vec}\,(M_3) \\ \mathrm{vec}\,(M_4) \end{bmatrix},$$

where

$$\mathcal{S} = \begin{bmatrix} \mathrm{diag}\,(\mathcal{I}_1, \mathcal{I}_1, \mathcal{I}_1, \mathcal{I}_1) \\ \mathrm{diag}\,(\mathcal{I}_2, \mathcal{I}_2, \mathcal{I}_2, \mathcal{I}_2) \\ \vdots \\ \mathrm{diag}\,(\mathcal{I}_l, \mathcal{I}_l, \mathcal{I}_l, \mathcal{I}_l) \end{bmatrix} \in \mathbb{R}^{4l^2\times 4l^2},$$

with $\mathcal{I}_1 = [I_l, \mathbf{0}, \ldots, \mathbf{0}]$, $\mathcal{I}_2 = [\mathbf{0}, I_l, \ldots, \mathbf{0}]$, $\ldots$, $\mathcal{I}_l = [\mathbf{0}, \mathbf{0}, \ldots, I_l] \in \mathbb{R}^{l\times l^2}$.

Now, we describe the structural properties of bi-Hermitian and skew bi-Hermitian matrices.

Definition 2.1. *([24]) Let $M \in \mathbb{R}^{n\times n}$; then, vector* $\text{vec}_B(M)$ *is defined by the following:*

$$\text{vec}_B(M) = \begin{cases} [m_1, m_2, \ldots, m_l]^T, & n = 2l, \\ [m_1, m_2, \ldots, m_l, m_{l+1}]^T, & n = 2l+1, \end{cases} \tag{2.4}$$

where $m_1 = M(1:n,1)^T$, $m_2 = M(2:n-1,2)^T$, $\ldots$, $m_l = M(l:n-l+1,l)^T$, *and* $m_{l+1} = M(l+1,l+1)^T$.

Definition 2.2. *([24]) Let $M \in \mathbb{R}^{n\times n}$; then, the vector* $\text{vec}_{SB}(M)$ *is defined by the following:*

$$\text{vec}_{SB}(M) = \begin{cases} [m_1, m_2, \ldots, m_{l-1}]^T, & n = 2l, \\ [m_1, m_2, \ldots, m_{l-1}, m_l]^T, & n = 2l+1, \end{cases} \tag{2.5}$$

where $m_1 = M(2:n-1,1)^T$, $m_2 = M(3:n-2,2)^T$,..., $m_{l-1} = M(l:n-l+1,l-1)^T$, *and* $m_l = M(l+1,l)^T$.

For a real bi-symmetric matrix M, the following relation holds between $\text{vec}(M)$ and $\text{vec}_B(M)$.

Lemma 2.4. *([24]) Let $M \in \mathbb{R}^{n\times n}$. The operator* $\text{vec}_B(M)$ *is given by* (2.4). *Then,*

$$M \in \mathbb{BR}^{n\times n} \quad \textit{if and only if} \quad \text{vec}(M) = V_{(n)}\,\text{vec}_B(M), \tag{2.6}$$

with

$$V_{(n)} = \begin{cases} [V_1\ V_2 \cdots V_{l-1}\ V_l] \in \mathbb{R}^{n^2\times l(l+1)}, & n = 2l, \\ [V_1\ V_2 \cdots V_l\ V_{l+1}] \in \mathbb{R}^{n^2\times (l+1)^2}, & n = 2l+1, \end{cases} \tag{2.7}$$

where V_η is defined as

$$\begin{bmatrix} \mathbf{0}_{n(\eta-1)\times(n-2(\eta-1))} \\ (I_{n-2(\eta-1)} \otimes e_\eta) + (J_{n-2(\eta-1)} \otimes e_{\,n-(\eta-1)}) \\ \mathbf{0}_{n(\eta-1)\times(n-2(\eta-1))} \end{bmatrix} + \begin{bmatrix} \mathbf{0}_{(n(\eta-1)+\eta)\times 1} & \mathbf{0}_{(n(\eta-1)+\eta)\times(n-2\eta)} & \mathbf{0}_{(n(\eta-1)+\eta)\times 1} \\ \mathbf{0}_{(n-2\eta)\times 1} & I_{(n-2\eta)} & \mathbf{0}_{(n-2\eta)\times 1} \\ \mathbf{0}_{((n-2\eta)n+2\eta)\times 1} & \mathbf{0}_{((n-2\eta)n+2\eta)\times(n-2\eta)} & \mathbf{0}_{((n-2\eta)n+2\eta)\times 1} \\ \mathbf{0}_{(n-2\eta)\times 1} & J_{(n-2\eta)} & \mathbf{0}_{(n-2\eta)\times 1} \\ \mathbf{0}_{(n(\eta-1)+\eta)\times 1} & \mathbf{0}_{(n(\eta-1)+\eta)\times(n-2\eta)} & \mathbf{0}_{(n(\eta-1)+\eta)\times 1} \end{bmatrix}$$

for $\eta = 1, 2, \ldots, l$, and

$$V_{l+1} = \begin{bmatrix} \mathbf{0}_{\,l(2l+1)\times 1} \\ I_1 \otimes e_{l+1} \\ \mathbf{0}_{\,l(2l+1)\times 1} \end{bmatrix}.$$

Likewise, for a skew bi-symmetric matrix $M \in \mathbb{R}^{n\times n}$, the relation that involves $\text{vec}(M)$ and $\text{vec}_{SB}(M)$ is given below.

Lemma 2.5. *([24]) Let $M \in \mathbb{R}^{n\times n}$ and* $\text{vec}_{SB}(M)$ *be defined as in (2.5). Then,*

$$M \in \mathbb{SBR}^{n\times n} \quad \textit{if and only if} \quad \text{vec}(M) = W_{(n)}\,\text{vec}_{SB}(M), \tag{2.8}$$

where

$$W_{(n)} = \begin{cases} [W_1\ \ W_2 \ldots W_{l-1}] \in \mathbb{R}^{n^2\times l(l-1)}, & n = 2l, \\ [W_1\ \ W_2 \ldots W_l] \in \mathbb{R}^{n^2\times l^2}, & n = 2l+1, \end{cases} \tag{2.9}$$

with

$$W_\eta = \begin{bmatrix} \mathbf{0}_{(\,n(\eta-1)+\eta\,)\times(n-2\eta)} \\ I_{(n-2\eta)} \\ \mathbf{0}_{\eta\times(n-2\eta)} \\ I_{(n-2\eta)}\otimes(-e_\eta) + J_{(n-2\eta)}\otimes(-e_{\,n-(\eta-1)\,}) \\ \mathbf{0}_{\eta\times(n-2\eta)} \\ J_{(n-2\eta)} \\ \mathbf{0}_{(\,n(\eta-1)+p\,)\times(n-2\eta)} \end{bmatrix}$$

for $\eta = 1, 2, \ldots, l$.

A (skew) bi-symmetric matrix is a square matrix characterized by (skew) symmetry across both its main and anti-diagonals. Consequently, the relations in (2.6) and (2.8) express the full vectorization through the reduced-dimension vectors $\mathrm{vec}_B(M)$ and $\mathrm{vec}_{SB}(M)$, respectively. The matrices $V_{(n)}$ and $W_{(n)}$ provide a systematic compression of the parameter space by identifying the independent entries and mapping them linearly into the full $n \times n$ matrix. In doing so, the (skew) bi-symmetric constraints are automatically enforced, which ensures that dependent entries are correctly positioned. As a result, the original n^2-dimensional space is reduced to a lower-dimensional subspace that only represents the true degrees of freedom, which allows for a consistent reconstruction of the matrix according to its inherent symmetry.

Lemma 2.6. *([18]) Let* $M = M_1 + M_2 i + M_3 j + M_4 k \in \Omega_n$. *Considering* $V_{(n)}$ *and* $W_{(n)}$ *as defined in equalities* (2.7) *and* (2.9), *respectively the following relations hold:*

$$M \in \mathbb{BQ}^{n\times n} \iff \mathrm{vec}\left(M_c^R\right) = \mathcal{T} \begin{bmatrix} \mathrm{vec}_B(M_1) \\ \mathrm{vec}_{SB}(M_2) \\ \mathrm{vec}_{SB}(M_3) \\ \mathrm{vec}_{SB}(M_4) \end{bmatrix}$$

and

$$M \in \mathbb{SBQ}^{n\times n} \iff \mathrm{vec}\left(M_c^R\right) = \widetilde{\mathcal{T}} \begin{bmatrix} \mathrm{vec}_{SB}(M_1) \\ \mathrm{vec}_B(M_2) \\ \mathrm{vec}_B(M_3) \\ \mathrm{vec}_B(M_4) \end{bmatrix},$$

where $\mathcal{T} = diag\,(V_{(n)}, W_{(n)}, W_{(n)}, W_{(n)})$, *and* $\widetilde{\mathcal{T}} = diag\,(W_{(n)}, V_{(n)}, V_{(n)}, V_{(n)})$.

A well-known lemma is stated below.

Lemma 2.7. *([21]) For* $M \in \mathbb{R}^{m\times n}$ *and* $g \in \mathbb{R}^m$, *the system* $Mx = g$ *has a solution if and only if* $MM^\dagger g = g$. *When this holds, the general solution can be expressed as follows:*

$$x = M^\dagger g + (I - M^\dagger M)h,$$

with arbitrary $h \in \mathbb{R}^n$. *The minimum norm least-squares solution of* $Mx = g$ *is uniquely given by* $x = M^\dagger g$.

For the sake of convenience, let us introduce some notations to be used from this point onward. Let $\mathcal{P}_X$ and $\mathcal{P}_Y$ denote the matrices derived from $\mathcal{P}$ in Lemma 2.2, associated with the matrices X and Y, respectively, with $\mathcal{P}_X \in \mathbb{R}^{16p^2\times 4p^2}$ and $\mathcal{P}_Y \in \mathbb{R}^{16q^2\times 4q^2}$. Similarly, $\mathcal{S}_X$ and $\mathcal{S}_Y$ denote the matrices derived from $\mathcal{S}$ in Lemma 2.3, associated with the matrices X and Y, respectively, with $\mathcal{S}_X \in \mathbb{R}^{4p^2\times 4p^2}$ and $\mathcal{S}_Y \in \mathbb{R}^{4q^2\times 4q^2}$.

Additionally, define $\text{diag}(\mathcal{P}_X, \mathcal{P}_Y) := \boldsymbol{\mathcal{P}}$ and $\text{diag}(\mathcal{S}_X, \mathcal{S}_Y) := \boldsymbol{\mathcal{S}}$. Let $V_{(p)}$ and $V_{(q)}$ be the matrices defined in (2.7), and $W_{(p)}$ and $W_{(q)}$ be the matrices defined in (2.9). Then, as in Lemma 2.6, let $\mathcal{T}_X = \text{diag}(V_{(p)}, W_{(p)}, W_{(p)}, W_{(p)})$, $\mathcal{T}_Y = \text{diag}(V_{(q)}, W_{(q)}, W_{(q)}, W_{(q)})$, $\widetilde{\mathcal{T}_X} = \text{diag}(W_{(p)}, V_{(p)}, V_{(p)}, V_{(p)})$, and $\widetilde{\mathcal{T}_Y} = \text{diag}(W_{(q)}, V_{(q)}, V_{(q)}, V_{(q)})$. Additionally, define $\text{diag}(\mathcal{T}_X, \mathcal{T}_Y) := \boldsymbol{\mathcal{T}}$, $\text{diag}(\widetilde{\mathcal{T}_X}, \widetilde{\mathcal{T}_Y}) := \dot{\boldsymbol{\mathcal{T}}}$ and $\text{diag}(\mathcal{T}_X, \widetilde{\mathcal{T}_Y}) := \acute{\boldsymbol{\mathcal{T}}}$.

3. Theoretical solution for Problem 1.1

This section addresses Problem 1.1 through the use of real-form quaternion matrix representations, supported by fundamental results from the Moore–Penrose inverse and Kronecker product theory.

Theorem 3.1. *Let $K \in \mathbb{Q}^{m\times p}$, $L \in \mathbb{Q}^{p\times n}$, $M \in \mathbb{Q}^{m\times q}$, $N \in \mathbb{Q}^{q\times n}$, and $O \in \mathbb{Q}^{m\times n}$. Then, for an arbitrary vector h of a compatible dimension, the least-squares solutions X and Y of the quaternion matrix Eq* (1.1) *which satisfy the bi-Hermitian and/or skew bi-Hermitian constraints are characterized as follows:*

$$S_\Omega = \left\{(X,Y) : \begin{bmatrix} \check{x} \\ \check{y} \end{bmatrix} = \boldsymbol{\mathcal{U}}\left[(\boldsymbol{\mathcal{RPSU}})^\dagger \text{vec}\left(O_c^R\right) + \left[I - (\boldsymbol{\mathcal{RPSU}})^\dagger(\boldsymbol{\mathcal{RPSU}})\right]h\right]\right\}, \tag{3.1}$$

where

$$X = X_1 + X_2 i + X_3 j + X_4 k \in \Omega_p, \quad Y = Y_1 + Y_2 i + Y_3 j + Y_4 k \in \Omega_q,$$

and

$$\boldsymbol{\mathcal{R}} = \left[\left(L_c^R\right)^T \otimes K^R \ \left(N_c^R\right)^T \otimes M^R\right], \quad \check{x} = \begin{bmatrix} \text{vec}\,(X_1) \\ \text{vec}\,(X_2) \\ \text{vec}\,(X_3) \\ \text{vec}\,(X_4) \end{bmatrix}$$

and

$$\check{y} = \begin{bmatrix} \text{vec}\,(Y_1) \\ \text{vec}\,(Y_2) \\ \text{vec}\,(Y_3) \\ \text{vec}\,(Y_4) \end{bmatrix}.$$

In particular, the minimum norm least-squares bi-Hermitian and/or skew bi-Hermitian solution $\left(\widehat{X}, \widehat{Y}\right)$ of the quaternion matrix equation satisfies the following:

$$\begin{bmatrix} \check{x} \\ \check{y} \end{bmatrix} = \boldsymbol{\mathcal{U}}(\boldsymbol{\mathcal{RPSU}})^\dagger \text{vec}\left(O_c^R\right), \tag{3.2}$$

where the matrix $\boldsymbol{\mathcal{U}}$ denotes the matrices $\boldsymbol{\mathcal{T}}$, $\dot{\boldsymbol{\mathcal{T}}}$, and $\acute{\boldsymbol{\mathcal{T}}}$ in the cases when both X and Y are bi-Hermitian, both are skew bi-Hermitian, or one of them (say X) is bi-Hermitian and the other (Y) is skew bi-Hermitian, respectively.

Proof. The proof will only be carried out only for the case when both the matrices X and Y are bi-Hermitian, that is, when $\mathcal{U} = \mathcal{T}$. The proofs for the other cases are entirely analogous.

By using Lemma 2.1, we have the following:

$$\begin{aligned}\|KXL + MYN - O\| &= \left\|(KXL)_c^R + (MYN)_c^R - O_c^R\right\| \\ &= \left\|K^R X^R L_c^R + M^R Y^R N_c^R - O_c^R\right\| \\ &= \left\|\operatorname{vec}\left(K^R X^R L_c^R\right) + \operatorname{vec}\left(M^R Y^R N_c^R\right) - \operatorname{vec}\left(O_c^R\right)\right\|.\end{aligned}$$

Furthermore, by utilizing equality (2.3), it follows that

$$\|KXL + MYN - O\| = \left\|\left(\left(L_c^R\right)^T \otimes K^R\right)\operatorname{vec}\left(X^R\right) + \left(\left(N_c^R\right)^T \otimes M^R\right)\operatorname{vec}\left(Y^R\right) - \operatorname{vec}\left(O_c^R\right)\right\|.$$

Defining

$$\mathcal{R} = \begin{bmatrix}\left(L_c^R\right)^T \otimes K^R & \left(N_c^R\right)^T \otimes M^R\end{bmatrix},$$

then the following is obtained:

$$\|KXL + MYN - O\| = \left\|\mathcal{R}\begin{bmatrix}\operatorname{vec}(X^R) \\ \operatorname{vec}(Y^R)\end{bmatrix} - \operatorname{vec}(O_c^R)\right\|.$$

Hence, using Lemma 2.2, it is derived that

$$\|KXL + MYN - O\| = \left\|\mathcal{R}\mathcal{P}\begin{bmatrix}\operatorname{vec}\left(X_c^R\right) \\ \operatorname{vec}\left(Y_c^R\right)\end{bmatrix} - \operatorname{vec}\left(O_c^R\right)\right\|.$$

If Lemma 2.3 is taken into account, then

$$\|KXL + MYN - O\| = \left\|\mathcal{R}\mathcal{P}\mathcal{S}\left(\begin{bmatrix}\operatorname{vec}(X_1) \\ \operatorname{vec}(X_2) \\ \operatorname{vec}(X_3) \\ \operatorname{vec}(X_4)\end{bmatrix} \atop \begin{bmatrix}\operatorname{vec}(Y_1) \\ \operatorname{vec}(Y_2) \\ \operatorname{vec}(Y_3) \\ \operatorname{vec}(Y_4)\end{bmatrix}\right) - \operatorname{vec}\left(O_c^R\right)\right\| \tag{3.3}$$

is acquired.

Since $X = X_1 + X_2 i + X_3 j + X_4 k \in \mathbb{BQ}^{p\times p}$, it follows that $X_1 \in \mathbb{BR}^{p\times p}$ and X_2, X_3, $X_4 \in \mathbb{SBR}^{p\times p}$. From Lemmas 2.4 and 2.5, we have $\operatorname{vec}(X_1) = V_{(p)}\operatorname{vec}_B(X_1)$, $\operatorname{vec}(X_2) = W_{(p)}\operatorname{vec}_{SB}(X_2)$, $\operatorname{vec}(X_3) = W_{(p)}\operatorname{vec}_{SB}(X_3)$, and $\operatorname{vec}(X_4) = W_{(p)}\operatorname{vec}_{SB}(X_4)$. Thus,

$$\begin{bmatrix}\operatorname{vec}(X_1) \\ \operatorname{vec}(X_2) \\ \operatorname{vec}(X_3) \\ \operatorname{vec}(X_4)\end{bmatrix} = \mathcal{T}_X \begin{bmatrix}\operatorname{vec}_B(X_1) \\ \operatorname{vec}_{SB}(X_2) \\ \operatorname{vec}_{SB}(X_3) \\ \operatorname{vec}_{SB}(X_4)\end{bmatrix}. \tag{3.4}$$

Similarly, by using Lemmas 2.4 and 2.5 again, for

$$Y = Y_1 + Y_2 i + Y_3 j + Y_4 k \in \mathbb{BQ}^{q\times q},$$

we obtain the following:

$$\begin{bmatrix} \mathrm{vec}(Y_1) \\ \mathrm{vec}(Y_2) \\ \mathrm{vec}(Y_3) \\ \mathrm{vec}(Y_4) \end{bmatrix} = \mathcal{T}_Y \begin{bmatrix} \mathrm{vec}_B(Y_1) \\ \mathrm{vec}_{SB}(Y_2) \\ \mathrm{vec}_{SB}(Y_3) \\ \mathrm{vec}_{SB}(Y_4) \end{bmatrix}. \tag{3.5}$$

Substituting equalities (3.4) and (3.5) into (3.3), we have the following:

$$\|KXL + MYN - O\| = \left\| \mathcal{RPST} \begin{pmatrix} \begin{bmatrix} \mathrm{vec}_B(X_1) \\ \mathrm{vec}_{SB}(X_2) \\ \mathrm{vec}_{SB}(X_3) \\ \mathrm{vec}_{SB}(X_4) \end{bmatrix} \\ \begin{bmatrix} \mathrm{vec}_B(Y_1) \\ \mathrm{vec}_{SB}(Y_2) \\ \mathrm{vec}_{SB}(Y_3) \\ \mathrm{vec}_{SB}(Y_4) \end{bmatrix} \end{pmatrix} - \mathrm{vec}\left(O_c^R\right) \right\|. \tag{3.6}$$

Therefore, we obtain the following real linear system:

$$\mathcal{RPST} \begin{pmatrix} \begin{bmatrix} \mathrm{vec}_B(X_1) \\ \mathrm{vec}_{SB}(X_2) \\ \mathrm{vec}_{SB}(X_3) \\ \mathrm{vec}_{SB}(X_4) \end{bmatrix} \\ \begin{bmatrix} \mathrm{vec}_B(Y_1) \\ \mathrm{vec}_{SB}(Y_2) \\ \mathrm{vec}_{SB}(Y_3) \\ \mathrm{vec}_{SB}(Y_4) \end{bmatrix} \end{pmatrix} = \mathrm{vec}\left(O_c^R\right). \tag{3.7}$$

Applying Lemma 2.7 to the real linear system above yields the following corresponding solution in the least-squares sense:

$$\begin{pmatrix} \begin{bmatrix} \mathrm{vec}_B(X_1) \\ \mathrm{vec}_{SB}(X_2) \\ \mathrm{vec}_{SB}(X_3) \\ \mathrm{vec}_{SB}(X_4) \end{bmatrix} \\ \begin{bmatrix} \mathrm{vec}_B(Y_1) \\ \mathrm{vec}_{SB}(Y_2) \\ \mathrm{vec}_{SB}(Y_3) \\ \mathrm{vec}_{SB}(Y_4) \end{bmatrix} \end{pmatrix} = (\mathcal{RPST})^{\dagger}\, \mathrm{vec}\left(O_c^R\right) + \left[I - (\mathcal{RPST})^{\dagger}\, (\mathcal{RPST})\right] h.$$

Then, multiplying both sides by $\mathcal{T}$ and considering equalities (3.4) and (3.5), we obtain the following:

$$\left(\begin{bmatrix}\mathrm{vec}(X_1)\\ \mathrm{vec}(X_2)\\ \mathrm{vec}(X_3)\\ \mathrm{vec}(X_4)\end{bmatrix}\atop\begin{bmatrix}\mathrm{vec}(Y_1)\\ \mathrm{vec}(Y_2)\\ \mathrm{vec}(Y_3)\\ \mathrm{vec}(Y_4)\end{bmatrix}\right)=\mathcal{T}\Big[(\mathcal{RPST})^{\dagger}\,\mathrm{vec}\Big(O_c^R\Big)+\Big[I-(\mathcal{RPST})^{\dagger}\,(\mathcal{RPST})\Big]h\Big].$$

Hence, the desired result is obtained for $\mathcal{U}=\mathcal{T}$. Consequently, the minimum norm least-squares bi-Hermitian solution $(\widehat{X},\widehat{Y})$ of Eq (1.1) satisfies the equality

$$\begin{bmatrix}\check{x}\\ \check{y}\end{bmatrix}=\mathcal{T}\Big(\mathcal{RPST}\Big)^{\dagger}\mathrm{vec}\Big(O_c^R\Big),$$

thus completing the proof.

Corollary 3.1. *Let* $K\in\mathbb{Q}^{m\times p}$*,* $L\in\mathbb{Q}^{p\times n}$*,* $M\in\mathbb{Q}^{m\times q}$*,* $N\in\mathbb{Q}^{q\times n}$*,* $O\in\mathbb{Q}^{m\times n}$*,* $X=X_1+X_2i+X_3j+X_4k\in\Omega_p$*,* $Y=Y_1+Y_2i+Y_3j+Y_4k\in\Omega_q$*, and* $\mathcal{R}=\Big[(L_c^R)^T\otimes K^R\quad (N_c^R)^T\otimes M^R\Big]$*. Then, Eq* (1.1) *has bi-Hermitian and/or skew bi-Hermitian solutions if and only if*

$$\Big[I_{4mn}-(\mathcal{RPSU})(\mathcal{RPSU})^{\dagger}\Big]\mathrm{vec}\left(O_c^R\right)=0. \tag{3.8}$$

In this case, the set of bi-Hermitian and/or skew bi-Hermitian solutions of Eq (1.1) is given by the set (3.1). Here, the matrix $\mathcal{U}$ is the matrix defined in Theorem 3.1.

Proof. Again, the proof will only be carried out for the case when both the matrices X and Y are bi-Hermitian, that is, when $\mathcal{U}=\mathcal{T}$. The proofs for the other cases are completely similar. In order for Eq (1.1) to have bi-Hermitian solutions, it is necessary and sufficient that there exist matrices $X\in\mathbb{BQ}^{p\times p}$ and $Y\in\mathbb{BQ}^{q\times q}$ such that $\|KXL+MYN-O\|=0$. By applying the properties of the Moore–Penrose inverse and using equalities (3.6) and (3.7), we have the following:

$$\|KXL+MYN-O\|=\left\|(\mathcal{RPST})\,(\mathcal{RPST})^{\dagger}\,(\mathcal{RPST})\left(\begin{bmatrix}\mathrm{vec}_B(X_1)\\ \mathrm{vec}_{SB}(X_2)\\ \mathrm{vec}_{SB}(X_3)\\ \mathrm{vec}_{SB}(X_4)\end{bmatrix}\atop\begin{bmatrix}\mathrm{vec}_B(Y_1)\\ \mathrm{vec}_{SB}(Y_2)\\ \mathrm{vec}_{SB}(Y_3)\\ \mathrm{vec}_{SB}(Y_4)\end{bmatrix}\right)-\mathrm{vec}\Big(O_c^R\Big)\right\|$$

$$=\Big\|(\mathcal{RPST})\,(\mathcal{RPST})^{\dagger}\,\mathrm{vec}\Big(O_c^R\Big)-\mathrm{vec}\Big(O_c^R\Big)\Big\|$$

$$=\Big\|\Big[I_{4mn}-(\mathcal{RPST})\,(\mathcal{RPST})^{\dagger}\Big]\mathrm{vec}\Big(O_c^R\Big)\Big\|.$$

Therefore, a necessary and sufficient condition for Eq (1.1) to possess a bi-Hermitian solution (X, Y) is that equality (3.8) holds. From this, the bi-Hermitian solution of (1.1) is obtained by solving the linear system (3.7).

4. Algorithm related to Problem 1.1

In this section, we present a brief pseudocode of the algorithm implemented in MATLAB R2024a, which details the solution steps for Problem 1.1 under consideration.

Algorithm 4.1.
Step 1. Generate the entries of quaternion matrices $K = K_1 + K_2 i + K_3 j + K_4 k \in \mathbb{Q}^{m\times p}$, $L = L_1 + L_2 i + L_3 j + L_4 k \in \mathbb{Q}^{p\times n}$, $M = M_1 + M_2 i + M_3 j + M_4 k \in \mathbb{Q}^{m\times q}$, $N = N_1 + N_2 i + N_3 j + N_4 k \in \mathbb{Q}^{q\times n}$, and $O = O_1 + O_2 i + O_3 j + O_4 k \in \mathbb{Q}^{m\times n}$ randomly and independently from a uniform distribution over (0, 1) using the following commands:

$$\begin{aligned} K &= quaternion(rand(m,p), rand(m,p), rand(m,p), rand(m,p)); \\ L &= quaternion(rand(p,n), rand(p,n), rand(p,n), rand(p,n)); \\ M &= quaternion(rand(m,q), rand(m,q), rand(m,q), rand(m,q)); \\ N &= quaternion(rand(q,n), rand(q,n), rand(q,n), rand(q,n)); \\ O &= quaternion(rand(m,n), rand(m,n), rand(m,n), rand(m,n)). \end{aligned}$$

Step 2. Construct the matrices $K^R \in \mathbb{Q}^{4m\times 4p}$, $M^R \in \mathbb{Q}^{4m\times 4q}$, $L_c^R \in \mathbb{Q}^{4p\times n}$, $N_c^R \in \mathbb{Q}^{4q\times n}$, and $O_c^R \in \mathbb{Q}^{4m\times n}$ according to equalities (2.1) and (2.2).

Step 3. Create the matrices $\mathcal{P}_X \in \mathbb{R}^{16p^2\times 4p^2}$ and $\mathcal{P}_Y \in \mathbb{R}^{16q^2\times 4q^2}$ derived from Lemma 2.2, and $\mathcal{S}_X \in \mathbb{R}^{4p^2\times 4p^2}$ and $\mathcal{S}_Y \in \mathbb{R}^{4q^2\times 4q^2}$ derived from Lemma 2.3, associated with the matrices X and Y, respectively. Then, calculate the matrices $\operatorname{diag}(\mathcal{P}_X, \mathcal{P}_Y) := \boldsymbol{\mathcal{P}}$ and $\operatorname{diag}(\mathcal{S}_X, \mathcal{S}_Y) := \boldsymbol{\mathcal{S}}$. Subsequently, the matrix $\boldsymbol{\mathcal{R}}$ is calculated as $\left[\left(L_c^R\right)^T \otimes K^R \quad \left(N_c^R\right)^T \otimes M^R\right]$ and creates the vector $\operatorname{vec}\left(O_c^R\right)$.

Step 4. According to whether p and q are odd or even, compute the matrices $V_{(p)}$, $V_{(q)}$, $W_{(p)}$, and $W_{(q)}$ according to equalities (2.7) and (2.9).

Step 5. If we seek $\widehat{X} \in \mathbb{BQ}^{p\times p}$, form $\mathcal{T}_X = \operatorname{diag}(V_{(p)}, W_{(p)}, W_{(p)}, W_{(p)})$; if we seek $\widehat{Y} \in \mathbb{BQ}^{q\times q}$, form $\mathcal{T}_Y = \operatorname{diag}(V_{(q)}, W_{(q)}, W_{(q)}, W_{(q)})$; if we seek $\widehat{X} \in \mathbb{SBQ}^{p\times p}$, form $\widetilde{\mathcal{T}_X} = \operatorname{diag}(W_{(p)}, V_{(p)}, V_{(p)}, V_{(p)})$; and if we seek $\widehat{Y} \in \mathbb{SBQ}^{q\times q}$, form $\widetilde{\mathcal{T}_Y} = \operatorname{diag}(W_{(q)}, V_{(q)}, V_{(q)}, V_{(q)})$. Then, create the matrices $\operatorname{diag}(\mathcal{T}_X, \mathcal{T}_Y) := \boldsymbol{\mathcal{T}}$ for both matrices X and Y if they are bi-Hermitian, $\operatorname{diag}(\widetilde{\mathcal{T}_X}, \widetilde{\mathcal{T}_Y}) := \dot{\boldsymbol{\mathcal{T}}}$ for both matrices X and Y if they are skew bi-Hermitian, and $\operatorname{diag}(\mathcal{T}_X, \widetilde{\mathcal{T}_Y}) := \acute{\boldsymbol{\mathcal{T}}}$ for one of them (say X) if it is bi-Hermitian and the other (Y) if it is skew bi-Hermitian.

Step 6. Compute the minimum norm least-squares solution vector as follows:

$$\begin{bmatrix} \check{x} \\ \check{y} \end{bmatrix} = \boldsymbol{\mathcal{U}}(\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{P}}\boldsymbol{\mathcal{S}}\boldsymbol{\mathcal{U}})^{\dagger} \operatorname{vec}\left(O_c^R\right),$$

where the matrix $\boldsymbol{\mathcal{U}}$ replaces the matrix $\boldsymbol{\mathcal{T}}$, $\dot{\boldsymbol{\mathcal{T}}}$, or $\acute{\boldsymbol{\mathcal{T}}}$ if both the matrices X and Y are bi-Hermitian, both are skew bi-Hermitian, or one of them (say X) is bi-Hermitian and the other (Y) is skew bi-Hermitian, respectively.

Step 7. As a final step, calculate the desired minimum norm least-squares solutions of the quaternion matrix Eq (1.1) using the "reshape" function in MATLAB R2024a.

5. Numerical examples

Now, some numerical examples will be given using Algorithm 4.1. These examples include the cases where both matrices X and Y are bi-Hermitian, both are skew bi-Hermitian, or one of them (say X) is bi-Hermitian and the other (Y) is skew bi-Hermitian.

In each example of the quaternion matrix Eq (1.1), the quaternion matrices $K \in \mathbb{Q}^{m\times p}$, $L \in \mathbb{Q}^{p\times n}$, $M \in \mathbb{Q}^{m\times q}$, $N \in \mathbb{Q}^{q\times n}$, and $O \in \mathbb{Q}^{m\times n}$ were generated using the 'rand' function in MATLAB R2024a. In other words, each entry of these matrices was independently selected from the uniform distribution on interval (0,1). Also, note that the entries of the matrices were rounded to two decimal digits; the precise data and generated matrices are provided on the web*.

Example 5.1. *Consider the quaternion matrix Eq* (1.1) *with* $m = 2$, $n = 2$, $p = 3$, *and* $q = 5$. *The coefficient matrices are given by the following:*

$$K = \begin{bmatrix} 0.40 + 0.41i + 0.33j + 0.24k & 0.23 + 0.90i + 0.36j + 0.09k & 0.18 + 0.49i + 0.78j + 0.94k \\ 0.07 + 0.04i + 0.90j + 0.40k & 0.12 + 0.94i + 0.11j + 0.13k & 0.23 + 0.48i + 0.38j + 0.95k \end{bmatrix},$$

$$L = \begin{bmatrix} 0.57 + 0.04i + 0.54j + 0.36k & 0.35 + 0.73i + 0.18j + 0.08k \\ 0.05 + 0.16i + 0.29j + 0.62k & 0.82 + 0.64i + 0.68j + 0.92k \\ 0.23 + 0.64i + 0.74j + 0.78k & 0.01 + 0.45i + 0.18j + 0.77k \end{bmatrix},$$

$$M = M_1 + M_2 i + M_3 j + M_4 k,$$

where

$$M_1 = \begin{bmatrix} 0.48 & 0.44 & 0.50 & 0.81 & 0.64 \\ 0.43 & 0.30 & 0.51 & 0.79 & 0.37 \end{bmatrix},$$

$$M_2 = \begin{bmatrix} 0.81 & 0.35 & 0.87 & 0.62 & 0.20 \\ 0.53 & 0.93 & 0.55 & 0.58 & 0.30 \end{bmatrix},$$

$$M_3 = \begin{bmatrix} 0.47 & 0.84 & 0.22 & 0.22 & 0.31 \\ 0.23 & 0.19 & 0.17 & 0.43 & 0.92 \end{bmatrix},$$

$$M_4 = \begin{bmatrix} 0.43 & 0.90 & 0.43 & 0.25 & 0.59 \\ 0.18 & 0.97 & 0.11 & 0.40 & 0.26 \end{bmatrix},$$

$$N = \begin{bmatrix} 0.60 + 0.80i + 0.52j + 0.91k & 0.31 + 0.57i + 0.39j + 0.67k \\ 0.71 + 0.02i + 0.23j + 0.79k & 0.42 + 0.23i + 0.36j + 0.13k \\ 0.22 + 0.92i + 0.48j + 0.09k & 0.50 + 0.45i + 0.98j + 0.72k \\ 0.11 + 0.73i + 0.62j + 0.26k & 0.08 + 0.96i + 0.03j + 0.10k \\ 0.29 + 0.48i + 0.67j + 0.33k & 0.26 + 0.54i + 0.88j + 0.65k \end{bmatrix},$$

$$O = \begin{bmatrix} 0.49 + 0.89i + 0.03j + 0.90k & 0.71 + 0.69i + 0.50j + 0.61k \\ 0.77 + 0.33i + 0.74j + 0.60k & 0.90 + 0.19i + 0.47j + 0.85k \end{bmatrix}.$$

*https://drive.google.com/drive/folders/1aggGkAiQ2qNa7UeGdBjbUGEnEictVuZC

The minimum norm least-squares solution $(\widehat{X}, \widehat{Y})$ *of the quaternion matrix Eq* (1.1)*, where both* $\widehat{X}$ *and* $\widehat{Y}$ *are bi-Hermitian matrices, is given by*

$$\widehat{X} = \begin{bmatrix} -0.12 & -0.17 + 0.25i + 0.08j - 0.46k & -0.13 \\ -0.17 - 0.25i - 0.08j + 0.46k & 0.11 & -0.17 - 0.25i - 0.08j + 0.46k \\ -0.13 & -0.17 + 0.25i + 0.08j - 0.46k & -0.12 \end{bmatrix}$$

and

$$\widehat{Y} = \begin{bmatrix} -0.00 & -0.07 - 0.40i + 0.32j - 0.02k & -0.01 - 0.25i + 0.09j - 0.10k & 0.21 + 0.17i - 0.33j + 0.01k & -0.00 \\ -0.07 + 0.40i - 0.32j + 0.02k & -0.18 & 0.35 + 0.11i - 0.21j + 0.10k & -0.07 & 0.21 - 0.17i + 0.33j - 0.01k \\ -0.01 + 0.25i - 0.09j + 0.10k & 0.35 - 0.11i + 0.21j - 0.10k & -0.02 & 0.35 - 0.11i + 0.21j - 0.10k & -0.01 + 0.25i - 0.09j + 0.10k \\ 0.21 - 0.17i + 0.33j - 0.01k & -0.07 & 0.35 + 0.11i - 0.21j + 0.10k & -0.18 & -0.07 + 0.40i - 0.32j + 0.02k \\ -0.00 & 0.21 + 0.17i - 0.33j + 0.01k & -0.01 - 0.25i + 0.09j - 0.10k & -0.07 - 0.40i + 0.32j - 0.02k & -0.00 \end{bmatrix}.$$

Moreover, the residual norm satisfies the following:

$$\left\| K\widehat{X}L + M\widehat{Y}N - O \right\| = 2.7811 \times 10^{-15}.$$

Example 5.2. *For the next example, the quaternion matrix Eq* (1.1) *is examined for* $m = 2$*,* $n = 2$*,* $p = 4$*, and* $q = 4$*. The corresponding coefficient matrices are given by* $K = K_1 + K_2 i + K_3 j + K_4 k$*, where*

$$K_1 = \begin{bmatrix} 0.80 & 0.18 & 0.88 & 0.48 \\ 0.57 & 0.23 & 0.02 & 0.16 \end{bmatrix},$$

$$K_2 = \begin{bmatrix} 0.97i & 0.50 & 0.05 & 0.04 \\ 0.71i & 0.47 & 0.68 & 0.07 \end{bmatrix},$$

$$K_3 = \begin{bmatrix} 0.52 & 0.81 & 0.72 & 0.65 \\ 0.09 & 0.81 & 0.14 & 0.51 \end{bmatrix},$$

$$K_4 = \begin{bmatrix} 0.97 & 0.80 & 0.43 & 0.08 \\ 0.64 & 0.45 & 0.82 & 0.13 \end{bmatrix},$$

$$L = \begin{bmatrix} 0.17 + 0.65i + 0.37j + 0.26k & 0.06 + 0.01i + 0.95j + 0.41k \\ 0.39 + 0.62i + 0.19j + 0.42k & 0.39 + 0.98i + 0.92j + 0.98k \\ 0.83 + 0.29i + 0.48j + 0.54k & 0.52 + 0.16i + 0.05j + 0.30k \\ 0.80 + 0.43i + 0.33j + 0.94k & 0.41 + 0.10i + 0.73j + 0.70k \end{bmatrix},$$

$$M = M_1 + M_2 i + M_3 j + M_4 k,$$

where

$$M_1 = \begin{bmatrix} 0.66 & 0.69 & 0.17 & 0.99 \\ 0.53 & 0.66 & 0.12 & 0.17 \end{bmatrix},$$

$$M_2 = \begin{bmatrix} 0.03 & 0.88 & 0.19 & 0.46 \\ 0.56 & 0.66 & 0.36 & 0.98 \end{bmatrix},$$

$$M_3 = \begin{bmatrix} 0.15 & 0.64 & 0.19 & 0.48 \\ 0.85 & 0.37 & 0.42 & 0.12 \end{bmatrix},$$

$$M_4 = \begin{bmatrix} 0.58 & 0.38 & 0.25 & 0.61 \\ 0.22 & 0.58 & 0.29 & 0.26 \end{bmatrix},$$

$$N = \begin{bmatrix} 0.82 + 0.81i + 0.42j + 0.06k & 0.58 + 0.42i + 0.69j + 0.40k \\ 0.98 + 0.26i + 0.09j + 0.31k & 0.10 + 0.31i + 0.69j + 0.81k \\ 0.73 + 0.59i + 0.59j + 0.53k & 0.90 + 0.16i + 0.63j + 0.71k \\ 0.34 + 0.02i + 0.47j + 0.65k & 0.87 + 0.17i + 0.03j + 0.96k \end{bmatrix},$$

$$O = \begin{bmatrix} 0.53 + 0.77i + 0.15j + 0.45k & 0.10 + 0.09i + 0.44j + 0.51k \\ 0.32 + 0.42i + 0.28j + 0.87k & 0.61 + 0.26i + 0.52j + 0.94k \end{bmatrix}.$$

The minimum norm least-squares solution $(\widehat{X}, \widehat{Y})$ *of the quaternion matrix Eq* (1.1)*, where both* $\widehat{X}$ *and* $\widehat{Y}$ *are skew bi-Hermitian matrices, is given by the following:*

$$\widehat{X} = \begin{bmatrix} -0.06i - 0.05j - 0.02k & 0.14 - 0.03i + 0.01j + 0.06k & -0.26 - 0.05i - 0.25j - 0.11k & 0.11i + 0.17j - 0.15k \\ -0.14 - 0.03i + 0.01j + 0.06k & 0.10i - 0.15j - 0.19k & -0.18i - 0.10j + 0.04k & 0.26 - 0.05i - 0.25j - 0.11k \\ 0.26 - 0.05i - 0.25j - 0.11k & -0.18i - 0.10j + 0.04k & 0.10i - 0.15j - 0.19k & -0.14 - 0.03i + 0.01j + 0.06k \\ 0.11i + 0.17j - 0.15k & -0.26 - 0.05i - 0.25j - 0.11k & 0.14 - 0.03i + 0.01j + 0.06k & -0.06i - 0.05j - 0.02k \end{bmatrix}$$

and

$$\widehat{Y} = \begin{bmatrix} -0.19i + 0.04j - 0.02k & -0.28 + 0.01i + 0.04j + 0.01k & -0.07 + 0.13i + 0.05j + 0.10k & -0.03i - 0.13j - 0.00k \\ 0.28 + 0.01i + 0.04j + 0.01k & -0.05i + 0.15j + 0.02k & -0.07i + 0.00j - 0.02k & 0.07 + 0.13i + 0.05j + 0.10k \\ 0.07 + 0.13i + 0.05j + 0.10k & -0.07i + 0.00j - 0.02k & -0.05i + 0.15j + 0.02k & 0.28 + 0.01i + 0.04j + 0.01k \\ -0.03i - 0.13j - 0.00k & -0.07 + 0.13i + 0.05j + 0.10k & -0.28 + 0.01i + 0.04j + 0.01k & -0.19i + 0.04j - 0.02k \end{bmatrix}.$$

The residual norm corresponding to this solution satisfies the following:

$$\|K\widehat{X}L + M\widehat{Y}N - O\| = 5.0652 \times 10^{-15}.$$

Example 5.3. *In this example, we consider the quaternion matrix Eq* (1.1) *for the case when* $m = 2$*,* $n = 3$*,* $p = 4$*, and* $q = 5$*. The coefficient matrices involved in the equation are given by* $K = K_1 + K_2 i + K_3 j + K_4 k$*, where*

$$K_1 = \begin{bmatrix} 0.63 & 0.24 & 0.28 & 0.69 \\ 0.95 & 0.67 & 0.67 & 0.06 \end{bmatrix},$$

$$K_2 = \begin{bmatrix} 0.25 & 0.66 & 0.34 & 0.67 \\ 0.22 & 0.84 & 0.78 & 0.00 \end{bmatrix},$$

$$K_3 = \begin{bmatrix} 0.60 & 0.91 & 0.46 & 0.46 \\ 0.38 & 0.00 & 0.42 & 0.77 \end{bmatrix},$$

$$K_4 = \begin{bmatrix} 0.32 & 0.47 & 0.17 & 0.47 \\ 0.78 & 0.03 & 0.72 & 0.15 \end{bmatrix},$$

$$L = \begin{bmatrix} 0.34 + 0.68i + 0.60j + 0.58k & 0.24 + 0.64i + 0.77j + 0.31k & 0.18 + 0.20i + 0.84j + 0.47k \\ 0.60 + 0.54i + 0.45j + 0.54k & 0.91 + 0.67i + 0.35j + 0.11k & 0.28 + 0.70i + 0.83j + 0.63k \\ 0.19 + 0.42i + 0.45j + 0.86k & 0.26 + 0.63i + 0.66j + 0.93k & 0.09 + 0.23i + 0.25j + 0.54k \\ 0.73 + 0.64i + 0.66j + 0.26k & 0.76 + 0.94i + 0.41j + 0.64k & 0.57 + 0.11i + 0.61j + 0.64k \end{bmatrix},$$

and

$$M = M_1 + M_2 i + M_3 j + M_4 k,$$

where

$$M_1 = \begin{bmatrix} 0.54 & 0.52 & 0.21 & 0.10 & 0.40 \\ 0.72 & 0.99 & 0.10 & 0.06 & 0.44 \end{bmatrix},$$

$$M_2 = \begin{bmatrix} 0.36 & 0.62 & 0.93 & 0.19 & 0.69 \\ 0.76 & 0.77 & 0.97 & 0.13 & 0.09 \end{bmatrix},$$

$$M_3 = \begin{bmatrix} 0.52 & 0.86 & 0.39 & 0.74 & 0.34 \\ 0.53 & 0.48 & 0.67 & 0.52 & 0.15 \end{bmatrix},$$

$$M_4 = \begin{bmatrix} 0.58k & 0.04k & 0.24k & 0.68k & 0.73k \\ 0.26k & 0.75k & 0.44k & 0.35k & 0.39k \end{bmatrix},$$

$$N = \begin{bmatrix} 0.68 + 0.75i + 0.58j + 0.49k & 0.42 + 0.32i + 0.82j + 0.92k & 0.88 + 0.16i + 0.11j + 0.98k \\ 0.70 + 0.37i + 0.15j + 0.14k & 0.27 + 0.67i + 0.78j + 0.69k & 0.39 + 0.86i + 0.13j + 0.00k \\ 0.44 + 0.21i + 0.19j + 0.05k & 0.19 + 0.43i + 0.31j + 0.58k & 0.76 + 0.98i + 0.67j + 0.86k \\ 0.01 + 0.79i + 0.40j + 0.85k & 0.82 + 0.83i + 0.53j + 0.81k & 0.39 + 0.51i + 0.49j + 0.61k \\ 0.33 + 0.94i + 0.74j + 0.56k & 0.42 + 0.76i + 0.08j + 0.87k & 0.80 + 0.88i + 0.18j + 0.98k \end{bmatrix}$$

and

$$O = \begin{bmatrix} 0.52 + 0.57i + 0.08j + 0.76k & 0.80 + 0.73i + 0.66j + 0.92k & 0.49 + 0.24i + 0.89j + 0.01k \\ 0.47 + 0.84i + 0.62j + 0.58k & 0.22 + 0.58i + 0.72j + 0.58k & 0.90 + 0.66i + 0.98j + 0.12k \end{bmatrix}.$$

For these data matrices, the minimum norm least-squares pair $(\widehat{X}, \widehat{Y})$ which satisfies the structural constraints that $\widehat{X}$ is bi-Hermitian and $\widehat{Y}$ is skew bi-Hermitian is obtained as follows:

$$\widehat{X} = \begin{bmatrix} -0.11 & 0.02 - 0.01i - 0.06j - 0.01k & 0.14 - 0.09i - 0.04j + 0.25k & 0.04 \\ 0.02 + 0.01i + 0.06j + 0.01k & 0.04 & -0.18 & 0.14 + 0.09i + 0.04j - 0.25k \\ 0.14 + 0.09i + 0.04j - 0.25k & -0.18 & 0.04 & 0.02 + 0.01i + 0.06j + 0.01k \\ 0.04 & 0.14 - 0.09i - 0.04j + 0.25k & 0.02 - 0.01i - 0.06j - 0.01k & -0.11 \end{bmatrix},$$

$$\widehat{Y} = \begin{bmatrix} -0.05i - 0.10j - 0.00k & 0.14 + 0.10i + 0.01j + 0.05k & 0.10 + 0.01i - 0.02j - 0.07k & -0.00 - 0.04i - 0.15j - 0.10k & -0.06i - 0.20j + 0.19k \\ -0.14 + 0.10i + 0.01j + 0.05k & 0.19i - 0.01j + 0.00k & 0.02 - 0.14i + 0.15j - 0.25k & -0.15i - 0.13j + 0.09k & 0.00 - 0.04i - 0.15j - 0.10k \\ -0.10 + 0.01i - 0.02j - 0.07k & -0.02 - 0.14i + 0.15j - 0.25k & 0.00i - 0.01j - 0.10k & -0.02 - 0.14i + 0.15j - 0.25k & -0.10 + 0.01i - 0.02j - 0.07k \\ 0.00 - 0.04i - 0.15j - 0.10k & -0.15i - 0.13j + 0.09k & 0.02 - 0.14i + 0.15j - 0.25k & 0.19i - 0.01j + 0.00k & -0.14 + 0.10i + 0.01j + 0.05k \\ -0.06i - 0.20j + 0.19k & -0.00 - 0.04i - 0.15j - 0.10k & 0.10 + 0.01i - 0.02j - 0.07k & 0.14 + 0.10i + 0.01j + 0.05k & -0.05i - 0.10j - 0.00k \end{bmatrix}.$$

We evaluate the residual of the equation to assess the accuracy of the computed solution. The corresponding residual norm is as follows:

$$\|K\widehat{X}L + M\widehat{Y}N - O\| = 4.8021 \times 10^{-15}.$$

6. Conclusions

We considered a least-squares problem for a generalized Sylvester-type matrix equation $KXL + MYN = O$ over quaternions, subject to bi-Hermitian or skew bi-Hermitian constraints. A direct real-valued reformulation leads to a fourfold increase in dimension; see (2.1), and the computational cost. To mitigate this, we exploited the fourth item of Lemma 2.1 and equality (2.2) to partially reduce the dimensional growth. Using this reduced-dimension representation, we obtained bi-Hermitian and/or skew bi-Hermitian solutions of the real matrix equation $K^R X^R L_c^R + M^R Y^R N_c^R = O_c^R$ via the vec operator, the Kronecker product, and the Moore–Penrose inverse. Then, the resulting real-valued solution was converted back to the corresponding quaternion solution of the original equation. Numerical implementation in MATLAB R2024a confirmed the effectiveness of the approach, thereby yielding residual norms that are nearly zero.

This study was inspired by the relationship between the standard vec operator and a customized vec operator for bi-Hermitian/skew bi-Hermitian real matrices presented in [18]. We extended this framework by providing compact forms for these matrices and solving the more general (1.1) matrix equation. Notably, to the best of our knowledge, the solutions established in this work for the generalized Sylvester-type quaternion matrix equation were reported in the literature for the first time.

Author contributions

Sinem Şimşek: contributed to the conceptualization, methodology, software, validation; Tuğba Demirkol: contributed to the conceptualization, methodology, writing–review and editing; Yıldız Kulaç: contributed methodology, software, writing–original draft, validation; Halim Özdemir: contributed to the conceptualization, supervision, and project administration. All authors confirm that they have read and approved the published version of the manuscript.

Use of Generative-AI tools declaration

Artificial intelligence tools were used only to improve the academic wording of a few sentences in the manuscript. The authors take full responsibility for the content of the manuscript.

Acknowledgements

The authors sincerely thank the reviewers for their valuable comments and constructive suggestions, which have significantly improved the quality of this manuscript.

Conflict of interest

The authors confirm that there are no conflicts of interest associated with this manuscript.

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