AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(4): 9146–9165.
DOI: 10.3934/math.2026377
Received: 22 February 2026
Revised: 21 March 2026
Accepted: 30 March 2026
Published: 02 April 2026

Research article

On weakly orthogonally invariant Finsler metrics with vanishing Douglas curvature

Newton Solórzano[1], **Dik Lujerio**[2], **Víctor León**[1,*] **and Alexis Rodriguez Carranza**[3]

[1] ILACVN - CICN, Universidade Federal da Integração Latino-Americana-UNILA, Itaipu Parquetec, Foz do Iguaçu-PR, 85867-970, Brazil

[2] Departamento Académico de Matemática de la Facultad de Ciencias, Universidad Nacional Santiago Antúnez de Mayolo-UNASAM, Av. Universitaria s/n, Shancayán, distrito de Independencia, Huaraz, Peru

[3] Departamento de Ciencias, Universidad Privada del Norte, UPN, Sede San Isidro, Av. El Ejercito 920, Trujillo, Peru

* **Correspondence:** Email: vicaml19@gmail.com.

Abstract: In this paper, we consider the *weakly orthogonally invariant Finsler metrics*, and we obtain their Douglas curvature. Furthermore, we derive the system of differential equations for weakly orthogonally invariant Finsler metrics with vanishing Douglas curvature. Many examples are included.

Keywords: Finsler metric; weakly orthogonally invariant; warped product; Douglas metric
Mathematics Subject Classification: 53B40, 53C60

1. Introduction

The Douglas curvature, introduced by J. Douglas [4] in 1927, is an important projective invariant in Finsler geometry. That is, if two Finsler metrics F and $\bar{F}$ are projectively equivalent, then F and $\bar{F}$ have the same Douglas curvature. A Finsler metric is called *Douglas metric* if their Douglas curvature vanishes. Douglas metrics are rich in the sense that all Rieman metrics and projectively flat metrics are also Douglas metrics. Besides, there are a lot of examples of Douglas metrics that are not Riemannian nor projectively flat. For instance, a Randers metric $F = \alpha + \beta$ is a Douglas metric if and only if β is closed ([1]). Other important contributions to the theory of Douglas metrics can be found in Li, Shen, and Shen [8], where a special class of Douglas metrics was studied.

On the other hand, there exist important Finsler metrics in the literature that satisfy

$$F\left((x^0, O\overline{x}), (y^0, O\overline{y})\right) = F\left((x^0, \overline{x}), (y^0, \overline{y})\right), \text{ for every } O \in O(n), \tag{1.1}$$

where $x = (x^0, \bar{x}) = (x^0, x^1, \ldots, x^n) \in M = I \times \mathbb{R}^n$ and $y = (y^0, \bar{y}) = (y^0, y^1, \ldots, y^n) \in T_xM$, like Shen's fish tank metric on $\Omega = \mathbb{B}^2 \times \mathbb{R} \subset \mathbb{R}^3$:

$$F = \frac{\sqrt{(-x^2y^1 + x^1y^2)^2 + ((y^1)^2 + (y^2)^2 + (y^3)^2)(1 - (x^1)^2 - (x^2)^2)}}{1 - (x^1)^2 - (x^2)^2} - \frac{x^2y^1 - x^1y^2}{1 - (x^1)^2 - (x^2)^2},$$

where $x = (x^1, x^2, x^3) \in \mathbb{B}^2 \times \mathbb{R}$ and $y = (y^1, y^2, y^3) \in T_xO$, or the spherically symmetric (or orthogonal invariance) Finsler metric [6, 14] :

$$F = |y|\phi\left(|x|, \frac{\langle x, y\rangle}{|y|}\right),$$

where $x \in M = \mathbb{R}^{n+1}$ and $y \in T_xM$, or the warped metrics [3, 7, 9, 12] defined on $I \times \mathbb{R}^n$ of the form

$$F = |\bar{y}|\phi\left(x^0, \frac{y^0}{|\bar{y}|}\right), \qquad F = |\bar{y}|\phi\left(\frac{y^0}{|\bar{y}|}, |\bar{x}|\right).$$

A Finsler metric F is called *weakly orthogonally invariant* ([11]) (or cylindrically symmetric in an alternative terminology in [13]) if F satisfies (1.1). In [11], Liu, Mo, and Zhu showed that weakly orthogonally invariant metrics are non-trivial in the sense that this type of metric is not of orthogonal invariance (see Proposition 2.2 in [11]).

In [2], Chávez showed that every weakly orthogonally invariant Finsler metric can be written as

$$F(x, y) = |\bar{y}|\phi\left(x^0, |\bar{x}|, \frac{\langle \bar{x}, \bar{y}\rangle}{|\bar{y}|}, \frac{y^0}{|\bar{y}|}\right),$$

where $|\cdot|$ and $\langle\cdot,\cdot\rangle$ are, respectively, the standard Euclidean norm and inner product on $\mathbb{R}^n$. Furthermore, Solórzano and León [13] provided necessary and sufficient conditions for $F = |\bar{y}|\phi$ to be a Finsler metric (Theorem 1 in [13]).

In [10], Liu and Mo obtained trough partial differential equations (PDEs), a characterization of weakly orthogonally invariant Finsler metrics that are Douglas type (Theorem 1.1 in [10]), and gave Douglas-Randers-type metrics as examples. In contrast to the work in [10], in this paper, we give the explicit Douglas curvature (Theorem 3) using the operator Ψ in (3.13) and its derivatives. This notation considerably reduces the expression of the Douglas curvature. Additionally, we give an equivalence of Theorem 1.1 in [10], which could be more manageable, using an alternative technique (see the proof of Lemma 2 in Section 3).

We set $M = I \times \mathbb{B}^n(\rho) \subset \mathbb{R} \times \mathbb{R}^n$, with coordinates on TM:

$$x = (x^0, \bar{x}), \quad \bar{x} = (x^1, \ldots, x^n), \tag{1.2}$$

$$y = (y^0, \bar{y}), \quad \bar{y} = (y^1, \ldots, y^n). \tag{1.3}$$

Throughout our work, the following convention for indices is adopted:

$$0 \leq A, B, \ldots \leq n,$$
$$1 \leq i, j, \ldots \leq n.$$

We introduce the notations

$$r := |\bar{x}|, \qquad s := \frac{\langle \bar{x}, \bar{y}\rangle}{|\bar{y}|}, \qquad z := \frac{y^0}{|\bar{y}|}, \tag{1.4}$$

where $|\cdot|$ and $\langle\cdot,\cdot\rangle$ are, respectively, the standard Euclidean norm and inner product on $\mathbb{R}^n$. In Section 3, we prove the following:

Theorem 1. Let $F(x,y) = |\overline{y}|\phi(x^0, r, s, z)$, be a Finsler metric defined on $M = I \times \mathbb{B}^n(\rho)$, $n \geq 3$, where $z = \frac{y^0}{|\overline{y}|}$, $r = |\overline{x}|$, and $s = \frac{\langle \overline{x}, \overline{y} \rangle}{|\overline{y}|}$, and TM defined with coordinates (1.2)–(1.3). Then F has vanishing Douglas curvature if and only if the positive function ϕ satisfies

$$z\phi_{x^0 s} - \frac{1}{r}\phi_r + \frac{s}{r}\phi_{rs} + \phi_{ss} = 2\left[\left(\phi - s\phi_s - z\phi_z + (r^2 - s^2)\phi_{ss}\right)U + \phi_{sz}L\right], \tag{1.5}$$

$$z\phi_{x^0 z} - \phi_{x^0} + \frac{s}{r}\phi_{rz} + \phi_{sz} = 2\left[\phi_{zz}L + (r^2 - s^2)\phi_{sz}U\right], \tag{1.6}$$

where

$$U = f_1\frac{s^2}{2} + f_2 sz + f_3\frac{z^2}{2} + f_4,$$
$$L = f_5\frac{s^2}{2} + f_6 sz + f_7\frac{z^2}{2} + f_8 - sz\left(f_1\frac{s^2}{2} + f_2 sz + f_3\frac{z^2}{2}\right),$$

and f_i $(i = 1, \ldots, 8)$ are arbitrary differentiable functions of (x^0, r). Furthermore, F is projectively flat if and only if $L = U = 0$.

Considering $U = 0$ and a nonzero constant $L = L_0$ in Theorem 1, in Section 3, we prove

Theorem 2. Let $L_0 \neq 0$ be a real constant. The family of functions given by

$$\begin{aligned}\phi =& C_1 + s g_1(r) + z g_2\left(x^0\right) + g_3(z + 2L_0 s) + \\ &+ I(x^0, z) + J(r, s) - \frac{1}{2}\int_0^{4L_0 x^0} g_4(u)du + \frac{1}{2}\int_0^{r^2} g_5(u)du,\end{aligned} \tag{1.7}$$

where

$$I(x^0, z) = \int_0^z \int_0^{\bar{z}} g_4\left(\zeta^2 + 4L_0 x^0\right) d\zeta d\bar{z}, \tag{1.8}$$

$$J(r, s) = \int_0^s \int_0^{\bar{s}} g_5\left(r^2 - \sigma^2\right) d\sigma d\bar{s}, \tag{1.9}$$

g_i, $i = 1, .., 5$ are arbitrary real differentiable functions, and C_1 is a constants real number, are solutions of the system (1.5)–(1.6) considering $U = 0$ and $L = L_0$. Moreover, any Finsler metric on $M = I \times \mathbb{B}^n(\rho)$ defined by

$$F(x, y) = |\overline{y}|\bar{\phi}\left(x^0, |\overline{x}|, \frac{\langle \overline{x}, \overline{y} \rangle}{|\overline{y}|}, \frac{y^0}{|\overline{y}|}\right),$$

where the positive function ϕ is given by (1.7) with $g_i, i = 3, 4, 5$ satisfying

$$\begin{aligned}\Lambda = &\,\Omega(g_3''(Z) + g_4(Z^2)) \\ &+ (r^2 - s^2)\left[g_3''(Z)(4L_0^2 g_4(Z^2) + g_5(r^2 - s^2)) + g_4(Z^2)\, g_5(r^2 - s^2)\right] > 0\end{aligned}$$

when $n = 2$, with additional inequality

$$\Omega = C_1 + \left(g_3(Z) - (Z)g_3'(Z)\right) - \frac{1}{2}\int_0^{Z^2} g_4(u)\,du + \frac{1}{2}\int_0^{r^2 - s^2} g_5(u)\,du > 0$$

when $n \geq 3$, is a non-locally projectively flat Douglas-type metric. Here, $Z = z + 2L_0 s$ and $Z^2 = z^2 + 4L_0 x^0$.

Remark 1. The conditions $\Omega > 0$ and $\Lambda > 0$ ensure that the Hessian matrix (g_{AB}) is positive definite, which guarantees that F defines a Finsler metric. In particular, $\Omega > 0$ and $\Lambda > 0$ are related to the strong convexity of F.

In Section 2, we give some preliminaries and recall some recent results about weakly orthogonally invariant Finsler metrics. In Section 3, we obtain their Douglas curvature (see Theorem 3) and demonstrate Theorems 1 and 2. Finally, in Section 4, we give some corollaries and examples.

2. Preliminaries

In this section, we give some notations, definitions, and lemmas that will be used in the proof of our main results. Let M be a manifold, and let $TM = \cup_{x\in M} T_xM$ be the tangent bundle of M, where T_xM is the tangent space at $x \in M$. We set $TM_o := TM \setminus \{0\}$, where $\{0\}$ stands for $\{(x, 0)|\, x \in M,\ 0 \in T_xM\}$. A *Finsler metric* on M is a function $F\colon TM \to [0, \infty)$ with the following properties:

(a) F is C^∞ on TM_o.
(b) At each point $x \in M$, the restriction $F_x := F|_{T_xM}$ is a Minkowski norm on T_xM.

Let $\mathbb{B}^n(\rho) \subset \mathbb{R}^n$ be the n-dimensional open ball of radius ρ and centered at the origin ($n \geq 2$).

In [2], Chávez proved that if the Finsler metric F satisfies (1.1), then there exists a positive function $\phi\colon \mathbb{R}^4 \to \mathbb{R}$ such that

$$F(x, y) = |\overline{y}|\phi(x^0, r, s, z). \tag{2.1}$$

On the other hand, defining Ω and Λ as,

$$\Omega :=\phi - s\phi_s - z\phi_z, \tag{2.2}$$
$$\Lambda :=\Omega\phi_{zz} + (r^2 - s^2)(\phi_{ss}\phi_{zz} - \phi_{sz}^2), \tag{2.3}$$

where the sub-indexes s, z are the partial derivatives with respect to s and z, respectively, the Hessian matrix $(g_{AB}) = \frac{1}{2}[F^2]_{y^A y^B} = \left(\begin{array}{c|c} g_{00} & g_{0j} \\ \hline g_{i0} & g_{ij} \end{array}\right)$ is given by

$$\begin{aligned} g_{00} &=\phi_z^2 + \phi\phi_{zz},\\ g_{i0} &=g_{0i} = (\phi\Omega)_z u^i + (\phi_s\phi_z + \phi\phi_{sz})x^i,\\ g_{ij} &=\phi\Omega\delta_{ij} + X_{ij}, \end{aligned}$$

where $X_{ij} = (u^i, x^i)\begin{pmatrix} -(s(\phi\Omega)_s + z(\phi\Omega)_z) & (\phi\Omega)_s \\ (\phi\Omega)_s & (\phi_s^2 + \phi\phi_{ss}) \end{pmatrix}\begin{pmatrix} u^j \\ x^j \end{pmatrix}$, with $u^j = \dfrac{y^j}{|\overline{y}|}$.

Note that the determinant of g_{AB} is given by

$$\det(g_{AB}) = \phi^{n+2}\Omega^{n-2}\Lambda.$$

With this, we recall the next result about the necessary and sufficiency condition for the function $F = |\overline{y}|\phi(x^0, r, s, z)$ to be a Finsler metric [13].

Proposition 1. Let $F = |\bar{y}|\phi(x^0, r, s, z)$ be a Finsler metric defined on M, where $z = \frac{y^0}{|\bar{y}|}$, $r = |\bar{x}|$, and $s = \frac{\langle \bar{x}, \bar{y} \rangle}{|\bar{y}|}$, and TM defined with coordinates (1.2)–(1.3). Then F is a Finsler metric if and only if the positive function ϕ satisfies $\Lambda > 0$ for $n = 2$ with the additional inequality $\Omega > 0$ for $n \geq 3$.

Remark 2. Note that through the paper, we are supposing $\phi_z \neq 0$.

The next proposition gives us one the most important quantities in Finsler geometry: The geodesic coefficients

$$G^A = Py^A + Q^A,$$

where

$$P := \frac{F_{x^C} y^C}{2F}, \qquad Q^A := \frac{F}{2} g^{AB} \left\{ F_{x^C y^B} y^C - F_{x^B} \right\},$$

where g^{AB} is the inverse of the matrix g_{AB} (see details in [13]) and has adopted the Einstein summation convention.

Proposition 2. Let $F = |\bar{y}|\phi(x^0, r, s, z)$ be a Finsler metric defined on M, where $z = \frac{y^0}{|\bar{y}|}$, $r = |\bar{x}|$, and $s = \frac{\langle \bar{x}, \bar{y} \rangle}{|\bar{y}|}$, and TM defined with coordinates (1.2)–(1.3). Then the geodesic spray coefficients G^A are given by

$$G^0 = u^2 \left\{ z(W + sU) + L \right\}, \tag{2.4}$$
$$G^i = u^2 W u_i + u^2 U x^i, \tag{2.5}$$

where $u = |\bar{y}|$, $u_i = \frac{y^i}{u}$, and Ω, Λ are given in (2.2)–(2.3), respectively, and

$$W := \frac{1}{\phi} \left\{ \frac{\varphi}{2} - s\phi U - \phi_z L - (r^2 - s^2)\phi_s U \right\},$$

with

$$\varphi := z\phi_{x^0} + \frac{s}{r}\phi_r + \phi_s,$$
$$U := \frac{1}{2\Lambda} \left\{ \left(\varphi_s - \frac{2}{r}\phi_r \right) \phi_{zz} - (\varphi_z - 2\phi_{x^0})\, \phi_{sz} \right\}, \tag{2.6}$$
$$V := \frac{1}{2\Lambda} \left\{ \left(\varphi_s - \frac{2}{r}\phi_r \right) \phi_{sz} - (\varphi_z - 2\phi_{x^0})\, \phi_{ss} \right\},$$
$$L := \frac{\Omega}{2\Lambda} (\varphi_z - 2\phi_{x^0}) - (r^2 - s^2) V. \tag{2.7}$$

3. Douglas curvature

A Finsler metric on an n-dimensional manifold N is called a *Douglas metric* if its geodesic coefficients $G^i = G^i(x, y)$ are given in the following form:

$$G^i = \frac{1}{2}\Gamma^i_{jk}(x) y^j y^k + P(x, y) y^i,$$

where $\Gamma^i_{jk}(x)$ are functions on N in local coordinates and $P(x, y)$ is a local positively y-homogeneous function of degree 1.

In [4], Douglas introduced the local functions $D_j{}^i{}_{kl}$ on TN^n defined by

$$D_j{}^i{}_{kl} := \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(G^i - \frac{1}{n+1}\sum_m \frac{\partial G^m}{\partial y^m} y^i\right)$$

in local coordinates $x^1, \ldots, x^n$ and $y = \sum_i y^i \partial/\partial x^i$. These functions are called *Douglas curvature* [4], and a Finsler metric F with $D_j{}^i{}_{kl} = 0$ is called *Douglas metric*.

Before we obtain the Douglas curvature for a weakly orthogonally invariant Finsler metric $F = |\bar{y}|\phi(x^0, r, s, z)$, we claim the next.

Lemma 1. Under the assumptions of Proposition 2, we have the following equalities:

$$\begin{aligned} G^0 - \frac{y^0}{n+2}\frac{\partial G^A}{\partial y^A} =& u^2 R, \\ G^i - \frac{y^i}{n+2}\frac{\partial G^A}{\partial y^A} =& u^2 U x^i - u^2 T u_i, \end{aligned}$$

where

$$R = L - \frac{z}{n+2}\left[L_z - (n-1)sU + (r^2 - s^2)U_s\right], \tag{3.1}$$

$$T = \frac{1}{n+2}\left\{3sU + L_z + (r^2 - s^2)U_s\right\}. \tag{3.2}$$

Proof. From (1.4), we have the partial derivatives of $u = |\bar{y}|$, s, and z, with respect to y^i:

$$u_j = \frac{y^j}{u}, \tag{3.3}$$

$$u_{jk} = \frac{1}{u}\left(\delta_{jk} - u_j u_k\right),$$

$$s_j = \frac{1}{u}\left(x^j - s u_j\right), \tag{3.4}$$

$$z_l = -\frac{z}{u} u_l. \tag{3.5}$$

For consistency with the Einstein summation convention, we adopt the identifications $u_i = u^i$ and $s_i = s^i$. From (3.3)–(3.5),

$$u_i u^i = 1, \qquad u_i x^i = s, \tag{3.6}$$

$$s_i x^i = \frac{r^2 - s^2}{u}, \qquad s_i u^i = 0, \tag{3.7}$$

$$u z_i x^i = -sz, \qquad z_i u^i = -\frac{z}{u}, \tag{3.8}$$

$$s_i s^i = \frac{r^2 - s^2}{u^2}.$$

Additionally, deriving G^0 given by (2.4), in relation to y^0, we have

$$\frac{\partial G^0}{\partial y^0} = u\left\{(W + sU) + z(W_z + sU_z) + L_z\right\}. \tag{3.9}$$

Note that uW (in (2.5)) is positive homogeneous of degree 1 on $y = (y^0, \overline{y})$. From Euler's theorem for homogeneous functions,

$$\frac{\partial(uW)}{\partial y^0} y^0 + \frac{\partial(uW)}{\partial y^i} y^i = uW,$$

then

$$\frac{\partial(uW)}{\partial y^i} y^i = u(W - zW_z). \tag{3.10}$$

From the definition of G^i in (2.5),

$$\sum \frac{\partial G^i}{\partial y^i} = \frac{\partial(uW)}{\partial y^i} y^i + nuW + 2Uuu_i x^i + u^2(U_s s_i + U_z z_i)x^i.$$

By using the identities (3.6)–(3.8) and (3.10), we obtain

$$\sum \frac{\partial G^i}{\partial y^i} = u\Big\{(n+1)W - zW_z + 2sU - szU_z + (r^2 - s^2)U_s\Big\}, \tag{3.11}$$

and consequently from (3.9) and (3.11), we have

$$\begin{aligned}\sum \frac{\partial G^A}{\partial y^A} &= \frac{\partial G^0}{\partial y^0} + \sum \frac{\partial G^i}{\partial y^i}\\ &= u\Big\{(n+2)W + 3sU + L_z + (r^2 - s^2)U_s\Big\}.\end{aligned} \tag{3.12}$$

Finally, using (2.4), (2.5), and (3.12), we obtain the result. □

To obtain the Douglas curvature of the weakly orthogonally invariant Finsler metric (2.1), for any differentiable function $\Theta = \Theta(s, z)$, we adopt the notation

$$\Psi(\Theta) = -s\Theta_s - z\Theta_z. \tag{3.13}$$

Remark 3. The operator Ψ arises in the computation of derivatives with respect to y^0 or y^i and provides an efficient way to reduce and organize the expressions associated with the variables $s = \frac{\langle \overline{x}, \overline{y}\rangle}{|\overline{y}|}$ and $z = \frac{y^0}{|\overline{y}|}$. The following identities for the operator Ψ will be frequently used in the computations of the Douglas curvature. They follow by direct differentiation and the definition of Ψ:

$$\begin{aligned}\Psi(\Psi(\Theta)) &= -\Psi(\Theta) - s\Psi(\Theta_s) - z\Psi(\Theta_z),\\ \frac{\Psi(z^m\Theta)}{z^m} &= \Psi(\Theta) - m\Theta,\ m \in \mathbb{Z}^*,\end{aligned}$$

$$\frac{\Psi\left(z^m\Theta\right)}{z^m} = \frac{\Psi\left(z^{m-1}\Theta\right)}{z^{m-1}} - \Theta,\ m \in \mathbb{Z}^*,$$

$$\Psi\left(z^2\Psi\left(\frac{\Theta}{z^2}\right)\right) = -sz\Psi\left(\frac{\Theta_s}{z}\right) - z^2\Psi\left(\frac{\Theta_z}{z}\right), \tag{3.14}$$

$$\frac{1}{z}\Psi\left(z^2\Psi\left(\frac{\Theta}{z}\right)\right) = -s\Psi(\Theta_s) - z\Psi(\Theta_z) - z\Psi\left(\frac{\Theta}{z}\right),$$

$$\Psi\left(\Theta_z\right) = \Psi_z(\Theta) + \Theta_z, \tag{3.15}$$

$$z\Psi_z(\Theta) = \Psi\left(z\Theta_z\right), \tag{3.16}$$

$$\Psi_s(\Theta) = \Psi(\Theta_s) - \Theta_s,$$

$$z\Psi_s\left(\frac{\Theta}{z}\right) = \Psi(\Theta_s), \tag{3.17}$$

$$\left(z\Psi\left(\frac{\Theta}{z}\right)\right)_z = \Psi(\Theta_z), \tag{3.18}$$

$$\Psi\left(z^2\Psi\left(\frac{\Theta_s}{z}\right)\right) = z\Psi_s\left(z^2\Psi\left(\frac{\Theta}{z^2}\right)\right),$$

$$\Psi_s\left(z^2\Psi\left(\frac{\Theta}{z^2}\right)\right) = \Psi\left(z\Psi\left(\frac{\Theta_s}{z}\right)\right) - z\Psi\left(\frac{\Theta_s}{z}\right).$$

As a consequence of the above identities, we derive the following expressions for the derivatives of Θ:

$$\frac{\partial\Theta}{\partial y^0} = \frac{\Theta_z}{u}, \tag{3.19}$$

$$u\frac{\partial\Theta}{\partial y^l} = \Theta_s x^l + \Psi(\Theta)u_l, \tag{3.20}$$

$$u\frac{\partial}{\partial y^k}\left(\Theta u_l\right) = \Theta\delta_{kl} + \Theta_s x^k u_l + \frac{1}{z}\Psi\left(z\Theta\right)u_k u_l, \tag{3.21}$$

$$u\frac{\partial}{\partial y^j}\left(\Theta u_k u_l\right) = \Theta\left(\delta_{jk}u_l\right)_{\overrightarrow{kl}} + \Theta_s x^j u_k u_l + \frac{1}{z^2}\Psi\left(z^2\Theta\right)u_j u_k u_l, \tag{3.22}$$

$$u\frac{\partial}{\partial y^j}\left(\Theta u_k u_l u_i\right) = \Theta(\delta_{jk}u_l u_i)_{\overrightarrow{kli}} + \Theta_s x^j u_k u_l u_i + \frac{1}{z^3}\Psi\left(z^3\Theta\right)u_j u_k u_l u_i, \tag{3.23}$$

where $(.)_{jkl}$ denotes the cyclic permutation (ex.: $(\delta_{jk}u_l u_i)_{\overrightarrow{kli}} = \delta_{jk}u_l u_i + \delta_{jl}u_i u_k + \delta_{ji}u_k u_l$).

Theorem 3. Let $F = |\overline{y}|\phi(x^0, r, s, z)$ be a Finsler metric defined on M, where $z = \frac{y^0}{|\overline{y}|}$, $r = |\overline{x}|$, and $s = \frac{\langle\overline{x},\overline{y}\rangle}{|\overline{y}|}$, and TM defined with coordinates (1.2)–(1.3). Then the Douglas curvature of F is given by

$$D^0_{000} = \frac{1}{u}R_{zzz},$$

$$D^0_{00l} = \frac{1}{u}\left\{R_{szz}x^l + \Psi(R_{zz})u_l\right\},$$

$$D^0_{0kl} = \frac{1}{u}\left\{R_{ssz}x^k x^l + \Psi\left(R_{sz}\right)\left(x^l u_k\right)_{\overrightarrow{lk}} + z\Psi\left(\frac{R_z}{z}\right)\delta_{kl} + \frac{1}{z}\Psi\left(z^2\Psi\left(\frac{R_z}{z}\right)\right)u_k u_l\right\},$$

$$D^0_{jkl} = \frac{1}{u}\left\{\frac{R_{sss}}{3}x^j x^k x^l + \Psi(R_{ss})x^j x^k u_l + z\Psi\left(\frac{R_s}{z}\right)x^j\delta_{kl} + \Psi\left(z^2\Psi\left(\frac{R}{z^2}\right)\right)u_j\delta_{kl}\right.$$
$$\left. + \frac{1}{z}\Psi\left(z^2\Psi\left(\frac{R_s}{z}\right)\right)x^j u_k u_l + \frac{1}{3z^2}\Psi\left(z^2\Psi\left(z^2\Psi\left(\frac{R}{z^2}\right)\right)\right)u_j u_k u_l\right\}_{\overrightarrow{jkl}},$$
$$D^i_{000} = \frac{1}{u}\left\{U_{zzz}x^i - T_{zzz}u_i\right\},$$
$$D^i_{00l} = \frac{1}{u}\left\{U_{szz}x^l x^i + \Psi(U_{zz})x^i u_l - T_{zz}\delta_{il} - T_{szz}x^l u_i - \Psi_z(T_z)u_l u_i\right\},$$
$$D^i_{0kl} = \frac{1}{u}\left\{U_{ssz}x^k x^l x^i + z\Psi\left(\frac{U_z}{z}\right)\delta_{kl}x^i + \frac{1}{z}\Psi\left(z^2\Psi\left(\frac{U_z}{z}\right)\right)u_k u_l x^i\right.$$
$$\left. - T_{ssz}x^k x^l u_i - \frac{1}{z^2}\Psi\left(z^2\Psi(T_z)\right)u_k u_l u_i\right\}$$
$$+ \frac{1}{u}\left\{\Psi(U_{sz})u_k x^l x^i - T_{sz}x^k\delta_{li} - \frac{1}{z}\Psi(zT_{sz})x^l u_k u_i\right\}_{\overrightarrow{kl}} - \frac{1}{u}\Psi(T_z)(\delta_{il}u_k)_{\overrightarrow{ikl}},$$

$$D^i_{jkl} = \frac{1}{u}\left\{U_{sss}x^j x^k x^l + \frac{1}{z^2}\Psi\left(z^2\Psi\left(z^2\Psi\left(\frac{U}{z^2}\right)\right)\right)u_j u_k u_l\right.$$
$$\left. + \left[\Psi(U_{ss})u_j x^k x^l + z\Psi\left(\frac{U_s}{z}\right)\delta_{jk}x^l + \Psi_s\left(z^2\Psi\left(\frac{U}{z^2}\right)\right)u_j u_k x^l + \Psi\left(z^2\Psi\left(\frac{U}{z^2}\right)\right)\delta_{jk}u_l\right]_{\overrightarrow{jkl}}\right\}x^i$$
$$- \frac{1}{u}\left\{\left[T_{ss}\delta_{ij}x^k x^l + \Psi_s(T_s)u_i u_j x^k x^l + \frac{1}{z^2}\Psi\left(z^2\Psi(T_s)\right)x^j u_k u_l u_i\right]_{\overrightarrow{jkl}}\right.$$
$$+ z\Psi\left(\frac{T}{z}\right)(\delta_{ji}\delta_{kl})_{\overrightarrow{ikl}} + \Psi(T_s)\left(x^j(u_i\delta_{kl})_{\overrightarrow{ikl}} + x^k(u_j\delta_{il})_{\overrightarrow{ijl}} + x^l(u_i\delta_{jk})_{\overrightarrow{ijk}}\right)$$
$$\left. + \frac{1}{z}\Psi\left(z^2\Psi\left(\frac{T}{z}\right)\right)(\delta_{ji}u_k u_l + \delta_{ik}u_l u_j)_{\overrightarrow{ikl}} + T_{sss}x^j x^k x^l u_i + \frac{1}{z^3}\Psi\left(z^2\Psi\left(z^2\Psi\left(\frac{T}{z}\right)\right)\right)u_j u_k u_l u_i\right\},$$

where $\Psi(\Theta) = -s\Theta_s - z\Theta_z$, $u = |\bar{y}|$, $u_i = \frac{\partial u}{\partial y^i} = \frac{y^i}{u}$, and $(\cdot)_{\overrightarrow{jkl}}$ denotes cyclic permutation.

Proof. By Lemma 1 and from (3.19)–(3.22), we have,

$$D_0{}^0{}_{00} = \frac{\partial^3}{\partial y^0\partial y^0\partial y^0}(u^2 R) = \frac{\partial^2}{\partial y^0\partial y^0}(u^2\frac{R_z}{u}) = \frac{\partial}{\partial y^0}(R_{zz}) = \frac{R_{zzz}}{u},$$
$$D_0{}^0{}_{0l} = \frac{\partial^3}{\partial y^0\partial y^0\partial y^l}(u^2 R) = \frac{\partial}{\partial y^l}(R_{zz}) = \frac{1}{u}\left[R_{szz}x^l + \Psi(R_{zz})u_l\right],$$

and using the identity $\Psi(R_{sz}) = z\Psi_s\left(\frac{R_z}{z}\right)$, where the sub-index s represents the partial derivative in s, we obtain

$$D_0{}^0{}_{kl} = \frac{\partial^2}{\partial y^k\partial y^l}(uR_z) = y^0\frac{\partial^2}{\partial y^k\partial y^l}\left(\frac{R_z}{z}\right) = \frac{\partial}{\partial y^k}\left(R_{sz}x^l + z\Psi\left(\frac{R_z}{z}\right)u_l\right)$$
$$= \frac{1}{u}\left\{R_{ssz}x^k x^l + \Psi(R_{sz})(x^l u_k)_{\overrightarrow{lk}} + z\Psi\left(\frac{R_z}{z}\right)\delta_{kl} + \frac{1}{z}\Psi\left(z^2\Psi\left(\frac{R_z}{z}\right)\right)u_k u_l\right\},$$

$$D_j{}^0{}_{kl} = (y^0)^2 \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(\frac{R}{z^2}\right) = y^0 \frac{\partial^2}{\partial y^j \partial y^k}\left(\frac{R_s}{z}x^l + z\Psi\left(\frac{R}{z^2}\right)u_l\right)$$
$$= \frac{\partial}{\partial y^j}\left[R_{ss}x^k x^l + z\Psi\left(\frac{R_s}{z}\right)(x^l u_k)_{\overrightarrow{kl}} + z^2\Psi\left(\frac{R}{z^2}\right)\delta_{kl} + \Psi\left(z^2\Psi\left(\frac{R}{z^2}\right)\right)u_k u_l\right].$$

From (2.5), (3.20), (3.21), and (3.16), $D_0{}^i{}_{00}$ and $D_0{}^i{}_{0l}$ are directly obtained. Using the properties of Ψ, we have

$$D_0{}^i{}_{kl} = y^0 \frac{\partial^2}{\partial y^k \partial y^l}\left(\frac{U_z}{z}x^i - \frac{T_z}{z}u_i\right)$$
$$= \frac{\partial}{\partial y^k}\left(U_{sz}x^l x^i + z\Psi\left(\frac{U_z}{z}\right)u_l x^i - T_z\delta_{li} - T_{sz}x^l u_i - \Psi(T_z)u_l u_i\right).$$

Analogous to the previous cases, using (3.23), we have

$$D_j{}^i{}_{kl} = \frac{\partial^2}{\partial y^j \partial y^k}\left(2Uuu_l x^i + uU_s x^l x^i + u\Psi(U)u_l x^i - 2uTu_l u_i - uT\delta_{li} - uT_s x^l u_i - \frac{\Psi(zT)}{z}uu_l u_i\right)$$
$$= \frac{\partial}{\partial y^j}\left\{\Psi\left(z^2\Psi\left(\frac{U}{z^2}\right)\right)u_k u_l x^i + z^2\Psi\left(\frac{U}{z^2}\right)\delta_{kl}x^i + z\Psi\left(\frac{U_s}{z}\right)(x^k u_l)_{\overrightarrow{kl}}x^i\right.$$
$$-\frac{1}{z}\Psi\left(z^2\Psi\left(\frac{T}{z}\right)\right)u_k u_l u_i - z\Psi\left(\frac{T}{z}\right)(\delta_{kl}u_i)_{\overrightarrow{kli}} - \Psi(T_s)(x^k u_l)_{\overrightarrow{kl}}u_i + U_{ss}x^l x^k x^i$$
$$\left.-T_{ss}u_i x^k x^l - T_s(x^k\delta_{li})_{\overrightarrow{kl}}\right\}.$$

□

Lemma 2. Let $F = |\overline{y}|\phi(x^0, r, s, z)$ be a Finsler metric defined on $I \times \mathbb{B}^n(\rho)$, $n \geq 3$, where $z = \frac{y^0}{|\overline{y}|}$, $r = |\overline{x}|$, and $s = \frac{\langle \overline{x}, \overline{y}\rangle}{|\overline{y}|}$, and TM defined with coordinates (1.2)–(1.3). Then F has vanishing Douglas curvature if and only if ϕ satisfies

$$(a)\quad z\Psi\left(\frac{U_s}{z}\right) = 0, \qquad (b)\quad z\Psi\left(\frac{U_z}{z}\right) = 0, \qquad (c)\quad U_{zzz} = 0, \tag{3.24}$$
$$(a)\quad z\Psi\left(\frac{R_s}{z}\right) = 0, \qquad (b)\quad z\Psi\left(\frac{R_z}{z}\right) = 0, \qquad (c)\quad R_{zzz} = 0, \tag{3.25}$$
$$(a)\quad z\Psi\left(\frac{T}{z}\right) = 0, \qquad (b)\quad T_{zz} = 0, \tag{3.26}$$

where $\Psi(\Theta) = -s\Theta_s - z\Theta_z$, and U, R, and T are given in (2.6), (3.1), and (3.2), respectively.

Proof. Suppose F has vanishing Douglas curvature. Consider the orthonormal matrix $O \in O(n)$ (see the proof of Proposition 1.3.1 in [5] or the proof of Lemma 1 in [2]) such that

$$\tilde{x} = O\overline{x} = (|\overline{x}|, 0, \ldots, 0),$$
$$\tilde{y} = O\overline{y} = \left(\frac{\langle \overline{x}, \overline{y}\rangle}{|\overline{x}|}, \frac{\sqrt{|\overline{x}|^2|\overline{y}|^2 - \langle \overline{x}, \overline{y}\rangle^2}}{|\overline{x}|}, 0, \ldots, 0\right).$$

For the invariance of r, s, and z under the action O, from $D_0{}^0{}_{00} = 0$, we obtain $R_{zzz} = 0$. From $D_0{}^0{}_{33} = 0$, we get

$$z\Psi\left(\frac{R_z}{z}\right) = 0. \tag{3.27}$$

Using property (3.14) and (3.27), we have

$$\Psi\left(z^2\Psi\left(\frac{R}{z^2}\right)\right) = -sz\Psi\left(\frac{R_s}{z}\right). \tag{3.28}$$

Due to the condition $D_1{}^0{}_{33} = 0$ and after performing a change of variables induced by an orthonormal matrix O, we may assume, without loss of generality, that the new coordinates (still denoted by x and y to simplify notation) satisfy $x^3 = 0$ and $u_3 = \frac{y^3}{u} = 0$. Consequently, in the expression of $D_1{}^0{}_{33}$, the only non-vanishing terms are those corresponding to the coefficients of $x^1\delta_{33}$ and $u_1\delta_{33}$. Hence,

$$rz\Psi\left(\frac{R_s}{z}\right) + \frac{s}{r}\Psi\left(z^2\Psi\left(\frac{R}{z^2}\right)\right) = 0. \tag{3.29}$$

Substituting (3.28) into (3.29), we get

$$\left(\frac{r^2 - s^2}{r}\right)\left(z\Psi\left(\frac{R_s}{z}\right)\right) = 0.$$

Hence,

$$z\Psi\left(\frac{R_s}{z}\right) = 0.$$

Thus, (3.25) is satisfied.

From $D_0{}^3{}_{03} = 0$ and $D_3{}^3{}_{33} = 0$, we have $T_{zz} = 0$ and $z\Psi\left(\frac{T}{z}\right) = 0$. Thus, (3.26) is satisfied.

From $D_0{}^1{}_{00} = 0$ and $T_{zz} = 0$, we get $U_{zzz} = 0$. From $D_0{}^1{}_{33} = 0$, we have

$$z\Psi\left(\frac{U_z}{z}\right) = 0.$$

From $D_3{}^1{}_{31} = 0$, we obtain

$$rz\Psi\left(\frac{U_s}{z}\right) + \frac{s}{r}\Psi\left(z^2\Psi\left(\frac{U}{z^2}\right)\right) = 0.$$

Similarly, as in the case of R, we conclude

$$\left(\frac{r^2 - s^2}{r}\right)\left(z\Psi\left(\frac{U_s}{z}\right)\right) = 0.$$

Therefore,

$$z\Psi\left(\frac{U_s}{z}\right) = 0,$$

and thus (3.24) is satisfied.

Conversely, assume that ϕ satisfies (3.24)–(3.26). From (3.25) (c), we get $D_0{}^0{}_{00} = 0$.

Using property (3.17) and (3.25) (a), (b), we obtain

$$\Psi(R_{ss}) = z\Psi_s\left(\frac{R_s}{z}\right) = 0 \tag{3.30}$$

and

$$\Psi(R_{sz}) = z\Psi_s\left(\frac{R_z}{z}\right) = 0. \tag{3.31}$$

By property (3.18) and (3.25) (b), we have

$$\Psi(R_{zz}) = \left(z\Psi\left(\frac{R_z}{z}\right)\right)_z = 0. \tag{3.32}$$

From (3.32) and (3.25) (c), we get

$$R_{zzs} = 0. \tag{3.33}$$

Therefore, by (3.32) and (3.33), we obtain ${D_0}^0{}_{0l} = 0$.

From (3.31) and (3.33), we also have

$$R_{zss} = 0. \tag{3.34}$$

Consequently, by (3.34), (3.31), and (3.25) (b), we have ${D_0}^0{}_{kl} = 0$.

From (3.30) and (3.34), we obtain

$$R_{sss} = 0. \tag{3.35}$$

Also, by property (3.14) and (3.25) (a), (b), we have

$$\Psi\left(z^2\Psi\left(\frac{R}{z^2}\right)\right) = -sz\Psi\left(\frac{R_s}{z}\right) - z^2\Psi\left(\frac{R_z}{z}\right) = 0. \tag{3.36}$$

Thus, by (3.35), (3.30), (3.25) (a), and (3.36), we conclude that ${D_j}^0{}_{kl} = 0$.

From (3.24) (c) and (3.26) (b), we have ${D_0}^i{}_{00} = 0$.

Now, by (3.24), analogously as in the case of R, we obtain

$$\Psi(U_{ss}) = \Psi(U_{sz}) = \Psi(U_{zz}) = 0, \tag{3.37}$$

$$U_{zss} = U_{zzs} = U_{sss} = 0, \quad \Psi\left(z^2\Psi\left(\frac{U}{z^2}\right)\right) = 0. \tag{3.38}$$

On the other hand, by property (3.15) and (3.26) (b), we have

$$\Psi_z(T_z) = \Psi(T_{zz}) - T_{zz} = 0. \tag{3.39}$$

Therefore, by (3.37), (3.38), (3.26) (b), and (3.39), we obtain ${D_0}^i{}_{0l} = 0$.

By property (3.18) and (3.26) (a), we get

$$\Psi(T_z) = \left(z\Psi\left(\frac{T}{z}\right)\right)_z = 0. \tag{3.40}$$

From (3.40) and (3.26) (b), we obtain

$$T_{sz} = 0. \tag{3.41}$$

Consequently, by (3.37), (3.38), (3.24) (b), (3.40), and (3.41), we get ${D_0}^i{}_{kl} = 0$.

By property (3.17) and (3.26) (a), we obtain

$$\Psi(T_s) = z\Psi_s\left(\frac{T}{z}\right) = 0. \tag{3.42}$$

From (3.42) and (3.41), we have

$$T_{ss} = 0. \tag{3.43}$$

Therefore, by (3.37), (3.38), (3.24) (a), (3.26) (a), (3.42), and (3.43), we obtain ${D_j}^i{}_{kl} = 0$. □

Proof of Theorem 1:

Proof. From (3.24)–(3.26), we have that there are differentiable functions $g_i = g_i(x^0, r)$ such that

$$\begin{aligned} U &= g_1\frac{s^2}{2} + g_2 sz + g_3\frac{z^2}{2} + g_4, \\ R &= g_5\frac{s^2}{2} + g_6 sz + g_7\frac{z^2}{2} + g_8, \\ T &= g_9 s + g_{10} z. \end{aligned}$$

From (3.1) and (3.2), we have that $R + zT = L + szU$, and then

$$\begin{aligned} L &= R + zT - szU \\ &= g_5\frac{s^2}{2} + g_6 sz + g_7\frac{z^2}{2} + g_8 + z(g_9 s + g_{10} z) - sz(f_1\frac{s^2}{2} + f_2 sz + f_3\frac{z^2}{2} + f_4). \end{aligned}$$

From definition of U and L in (2.6) and (2.7), we have

$$\phi_{zz}p_1 - \phi_{sz}p_2 = 2\Lambda U, \tag{3.44}$$

$$-(r^2 - s^2)\phi_{sz}p_1 + (\Omega + (r^2 - s^2)\phi_{ss})p_2 = 2\Lambda L, \tag{3.45}$$

where

$$p_1 := \left(\varphi_s - \frac{2}{r}\phi_r\right) = z\phi_{x^0 s} - \frac{1}{r}\phi_r + \frac{s}{r}\phi_{rs} + \phi_{ss}, \tag{3.46}$$

$$p_2 := (\varphi_z - 2\phi_{x^0}) = z\phi_{x^0 z} - \phi_{x^0} + \frac{s}{r}\phi_{rz} + \phi_{sz}, \tag{3.47}$$

and Ω, Λ are given in (2.2) and (2.3), respectively. Due to the fact $\Lambda \neq 0$, the system (3.44)–(3.45) is equivalent to the system (1.5)–(1.6).

On the other hand, suppose that

$$\begin{aligned} U &= f_1\frac{s^2}{2} + f_2 sz + f_3\frac{z^2}{2} + f_4, \\ L &= f_5\frac{s^2}{2} + f_6 sz + f_7\frac{z^2}{2} + f_8 - sz\left(f_1\frac{s^2}{2} + f_2 sz + f_3\frac{z^2}{2}\right). \end{aligned}$$

Then, from the definition of R and T in (3.1) and (3.2), we have

$$\begin{aligned} T &= \frac{3f_4 + f_6 + f_1 r^2}{n+2}\, s \;+\; \frac{f_7 + f_2 r^2}{n+2}\, z, \\ R &= f_8 + \frac{f_5}{2}\, s^2 + \frac{(n+1)f_6 + (n-1)f_4 - f_1 r^2}{n+2}\, sz + \frac{\frac{n}{2}f_7 - f_2 r^2}{n+2}\, z^2. \end{aligned}$$

□

Remark 4. From (1.5)–(1.6) and due to the fact a weakly orthogonally invariant Finsler metric $F = |\bar{y}|\phi(x^0, r, s, z)$ is projectively flat (see Theorem 1.1 in [11]) if and only if $p_1 = p_2 = 0$, then F is projectively flat if and only if $L = U = 0$.

Proof of Theorem 2. To construct this family, in Theorem 1, we suppose $U = 0$ and $L = L_0$ are real constants. Then, deriving (1.5) and (1.6) with respect to z and s, respectively, and subtracting them, we have

$$\phi_{rz} = r\phi_{x^0 s}. \tag{3.48}$$

Using this identity, Eqs (1.5) and (1.6) become

$$\Omega_r - r\phi_{ss} + 2rL_0\phi_{sz} = 0, \tag{3.49}$$
$$\Omega_{x^0} - \phi_{sz} + 2L_0\phi_{zz} = 0. \tag{3.50}$$

Now, note that ϕ given by

$$\phi(x^0, r, s, z) = \varepsilon_1(x^0, r) + \varepsilon_2(x^0, z) + \varepsilon_3(r, s) + \varepsilon_4(s, z) \tag{3.51}$$

satisfies Eq (3.48). Equations (3.49) and (3.50), in this case, become

$$[\varepsilon_1]_r + [\varepsilon_3]_r - s\,[\varepsilon_3]_{rs} - r\,[\varepsilon_3 + \varepsilon_4]_{ss} + 2rL_0\,[\varepsilon_4]_{sz} = 0, \tag{3.52}$$
$$[\varepsilon_1]_{x^0} + [\varepsilon_2]_{x^0} - z\,[\varepsilon_2]_{x^0 z} - [\varepsilon_4]_{sz} + 2L_0\,[\varepsilon_2 + \varepsilon_4]_{zz} = 0. \tag{3.53}$$

Notice that, throughout the proof, $h_i \colon \mathbb{R} \to \mathbb{R}$ are some differentiable real functions. Deriving (3.53) with respect to r,

$$[\varepsilon_1]_{x^0 r} = 0,$$

therefore,

$$\varepsilon_1 = h_1(x^0) + h_2(r).$$

Now, taking the derivative of (3.53) with respect to x^0 and z,

$$-z\left([\varepsilon_2]_{x^0 zz}\right)_{x^0} + 2L_0\left([\varepsilon_2]_{x^0 zz}\right)_z = 0,$$

and by the characteristics method, we have

$$[\varepsilon_2]_{x^0 zz} = h_3\left(z^2 + 4L_0 x^0\right).$$

Integrating and using $4L_0 h_4' = h_3$, we have

$$\varepsilon_2 = \iint h_4\left(z^2 + 4Lx^0\right) dz dz + h_5(z) + z \cdot h_6\left(x^0\right) + h_7\left(x^0\right).$$

Taking the derivative of (3.53) with respect to s,

$$\left([\varepsilon_4]_{sz}\right)_s - 2L\left([\varepsilon_4]_{sz}\right)_z = 0,$$

thus we have

$$\varepsilon_4 = h_8(z + 2Ls) - h_9(s) + h_{10}(z).$$

Taking the derivative of (3.52) with respect to s,

$$s\left([\varepsilon_3]_{ss}\right)_r + r\left([\varepsilon_3]_{ss}\right)_s = rh_9'''(s)$$

thus we have

$$\varepsilon_3 = \iint h_{11}(r^2 - s^2)dsds + h_9(s) + sh_{12}(r) + h_{13}(r).$$

We use the notations

$$\begin{aligned} I(x^0, z) &= \int_0^z \int_0^{\bar{z}} h_4\left(\zeta^2 + 4L_0x^0\right) d\zeta d\bar{z}, \\ J(r, s) &= \int_0^s \int_0^{\bar{s}} h_{11}\left(r^2 - \sigma^2\right) d\sigma d\bar{s}. \end{aligned}$$

From (3.52) and (3.53),

$$\begin{aligned} h_2'(r) + h_{12}'(r) + J_r - sJ_{rs} - rJ_{ss} = 0, \\ h_1'(x^0) + h_7'(x^0) + I_{x^0} - zI_{x^0z} + 2L_0I_{zz} + 2L_0[h_5''(z) + h_{10}''(z)] = 0. \end{aligned}$$

Note that $J_r - sJ_{rs} - rJ_{ss}$ and $I_{x^0} - zI_{x^0z} + 2L_0I_{zz}$ do not depend on s and z, respectively. Then,

$$\begin{aligned} h_2(r) + h_{13}(r) &= h_{14}(r), \\ h_1(x^0) + h_7(x^0) &= h_{15}(x^0) + Cx^0, \\ h_5(z) + h_{10}(z) &= C\frac{z^2}{4L_0} + c_2z + c_3, \end{aligned}$$

for some real constants C, c_1, c_2, and c_3. With this, from (3.51) and (1.5)–(1.6) and using the integral representations

$$I(x^0, z) = \int_0^z (z - \zeta)\, g_4(\zeta^2 + 4L_0x^0)\, d\zeta, \qquad J(r, s) = \int_0^s (s - \sigma)\, g_5(r^2 - \sigma^2)\, d\sigma,$$

we have the result. □

4. Douglas metric examples

Remark 5. In (1.8) and (1.9), the integrands are compositions of smooth functions, hence continuous. Therefore, Fubini's theorem applies and justifies the interchange of the order of integration. Consequently,

$$\begin{aligned} I(x^0, z) &= \int_0^z (z - \zeta)\, g_4(\zeta^2 + 4L_0x^0)\, d\zeta, \\ J(r, s) &= \int_0^s (s - \sigma)\, g_5(r^2 - \sigma^2)\, d\sigma. \end{aligned}$$

Example 1. Let $L_0 \neq 0$. Considering $g_3 = g_5 = 0$ and $g_4(u) = ae^{-u}$ with $a > 0$ and $C_1 > \frac{a}{2}$ in Theorem 2, using Remark 5, we have

$$I(x^0, z) = ae^{-4L_0x^0} \int_0^z (z-\zeta)e^{-\zeta^2}\, d\zeta.$$

Evaluating the integral

$$\int_0^z (z-\zeta)e^{-\zeta^2}\, d\zeta = z\int_0^z e^{-\zeta^2}\, d\zeta - \int_0^z \zeta e^{-\zeta^2}\, d\zeta = \frac{\sqrt{\pi}}{2} z \operatorname{erf}(z) - \frac{1}{2}(1-e^{-z^2}),$$

we obtain

$$\phi(x^0, r, s, z) = C_1 + s\, g_1(r) + z\, g_2(x^0) + \frac{a}{2}\Big(e^{-4L_0x^0}\big(\sqrt{\pi}\, z \operatorname{erf}(z) + e^{-z^2}\big) - 1\Big), \tag{4.1}$$

where $\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2}\, dt$ is the Gauss error function. With this,

$$\Omega = C_1 + \tfrac{a}{2}(e^{-4L_0x^0}e^{-z^2} - 1) = \Big(C_1 - \tfrac{a}{2}\Big) + \tfrac{a}{2}e^{-(z^2+4L_0x^0)}.$$

Since $a > 0$ and $C_1 > a/2$, we have $\Omega > 0$. Finally, because $\phi_{ss} = \phi_{sz} = 0$,

$$\Lambda = \Omega\, \phi_{zz} = \Omega \cdot a\, e^{-(z^2+4L_0x^0)}.$$

As $\phi_{zz} > 0$ and $\Omega > 0$, we obtain $\Lambda > 0$. Thus, considering $g_1 = g_2 = 0$, the Finsler metric

$$F(x,y) = |\overline{y}|\bar{\phi}\left(x^0, |\overline{x}|, \frac{\langle \overline{x}, \overline{y}\rangle}{|\overline{y}|}, \frac{y^0}{|\overline{y}|}\right),$$

where ϕ is given by (4.1), is Douglas type.

Corollary 1. Let $L_0 \neq 0$. Define

$$\phi(x^0, r, s, z) = C_1 + \frac{b}{2}(r^2 + s^2) + \frac{a}{2}\Big(e^{-4L_0x^0}\big(\sqrt{\pi}\, z \operatorname{erf}(z) + e^{-z^2}\big) - 1\Big), \tag{4.2}$$

with constants $a > 0$, $b \geq 0$, and $C_1 > a/2$. The Finsler metric

$$F(x,y) = |\overline{y}|\bar{\phi}\left(x^0, |\overline{x}|, \frac{\langle \overline{x}, \overline{y}\rangle}{|\overline{y}|}, \frac{y^0}{|\overline{y}|}\right),$$

where ϕ is given by (4.2), is Douglas type.

Proof. In Theorem 2, considering $g_4(u) = ae^{-u}$ and $g_5(u) = b$ ($b > 0$ is a positive constant), we have

$$\phi_s = bs, \quad \phi_{ss} = b, \quad \phi_z = \frac{a}{2}\sqrt{\pi}\, e^{-4L_0x^0} \operatorname{erf}(z), \quad \phi_{zz} = ae^{-(z^2+4L_0x^0)}, \quad \phi_{sz} = 0.$$

Then

$$\Omega = \phi - s\phi_s - z\phi_z = C_1 - \frac{a}{2} + \frac{b}{2}(r^2 - s^2) + \frac{a}{2}e^{-(z^2+4L_0x^0)} > 0,$$

$$\Lambda = \Omega\phi_{zz} + (r^2 - s^2)(\phi_{ss}\phi_{zz} - \phi_{sz}^2) = ae^{-(z^2+4L_0x^0)}(\Omega + b(r^2 - s^2)) > 0.$$

□

Example 2. Choosing $a = 1$, $b = 1$, and $C_1 = 1$ in Corollary 1, we have

$$\phi(x^0, r, s, z) = \frac{1}{2} + \frac{1}{2}(r^2 + s^2) + \frac{1}{2}e^{-4L_0x^0}\left(\sqrt{\pi}\, z\ \mathrm{erf}(z) + e^{-z^2}\right),$$

and for this ϕ, we have

$$\Omega = \frac{1}{2} + \frac{1}{2}e^{-(z^2+4L_0x^0)} + \frac{1}{2}(r^2 - s^2) > 0,$$
$$\Lambda = e^{-(z^2+4L_0x^0)}\Big(\frac{1}{2} + \frac{1}{2}e^{-(z^2+4L_0x^0)} + \frac{3}{2}(r^2 - s^2)\Big) > 0.$$

Corollary 2. Let $L_0 \neq 0$. Define

$$\phi(x^0, r, s, z) = C_1 + \frac{a}{2}\Big(e^{-4L_0x^0}(\sqrt{\pi}z\,\mathrm{erf}(z) + e^{-z^2}) - 1\Big) + \frac{b}{2}\Big(e^{r^2}(\sqrt{\pi}s\,\mathrm{erf}(s) + e^{-s^2}) - 1\Big), \tag{4.3}$$

with constants $a > 0$, $b > 0$, and C_1 sufficiently large. Then, the Finsler metric

$$F(x, y) = |\overline{y}|\bar{\phi}\left(x^0, |\overline{x}|, \frac{\langle\overline{x}, \overline{y}\rangle}{|\overline{y}|}, \frac{y^0}{|\overline{y}|}\right),$$

where ϕ is given by (4.3), is Douglas type.

Proof. Compute derivatives

$$\phi_s = \frac{b}{2}e^{r^2}(\sqrt{\pi}\,\mathrm{erf}(s) + 2se^{-s^2}), \quad \phi_{ss} = \frac{b}{2}e^{r^2}\frac{d^2}{ds^2}(\sqrt{\pi}s\,\mathrm{erf}(s) + e^{-s^2}),$$
$$\phi_z = \frac{a}{2}e^{-4L_0x^0}(\sqrt{\pi}\,\mathrm{erf}(z) - 2ze^{-z^2}), \quad \phi_{zz} = ae^{-4L_0x^0-z^2}, \quad \phi_{sz} = 0.$$

Hence,

$$\Omega = \phi - s\phi_s - z\phi_z = C_1 - \frac{a}{2} - \frac{b}{2} + \frac{a}{2}e^{-4L_0x^0-z^2} + \frac{b}{2}e^{r^2-s^2} > 0$$

for C_1 large enough. Then

$$\Lambda = \Omega\phi_{zz} + (r^2 - s^2)(\phi_{ss}\phi_{zz} - \phi_{sz}^2) \geq \Omega\phi_{zz} > 0.$$

□

Example 3. Taking $a = 1$, $b = 1$, and $C_1 = 2$ in Corollary 2, we have

$$\phi(x^0, r, s, z) = 2 + \frac{1}{2}\Big(e^{-4L_0x^0}(\sqrt{\pi}z\,\mathrm{erf}(z) + e^{-z^2}) - 1\Big) + \frac{1}{2}\Big(e^{r^2}(\sqrt{\pi}s\,\mathrm{erf}(s) + e^{-s^2}) - 1\Big),$$

and for this ϕ,

$$\Omega = \frac{1}{2}\Big(e^{-4L_0x^0-z^2} + e^{r^2-s^2}\Big) > 0, \qquad \Lambda > 0.$$

Corollary 3. Suppose that $F = |\overline{y}|\phi(x^0, r, s, z)$ has vanishing Douglas curvature, then ϕ satisfies

$$z\psi_{x^0} + \frac{s}{r}\psi_r + \Big[1 - 2(r^2 - s^2)U\Big]\psi_s - 2L\psi_z = 0, \tag{4.4}$$

where $\psi = \sqrt{r^2 - s^2}\Omega = \sqrt{r^2 - s^2}(\phi - s\phi_s - z\phi_z)$.

Proof. From the definition of p_1 and p_2 in (3.46) and (3.47), we have, $sp_1 + zp_2 = -z\Omega_{x^0} - \frac{s}{r}\Omega_r - \Omega_s$. Using (3.44)–(3.45), we obtain

$$2sU\Omega + z\Omega_{x^0} + \frac{s}{r}\Omega_r + \left[1 - 2(r^2 - s^2)U\right]\Omega_s - 2L\Omega_z = 0,$$

which is equivalent to (4.4), using the substitution $\psi = \sqrt{r^2 - s^2}\Omega$. □

Example 4 ([2]). Consider $U = \frac{1}{2}\frac{g'(r)}{rg(r)}$ and $L = \frac{1}{2}\frac{szg'(r)}{rg(r)}$, where g is a positive arbitrary differentiable function. The function $\psi = G\left(\frac{r^2-s^2}{g(r)^2}, zg(r)\right)$ solves Eq (4.4). Then, if $G = \frac{\sqrt{r^2-s^2}}{g(r)\sqrt{g(r)^2z^2+1}}$, the PDE

$$\Omega = -s^2\left[\frac{\phi}{s}\right]_s - sz\left[\frac{\phi}{s}\right]_z = \frac{1}{g(r)\sqrt{g(r)^2z^2+1}}$$

gives us

$$\phi = \frac{\sqrt{g(r)^2z^2+1}}{g(r)} + h(x^0, r, \frac{z}{s})s,$$

where h is an arbitrary differentiable function of $(x^0, r, \frac{z}{s})$.

In particular, considering $h(x, r, \frac{z}{s}) = h(x^0)\frac{z}{s}$, we obtain the next weakly orthogonally invariant Douglas metric

$$F(x,y) = h(x^0)y^0 + \frac{\sqrt{g(|\overline{x}|)^2(y^0)^2+1}}{g(|\overline{x}|)},$$

where $|h(x^0)| < 1$.

Example 5. Similar to the previous example, considering $U = \frac{1}{2}\frac{g'(r)}{rg(r)}$, $L = \frac{1}{2}\frac{szg'(r)}{rg(r)}$, and $G = \frac{\sqrt{r^2-s^2}}{g(r)}\left(1 + \frac{1}{(g(r)^2z^2+1)^{3/2}}\right)$, we get

$$\phi = h(x^0, r, \frac{z}{s})s + \frac{1}{g(r)}\left(1 + \frac{2g(r)^2z^2+1}{\sqrt{g(r)^2z^2+1}}\right).$$

If $h(x^0, r, \frac{z}{s})s = h(x^0)z$, we obtain the next weakly orthogonally invariant Douglas metric

$$F(x,y) = h(x^0)y^0 + \frac{1}{g(|\overline{x}|)}\left(|\overline{y}| + \sqrt{g(|\overline{x}|)^2(y^0)^2 + |\overline{y}|^2} + \frac{g(|\overline{x}|)^2(y^0)^2}{\sqrt{g(|\overline{x}|)^2(y^0)^2 + |\overline{y}|^2}}\right),$$

where $|h(x^0)| < 1$ and $g(r) > 0$.

Example 6. Let $|h(x^0)| < 1$, $g(r) > 0$, and $f(x^0) > 0$ be differentiable functions. Motivated by Example 5, the next weakly orthogonally invariant Finsler metric

$$F(x,y) = h(x^0)y^0 + \frac{1}{g(|\overline{x}|)}\left(|\overline{y}| + \frac{2g(|\overline{x}|)^2(y^0)^2 + f(x^0)|\overline{y}|^2}{\sqrt{g(|\overline{x}|)^2(y^0)^2 + f(x^0)|\overline{y}|^2}}\right)$$

has vanishing Douglas curvature with

$$U = \frac{1}{2}\frac{g'(r)}{rg(r)},$$
$$L = -\frac{f'}{3f}z^2 + \frac{g'}{2rg}sz - \frac{f'}{12\,g^2}.$$

Remark 6. In Examples 4–6, the function h is chosen in such a way that the positivity of F can be controlled.

5. Conclusions

In this paper, we derived an explicit formula for the Douglas curvature of weakly orthogonally invariant Finsler metrics, expressed in a compact form via the operator Ψ. This approach significantly simplifies the computation and reveals the underlying structure.

We proved that the vanishing of the Douglas curvature is equivalent to a system of partial differential equations for $\phi(x^0, r, s, z)$, which can be reduced to a tractable form involving auxiliary functions U and L, providing a more manageable formulation of existing results.

Furthermore, we constructed explicit families of solutions, including non-locally projectively flat Douglas metrics, and identified the conditions ensuring the Finslerian character through the positivity of Ω and Λ.

These results broaden the class of known Douglas-type metrics and contribute to the understanding of invariant structures in Finsler geometry.

Author contributions

Newton Solórzano: Formal analysis, conceptualization, writing - original draft, writing - review & editing, investigation; Dik Lujerio, Víctor León and Alexis Rodriguez Carranza: Conceptualization, writing - original draft, writing - review & editing, investigation. All authors have read and agreed to the published version of the manuscript.

Acknowledgments

Alexis R. C. and Dik L. acknowledge the financial support provided by the National Council for Science, Technology and Technological Innovation (CONCYTEC) and the National Program for Scientific Research and Advanced Studies (PROCIENCIA), within the framework of the call "E041-2025-01 Basic Research Projects," under Contract No. PE501094767-2025-PROCIENCIA.

The authors express their sincere appreciation to the anonymous reviewers for their careful reading of the manuscript and for their valuable comments and suggestions, which have substantially improved the clarity and quality of this work.

Conflict of interest

The authors declare that they have no conflict of interest.

References

1. S. Bácsó, M. Matsumoto, On Finsler spaces of Douglas type—a generalization of the notion of Berwald space, *Publ. Math. Debrecen,* **51** (1997), 385–406. https://doi.org/10.5486/pmd.1997.1975

2. N. M. S. Chávez, On Douglas warped product metrics, *Results Math.,* **77** (2022), 189. https://doi.org/10.1007/s00025-022-01724-2
3. B. Chen, Z. Shen, L. L. Zhao, Constructions of Einstein Finsler metrics by warped product, *Int. J. Math.,* **29** (2018), 1850081, 15 pp. https://doi.org/10.1142/S0129167X18500817
4. J. Douglas, The general geometry of paths, *Ann. Math.,* **29** (1927/28), 143–168. https://doi.org/10.2307/1967989
5. E. L. Guo, X. H. Mo, *Spherically symmetric Finsler metrics*, In: The Geometry of Spherically Symmetric Finsler Manifolds, Singapore: Springer Briefs in Mathematics-Springer, 2018. https://doi.org/10.1007/978-981-13-1598-5
6. L. B. Huang, X. H. Mo, On spherically symmetric Finsler metrics of scalar curvature, *J. Geom. Phys.,* **62** (2012), 2279–2287. https://doi.org/10.1016/j.geomphys.2012.07.005
7. L. Kozma, I. R. Peter, C. G. Varga, Warped product of Finsler manifolds, *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.,* **44** (2001), 157–170.
8. B. L. Li, Y. Shen, Z. Shen, On a class of Douglas metrics, *Stud. Sci. Math. Hung.,* **46** (2009), 355–365. https://doi.org/10.1556/sscmath.2009.1096
9. H. Liu, X. H. Mo, Finsler warped product metrics of Douglas type, *Canad. Math. Bull.,* **62** (2019), 119–130. https://doi.org/10.4153/CMB-2017-077-0
10. H. Liu, X. Mo, On a class of Finsler metrics of Douglas type, *Acta Math. Sin.,* **41** (2025), 1491–1507. https://doi.org/10.1007/s10114-025-3309-0
11. H. Liu, X. H. Mo, L. Zhu, A class of projectively flat Finsler metrics, *Results Math.,* **79** (2024), 226. https://doi.org/10.1007/s00025-024-02252-x
12. P. Marçal, Z. Shen, Ricci-flat Finsler metrics by warped product, *P. Am. Math. Soc.,* **151** (2023), 2169–2183. https://doi.org/10.1090/proc/16217
13. N. Solórzano, V. León, On cylindrical symmetric projectively flat Finsler metrics, *J. Geom. Phys.,* **186** (2023), 104777. https://doi.org/10.1016/j.geomphys.2023.104777
14. L. Zhou, Spherically symmetric Finsler metrics in R^n, *Publ. Math. Debrecen,* **80** (2012), 67–77. https://doi.org/10.1007/s10711-011-9639-3

AIMS Press