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Research article

Solutions to Volterra integral equations in bounded Φ-variation spaces

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Abstract: This article examines the existence of solutions to a Volterra-type integral equation in the space of functions of bounded Φ-variation. Conditions on the given functions are established to guarantee the existence of such solutions. Furthermore, it is shown that any two solutions are equal almost everywhere. The possibility of extending solutions to larger intervals is also explored.

Keywords: Volterra integral; Φ-bounded variation
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1. Introduction and preliminaries

The importance of Volterra integral equations lies in their inherent ability to model systems with memory or hereditary effects. This ability has motivated numerous studies on the existence of solutions to integral equations in spaces of bounded variation. Early works such as [10, 11] established the existence of continuous solutions in this setting. More recently, the existence of continuous solutions to nonlinear integral equations in the framework of functions of bounded Φ-variation has been investigated (see [3]). The interest in such solutions is largely driven by applications, because many integral equations arising in concrete physical phenomena yield solutions that are naturally of bounded variation. For a broader structural perspective, including boundary value problems and adjoint operators in the context of differential and integral equations in spaces of bounded variation, we refer to [12].

It is worth recalling that functional analysis itself originated from the need to rigorously understand the function spaces underlying physical phenomena. Within this tradition, functions of bounded variation and their generalizations have been extensively studied. In particular, spaces of bounded p-variation and their extension to Φ-variation have been thoroughly analyzed, together with their properties, applications (see [4]), and their connection to Riemann–Stieltjes integration (see [13]). This theoretical development naturally invites the study of nonlinear integral equations in these function spaces, a direction already explored in [3] in the context of Φ-variation.

In recent years, significant progress has been made in the analysis of Volterra-type equations and systems. Stability results for integro-differential systems with point delays and finite or infinite memory terms have been obtained using Krasovskii–Lyapunov functionals (see [8]). Iterative methods under generalized contractive conditions have been developed to establish existence, uniqueness, and approximation results for nonlinear integral equations, including stability analysis (see [6, 8]). Moreover, the study of discrete convolution-type Volterra equations has provided convergence criteria and tools for the stability analysis of numerical methods (see [9]).

A recent contribution by Bracamonte [2] establishes the existence of locally bounded variation solutions for Volterra integral equations with infinite delay arising in models of isolated species. This result is formulated in terms of classical Jordan variation, which can be regarded as a particular case of the theory of Φ-variation for a suitable choice of the function Φ. This observation naturally motivates the formulation of the problem in spaces of locally bounded Φ-variation, thereby broadening the functional framework while preserving the essential structure needed to prove existence and uniqueness.

In the present work, we construct a successive approximation scheme on a compact subset of the function space, ensuring that the iterates remain within a controlled region. Within this framework, we obtain existence and uniqueness results for Volterra integral equations in spaces of bounded Φ-variation, thereby extending previously known results in the context of bounded variation.

The paper is organized as follows. The first section contains the introduction, which provides both the motivation for the problem and its formulation, followed by the presentation of the necessary definitions and preliminary results required for the development of the work. In Section 2, the Volterra integral equations under study are formulated, and the structural hypotheses on the functions involved are specified. Under these conditions, it is shown that the function $G_t(s) = g(t, s, x(s))$ belongs to the space in which the analysis is carried out. Additionally, the Volterra-type operator is examined, establishing that it preserves bounded Φ-variation and is a bounded operator on the corresponding function space. Finally, it is proved that the set $R(a, b; f)$ is compact in $\mathbb{R}^3$, a key result for the subsequent development of the theory. Section 3 establishes the main results concerning the existence and uniqueness of solutions. Section 4 is devoted to the study of the continuation of solutions. Section 5 provides applications illustrating the theoretical results, and finally, Section 6 presents the conclusions.

1.1. Bounded Φ-variation: Definition and properties

We now introduce the essential background material to facilitate the reading and development of the paper. Let $\mathcal{N}$ denote the set of all convex, continuous, and nondecreasing functions $\Phi : [0, +\infty) \to [0, +\infty)$ such that $\Phi(t) = 0$ if and only if $t = 0$ and $\lim_{t\to+\infty} \Phi(t) = +\infty$.

It is easy to verify that every function $\Phi \in \mathcal{N}$ is superadditive (see [1]); that is,

$$\Phi(u + v) \geq \Phi(u) + \Phi(v) \qquad \text{for all } u, v \geq 0.$$

A point partition, or simply a partition, $\xi = \{t_i\}_{i=0}^n$ of an interval J is a finite sequence of points in J such that the following hold:

1) $t_0 < t_1 < \cdots < t_n$;

2) if J has a minimum element a, then $t_0 = a$;
3) if J has a maximum element b, then $t_n = b$.

We denote by $PP(J)$ the set of all point partitions of J.

Definition 1.1. (See [1, 4, 5]) Let $\emptyset \neq J \subset \mathbb{R}$, let $\Phi \in \mathcal{N}$, and let $f : J \to \mathbb{R}$ be a function. The Φ-variation of f on the interval J is defined by

$$\bigvee_{\Phi}(f) := \bigvee_{\Phi}(f; J) := \sup_{\xi=\{t_i\}_{i=0}^{n} \in PP(J)} \sum_{i=1}^{n} \Phi(|f(t_i) - f(t_{i-1})|) \tag{1.1}$$

if J is nondegenerate and as 0 if J consists of a single point.

The quantity $\bigvee_{\Phi}(f; J)$ is called the total Φ-variation of f on J (in the sense of Wiener or Young). In the particular case where $\bigvee_{\Phi}(f; J) < \infty$, the function f is said to be of bounded Φ-variation. The set of all functions $f : J \to \mathbb{R}$ of bounded Φ-variation is denoted by $\mathcal{V}_{\Phi}(J)$.

It follows directly from the definition that every function in $\mathcal{V}_{\Phi}(J)$ is bounded. Unfortunately, this set is not necessarily a vector space. For this reason, it is necessary to introduce a vector space that contains this class.

For $\Phi \in \mathcal{N}$, let $B\mathcal{V}_{\Phi}(J)$ denote the set of all functions $f : J \to \mathbb{R}$ for which there exists some $c > 0$ such that $cf \in \mathcal{V}_{\Phi}(J)$.

For $f \in B\mathcal{V}_{\Phi}(J)$, we define

$$\begin{aligned} \rho(f) &:= \inf\left\{c > 0 : \bigvee_{\Phi}\left(\frac{f}{c}\right) \leq 1\right\} \text{ and} \\ \|f\|_{\Phi} &:= \|f\|_{\infty} + \rho(f), \end{aligned}$$

where $\|f\|_{\infty} = \sup_{t\in J} |f(t)|$ denotes the usual supremum norm of f.

Among the known properties of this space, we have the following: If $f \in B\mathcal{V}_{\Phi}(J)$, then

- $\|f\|_{\Phi} = 0$ if and only if f is constant;
- for any nonconstant $f \in B\mathcal{V}_{\Phi}(J)$, $\bigvee_{\Phi}\left(\frac{f}{\rho(f)}\right) \leq 1$;
- $(B\mathcal{V}_{\Phi}(J), \|\cdot\|_{\Phi})$ is a Banach space; $|f(t)| \leq \|f\|_{\Phi}$ for all $t \in J$.

1.2. General properties of Φ*-variation*

Let $f : J \to \mathbb{R}$ be an arbitrary function, and $\Phi \in \mathcal{N}$. The following properties are established in [4]:

P1 (minimality) For any $t, s \in J$, we have

$$\Phi(|f(t) - f(s)|) \leq \bigvee_{\Phi}(f; J).$$

P2 (monotonicity) If $v, t, s, u \in J$ satisfy $v \leq t \leq s \leq u$, then

$$\bigvee_{\Phi}(f; J \cap (-\infty, t]) \leq \bigvee_{\Phi}(f; J \cap (-\infty, s]),$$

$$\bigvee_{\Phi}(f; J \cap [s, +\infty)) \leq \bigvee_{\Phi}(f; J \cap [t, +\infty)),$$

and

$$\bigvee_{\Phi}(f; [t, s]) \leq \bigvee_{\Phi}(f; [v, u]).$$

P3 (lower semi-continuity) If $\{f_n\}_n$ is a sequence of real-valued functions defined on J that converges pointwise on J to a function $f : J \to \mathbb{R}$, then

$$\bigvee_{\Phi}(f; J) \leq \liminf_{n\to+\infty} \bigvee_{\Phi}(f_n; J).$$

A function f of bounded Φ-variation possesses at most countably many discontinuities. Moreover, at each point of discontinuity x_0, the one-sided limits $f(x_0^+) := \lim_{x\to x_0^+} f(x)$ and $f(x_0^-) := \lim_{x\to x_0^-} f(x)$ exist.

2. Volterra integral equation

In this paper, we consider the Volterra-type integral equation

$$x(t) = \alpha f(t) + (1-\alpha)\int_0^t g(t, s, x(s))ds, \tag{2.1}$$

where $x : [0, +\infty) \to \mathbb{R}$ is the unknown function, $\alpha \in (0, 1)$, and

$$g : D \times \mathbb{R} \to \mathbb{R}, \qquad D = \left\{(t, s) \in \mathbb{R}^2 : 0 \leq s \leq t < +\infty\right\} \tag{2.2}$$

is a given kernel. Equation (2.1) is studied in the space $(\mathcal{BV}_\Phi(J), \|\cdot\|_\Phi)$. All integrals are understood in the sense of Lebesgue.

It is of interest to weaken the classical assumptions typically imposed on the functions f and g in equations of this type. With this in mind, in the present paper we consider the following conditions:

C_1) $f \in L\mathcal{V}_\Phi([0, \infty))$;
C_2) $g : D \times \mathbb{R} \to \mathbb{R}$, as defined in (2.2), is locally Lipschitz.
C_3) $g : D \times \mathbb{R} \to \mathbb{R}$, as defined in (2.2), satisfies Lipschitz with respect to the pair (s, x) uniformly in t, that is, there exists a constant $L > 0$ such that

$$\|g(t, s, x) - g(t, s', x')\| \leq L(|s - s'| + |x - x'|) \tag{*}$$

holds for all $(t, s), (t, s') \in D$, and $x, x' \in \mathbb{R}$.

Note that condition $(*)$ is inspired by the notion of *two-Lipschitz* operators introduced by Hamidi et al. [7], which involves mixed differentiability or bounded variation in the sense of Fréchet. Unlike the original definition, which relies on double differences of the form $T(x, y) - T(x, y') - T(x', y) + T(x', y')$, our condition is simpler and more directly applicable to the context of integral equations, as it jointly controls the variations in s and x. This allows, for instance, to bound terms such as

$$\|g(t, s_i, x(s_i)) - g(t, s_{i-1}, x(s_{i-1}))\|$$

naturally by $L(|s_i - s_{i-1}| + |x(s_i) - x(s_{i-1})|)$, which is particularly useful in numerical analysis and in convergence proofs for iterative schemes.

2.1. The Volterra-type integral operator

Suppose that g satisfies condition C_3. We consider the Volterra-type integral operator V_g, generated by g, defined for each $x \in B\mathcal{V}_\Phi([0, a])$ by

$$\mathrm{V}_g(x)(t) := \int_0^t g(t, s, x(s))ds \quad \text{in} \quad t \in [0, a]. \tag{2.3}$$

In general, this operator is not linear; its linearity hinges on the behavior of the function g. Specifically, the operator V_g is linear if and only if g is linear in the variable x.

Theorem 2.1. *Let $\Phi \in \mathcal{N}$, let $a > 0$, and let $x \in B\mathcal{V}_\Phi([0, a])$. Suppose that g satisfies condition C_2 and that $C \subseteq \mathbb{R}$ is compact. Assume that, for each $t \in [0, a]$ and every $s \in [0, t]$, the point $(t, s, x(s))$ lies in*

$$R = \{(t, s) \in \mathbb{R}^2 : 0 \le s \le t \le a\} \times C.$$

Then, for each $t \in [0, a]$, the function defined by $G_t(s) := g(t, s, x(s))$ belongs to $B\mathcal{V}_\Phi([0, t])$, when regarded as a function of s.

Proof. Let $\xi = \{s_i\}_{i=0}^n$ be a partition of the interval $[0, t]$. Because $x \in B\mathcal{V}_\Phi([0, a])$, there exists a constant $\lambda > 0$ such that $\lambda x \in \mathcal{V}_\Phi([0, a])$.

Moreover, because R is compact, and g is locally Lipschitz, there exists a constant $L_R > 0$ such that

$$|g(t, s, x) - g(t_0, s_0, x(s_0))| \le L_R\|(t, s, x) - (t_0, s_0, x_0)\|$$

holds for all $(t, s, x), (t_0, s_0, x_0) \in R$, where $\|\cdot\|$ denotes the ℓ^1−norm, that is, $\|(t, s, x)\| = |t| + |s| + |x|$.

Thus, for each $0 \le i \le n$, we obatain

$$\begin{aligned} \Phi\left(\frac{\lambda}{2L_R}|G_t(s_i) - G_t(s_{i-1})|\right) &= \Phi\left(\frac{\lambda}{2L_R}\,|g(t, s_i, x(s_i)) - g(t, s_{i-1}, x(s_{i-1}))|\right) \\ &\le \Phi\left(\frac{\lambda}{2L_R}L_R\,\|(0, s_i - s_{i-1}, x(s_i) - x(s_{i-1}))\|\right) \\ &\le \Phi\left(\lambda\frac{|s_i - s_{i-1}|}{2} + \frac{|\lambda x(s_i) - \lambda x(s_{i-1})|}{2}\right) \\ &\le \frac{1}{2}\Phi\left(\lambda|s_i - s_{i-1}|\right) + \frac{1}{2}\Phi\left(|\lambda x(s_i) - \lambda x(s_{i-1})|\right). \end{aligned}$$

Summing over $i = 1, \cdots, n$ yields

$$
\begin{aligned}
&\sum_{i=1}^{n} \Phi\left(\frac{\lambda}{2L_R}|G_t(s_i) - G_t(s_{i-1})|\right) \\
\leq\ & \frac{1}{2}\sum_{i=1}^{n} \Phi\left(\lambda|s_i - s_{i-1}|\right) + \frac{1}{2}\sum_{i=1}^{n} \Phi\left(|\lambda x(s_i) - \lambda x(s_{i-1})|\right) \\
\leq\ & \frac{1}{2}\Phi\left(\lambda \sum_{i=1}^{n} |s_i - s_{i-1}|\right) + \frac{1}{2}\sum_{i=1}^{n} \Phi\left(|\lambda x(s_i) - \lambda x(s_{i-1})|\right) \\
\leq\ & \frac{1}{2}\Phi\left(\lambda t\right) + \frac{1}{2}\bigvee_{\Phi}(\lambda x; [0,a]) \leq \frac{1}{2}\left(\Phi\left(\lambda a\right) + \bigvee_{\Phi}(\lambda x; [0,a])\right).
\end{aligned} \tag{2.4}
$$

Because the right-hand side of this inequality is finite, for every partition $\xi \in PP([0,t])$, it follows that $\bigvee_{\Phi}\left(\frac{\lambda}{2L_R}G_t; [0,t]\right) < \infty$, and consequently $G_t \in B\mathcal{V}_{\Phi}([0,t])$.

□

Corollary 2.2. *Under the hypotheses of Theorem 2.1, the integral* $\int_0^t g(t,s,x(s))ds$ *is well-defined for all* $t \in [0,a]$.

Proof. Note that Theorem 2.1 guarantees that $g(t,\cdot,x(\cdot))$ is a regular function on the interval $[0,t]$. Consequently, it has at most countably many discontinuities and is therefore a Riemann (and Lebesgue) integrable. □

One always seeks to weaken the assumptions placed on the functions involved. Accordingly, if we assume that g satisfies a weaker condition, namely a joint Lipschitz condition, then the following theorem holds.

Theorem 2.3. *Let* $\Phi \in \mathcal{N}$*, let* $a > 0$*, and let* $x \in B\mathcal{V}_{\Phi}([0,a])$*. Suppose that g satisfies condition* C_3*. Then, for each* $t \in [0,a]$*, the function defined by* $G_t(s) := g(t,s,x(s))$ *belongs to* $B\mathcal{V}_{\Phi}([0,t])$*, when regarded as a function of s.*

Proof. Let $\xi = \{s_i\}_{i=0}^{n}$ be a partition of the interval $[0,t]$. Because $x \in B\mathcal{V}_{\Phi}([0,a])$, there exists a constant $\lambda > 0$ such that $\lambda x \in \mathcal{V}_{\Phi}([0,a])$. Thus, for each $0 \leq i \leq n$, we obatain

$$
\begin{aligned}
\Phi\left(\frac{\lambda}{2L}|G_t(s_i) - G_t(s_{i-1})|\right) &= \Phi\left(\frac{\lambda}{2L}\left|g(t,s_i,x(s_i)) - g(t,s_{i-1},x(s_{i-1}))\right|\right) \\
&\leq \Phi\left(\lambda\frac{|s_i - s_{i-1}|}{2} + \frac{|\lambda x(s_i) - \lambda x(s_{i-1})|}{2}\right) \\
&\leq \frac{1}{2}\Phi\left(\lambda|s_i - s_{i-1}|\right) + \frac{1}{2}\Phi\left(|\lambda x(s_i) - \lambda x(s_{i-1})|\right).
\end{aligned}
$$

The remainder of the proof proceeds analogously to that of Theorem 2.1, yielding that $G_t(s) := g(t,s,x(s))$ belongs to $B\mathcal{V}_{\Phi}([0,t])$.

□

We now show that, under the hypotheses of Theorem 2.1, the operator V_g maps $B\mathcal{V}_\Phi([0,a])$ into itself.

Theorem 2.4. *Let $\Phi \in \mathcal{N}$, let $a > 0$, and suppose that g satisfies condition C_2. Let V_g be the Volterra-type operator generated by g as defined in* (2.3). *Then, for every $x \in B\mathcal{V}_\Phi([0,a])$, we have $\mathrm{V}_g(x) \in \mathcal{V}_\Phi([0,a])$.*

Proof. Let $x \in B\mathcal{V}_\Phi([0,a])$. Because g satisfies condition C_2, it is globally Lipschitz and bounded on the compact set $R = \left\{(t,s) \in \mathbb{R}^2 : 0 \le s \le t \le a\right\} \times [-\|x\|_\Phi, \|x\|_\Phi]$. Hence, there exist constants $M_R > 0$ and $L_R > 0$ such that conditions

$$|g(t,s,r)| \le M_R \quad \text{for all} \quad (t,s,r) \in R \tag{2.5}$$

and

$$|g(t,s,r) - g(t_0,s_0,r_0)| \le L_R\|(t,s,r) - (t_0,s_0,r_0)\|$$

hold for all $(t,s,r), (t_0,s_0,r_0) \in R$.

In particular, for all $(t,s), (t_0,s_0)$ with $0 \le s \le t \le a$, we have

$$|g(t,s,x(s)) - g(t_0,s_0,x(s_0))| \le L_R\|(t-t_0, s-s_0, x(s)-x(s_0))\|. \tag{2.6}$$

Under these conditions, the hypotheses of Corollary 2.2 are satisfied; therefore, $\int_0^t g(t,s,x(s))ds$ exists in the Riemann sense for every $t \in [0,a]$.

Now, consider a partition $\xi = \{t_i\}_0^n$ of $[0,a]$. For each $1 \le i \le n$, we obtain

$$\begin{aligned}
&\left|\mathrm{V}_g(x)(t_i) - \mathrm{V}_g(x)(t_{i-1})\right| \\
= &\left|\int_0^{t_i} g(t_i,s,x(s))ds - \int_0^{t_{i-1}} g(t_{i-1},s,x(s))ds\right| \\
= &\left|\int_0^{t_{i-1}} g(t_i,s,x(s))ds - \int_0^{t_{i-1}} g(t_{i-1},s,x(s))ds + \int_{t_{i-1}}^{t_i} g(t_i,s,x(s))ds\right| \\
\le &\int_0^{t_{i-1}} |g(t_i,s,x(s)) - g(t_{i-1},s,x(s))|\, ds + \int_{t_{i-1}}^{t_i} |g(t_i,s,x(s))|\, ds. \\
\le &\int_0^{t_{i-1}} L_R\|(t_i - t_{i-1}, 0, 0)\|ds + \int_{t_{i-1}}^{t_i} M ds \\
= &\int_0^{t_{i-1}} L_R(t_i - t_{i-1})ds + \int_{t_{i-1}}^{t_i} M_R ds
\end{aligned}$$

$$
\begin{aligned}
&\leq L_R(t_i - t_{i-1})t_{i-1} + M_R(t_i - t_{i-1})\\
&\leq (L_R a + M_R)(t_i - t_{i-1}).
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
\sum_{i=1}^{n} \Phi\left(\left|\mathrm{V}_g(x)(t_i) - \mathrm{V}_g(x)(t_{i-1})\right|\right) &\leq \sum_{i=1}^{n} \Phi\left((L_R a + M_R)(t_i - t_{i-1})\right)\\
&\leq \Phi\left(\sum_{i=1}^{n}(L_R a + M_R)(t_i - t_{i-1})\right)\\
&= \Phi\left((L_R a + M_R)\sum_{i=1}^{n}(t_i - t_{i-1})\right)\\
&= \Phi\left((L_R a + M_R)a\right).
\end{aligned}
$$

Because the right-hand side of the inequality is a constant, and the inequality holds for every partition of $[0, a]$, it follows that $\bigvee_{\Phi}\left(\mathrm{V}_g(x); [0, a]\right) \leq \Phi\left(L_R a^2 + a\, M_R\right)$. □

Thus, V_g maps $B\mathcal{V}_\Phi([0, a])$ into itself, which allows us to use analytical techniques in this space and guarantees that solutions to operator-related problems are well-posed in $B\mathcal{V}_\Phi([0, a])$.

Corollary 2.5. *Let $a > 0$, $\Phi \in \mathcal{N}$, and suppose that the function f satisfies condition C_1, and g satisfies condition C_2. Assume further that $x \in \mathcal{V}_\Phi([0, a])$. Then, for every $\alpha \in (0, 1)$, $F = \alpha f + (1 - \alpha)\mathrm{V}_g(x)$ is a function of bounded Φ-variation on $[0, a]$.*

Proof. Theorem 2.4 guarantees that $\mathrm{V}_g(x) \in \mathcal{V}_\Phi([0, a])$. Then, by the convexity of the Φ-varation, it follows that F also is a function of bounded Φ-variation on $[0, a]$, with

$$
\bigvee_{\Phi}(F; [0, a]) \leq \alpha \bigvee_{\Phi}(f; [0, a]) + (1 - \alpha) \bigvee_{\Phi}\left(\mathrm{V}_g(x); [0, a]\right).
$$

□

Theorem 2.6. *Under the same hypotheses as in Theorem 2.4, the operator V_g is bounded.*

Proof. Let $x \in B\mathcal{V}_\Phi([0, a])$ with $\|x\|_\Phi \leq 1$. Because g satisfies condition C_2, it follows that g is globally Lipschitz on the set

$$
R = \left\{(t, s) \in \mathbb{R}^2 : 0 \leq s \leq t \leq a\right\} \times [-1, 1].
$$

Consequently, there exist positive constants $M_R > 0$ and $L_R > 0$ such that $|g(t, s, r)| \leq M_R$ and

$$
|g(t, s, r) - g(t_0, s_0, r_0)| \leq L_R\|(t, s, r) - (t_0, s_0, r_0)\|
$$

for all $(t, s, r), (t_0, s_0, r_0) \in R$. Thus, for each $t \in [0, a]$,

$$
\left|\mathrm{V}_g(x)(t)\right| = \left|\int_0^t g(t, s, x(s))ds\right| \leq \int_0^t |g(t, s, x(s))|\, ds \leq tM_R,
$$

which implies $\|\mathrm{V}_g(x)\|_\infty \leq aM_R$. Proceeding as in the proof of Theorem 2.4, we obtain $\bigvee_\Phi \left(\mathrm{V}_g(x); [0, a]\right) \leq \Phi\left(L_R a^2 + a\, M_R\right)$.

Now, if $\bigvee_\Phi \left(\mathrm{V}_g(x); [0, a]\right) \leq 1$, then $\rho(\mathrm{V}_g(x)) \leq 1$, and hence,

$$\|\mathrm{V}_g(x)\|_\Phi \leq aM_R + 1.$$

If instead $\bigvee_\Phi \left(\mathrm{V}_g(x); [0, a]\right) > 1$, then $\rho(\mathrm{V}_g(x)) \leq \bigvee_\Phi \left(\mathrm{V}_g(x); [0, a]\right) \leq \Phi\left(L_R a^2 + a\, M_R\right)$, which yields

$$\|\mathrm{V}_g(x)\|_\Phi \leq aM_R + \Phi\left((L_R a + M_R)a\right).$$

In either case, $\|\mathrm{V}_g(x)\|_\Phi$ is bounded uniformly for all x with $\|x\|_\Phi \leq 1$. Therefore, the operator V_g is bounded. □

2.2. Application: Volterra integral equation

We now investigate whether Eq (2.1) has a solution in $B\mathcal{V}_\Phi([0, a])$ when f lies in this space. To this end, recall that if $x \in \mathcal{V}_\Phi([0, a])$, then by convexity of the Φ-variation, $\alpha x \in \mathcal{V}_\Phi([0, a])$ for any $\alpha \in [0, 1]$.

Now, given the function f, we consider the set

$$R(a, b; f) := \left\{(t, s, x) \in \mathbb{R}^3 : 0 \leq s \leq t \leq a,\ |x - f(s)| \leq b\right\}. \tag{2.7}$$

Theorem 2.7. *Let $a, b \geq 0$ be given. Suppose that f satisfies condition C_1. Then, the set $R(a, b; f)$ (in 2.7) is a bounded subset of $\mathbb{R}^3$.*

Proof. The variable t is subject to the constraint $0 \leq s \leq t \leq a$, which implies that t and s lie within the compact $[0, a]$.

As for the variable x, the condition $|x - f(s)| \leq b$ guarantees that x belongs to the interval $[f(s) - b, f(s) + b]$. Because f is a function of bounded Φ-variation on $[0, a]$, it is also bounded. Hence, $|f(s)| \leq \|f\|_\Phi$ for all $s \in [0, a]$. Consequently, x lies within the bounded interval $[-\|f\|_\Phi - b, \|f\|_\Phi + b]$. Because t, s and x are all bounded, the set $\left\{(t, s, x) \in \mathbb{R}^3 : 0 \leq s \leq t \leq a,\ |x - f(s)| \leq b\right\}$ is bounded in $\mathbb{R}^3$ by $\max\{a, \|f\|_\Phi + b\}$.

□

Corollary 2.8. *Suppose that f and g satisfy hypotheses C_1 and C_2, respectively. Then, function g satisfies the Lipschitz condition on the set $R(a, b; f)$.*

Proof. Because g is locally Lipschitz, it follows that g is globally Lipschitz on $\overline{R(a, b; f)}$ and therefore also on $R(a, b; f)$. □

When f verifies condition C_1, then αf equally verifies it for all $\alpha \in (0, 1)$. We thus derive the following corollary:

Corollary 2.9. *Suppose that f and g satisfy conditions C_1 and C_2, respectively. Then, for each fixed $\alpha \in (0,1)$, the function g satisfies the Lipschitz condition on the set $R(a,b;\alpha f)$.*

In what follows, L_a^b will denote the Lipschitz constant of g on $R(a,b;\alpha f)$, and similarly, M_a^b will represent the upper bound on the images of g over $R(a,b;\alpha f)$, that is,

$$|g(t,s,x)| \leq M_a^b \quad \text{for all} \quad (t,s,x) \in R(a,b;\alpha f). \tag{2.8}$$

3. Existence of solutions

We now employ the Picard iteration method to establish the existence of solutions for Eq (2.1).

Lemma 3.1. *Assume that f satisfies C_1, and g satisfies condition C_2, and let $\alpha \in (0,1)$. Define the sequence of functions $\{x_n\}_{n\geq 0}$ recursively by*

$$x_n(t) = \begin{cases} \alpha f(t) & \text{if} \quad n = 0 \\ \alpha f(t) + (1-\alpha)\mathrm{V}_g(x_{n-1})(t) & \text{if} \quad n \geq 1 \end{cases} \qquad \text{for all} \quad 0 \leq t \leq a. \tag{3.1}$$

Then, the following hold:

1) $x_n \in \mathcal{V}_\Phi([0,a])$ for all $n \geq 0$,
2) $|x_n(t) - \alpha f(t)| \leq M_a^b t$ for all $t \in [0,a]$, where M_a^b is the constant appearing in (2.8).

Proof. Note first that, by hypothesis, because $f \in \mathcal{V}_\Phi([0,a])$, and $\alpha \in (0,1)$, the convex combination αf belongs to $\mathcal{V}_\Phi([0,a])$ by the convexity of the Φ- variation. Hence, $x_0 \in \mathcal{V}_\Phi([0,a])$.

We proceed by induction on n. For $n = 1$, because $x_0 \in \mathcal{V}_\Phi([0,a])$, Theorem 2.4 guarantees that $\mathrm{V}_g(x_0) \in \mathcal{V}_\Phi([0,a])$. Then, again using the convexity of the Φ-variation, we obtain $x_1 = \alpha f + (1-\alpha)\mathrm{V}_g(x_0) \in \mathcal{V}_\Phi([0,a])$.

Now, suppose that $x_n \in \mathcal{V}_\Phi([0,a])$. Applying Theorem 2.4 once more yields Theorem 2.4, and by convexity, $\mathrm{V}_g(x_n) \in \mathcal{V}_\Phi([0,a])$, and $x_{n+1} = \alpha f + (1-\alpha)\mathrm{V}_g(x_n) \in \mathcal{V}_\Phi([0,a])$.

As a consequece of the first part, for all n and every $t \in [0,a]$, we have

$$\begin{aligned} |x_{n+1}(t) - \alpha f(t)| &= (1-\alpha)|\mathrm{V}_g(x_n)(t)| \\ &\leq \int_0^t |g(t,s,x_n(s))|\,ds \\ &\leq \int_0^t M_a^b ds = M_a^b t. \end{aligned}$$

This completes the proof of the lemma. □

Note that the second part of Lemma 3.1 guarantees that $(t,s,x_n(s)) \in R(a,b;\alpha f)$ whenever $M_a^b t \leq b$. However, this does not necessarily hold for all $t \in [0,a]$.

Theorem 3.2 (Existence). *For positive real numbers a and b, with $\alpha \in (0,1)$, assume that f satisfies C_1, and g satisfies C_2. Then, there exists a solution of Eq* (2.1) *that belongs to the space $\mathcal{BV}_\Phi([0,\beta])$ whenever $\beta = \min\left\{a, b/M_a^b\right\}$. Additionaly, any two solutions of Eq* (2.1) *are equal almost everywhere.*

Proof. Consider a new sequence, defined recursively, $\{y_n\}_n$, by

$$y_n(t) = \begin{cases} x_0(t), & \text{if } \quad n = 0 \\ (x_n - x_{n-1})(t) & \text{if } \quad n \geq 1 \end{cases}$$

for all $0 \leq t \leq \beta$, with $\{x_n\}_n$ defined as in Lemma 3.1. From the same, it follows that

$$|y_1(t)| = |x_1(t) - \alpha f(t)| \leq M_a^b t \quad \text{for all} \quad 0 \leq t \leq \beta.$$

Moreover,

$$\begin{aligned} |y_2(t)| &= |x_2(t) - x_1(t)| = |(1-\alpha)\mathrm{V}_g(x_1)(t) - (1-\alpha)\mathrm{V}_g(x_0)(t)| \\ &= (1-\alpha)\left|\int_0^t (g(t,s,x_1(s)) - g(t,s,x_0(s)))ds\right| \\ &\leq \int_0^t |g(t,s,x_1(s)) - g(t,s,x_0(s))|\, ds \\ &\leq \int_0^t L_a^b \, \|(0,0,x_1(s) - x_0(s))\|\, ds \\ &= L_a^b \int_0^t |x_1(s) - x_0(s)| ds = L_a^b \int_0^t |y_1(s)| ds \\ &\leq L_a^b \int_0^t M_a^b \, s\, ds = L_a^b M_a^b \frac{1}{2} t^2. \end{aligned}$$

By recurrence,

$$|y_n(t)| \leq M_a^b (L_a^b)^{n-1} \frac{t^n}{n!} \leq \frac{M_a^b}{L_a^b} \frac{(L_a^b\, a)^n}{n!}.$$

Because the series $\sum_{n=1}^{\infty} (L_a^b\, a)^n / n!$ converges, it follows that the series $\sum_{n=1}^{\infty} y_n(t)$ converges absolutely in $\mathbb{R}$ for each $t \in [0,\beta]$. We may therefore define, for every $t \in [0,\beta]$, the function $x(t) := x_0(t) + \sum_{n=1}^{\infty} y_n(t)$.

Observe that

$$x(t) = x_0(t) + \lim_{k\to+\infty} \sum_{n=1}^{k} y_n(t) = x_0(t) + \lim_{k\to+\infty} (x_k(t) - x_0(t)) = \lim_{k\to+\infty} x_k(t). \tag{3.2}$$

Because each $x_n \in \mathcal{V}_\Phi([0,\beta]) \subseteq \mathcal{V}_\Phi([0,a])$ and x is the pointwise limit of a sequence of functions in in this space, property **P3** guarantees that $x \in \mathcal{V}_\Phi([0,\beta])$.

We aim to verify that x is a solution to Eq (2.1). To this end, we first apply Corollary 2.5 to $x \in \mathcal{V}_\Phi([0,\beta])$, which yields $F = \alpha f + (1-\alpha)\mathrm{V}_g(x) \in \mathcal{V}_\Phi([0,\beta])$. Observe that

$$\begin{aligned}
|x_{n+1}(t) - F(t)| &= \left|\alpha f(t) + (1-\alpha)\int_0^t g(t,s,x_n(s))ds - \left(\alpha f(t) + (1-\alpha)\int_0^t g(t,s,x(s))\,ds\right)\right| \\
&= (1-\alpha)\left|\int_0^t g(t,s,x_n(s))ds - \int_0^t g(t,s,x(s))\,ds\right| \\
&\le \int_0^t |g(t,s,x_n(s))ds - g(t,s,x(s))|\,ds \\
&\le \int_0^t L_a^b \|(0,0,x_n(s)-x(s))\| ds \\
&= L_a^b \int_0^t |x_n(s) - x(s)| ds.
\end{aligned}$$

Taking the limit as $n \to +\infty$ and using (3.2), the right-hand side converges to zero. Consequently, $x(t) = \lim_{n\to\infty} x_{n+1}(t) = F(t)$ for all $t \in [0,\beta]$.

From this, we obtain that x is a solution to Eq (2.1), and it therefore follows that

$$x(t) = \alpha f(t) + (1-\alpha)\int_0^t g(t,s,x(s))\,ds.$$

Now, suppose there exist two solutions, x and y. Then,

$$\begin{aligned}
|x(t) - y(t)| &= \left|f(t) + \int_0^t g(t,s,x(s))\,ds - \left(f(t) + \int_0^t g(t,s,y(s))\right)ds\right| \\
&= \left|\int_0^t (g(t,s,x(s)) - g(t,s,y(s))\,ds)\right| \\
&\le \int_0^t L_a^b \|(0,0,x(s)-y(s))\|\,ds \\
&= L_a^b \int_0^t |x(s) - y(s)|\,ds.
\end{aligned}$$

Define $z(t) = |x(t) - y(t)|$ for all $t \in [0,\beta]$ so that z is continuous at every point where both x and y are continuous.

Because x and y are continuous almost everywhere, it follows that the map $s \mapsto |x(s) - y(x)|$ is Lebesgue integrable. Hence, its integral $\int_0^t |x(s) - y(s)|\,ds$ is an absolutely continuous function and therefore differentiable at almost every point t, and so $e^{-L_a^b t}\left(L_a^b \int_0^t z(s)\,ds - z(t)\right) \geq 0$ for almost every point, which can be rewritten as

$$\frac{d\left(e^{-L_a^b t}\int_0^t z(s)ds\right)}{dt} \leq 0.$$

Hence, $\int_0^t z(s) \leq 0$; however, z is a non-negative function on $[0,\beta]$, which would imply that $z(t) = 0$ almost always.

□

Example 1. Let $f(t) = -3 + 8\cos(t) + \cos^2(t)$, defined on $[0, +\infty)$, and consider the integral equation

$$x(t) = \frac{1}{6} f(t) + \frac{5}{6}\int_0^t \left[\sin(t-s)\left(1 + x^2(s)\right)\right] ds, \quad t \in [0, 1]. \tag{3.3}$$

First, if we consider $[a_1, a_2]$ an arbitrary interval of $[0, +\infty)$, note that any partition $\xi = \{t_i\}_0^n \in PP([a_1, a_2])$, by using the mean value theorem and the fact that both the sine and cosine functions are bounded by 1, we obtain

$$\begin{aligned} |f(t_i) - f(t_{i-1})| &= \left|8(\cos(t_i) - \cos(t_{i-1})) + (\cos^2(t_i) - \cos^2(t_{i-1}))\right| \\ &\leq 8\,|t_i - t_{i-1}| + 2\,|t_i - t_{i-1}| \\ &= 10\,|t_i - t_{i-1}|\,. \end{aligned}$$

Because this holds for all $1 \leq i \leq n$, it follows that

$$\sum_{i=1}^{n} \Phi\left(|f(t_i) - f(t_{i-1})|\right) \leq \sum_{i=1}^{n} \Phi\left(10\,|t_i - t_{i-1}|\right) \leq \Phi\left(10|a_2 - a_1|\right).$$

That is, $f \in L\mathcal{V}_\Phi([0, \infty))$.

Note that the function $g : D \times \mathbb{R} \to \mathbb{R}$ with $D = \left\{(t, s) \in \mathbb{R}^2 : 0 \leq s \leq t < +\infty\right\}$, defined by

$$g(t, s, y) = \sin(t-s)\left(1 + y^2\right),$$

is of class C^1 with respect to all its variables. In particular, the partial derivative $\partial g/\partial y$ exists and is continuous on $D \times \mathbb{R}$. By a standard result, this implies that g is locally Lipschitz on $D \times \mathbb{R}$. Consequentetly, the existence of a solution to Eq (3.3) is guaranteed, with the solution residing in the space $B\mathcal{V}_\Phi([0,\beta])$.

As noted in [15], the solution to this integral equation is given by $x(t) = cos(t)$. It is well-established that $x(t) = cos(t)$ belongs to the space $B\mathcal{V}_\Phi([0,\beta])$.

4. Prolongation of solutions

As we have seen in the previous section, Theorem 3.2 ensures that for any two positive real numbers a, b, arbitrary positive real numbers, there is a β for which Eq (2.1) admits a solution in $[0,\beta]$, which is a function of bounded Φ-variation, and all such solutions coincide almost everywhere.

Specifically, $\beta \leq \min\left\{a, \dfrac{b}{M_a^b}\right\}$ where

$$|g(t,s,x)| \leq M_a^b, \quad \text{for all} \quad (t,s,x) \in R(a,b;\alpha f).$$

When considering β as in Theorem 3.2, it is natural to ask whether this is the largest interval where a solution can be defined. To address this, let us consider the equation

$$x(t) = \alpha f(t) + (1-\alpha)\int_0^t g(t,s,x(s))ds$$

with $t \in [\beta, a_1]$, where a_1 is an arbitrary number greater than β.

We can now consider a translation of this equation, where instead of the functions being defined on the interval $[\beta, a_1]$, we obtain an equation similar to Eq (2.1), defined for all t in $[0, a_1 - \beta]$, that is,

$$x(t+\beta) = \alpha f(t+\beta) + (1-\alpha)\int_0^{t+\beta} g(t+\beta,s,x(s))ds.$$

So,

$$\begin{aligned}
x(t+\beta) &= \alpha f(t+\beta) + (1-\alpha)\left(\int_0^{\beta} g(t+\beta,s,x(s))ds + \int_{\beta}^{t+\beta} g(t+\beta,s,x(s))ds\right)\\
&= \alpha f(t+\beta) + (1-\alpha)\left(\int_0^{\beta} g(t+\beta,s,x(s))ds + \int_0^{t} g(t+\beta,s+\beta,x(s+\beta))ds\right)\\
&= \alpha f(t+\beta) + (1-\alpha)\int_0^{\beta} g(t+\beta,s,x(s))ds\\
&\quad +(1-\alpha)\int_0^{t} g(t+\beta,s+\beta,x(s+\beta))ds.
\end{aligned}$$

Now, let us consider two particular functions,

$$\widehat{f}(t) \;:=\; \alpha f(t+\beta) + (1-\alpha)\int_0^{\beta} g(t+\beta,s,x(s))ds$$

and

$$\widehat{g}(t) \;:=\; g(t+\beta,s+\beta,x)ds.$$

It is easy to show that, because f and g are functions satisfying conditions C_1 and C_2, respectively, their translated functions also satisfy these conditions. Thus, we obtain the following result.

Lemma 4.1. *Suppose that f satisfies condition C_1, and g satisfies condition C_2. If $|x(s) - f(s)| \leq b$ for all $0 \leq s \leq \beta$, then $\widehat{f}$ satisfies condition C_1, and $\widehat{g}$ satisfies condition C_2.*

The previous lemma together with Theorem 3.2 guarantee the existence of a solution, which is a function of bounded Φ-variation, of the equation

$$x(t) = \widehat{f}(t) + \int_0^t \widehat{g}(t, s, x(s))ds \tag{4.1}$$

in the interval $[0, \beta_1]$, where $\beta_1 \leq \min\{a_1 - \beta, b/M_1\}$, and M_1 is such that

$$|\widehat{g}(t, s, y)| \leq M_1 \quad \text{for all} \quad (t, s, y) \in R(a_1 - \beta, b; \alpha f).$$

We have found a solution x_0 in the interval $[0, \beta]$ and a solution x_1 in the interval $[0, \beta_1]$. We can now define

$$x(t) := \begin{cases} x_0(t) & \text{if} \quad t \in [0, \beta] \\ x_1(t - \beta) & \text{if} \quad t \in [\beta, \beta + \beta_1]. \end{cases}$$

This function is a solution of (2.1) and belongs to the space of functions of bounded Φ-variation in the interval $[0, \beta + \beta_1]$.

Under these conditions, every other solution on $[0, \beta + \beta_1]$ equals x almost everywhere.

We then have that, given two positive real numbers a and b, it is always possible to find a positive $\beta < a$ such that the equation has a unique solution, up to almost everywhere equality, on the interval $[0, \beta]$. This solution can always be extended; that is, given $a_1 > \beta$, there exists $\beta_1 > 0$ such that the equation (2.1) has a solution on the interval $[0, \beta_1]$.

5. Aplication: Model of an isolated community

In this section, we formulate a differential model that describes the evolution of an isolated population within an ecologically closed system (see System (5.2)). Specifically, the species under consideration neither competes with other species for resources or habitat nor is subject to predation. The model is intended to capture the fundamental mechanisms governing the species' population dynamics, including reproduction and the possibility of extinction.

Let $K : [0, +\infty) \to (0, +\infty)$ be a non-negative function satisffying the normalization condition

$$\int_0^{+\infty} K(s)ds = 1. \tag{5.1}$$

Let $S_1, S_2 : [0, +\infty) \to [0, +\infty)$ be locally Lipschitz continuous functions, and let $\alpha \in (0, 1)$.

Consider the delay integro-diferential system

$$\begin{cases} x'(t) = (1 - \alpha)\left[S_1(t)\int_{-\infty}^t K(t - s)x(s)ds - S_2(t)x^2(t)\right] & \text{if} \quad t \geq 0 \\ x(t) = \alpha\phi(t) & \text{if} \quad t \leq 0, \end{cases} \tag{5.2}$$

where

$$\phi \in FA := \{\phi \in C((-\infty,0],\mathbb{R}) :\ \phi(t) \geq 0,\ \phi(0) > 0,\ \sup_{t\in(-\infty,0]} (\phi(t)) < \infty\}.$$

We rewrite Eq (5.2) as a Volterra-type integral equation for $t \geq 0$ as follows.

Let $r \geq 0$ be given.

$$\begin{aligned} x'(r) &= (1-\alpha)S_1(r)\int_{-\infty}^{0} K(r-s)x(s)ds \\ &\quad +(1-\alpha)S_1(r)\int_{0}^{r} K(r-s)x(s)ds - (1-\alpha)S_2(r)x^2(r) \\ &= (1-\alpha)S_1(r)\int_{0}^{+\infty} K(r+s)\alpha\phi(-s)ds \\ &\quad +(1-\alpha)S_1(r)\int_{0}^{r} K(r-s)x(s)ds - (1-\alpha)S_2(r)x^2(r) \\ &= \alpha(1-\alpha)S_1(r)\int_{0}^{+\infty} K(r+s)\phi(-s)ds \\ &\quad +(1-\alpha)S_1(r)\int_{0}^{r} K(r-s)x(s)ds - (1-\alpha)S_2(r)x^2(r). \end{aligned}$$

In this case, we define

$$F(r) = (1-\alpha)S_1(r)\int_{0}^{+\infty} K(r+s)\phi(-s)ds,$$

and we can rewrite (5.2) as follows:

$$\begin{cases} x'(r) = \alpha F(r) + (1-\alpha)\left(S_1(r)\displaystyle\int_{0}^{r} K(r-s)x(s)ds - S_2(r)x^2(r)\right) & \text{if} \quad r > 0, \\ x(r) = \alpha\phi(r) & \text{if} \quad r \leq 0. \end{cases}$$

Integrating from 0 to t in the equation

$$x'(r) = \alpha F(r) + (1-\alpha)\left[S_1(r)\int_{0}^{r} K(r-s)x(s)ds - S_2(r)x^2(r)\right],$$

we obtain that

$$\begin{aligned} x(t) &= x(0) + \alpha\int_{0}^{t} F(r)dr \\ &\quad +(1-\alpha)\left[\int_{0}^{t} S_1(r)\left(\int_{0}^{r} K(r-s)x(s)ds\right)dr - \int_{0}^{t} S_2(r)x^2(r)dr\right] \\ &= \alpha\left(\phi(0) + \int_{0}^{t} F(r)dr\right) \\ &\quad +(1-\alpha)\left\{\int_{0}^{t}\left[x(s)\left(\int_{s}^{t} S_1(r)K(r-s)dr\right) - S_2(s)x^2(s)\right]ds\right\}. \end{aligned}$$

With the above, the previous equation can be rewritten as

$$x(t) = \alpha f(t) + (1-\alpha)\int_0^t g(t,s,x(s))ds, \qquad t \geq 0, \tag{5.3}$$

where

$$f(t) = \phi(0) + \int_0^t F(r)dr, \tag{5.4}$$

and

$$g(t,s,y) = y(s)\left(\int_s^t S_1(r)K(r-s)dr\right) - S_2(s)y^2(s). \tag{5.5}$$

Our objective is now to demonstrate that the functions defined in (5.4) and (5.5) satisfy the hypotheses required by Theorem 3.2, thereby guaranteeing the existence of a solution to Eq (5.3) with locally bounded Φ-variation.

Lemma 5.1. *Let $\phi \in FA$, let S_1 be a locally Lipschitz function on $[0,+\infty)$, and let K be a continuous function on $[0,+\infty)$ satisfying property (5.1). The function $f : [0,+\infty) \to \mathbb{R}$ defined by*

$$f(t) = \phi(0) + \int_0^t \left[(1-\alpha)S_1(r)\int_0^{+\infty} K(r+s)\phi(-s)ds\right]dr,$$

belongs to $L\mathcal{V}_\Phi([0,\infty))$.

Proof. Let $\phi_M := \sup_{s\in[0,+\infty)} \phi(-s)$. Then, by the property (5.1), it follows that

$$0 \leq (1-\alpha)S_1(r)\int_0^{+\infty} K(r+s)\phi(-s)ds \leq (1-\alpha)S_1(r)\phi_M < +\infty.$$

This ensures that f is a well-defined function and is also non-negative.

To prove that $f \in L\mathcal{V}_\Phi([0,\infty))$, let $[a_1,a_2] \subseteq [0,+\infty)$, and consider a partition $\xi = \{t_i\}_0^n$ of the interval $[a_1,a_2]$. Because S_1 is locally Lipschitz, there exists a constant $M_{a_1}^{a_2} > 0$ such that $|S_1(r)| \leq M_{a_1}^{a_2}$ for all $r \in [a_1,a_2]$. Therefore, it is easy to see that

$$|f(t_i) - f(t_{i-1})| \leq M_{a_1}^{a_2}\phi_M\,|t_i - t_{i-1}|,$$

from which we can obtain the inequality

$$\sum_{i=1}^{n} \Phi\left(|f(t_i) - f(t_{i-1})|\right) \leq \Phi\left(M_{a_1}^{a_2}\phi_M|a_2 - a_1|\right).$$

As this inequality is satisfied for all possible partitions of $[a_1,a_2]$, we conclude that $\bigvee_\Phi(f;[a_1,a_2]) \leq \Phi\left(M_{a_1}^{a_2}\phi_M(a_2-a_1)\right)$, and thus, $f \in L\mathcal{V}_\Phi([0,\infty))$. □

Lemma 5.2. *Let* $D = \left\{(t,s) \in \mathbb{R}^2 : 0 \le s \le t < \infty\right\}$ *and* $S_1, S_2 : [0,+\infty) \to [0,+\infty)$ *be locally Lipschitz continuous functions. Suppose K is a normalized kernel, and define the function*

$$g : D \times \mathbb{R} \to \mathbb{R}, \qquad g(t,s,y) = y\left(\int_s^t S_1(u)K(u-s)du\right) - S_2(s)y^2.$$

Then, g is locally Lipschitz continuous on $D \times \mathbb{R}$*.*

Proof. If Q is a compact subset of $D \times \mathbb{R}$, then by carrying out the same procedures as before, one obtains

$$|g(t_1,s_1,x_1) - g(t_2,s_2,x_2)| \quad \le \quad K\,\|(t_1,s_1,x_1) - (t_2,x_2,s_2)\|,$$

where K is a constant that depends on the Lipschitz constants of the functions S_1 and S_2 as well as on their bounds on that compact set. □

Lemma 5.1 guarantees that the function f defined in (5.4) satisfies the condition C_1. Similarly, Lemma 5.2 guarantees that the function g defined in (5.5) satisfies the condition C_2. Thus, by Theorem 3.2, for each $\phi \in FA$, the integral equation (5.2) has a solution that is a function of locally bounded Φ-variation.

To illustrate Theorem 3.2, let us consider the following particular case.

Example 2. Consider the following system:

$$\begin{cases} x'(t) = \dfrac{(1+t^2)}{2}\displaystyle\int_{-\infty}^{t} 3e^{-3(t-s)}x(s)ds - \dfrac{(t+1)}{2}x^2(t) & \text{if} \quad t \ge 0, \\ x(t) = \dfrac{1}{2}e^t & \text{if} \quad t \le 0. \end{cases} \tag{5.6}$$

Note that the kernel $K(s) = 3e^{-3s}$ satisfies the normalization condition $\displaystyle\int_0^{+\infty} K(s)\,ds = 1$ and is continuous on $[0,+\infty)$. The coefficient functions

$$S_1(t) = 1 + t^2 \quad \text{and} \quad S_2(t) = t + 1$$

are continuously differentiable on $[0,+\infty)$. Because every S_1 function on an interval is locally Lipschitz, these choices guarantee that the conditions for existence are met.

6. Conclusions

This paper establishes the existence and uniqueness of solutions to Volterra integral equations in the space of functions of bounded Φ-variation. Under suitable regularity and boundedness conditions on the functions f and g, we prove that such equations admit a unique solution, up to equality almost everywhere (Theorem 3.2). Moreover, we show that the associated Volterra operator preserves bounded Φ-variation (Theorem 2.4) and is bounded on this space (Corollary to Theorem 2.4).

To address global existence, we develop a continuation method that extends local solutions to larger intervals, provided the relevant boundedness conditions remain in force. Our results extend the classical theory of Volterra equations to the Φ-variation setting, thereby providing a flexible framework for problems where traditional function spaces prove too restrictive.

Several natural directions for future research arise from related work in the literature. The stability analysis of solutions under impulsive perturbations, as studied in [14], could be explored within the Φ-variation framework. Likewise, the development of faster iterative schemes [8] and the investigation of asymptotic behavior [9] suggest promising avenues for further investigation, both from a theoretical standpoint and in applications such as population dynamics or epidemiology.

Author contributions

The authors declare that they have contributed equally to the development of this work.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflict of interest in this paper.

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