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Research article

A thorough examination of the bivariate Kimeldorf-Sampson extended Weibull family in light of versatile statistical applications

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Abstract: Families of bivariate distributions with weak dependence between their margins are of practical importance, despite being relatively uncommon. These families can be constructed using specific copulas. This paper introduces a new family, KSEW, developed using the bivariate Kimeldorf-Sampson (KS) copula and the extended Weibull (EW) distributions. The KSEW family is designed for applications requiring marginal distributions with low but significant levels of dependence, positioned above the threshold of independence. Key properties of the KSEW family were derived, including product moments and the correlation coefficient. The study also explored the uniform, power, Weibull, Rayleigh, exponential, and Lomax distributions as specific cases within the EW family, identifying the maximum correlation coefficient for each. Additionally, the reliability function in the dependent stress-strength model under the KS framework was developed. The concomitants of order statistics and associated information measures using the KSEW distribution were also investigated. The practical applicability of the KSEW family was demonstrated through analysis of two real-world datasets, highlighting its suitability for scenarios involving weakly dependent random variables.

Keywords: Kimeldorf-Sampson family; extended Weibull distribution; concomitants; simulation; order statistics; Shannon entropy; extropy
Mathematics Subject Classification: 60B12, 62G30

1. Introduction

Recently, the issue of modeling bivariate data has garnered the interest of numerous scholars, particularly, when there is previous knowledge available in the form of marginal distributions. Examining families of bivariate distributions that have predetermined marginal distributions can be beneficial. Many authors have extensively investigated the Farlie-Gumbel-Morgenstern (FGM) family of bivariate distributions. This family is represented by the bivariate cumulative distribution function (CDF) $F_{X,Y}(x,y) = F_X(x)F_Y(y)[1+\alpha(1-F_X(x))(1-F_Y(y))]$, $-1 \leq \alpha \leq 1$, where $F_X(x)$ and $F_Y(y)$ are the marginal CDFs of the two random variables (RVs) X and Y, respectively. The Pearson correlation coefficient ρ for the FGM family is bounded by $|\rho| \leq 1/3$, a defining characteristic of this family. Moreover, this maximum absolute correlation is attained precisely for the FGM copula, i.e., when the marginals are uniform (cf. Schucany et al. [1]). However, due to its limited dependence range, the FGM family is mainly suitable for modeling data with weak correlation. For applications of the FGM family, see Nelsen [2], Schucany et al. [1], and Drouet and Kotz [3]. More recent studies include Barmalzan and Vali [4], Arun et al. [5], and Blier-Wong [6], among others.

Many researchers have effectively extended the FGM family, enabling a greater range of component correlations. Among the early contributions in this area are Cambanis [7] and Huang and Kotz [8, 9], who investigated several theoretical aspects of dependence structures. Further developments were provided by Bairamov et al. [10] and Bekrizadeh et al. [11]. More recently, related results were obtained by Barakat et al. [12, 13]. These extensions have been widely used in recent studies in both theoretical and applied statistics. Examples include Abd Elgawad et al. [14] and Abd Elgawad and Alawady [15]. Additional applications were presented by Alawady et al. [16]. Further developments can be found in Husseiny et al. [17, 18] and Mansour et al. [19, 20].

Besides the need to obtain extensions of the FGM family with a correlation higher than $1/3$, there is a clear need to obtain families with a weaker correlation ρ than the FGM family, i.e., $\rho \leq \frac{1}{3}$. Indeed, choosing bivariate families with low maximum correlation is an effective way to mitigate the risks of over-fitting or exaggerating dependence, particularly, in cases where variables are weakly related or nearly independent. This ensures more realistic and interpretable models. Applications of such families include:

- Modeling random processes with minimal interaction, such as noise in uncorrelated physical systems.
- Investigating weak relationships between climate variables in some stable areas, such as temperature and humidity.
- Weakly dependent families are used as a null or alternative hypothesis when examining independence.
- Generating data for sensitivity analysis that is very close to independence.

Among bivariate distributions or copulas with a maximum correlation below $\frac{1}{3}$, the following are viable options:

(1) The Ali–Mikhail–Haq copula allows only weak dependence since Kendall's tau satisfies $-0.1817 \leq \tau \leq 1/3$ (see Balakrishnan and Lai, [21]).
(2) Modified Gaussian copula: correlation parameter ρ artificially constrained to $|\rho| < 1/3$.

(3) Mixtures that combine independent and weakly dependent structures, e.g., $G_{X,Y} = \theta F_X F_Y + (1 - \theta)F_{X,Y}$, where $0 < \theta < 1$ and $F_{X,Y}$ is the FGM family.

Kimeldorf and Sampson [22] presented several examples of bivariate uniform distributions. One of these models (the eighth model) is the focus of the present study. Its CDF and probability density function (PDF), respectively, are given by

$$F_{X,Y}(x,y) = F_X(x)F_Y(y)[1 + \alpha(1 - F_X(x))(1 - F_Y(y)) + \beta(1 - F_X^2(x))(1 - F_Y^2(y))] \tag{1.1}$$

and

$$f_{X,Y}(x,y) = f_X(x)f_Y(y)\left[1 + \alpha(1 - 2F_X(x))(1 - 2F_Y(y)) + \beta\left(1 - 3F_X^2(x)\right)\left(1 - 3F_Y^2(y)\right)\right], \tag{1.2}$$

where $|\alpha| \leq 1/2$ and $|\beta| \leq 1/8$. The constraints $|\alpha| \leq 1/2$ and $|\beta| \leq 1/8$ are necessary and sufficient for the joint density (1.2) to be non-negative for all (x, y); equivalently, they ensure that the copula density $c(u, v) = 1 + \alpha(1 - 2u)(1 - 2v) + \beta(1 - 3u^2)(1 - 3v^2)$ is non-negative for all $u, v \in [0, 1]$. A detailed derivation of these bounds can be found in Kimeldorf and Sampson [22]. At the boundary values, the density touches zero but never becomes negative.

This model, denoted by the Kimeldorf-Sampson (KS) family, reduces to a restricted version of the FGM family for $\beta = 0$, with a narrower admissible parameter space, $|\alpha| \leq 1/2$. Hence, the KS family may be viewed as a constrained FGM-type copula rather than a full generalization. The parameter α governs the classical linear dependence component of the FGM family, while β introduces higher-order polynomial terms that modify the dependence structure. However, due to the constraints on the admissible parameters, the overall attainable correlation for the copula in (1.1) is strictly lower than the FGM upper bound of $1/3$, with a maximum value of $25/96 \approx 0.260417$ (see Subsection 3.1). Since the KS family has not been studied previously, these features make it a compelling choice for constructing a new bivariate copula, particularly, in applications where weak dependence is a key characteristic.

We choose the EW family for suitable margins for the proposed bivariate model based on the KS copula since the Weibull distribution's versatility in adjusting to different forms and failure modes guarantees its central role in probabilistic modeling and risk analysis. A class of probability distributions known as the EW family expands upon the traditional Weibull distribution to allow for more adaptable data modeling. The Weibull distribution's versatility is occasionally insufficient for complex datasets, even though it is frequently employed in extreme value theory, survival analysis, and reliability analysis, because it can model a variety of failure rates, cf. Meng and Zhu [23] and Meng et al. [24]. By adding new parameters or transformations, extensions of the Weibull distribution increase its suitability for data from the real world. The EW family of distributions, which Gurvich et al. [25] created for modeling the random strength of brittle materials, is one of the most significant extensions of the Weibull family. The CDF and PDF of the EW$(a, \boldsymbol{\tau})$ distribution are, respectively, given by

$$F_X(x; a, \boldsymbol{\tau}) = 1 - e^{-aG(x;\boldsymbol{\tau})} \tag{1.3}$$

and

$$f_X(x; a, \boldsymbol{\tau}) = ag(x; \boldsymbol{\tau})e^{-aG(x;\boldsymbol{\tau})}, \tag{1.4}$$

where $a > 0$, $\boldsymbol{\tau}$ is a vector of parameters, and $G(x; \boldsymbol{\tau})$ is a non-negative, continuous, monotone increasing, and differentiable function of x that depends on the parameter vector $\boldsymbol{\tau}$. Also, $G(x; \boldsymbol{\tau}) \to 0^+$ as $x \to 0^+$ and $G(x; \boldsymbol{\tau}) \to +\infty$ as $x \to +\infty$. Further, $g(x; \boldsymbol{\tau})$ is the derivative of $G(x; \boldsymbol{\tau})$ with respect to x.

The EW distribution is a flexible model that includes several well-known lifetime distributions as special cases. Its properties and applications have been widely studied in the literature; see, for example, Jafari et al. [26], Pham and Lai [27], and Tahmasebi and Jafari [28]. More recent contributions include Elgawad et al. [29] and Nadarajah and Kotz [30]. These studies highlight that the extended Weibull (EW) distribution contains several important submodels, including the Weibull, Rayleigh, exponential, Fréchet, Pareto, Gompertz, Kies, and Lomax distributions. Six of these special cases are of interest in this study via the KS family with margins belonging to EW distribution. KSEW denotes this proposed family of bivariate distributions.

Concomitants of order statistics (COSs) arise naturally in the study of bivariate or multivariate RVs, particularly, when there is interest in the relationship between two variables and their joint behavior in ranked data. To give a rigorous mathematical definition of COSs, we assume that a random sample from a bivariate CDF $F_{X,Y}$ is (X_i, Y_i), $i = 1, 2, ..., n$. The Y-variate that corresponds to the rth order statistic (OS) $X_{r:n}, r = 1, ..., n$, is known as the concomitant of the rth OS and is represented as $Y_{[r:n]}$. This is the case if we arrange the sample according to the X-variate and obtain the OSs $X_{1:n} \leq X_{2:n} \leq .... \leq X_{n:n}$ for the X sample. COSs were first proposed by David [31]. The COSs are of interest in problems involving prediction and selection. An excellent examination of COSs is given by David et al. [32], Hanif [33], and David and Nagaraja [34]. Several papers have studied COSs based on generalizations of the FGM family. Early and foundational work includes David and Nagaraja [34] and Eryilmaz [35]. More recent studies include Alawady et al. [36, 37], and Husseiny et al. [17, 18]. The PDF of the rth concomitant, $Y_{[r:n]}$, of the rth OS, $X_{r:n}$, $1 \leq r \leq n$, is given by

$$f_{[r:n]}(y) = \int_{-\infty}^{\infty} f_{Y|X}(y|x) f_{r:n}(x) dx, \tag{1.5}$$

where $f_{r:n}(x)$ is the PDF of the rth OS and $f_{Y|X}(y|x)$ is the conditional PDF of Y given X (see, e.g., Barakat et al. [12, 13] and Abd Elgawad et al. [14]).

Entropy, introduced first by Shannon [38], expresses the average decrease in variability or uncertainty associated with an RV. Boltzmann [39] originated the notion of "information entropy", which Shannon [38] later formalized in the domains of information and communication. Higher entropy implies greater uncertainty or spread in the distribution of the given RV. For a continuous positive RV X with PDF $f_X(x)$, the (differential) entropy is given by (see, for example, Elgawad et al. [14])

$$H(X) = -E(\log f_X(X)) = -\int_0^{\infty} f_X(x) \log f_X(x) dx. \tag{1.6}$$

Extropy, introduced recently by Lad et al. [40], has found its place in academic literature as a complementary dual function of entropy. According to Lad et al. [40], if an RV X has a PDF $f_X(x)$, its extropy is defined as follows:

$$J(X) = -\frac{1}{2} E(f_X(X)) = -\frac{1}{2} \int_0^{\infty} f_X^2(x) dx. \tag{1.7}$$

Extropy quantifies the concentration or overlap of the PDF $f_X(x)$. Unlike entropy, which measures uncertainty, extropy focuses on the reliability or concentration of the distribution, further foundational studies on extropy can be found in Qiu [41]. Together, entropy and extropy for continuous positive RVs provide a more comprehensive understanding of the behavior of positive continuous RVs, enabling better decision-making in applications like reliability engineering, survival analysis, and risk management.

Many studies have focused on creating bivariate distributions by combining FGM-type copulas with flexible marginal families like Weibull and EW distributions. Notable examples include Jafari et al. [26] and similar research, where FGM or generalized FGM copulas are used to model weak dependence structures and derive related properties like moments, dependence measures, and information quantities.

Although there are surface similarities, this work is fundamentally different from those contributions in both copula construction and modeling goals. The key differences are summarized below.

(i) Copula structure and parameterizations. The classical FGM copula and many of its common extensions employ a single dependence parameter and are based on a bilinear perturbation of the product copula. In contrast, the KS copula considered in this paper is a genuine two-parameter copula, in which the dependence structure is governed by the parameters α and β through higher-order polynomial terms. These parameters play distinct and interpretable roles in shaping the joint dependence structure, which cannot be replicated by the classical FGM copula and its standard extensions.

(ii) Attainable dependence range and modeling philosophy. While many extensions of the FGM family aim to expand the allowable correlation range beyond the classical FGM limit, the KS copula is specifically designed to model ultra-weak dependence. Its maximum attainable correlation is below that of the FGM family under uniform margins. This characteristic is not a drawback but an advantage in scenarios where variables are nearly independent and overstating dependence is undesirable.

(iii) Interaction with EW margins. When paired with EW margins, the KS copula creates a bivariate family where the induced Pearson correlation depends on both the copula parameters and the shapes of the margins. This interaction produces dependence behavior that is qualitatively different from FGM-EW constructions, including instances where the induced correlation changes nonlinearly across specific EW subclasses. Such behavior does not occur in existing FGM-based models, including those reviewed by Jafari et al. [26].

(iv) Reliability modeling under weak dependence. This paper provides a clear development of the dependent stress-strength reliability function within the KS framework. To our knowledge, this kind of analysis has not been done for KS-based models with EW margins, nor is it available in FGM-EW constructions. The results show how weak dependence impacts system reliability without creating unrealistically strong joint tail behavior.

(v) Concomitants and information measures. While COSs and entropy-based measures have been examined for FGM-type families, deriving them under the KS copula results in structurally different expressions that reflect the higher-order dependence terms. The resulting entropy and extropy measures provide further insight into the behavior of ranked observations under controlled weak dependence.

In summary, unlike previous EW-based models that rely on FGM or generalized FGM copulas, this work is the first to combine the KS copula with the EW family. This combination produces a bivariate model with analytically manageable properties, two interpretable dependence parameters, and controllable ultra-weak dependence below the classical FGM upper bound. These features position the proposed KSEW family as a distinct and complementary modeling framework rather than just a minor variant of existing FGM-based approaches.

The rest of the paper is organized as follows. In Section 2, we derive important properties of the KSEW family, including moments and the correlation coefficient, and discuss parameter identifiability. In Section 3, we discuss the uniform, Weibull, Rayleigh, exponential, power, Lomax, and two-parameter Weibull distributions, which are special cases of the EW family. Also, the maximum correlation coefficient for each of these cases is determined, and the reliability in the dependent stress-strength KS model is discussed. Section 4 undertakes the study of COSs based on the KSEW family. Additionally, in this section, we examine the entropy and extropy of the COSs at specific parameter values to observe their behavior. Section 5 contains evaluations of two real-world data sets to examine the suitability of the KS family for them. Finally, Section 6 presents the study's conclusion.

2. Properties of the KSEW distribution

In this section, the KSEW model is introduced and some of its properties are concluded.

2.1. Distributional properties

Let $X \sim \mathrm{EW}(a, \boldsymbol{\tau}_1)$ and $Y \sim \mathrm{EW}(b, \boldsymbol{\tau}_2)$. Beginning with (1.2) and (1.4), the PDF of the KSEW family is deduced as follows:

$$\begin{aligned} f_{X,Y}(x,y) &= abg(x;\boldsymbol{\tau}_1)g(y;\boldsymbol{\tau}_2)e^{-aG(x;\boldsymbol{\tau}_1)}e^{-bG(y;\boldsymbol{\tau}_2)}\Big[1+\alpha\Big(2e^{-aG(x;\boldsymbol{\tau}_1)}-1\Big)\\ &\quad\times\Big(2e^{-bG(y;\boldsymbol{\tau}_2)}-1\Big)+\beta\Big(6e^{-aG(x;\boldsymbol{\tau}_1)}-3e^{-2aG(x;\boldsymbol{\tau}_1)}-2\Big)\\ &\quad\times\Big(6e^{-bG(y;\boldsymbol{\tau}_2)}-3e^{-2bG(y;\boldsymbol{\tau}_2)}-2\Big)\Big]. \end{aligned} \tag{2.1}$$

Suppose (X, Y) has the KSEW distribution with parameters $\alpha, \beta, a, b, \boldsymbol{\tau}_1$, and $\boldsymbol{\tau}_2$. Let U_1, U_2, V_1, and V_2 be independent RVs such that $U_i \sim \mathrm{EW}((i+1)a, \boldsymbol{\tau}_1)$ and $V_i \sim \mathrm{EW}((i+1)b, \boldsymbol{\tau}_2)$, for $i = 1, 2$. Then, we obtain the following statistical features:

(1) The joint PDF of the KSEW family that is concluded in (2.1) can be written by

$$\begin{aligned} f_{X,Y}(x,y) &= f_X(x)f_Y(y)+\alpha\left(f_{U_1}(x)-f_X(x)\right)\left(f_{V_1}(y)-f_Y(y)\right)\\ &\quad+\beta\left(3f_{U_1}(x)-f_{U_2}(x)-2f_X(x)\right)\left(3f_{V_1}(y)-f_{V_2}(y)-2f_Y(y)\right), \end{aligned} \tag{2.2}$$

where $f_X(.)$, $f_Y(.)$, $f_{U_i}(.)$, and $f_{V_i}(.)$ are the PDFs of X, Y, U_i, and V_i, respectively, $i = 1, 2$.

(2) The (n, m)th product moment of the KSEW family is

$$\begin{aligned} E\left[X^nY^m\right] &= \int_{y=0}^{\infty}\int_{x=0}^{\infty}x^ny^mf_{X,Y}(x,y)dxdy\\ &= \int_{x=0}^{\infty}x^nf_X(x)dx\int_{y=0}^{\infty}y^mf_Y(y)dy \end{aligned}$$

$$
\begin{aligned}
&+\alpha \int_{x=0}^{\infty} x^n \left(f_{U_1}(x) - f_X(x)\right) dx \int_{y=0}^{\infty} y^m \left(f_{V_1}(y) - f_Y(y)\right) dy \\
&+\beta \int_{x=0}^{\infty} x^n \left(3f_{U_1}(x) - f_{U_2}(x) - 2f_X(x)\right) dx \\
&\times \int_{y=0}^{\infty} y^m \left(3f_{V_1}(y) - f_{V_2}(y) - 2f_Y(y)\right) dy \\
&= E\left[X^n\right] E\left[Y^m\right] + \alpha \left(E\left[U_1^n\right] - E\left[X^n\right]\right)\left(E\left[V_1^m\right] - E\left[Y^m\right]\right) \\
&+\beta \left(3E\left[U_1^n\right] - E\left[U_2^n\right] - 2E\left[X^n\right]\right)\left(3E\left[V_1^m\right] - E\left[V_2^m\right] - 2E\left[Y^m\right]\right). \qquad (2.3)
\end{aligned}
$$

(3) The covariance of X and Y is given by

$$
\begin{aligned}
Cov(X, Y) = &\ \alpha \left(E\left[U_1\right] - E\left[X\right]\right)\left(E\left[V_1\right] - E\left[Y\right]\right) \\
&+\beta \left(3E\left[U_1\right] - E\left[U_2\right] - 2E\left[X\right]\right)\left(3E\left[V_1\right] - E\left[V_2\right] - 2E\left[Y\right]\right). \qquad (2.4)
\end{aligned}
$$

(4) The correlation coefficient of (X, Y) is

$$
\begin{aligned}
\rho_{X,Y} = &\ \alpha \left(\frac{1}{CV(X)} - \frac{E\left[U_1\right]}{\sqrt{Var(X)}}\right)\left(\frac{1}{CV(Y)} - \frac{E\left[V_1\right]}{\sqrt{Var(Y)}}\right) \\
&+\beta \left(\frac{2}{CV(X)} + \frac{E\left[U_2\right] - 3E\left[U_1\right]}{\sqrt{Var(X)}}\right)\left(\frac{2}{CV(Y)} + \frac{E\left[V_2\right] - 3E\left[V_1\right]}{\sqrt{Var(Y)}}\right), \qquad (2.5)
\end{aligned}
$$

where $CV(X)$ and $CV(Y)$ are the coefficient of variation of X and Y, respectively.

2.2. Parameter identifiability

A crucial aspect of any parametric model is the identifiability of its parameters. A model is said to be identifiable if distinct parameter values lead to distinct distributions; otherwise, estimation and inference become problematic. In the context of the KSEW family, we consider two sets of parameters: the copula parameters (α, β) and the marginal parameters $(a, b, \boldsymbol{\tau}_1, \boldsymbol{\tau}_2)$.

Identifiability of the copula parameters. The KS copula given in (1.1) has the form

$$
C(u, v) = uv[1 + \alpha(1 - u)(1 - v) + \beta(1 - u^2)(1 - v^2)],
$$

with $|\alpha| \leq 1/2$ and $|\beta| \leq 1/8$. The functions $(1 - u)(1 - v)$ and $(1 - u^2)(1 - v^2)$ are linearly independent on $(0, 1)^2$; hence, the parameters α and β are uniquely determined by the copula. This ensures that the dependence structure is identifiable under the usual regularity conditions.

Identifiability of the marginal parameters. The EW family defined by (1.3) and (1.4) is a flexible class that includes many common distributions. For a fixed choice of the baseline function $G(\cdot; \boldsymbol{\tau})$, the parameter $a > 0$ and the vector $\boldsymbol{\tau}$ are typically identifiable provided that the mapping $\boldsymbol{\tau} \mapsto G(x; \boldsymbol{\tau})$ is injective in $\boldsymbol{\tau}$ for each x. This holds for all the special cases considered in this paper (uniform, power, Weibull, Rayleigh, exponential, Lomax). Therefore, under standard regularity conditions, the full parameter vector $(\alpha, \beta, a, b, \boldsymbol{\tau}_1, \boldsymbol{\tau}_2)$ is identifiable.

Practical identifiability issues. Despite theoretical identifiability, practical problems can arise, especially, when the dependence is very weak or the sample size is small. In such cases, the likelihood surface may become nearly flat in certain directions, leading to large standard errors or convergence difficulties. For example, extremely large estimates of $\boldsymbol{\tau}_1$ or $\boldsymbol{\tau}_2$ (as observed for the KS Lomax (KSL) model in Tables 5 and 7) may indicate that the marginal distribution is approaching a simpler submodel (e.g., exponential), causing weak identifiability of the scale parameters. Similarly, when α and β are close to the boundaries of their admissible intervals, the information matrix may be ill-conditioned.

Handling identifiability in the empirical analysis. In our applications, we address potential identifiability concerns in several ways:

- We enforce the parameter constraints $|\alpha| \leq 1/2$ and $|\beta| \leq 1/8$ during maximum likelihood (ML) estimation, which prevents the optimizer from moving into inadmissible regions.
- Convergence of the optimization algorithm is checked by examining the gradient norm at the solution and the positive definiteness of the Hessian matrix.
- Standard errors are computed from the observed Fisher information matrix (for point estimates) and supplemented by bootstrap confidence intervals (Subsection 5.1). The bootstrap provides a robust assessment of parameter uncertainty without relying on asymptotic normality.
- For models with extremely large estimates (e.g., KSL in Tables 5 and 7), we interpret the results cautiously, noting that such values may signal weak identifiability and that a simpler marginal model could be more appropriate. The Akaike information criterion (AIC) values confirm that the better-fitting KS Weibull (KSW) model does not exhibit such extreme estimates.

Overall, the KSEW family is theoretically identifiable, and our empirical implementation includes diagnostic checks to detect and mitigate practical identifiability problems. This ensures that the reported estimates and model comparisons are reliable.

3. Special cases of the KSEW distribution and the reliability in the KS model

This section investigates some special cases of the KSEW family and their maximum correlation coefficient. Furthermore, the reliability in the KS model is discussed (Subsection 3.8).

3.1. Standard uniform distribution

The distributions $\mathrm{EW}(a, \boldsymbol{\tau}_i),\ i = 1, 2$, reduce to standard uniform distributions by putting $a = 1$ and $G(x; \boldsymbol{\tau}_i) = -\log(1 - x), 0 \leq x \leq 1$. In this case, we get the bivariate KS copula distribution as

$$F_{X,Y}(x, y) = xy\left[1 + \alpha(1 - x)(1 - y) + \beta(1 - x^2)(1 - y^2)\right],$$

and its properties mentioned in (2.2)–(2.5) become the following:

(1) The joint PDF is

$$f_{X,Y}(x, y) = 1 + \alpha(1 - 2x)(1 - 2y) + \beta(1 - 3x^2)(1 - 3y^2).$$

(2) The (n, m)th product moment is

$$E(X^nY^m) = \frac{1}{(n+1)(m+1)} + \alpha\left(\frac{1}{n+1} - \frac{2}{n+2}\right)\left(\frac{1}{m+1} - \frac{2}{m+2}\right) + \beta\left(\frac{1}{n+1} - \frac{3}{n+3}\right)\left(\frac{1}{m+1} - \frac{3}{m+3}\right).$$

(3) The covariance between X and Y is

$$Cov(X, Y) = \frac{\alpha}{36} + \frac{\beta}{16}.$$

(4) The correlation coefficient is

$$\rho_{X,Y} = \frac{\alpha}{3} + \frac{3\beta}{4},$$

which has the maximum value 0.260417 at $\alpha = 0.5$ and $\beta = 0.125$.

3.2. Power distribution

The power distribution is obtained from $\mathrm{EW}(a, \boldsymbol{\tau}_i)$, $i = 1, 2$, by putting $a = 1$ and $G(x; \boldsymbol{\tau}_i) = -\log(1 - x^{\tau_i}), 0 \le x \le 1, \tau_i \ge 0$. Thus, the KS power distribution (denoted by KSP) is given by

$$F_{X,Y}(x, y) = x^{\tau_1}y^{\tau_2}\left[1 + \alpha(1 - x^{\tau_1})(1 - y^{\tau_2}) + \beta\left(1 - x^{2\tau_1}\right)\left(1 - y^{2\tau_2}\right)\right].$$

Therefore, the statistical features of the KSP family mentioned in (2.2)–(2.5) are as follows:

(1) The joint PDF is

$$f_{X,Y}(x, y) = \tau_1\tau_2x^{\tau_1-1}y^{\tau_2-1}\left[1 + \alpha(1 - 2x^{\tau_1})(1 - 2y^{\tau_2}) + \beta\left(1 - 3x^{2\tau_1}\right)\left(1 - 3y^{2\tau_2}\right)\right].$$

(2) The (n, m)th product moment is

$$E(X^nY^m) = \tau_1\tau_2\left[\frac{1}{(\tau_1+n)(\tau_2+m)} + \alpha\left(\frac{1}{\tau_1+n} - \frac{2}{2\tau_1+n}\right)\left(\frac{1}{\tau_2+m} - \frac{2}{2\tau_2+m}\right) + \beta\left(\frac{1}{\tau_1+n} - \frac{3}{3\tau_1+n}\right)\left(\frac{1}{\tau_2+m} - \frac{3}{3\tau_2+m}\right)\right].$$

(3) The covariance between X and Y is

$$Cov(X, Y) = \tau_1\tau_2\left[\alpha\left(\frac{1}{\tau_1+1} - \frac{2}{2\tau_1+1}\right)\left(\frac{1}{\tau_2+1} - \frac{2}{2\tau_2+1}\right) + \beta\left(\frac{1}{\tau_1+1} - \frac{3}{3\tau_1+1}\right)\left(\frac{1}{\tau_2+1} - \frac{3}{3\tau_2+1}\right)\right].$$

(4) The correlation coefficient is

$$\rho_{X,Y} = \frac{\tau_1\tau_2}{\sqrt{\left[\frac{\tau_1}{\tau_1+2} - \frac{\tau_1^2}{(\tau_1+1)^2}\right]\left[\frac{\tau_2}{\tau_2+2} - \frac{\tau_2^2}{(\tau_2+1)^2}\right]}}\left[\alpha\left(\frac{1}{\tau_1+1} - \frac{2}{2\tau_1+1}\right) \times \left(\frac{1}{\tau_2+1} - \frac{2}{2\tau_2+1}\right) + \beta\left(\frac{1}{\tau_1+1} - \frac{3}{3\tau_1+1}\right)\left(\frac{1}{\tau_2+1} - \frac{3}{3\tau_2+1}\right)\right],$$

which has the maximum value 0.262642 at $\tau_1 = 0.76228$ and $\tau_2 = 0.76228$.

3.3. Weibull distribution

The Weibull distribution can be obtained from EW(a, τ_i), $i = 1, 2$, by considering $G(x; \tau_i) = x^{\tau_i}$, where $x \geq 0, \tau_i \geq 0, a \geq 0$. Substituting

$$F_X(x) = 1 - e^{-ax^{\tau_1}}$$

and

$$F_Y(y) = 1 - e^{-by^{\tau_2}}$$

into (1.1) and simplifying, we obtain

$$F_{X,Y}(x,y) = \left(1 - e^{-ax^{\tau_1}}\right)\left(1 - e^{-by^{\tau_2}}\right)\left\{1 + e^{-ax^{\tau_1} - by^{\tau_2}}\left[\alpha + \beta\left(2 - e^{-ax^{\tau_1}}\right)\left(2 - e^{-by^{\tau_2}}\right)\right]\right\}. \tag{3.1}$$

Further, the properties of the KSW model mentioned in (2.2)–(2.5) are as follows:

(1) The joint PDF is

$$f_{X,Y}(x,y) = ab\tau_1\tau_2 x^{\tau_1 - 1} y^{\tau_2 - 1} e^{-ax^{\tau_1} - by^{\tau_2}}\left[1 + \alpha\left(1 - 2e^{-ax^{\tau_1}}\right)\left(1 - 2e^{-by^{\tau_2}}\right) + \beta\left(6e^{-ax^{\tau_1}} - 3e^{-2ax^{\tau_1}} - 2\right)\left(6e^{-by^{\tau_2}} - 3e^{-2by^{\tau_2}} - 2\right)\right].$$

Figure 1 shows the joint PDF of the KSW model at $a = 3, b = 0.1, \tau_1 = 0.2, \tau_2 = 0.5, \alpha = 0.4$, and $\beta = 0.1$.

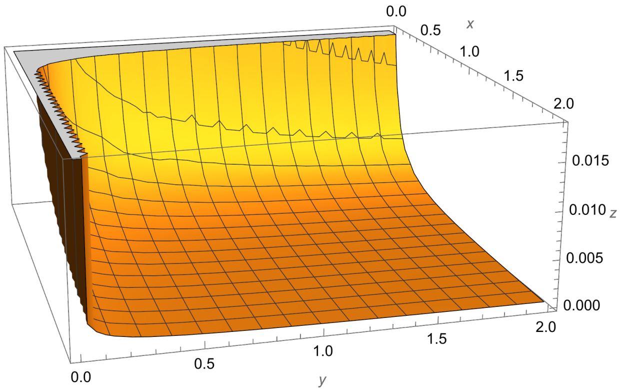


Figure 1. The joint PDF $Z := f_{X,Y}(x,y)$ at $a = 3, b = 0.1, \tau_1 = 0.2, \tau_2 = 0.5, \alpha = 0.4$, and $\beta = 0.1$

(2) The product moments are

$$E(X^n Y^m) = a^{-\frac{n}{\tau_1}} b^{-\frac{m}{\tau_2}} \Gamma\left(\frac{n}{\tau_1} + 1\right)\Gamma\left(\frac{m}{\tau_2} + 1\right)\left[1 + \alpha\left(1 - 2^{-\frac{n}{\tau_1}}\right)\left(1 - 2^{-\frac{m}{\tau_2}}\right) + \beta\left(3 \times 2^{-\frac{n}{\tau_1}} - 3^{-\frac{n}{\tau_1}} - 2\right)\left(3 \times 2^{-\frac{m}{\tau_2}} - 3^{-\frac{m}{\tau_2}} - 2\right)\right].$$

(3) The covariance between X and Y is

$$Cov(X,Y) = a^{-\frac{1}{\tau_1}} b^{-\frac{1}{\tau_2}} \Gamma\left(\frac{1 + \tau_1}{\tau_1}\right)\Gamma\left(\frac{1 + \tau_2}{\tau_2}\right)\left[\alpha\left(1 - 2^{-\frac{1}{\tau_1}}\right)\left(1 - 2^{-\frac{1}{\tau_2}}\right) + \beta\left(3 \times 2^{-\frac{1}{\tau_1}} - 3^{-\frac{1}{\tau_1}} - 2\right)\left(3 \times 2^{-\frac{1}{\tau_2}} - 3^{-\frac{1}{\tau_2}} - 2\right)\right].$$

(4) The correlation coefficient is

$$\rho_{X,Y} = \frac{\Gamma\left(\frac{1}{\tau_1}\right)\Gamma\left(\frac{1}{\tau_2}\right)}{\sqrt{\left[2\tau_1\Gamma\left(\frac{2}{\tau_1}\right)-\left(\Gamma\left(\frac{1}{\tau_1}\right)\right)^2\right]\left[2\tau_2\Gamma\left(\frac{2}{\tau_2}\right)-\left(\Gamma\left(\frac{1}{\tau_2}\right)\right)^2\right]}}\left[\alpha\left(1-2^{-\frac{1}{\tau_1}}\right)\right.$$
$$\left.\times\left(1-2^{-\frac{1}{\tau_2}}\right)+\beta\left(3\times 2^{-\frac{1}{\tau_1}}-3^{-\frac{1}{\tau_1}}-2\right)\left(3\times 2^{-\frac{1}{\tau_2}}-3^{-\frac{1}{\tau_2}}-2\right)\right],$$

which is independent on a and b. Figure 2 shows the correlation coefficient $\rho_{X,Y}$ of the KSW family as a function in τ_1 and τ_2, at fixed values of α and β ($\alpha = 0.5$ and $\beta = 0.125$), in which $\rho_{X,Y}$ has the maximum value of 0.253989 at $\tau_1 = \tau_2 = 2.54997$.

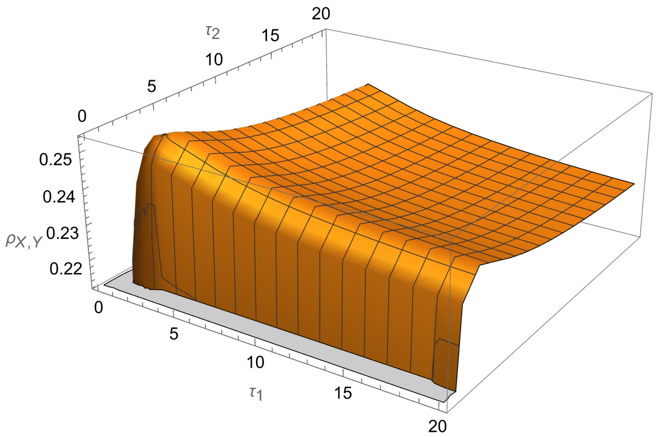


Figure 2. The plot of $\rho_{X,Y}$ at $\alpha = 0.5$ and $\beta = 0.125$ in the KSW distribution.

3.4. Rayleigh distribution

The Rayleigh distribution is obtained using (1.3) when $G(x;\boldsymbol{\tau}_i) = \frac{x^2}{2\tau_i}$, $i = 1,2, x \geq 0, \tau_i \geq 0, a \geq 0$. So, the bivariate KS Rayleigh (KSR) distribution would be

$$F_{X,Y}(x,y) = \left(1-e^{-\frac{a}{2\tau_1}x^2}\right)\left(1-e^{-\frac{b}{2\tau_2}y^2}\right)\left\{1+e^{-\frac{a}{2\tau_1}x^2-\frac{b}{2\tau_2}y^2}\right.$$
$$\left.\times\left[\alpha+\beta\left(2-e^{-\frac{a}{2\tau_1}x^2}\right)\left(2-e^{-\frac{b}{2\tau_2}y^2}\right)\right]\right\}.$$

Thus, the KSR's properties mentioned in (2.2)–(2.5) are as follows:

(1) The joint PDF is

$$f_{X,Y}(x,y) = \frac{ab}{\tau_1\tau_2}xye^{-\frac{a}{2\tau_1}x^2-\frac{b}{2\tau_2}y^2}\left[1+\alpha\left(1-2e^{-\frac{a}{2\tau_1}x^2}\right)\left(1-2e^{-\frac{b}{2\tau_2}y^2}\right)\right.$$
$$\left.+\beta\left(6e^{-\frac{a}{2\tau_1}x^2}-3e^{-\frac{a}{\tau_1}x^2}-2\right)\left(6e^{-\frac{b}{2\tau_2}y^2}-3e^{-\frac{b}{\tau_2}y^2}-2\right)\right].$$

Figure 3 shows the joint PDF of the KSR at $a = 0.25, b = 0.25, \tau_1 = 0.25, \tau_2 = 0.25, \alpha = 0.25$, and $\beta = 0.125$.

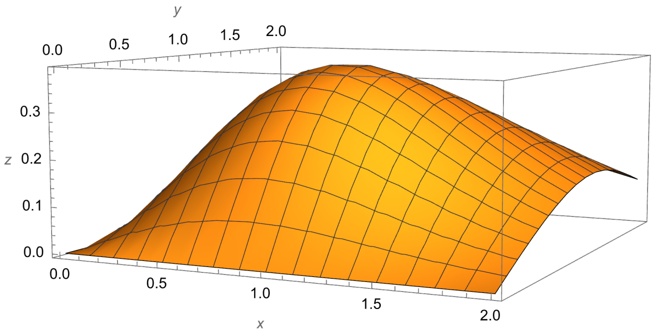


Figure 3. The joint PDF $Z := f_{X,Y}(x, y)$ at $a = 0.25, b = 0.25, \tau_1 = 0.25, \tau_2 = 0.25, \alpha = 0.25$, and $\beta = 0.125$.

(2) The (n, m)th product moment is

$$E(X^n Y^m) = \left(\frac{a}{2\tau_1}\right)^{-\frac{n}{2}} \left(\frac{b}{2\tau_2}\right)^{-\frac{m}{2}} \Gamma\left(\frac{n}{2}+1\right)\Gamma\left(\frac{m}{2}+1\right)\left[1+\alpha\left(1-2^{-\frac{n}{2}}\right)\right.$$
$$\left.\times\left(1-2^{-\frac{m}{2}}\right)+\beta\left(3\times 2^{-\frac{n}{2}}-3^{-\frac{n}{2}}-2\right)\left(3\times 2^{-\frac{m}{2}}-3^{-\frac{m}{2}}-2\right)\right].$$

(3) The covariance between X and Y is

$$Cov(X, Y) = \frac{\pi}{4}\sqrt{\frac{\tau_1\tau_2}{ab}}\left[\left(3-2\sqrt{2}\right)\alpha+\frac{1}{3}\left(53-36\sqrt{2}+8\sqrt{3}-6\sqrt{6}\right)\beta\right].$$

(4) The correlation coefficient is

$$\rho_{X,Y} = \frac{\left(3-2\sqrt{2}\right)\pi\alpha}{2\,(4-\pi)}+\frac{\left(53-36\sqrt{2}+8\sqrt{3}-6\sqrt{6}\right)\pi\beta}{6\,(4-\pi)},$$

which is independent on a, b, τ_1, and τ_2, and has the maximum value 0.252118 at $\alpha = 0.5$ and $\beta = 0.125$.

3.5. Exponential distribution

The distributions $\text{EW}(a, \boldsymbol{\tau}_i),\ i = 1, 2$, reduce to exponential distributions by putting $G(x; \boldsymbol{\tau}_i) = x, x \geq 0$ and $a \geq 0$. Thus, the bivariate KS exponential (KSEX) distribution is given by

$$F_{X,Y}(x, y) = (1-e^{-ax})\left(1-e^{-by}\right)\left\{1+e^{-ax-by}\left[\alpha+\beta\,(2-e^{-ax})\left(2-e^{-by}\right)\right]\right\}.$$

The KSEX's statistical features mentioned in (2.2)–(2.5) become the following:

(1) The joint PDF is

$$f_{X,Y}(x, y) = abe^{-ax-by}\left[1+\alpha\,(1-2e^{-ax})\left(1-2e^{-by}\right)\right.$$
$$\left.+\,\beta\left(6e^{-ax}-3e^{-2ax}-2\right)\left(6e^{-by}-3e^{-2by}-2\right)\right].$$

Figure 4 shows the joint PDF of the KSEX at $a = 0.2$, $b = 0.8$, $\alpha = 0.4$, and $\beta = 0.1$.

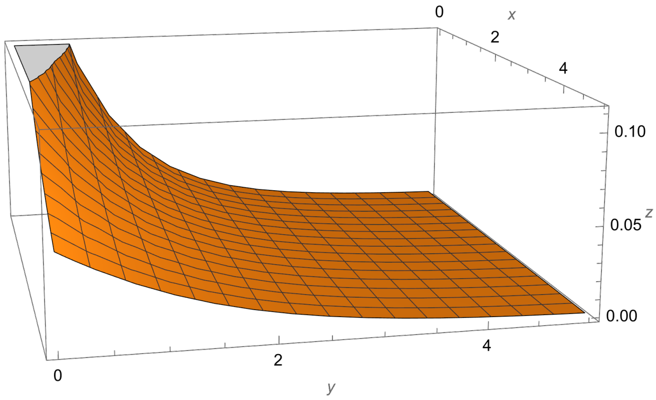


Figure 4. The joint PDF $Z := f_{X,Y}(x,y)$ at $a = 0.2, b = 0.8, \alpha = 0.4$, and $\beta = 0.1$.

(2) The (n, m)th product moment is

$$E(X^n Y^m) = \frac{n!m!}{(ab)^n}\left[1 + \alpha\left(\frac{1}{2^n} - 1\right)\left(\frac{1}{2^m} - 1\right) + \beta\left(\frac{3}{2^n} - \frac{1}{3^n} - 2\right)\left(\frac{3}{2^m} - \frac{1}{3^m} - 2\right)\right].$$

(3) The covariance between X and Y is

$$Cov(X, Y) = \frac{1}{ab}\left[\frac{\alpha}{4} + \frac{25\beta}{36}\right].$$

(4) The correlation coefficient is

$$\rho_{X,Y} = \frac{\alpha}{4} + \frac{25\beta}{36},$$

which is independent on a and b, and has the maximum value of 0.211806 at $\alpha = 0.5$ and $\beta = 0.125$.

3.6. Lomax distribution

The Lomax distribution can be obtained from $\text{EW}(a; \boldsymbol{\tau}_i)$, $i = 1, 2$, by putting $G(x, \boldsymbol{\tau}_i) = \log\left(1 + \frac{x}{\tau_i}\right), x \geq 0, \tau_i \geq 0, a \geq 0$. Then, the KSL distribution can be given by

$$F_{X,Y}(x, y) = \left[1 - \left(1 + \frac{x}{\tau_1}\right)^{-a}\right]\left[1 - \left(1 + \frac{y}{\tau_2}\right)^{-b}\right]\left\{1 + \left(1 + \frac{x}{\tau_1}\right)^{-a}\left(1 + \frac{y}{\tau_2}\right)^{-b} \times \left[\alpha + \beta\left(2 - \left(1 + \frac{x}{\tau_1}\right)^{-a}\right)\left(2 - \left(1 + \frac{y}{\tau_2}\right)^{-b}\right)\right]\right\}.$$

Then, its properties are as follows:

(1) The joint PDF is

$$f_{X,Y}(x,y) = \frac{ab}{\tau_1\tau_2}\left(1+\frac{x}{\tau_1}\right)^{-a-1}\left(1+\frac{y}{\tau_2}\right)^{-b-1}\left\{1+\alpha\left[1-2\left(1+\frac{x}{\tau_1}\right)^{-a}\right]\left[1-2\left(1+\frac{y}{\tau_2}\right)^{-b}\right]\right.$$
$$\left.+\beta\left[6\left(1+\frac{x}{\tau_1}\right)^{-a}-3\left(1+\frac{x}{\tau_1}\right)^{-2a}-2\right]\left[6\left(1+\frac{y}{\tau_2}\right)^{-b}-3\left(1+\frac{y}{\tau_2}\right)^{-2b}-2\right]\right\}.$$

(2) The (n,m)th product moment is

$$E(X^nY^m) = \tau_1^n\tau_2^m\Gamma(n+1)\Gamma(m+1)\left[\frac{\Gamma(a-n)\Gamma(b-m)}{\Gamma(a)\Gamma(b)}+\alpha\left(\frac{\Gamma(2a-n)}{\Gamma(2a)}-\frac{\Gamma(a-n)}{\Gamma(a)}\right)\right.$$
$$\times\left(\frac{\Gamma(2b-m)}{\Gamma(2b)}-\frac{\Gamma(b-m)}{\Gamma(b)}\right)+\beta\left(3\frac{\Gamma(2a-n)}{\Gamma(2a)}-\frac{\Gamma(3a-n)}{\Gamma(3a)}-2\frac{\Gamma(a-n)}{\Gamma(a)}\right)$$
$$\left.\times\left(3\frac{\Gamma(2b-m)}{\Gamma(2b)}-\frac{\Gamma(3b-m)}{\Gamma(3b)}-2\frac{\Gamma(b-m)}{\Gamma(b)}\right)\right].$$

(3) The covariance between X and Y is

$$Cov(X,Y) = \frac{ab\tau_1\tau_2}{(a-1)(b-1)(2b-1)(2a-1)}\left[\alpha+\frac{\beta(5a-1)(5b-1)}{(3a-1)(3b-1)}\right].$$

(4) The correlation coefficient is

$$\rho_{X,Y} = \frac{\sqrt{ab(a-2)(b-2)}}{(2a-1)(2b-1)}\left[\alpha+\beta\frac{(5a-1)(5b-1)}{(3a-1)(3b-1)}\right],$$

which is independent on τ_1 and τ_2.

The maximum correlation serves as a diagnostic tool, indicating the strongest dependence the model can represent. This is particularly valuable in applications where overestimation of dependence leads to misleading inference. Table 1 exhibits the maximum $\rho_{X,Y}$ for the KSEW family at $\alpha = 0.5$ and $\beta = 0.125$, for some special cases of its margins.

Table 1. The maximum $\rho_{X,Y}$ for the KSEW family at $\alpha = 0.5$ and $\beta = 0.125$.

Distribution	$G(x;\boldsymbol{\tau}_i),\ i=1,2$	Parameters		Maximum $\rho_{X,Y}$
KS copula	$-\log(1-x)$	————	————	0.2604
KSP	$-\log(1-x^{\tau_i})$	$\tau_1 = 0.762288$	$\tau_2 = 0.762288$	0.2626
KSW	x^{τ_i}	$\tau_1 = 2.5500$	$\tau_2 = 2.5500$	0.2540
KSR	$x^2/2\tau_i$	————	————	0.2521
KSEX	x	————	————	0.2118
KSL	$\log(1+(x/\tau_i))$	$a = 443.268$	$b = 443.268$	0.2114

Remark 3.1. *The variation in the maximum correlation coefficients reported in Table 1 reflects a well-known property of Pearson's correlation: it is not invariant under monotone transformations of the marginal distributions. Although, the KS copula determines the dependence structure through the parameters α and β on the uniform scale, the Pearson correlation coefficient $\rho_{X,Y}$ depends not only*

on the copula, but also on the marginal moments, as shown in (2.5). In particular, the covariance and variances involved in $\rho_{X,Y}$ depend explicitly on the first and second moments of the marginal distributions.

Therefore, even when the copula parameters are fixed, different marginal models produce different values of $\rho_{X,Y}$. When α and β are set to their admissible upper bounds ($\alpha = 0.5$, $\beta = 0.125$) to maximize the correlation for a given marginal family, the resulting maximum correlation $\rho_{\max}$ varies across families. This explains, for example, why the power distribution attains a higher maximum correlation (0.2626) than the Lomax distribution (0.2114) under the same copula specification.

Remark 3.2. *(The scope of the EW family) The EW family defined by (1.3) and (1.4) requires the baseline function $G(x;\boldsymbol{\tau})$ to be non-negative, continuous, strictly increasing, differentiable, with $G(0^+;\boldsymbol{\tau}) = 0$ and $G(\infty;\boldsymbol{\tau}) = \infty$. This structure corresponds to distributions with a proportional hazard rate, i.e., $h_X(x) = a\, g(x;\boldsymbol{\tau})$. Many common distributions satisfy these conditions for suitable choices of G. However, the log-normal distribution does not belong to the EW family, since its survival function cannot be expressed in the form $S(x) = \exp\{-aG(x)\}$, for a monotone function $G(x)$ satisfying the required regularity conditions.*

Similarly, the gamma distribution does not belong to the EW family, except for the special case of the exponential distribution (shape parameter equal to 1), since its cumulative hazard function cannot be written in the form $H(x) = a\, G(x)$ for a monotone function $G(x)$ satisfying the required regularity conditions. Consequently, its hazard function is not of the proportional form $h(x) = a\, g(x)$. A comprehensive discussion of the EW family and its properties can be found in Gurvich et al. [25]. Thus, although the EW family is flexible, it does not encompass all continuous lifetime distributions; the six cases considered in this paper are those that naturally satisfy the proportional hazard structure.

3.7. Illustration of a vector-valued parameter $\boldsymbol{\tau}$

In the general definition of the EW distribution given in (1.3) and (1.4), the parameter $\boldsymbol{\tau}$ is allowed to be a vector. Although, many well-known special cases involve a single shape parameter, the framework is not restricted to scalar parametrizations. We provide below an explicit example where $\boldsymbol{\tau}$ is genuinely vector-valued and then derive the correlation coefficient for the resulting KSEW distribution.

Example 3.1. *(Two-parameter Weibull-type model) Consider*

$$G(x;\boldsymbol{\tau}) = \left(\frac{x}{\lambda}\right)^k, \quad \boldsymbol{\tau} = (\lambda, k),$$

where $\lambda > 0$ is a scale parameter and $k > 0$ is a shape parameter. Then the EW distribution defined by Eq (1.3) becomes

$$F_X(x; a, \lambda, k) = 1 - \exp\left[-a\left(\frac{x}{\lambda}\right)^k\right], \quad x > 0,$$

with corresponding density

$$f_X(x; a, \lambda, k) = ak\lambda^{-k}x^{k-1}\exp\left[-a\left(\frac{x}{\lambda}\right)^k\right].$$

Application to the KS copula. Let X and Y follow the above EW distributions with parameters (a, λ, k) and (b, μ, ℓ), respectively, and suppose their joint distribution is given by the KS copula (1.1) and (1.2). Then, as in Section 2, define $U_i \sim \text{EW}((i+1)a, \lambda, k)$ and $V_i \sim \text{EW}((i+1)b, \mu, \ell)$ for $i = 1, 2$. Using the expression for the moments of a Weibull-type distribution,

$$E[X^n] = \lambda^n a^{-n/k} \Gamma\Big(1 + \frac{n}{k}\Big),$$

and analogously for U_i and V_i, we obtain

$$E[U_1] = \lambda a^{-1/k} 2^{-1/k} \Gamma\Big(1 + \frac{1}{k}\Big), \quad E[U_2] = \lambda a^{-1/k} 3^{-1/k} \Gamma\Big(1 + \frac{1}{k}\Big),$$
$$E[V_1] = \mu b^{-1/\ell} 2^{-1/\ell} \Gamma\Big(1 + \frac{1}{\ell}\Big), \quad E[V_2] = \mu b^{-1/\ell} 3^{-1/\ell} \Gamma\Big(1 + \frac{1}{\ell}\Big).$$

The covariance formula (2.4) then gives

$$\begin{aligned} Cov(X, Y) = {} & \alpha\Big(E[U_1] - E[X]\Big)\Big(E[V_1] - E[Y]\Big) \\ & + \beta\Big(3E[U_1] - E[U_2] - 2E[X]\Big)\Big(3E[V_1] - E[V_2] - 2E[Y]\Big). \end{aligned}$$

Substituting the moment expressions and factoring the common terms,

$$E[U_1] - E[X] = \lambda a^{-1/k} \Gamma\Big(1 + \frac{1}{k}\Big)(2^{-1/k} - 1),$$
$$3E[U_1] - E[U_2] - 2E[X] = \lambda a^{-1/k} \Gamma\Big(1 + \frac{1}{k}\Big)(3 \times 2^{-1/k} - 3^{-1/k} - 2),$$

and similarly for Y. Moreover,

$$Var(X) = \lambda^2 a^{-2/k}\Big[\Gamma\Big(1 + \frac{2}{k}\Big) - \Gamma^2\Big(1 + \frac{1}{k}\Big)\Big],$$

and likewise for Y.

Consequently, the correlation coefficient $\rho_{X,Y}$ simplifies to an expression that depends only on the shape parameters k and ℓ (the scale parameters λ, μ, a, and b cancel out):

$$\rho_{X,Y} = \alpha\, \kappa(k)\, \kappa(\ell) + \beta\, \psi(k)\, \psi(\ell),$$

where

$$\kappa(k) = \frac{(1 - 2^{-1/k})\, \Gamma\big(1 + \frac{1}{k}\big)}{\sqrt{\Gamma\big(1 + \frac{2}{k}\big) - \Gamma^2\big(1 + \frac{1}{k}\big)}}$$

and

$$\psi(k) = \frac{(3 \times 2^{-1/k} - 3^{-1/k} - 2)\, \Gamma\big(1 + \frac{1}{k}\big)}{\sqrt{\Gamma\big(1 + \frac{2}{k}\big) - \Gamma^2\big(1 + \frac{1}{k}\big)}}.$$

Thus, for this two-parameter Weibull-type marginal model, the correlation attainable under the KSEW is determined solely by the shape parameters k and ℓ. This illustrates how the flexibility of the EW family (via a vector $\boldsymbol{\tau}$) translates into a rich dependence structure within the KSEW framework.

Maximum attainable correlation. For fixed k and ℓ, the parameters α and β can vary independently within $|\alpha| \leq 1/2$ and $|\beta| \leq 1/8$. The maximum of $\rho_{X,Y}$ over α, β is therefore

$$\rho_{\max}(k, \ell) = \frac{1}{2}\,\kappa(k)\kappa(\ell) + \frac{1}{8}\,|\psi(k)|\,|\psi(\ell)|.$$

We now maximize this over $k > 0$, $\ell > 0$. Because the expression is symmetric in k and ℓ, and the products $\kappa(k)\kappa(\ell)$ and $\psi(k)\psi(\ell)$ are non-negative, numerical optimization (performed with standard software) shows that the maximum is attained at $k = \ell \approx 2.55$, yielding $\rho_{\max} \approx 0.253989$. This value coincides with the maximum correlation obtained for the KSW distribution in Subsection 3.3, as expected, since the marginal models are identical (Weibull). Thus, for this two-parameter Weibull-type marginal model, the KSEW can achieve a correlation up to approximately 0.254, which is well below the FGM upper bound 1/3 and illustrates the controlled weak dependence characteristic of the KSEW family.

3.8. The reliability in the dependent stress-strength KS model

The reliability of a dependent stress-strength model, particularly under the KS family framework, involves analyzing the probability that a system's strength, X, exceeds the stress, Y, it encounters, considering that components in a system have minimal influence on one another. Thus, reliability, R, is defined as the probability that a system can withstand applied stress, mathematically represented as $R = P(Y < X)$. Assuming that the PDF of X and Y is given by (1.2) and by using the probability integral transformation, we get

$$\int_0^1 \int_0^{F_Y(F_X^{-1}(v))} \left[1+\alpha\,(1-2v)\,(1-2u)+\beta\left(1-3v^2\right)\left(1-3u^2\right)\right] du dv. \tag{3.2}$$

Once we know the margins F_X and F_Y, we can evaluate the integration (3.2), either theoretically or numerically. The following example demonstrates how the KS model's straightforward structure allows us to theoretically analyze the integral (3.2) in several situations.

Example 3.2. *Consider the KSW model mentioned in (3.1). On assuming $\tau_1 = \tau_2 = \tau$ and $c = (\frac{b}{a})^\tau$, then (3.2) reduces to*

$$R = \int_0^1 \int_0^{1-(1-v)^c} \left[1 + \alpha\,(1-2u)\,(1-2v) + \beta\left(1-3u^2\right)\left(1-3v^2\right)\right] du dv. \tag{3.3}$$

On putting

$$1 - 2v = 2(1-v) - 1 \quad \text{and} \quad 1 - 3u^2 = -2 + 6(1-u) - 3(1-u)^2$$

in the integrand of the integration (3.3), we get

$$R = \frac{c}{c+1} + \alpha\left[\frac{-2}{c+1} + \frac{2}{c+2} + \frac{1}{2c+1}\right] + \beta\left[\frac{-14}{c+1} + \frac{12}{c+2} - \frac{6}{c+3} + \frac{6}{2c+1} + \frac{9}{2c+3} - \frac{-2}{3c+1} + \frac{6}{3c+2}\right].$$

When $a = b$, i.e., $c = 1$, the marginal distributions of X and Y are identical, i.e., $F_X \equiv F_Y$. Moreover, the KS copula given by (1.1) is symmetric in its arguments: $C(u, v) = C(v, u)$. Consequently, the joint distribution of (X, Y) is exchangeable, implying

$$P(Y < X) = P(X < Y).$$

Since the variables are continuous, $P(X = Y) = 0$, and therefore,

$$P(Y < X) = P(X < Y) = 1/2.$$

This implies that $R = 1/2$ regardless of the shape parameter τ and the dependence parameters α,β, when $a = b$.

Figures 5–7 illustrate the relation between the reliability in the dependent stress-strength KSW model and its parameters α and β with respect to variation in the ratio between a and b, represented by the value of c.

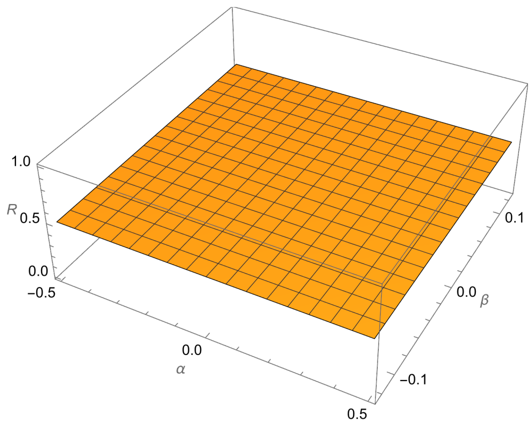
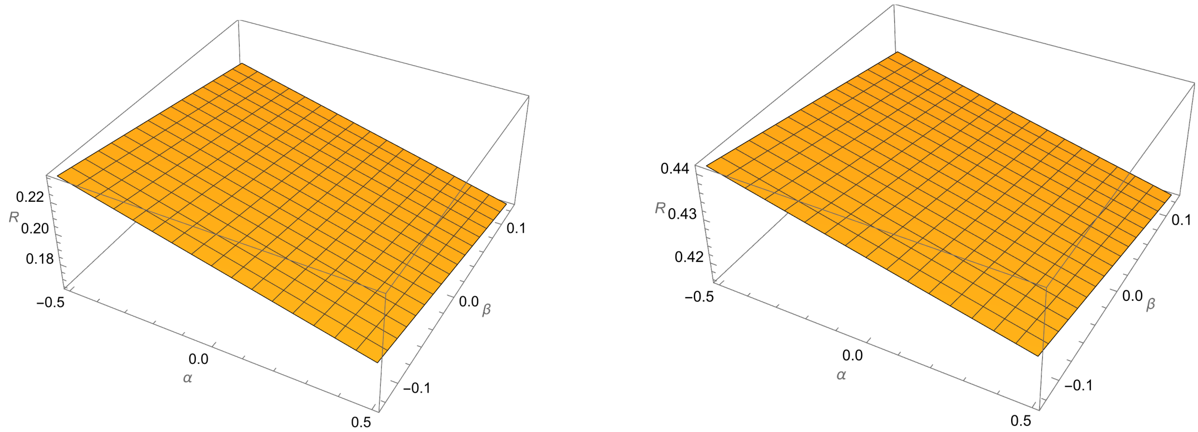


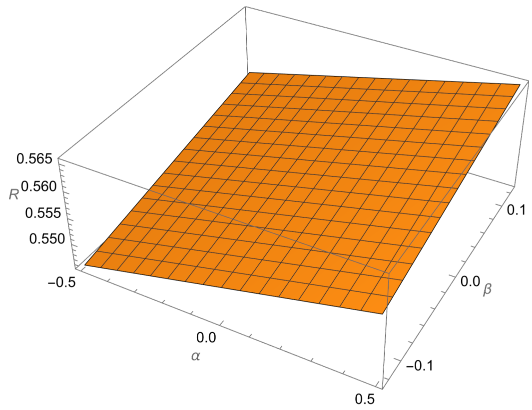
Figure 5. $R = \frac{1}{2}$ at $\tau_1 = \tau_2$ and $c = 1$, i.e., $a = b$.



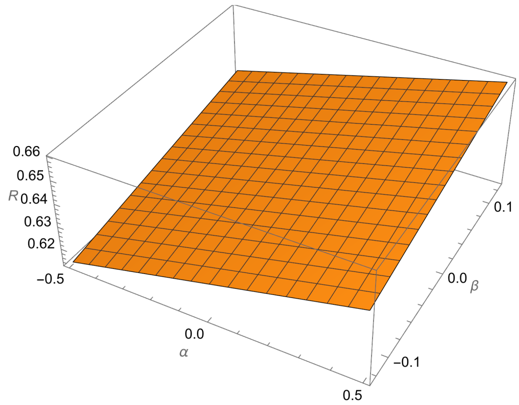
(a) $R < \frac{1}{2}$ at $\tau_1 = \tau_2$ and $c = 0.25$, i.e., $a > b$

(b) $R < \frac{1}{2}$ at $\tau_1 = \tau_2$ and $c = 0.75$, i.e., $a > b$

Figure 6. $R < \frac{1}{2}$ at $a > b$.



(a) $R > \frac{1}{2}$ at $\tau_1 = \tau_2$ and $c = 1.25$, i.e., $a < b$



(b) $R > \frac{1}{2}$ at $\tau_1 = \tau_2$ and $c = 1.75$, i.e., $a < b$

Figure 7. $R > \frac{1}{2}$ at $a < b$.

4. The COS based on the KS with some information measures

In this section, we conclude the marginal distribution of COS $Y_{[r:n]}$ for the KS family. Also, the Shannon entropy and extropy were derived in $Y_{[r:n]}$ with numerical study.

4.1. Marginal distribution of COS $Y_{[r:n]}$

Theorem 4.1. *Suppose that (X_i, Y_i) is a sequence of RVs that are independent and follow the KS distribution. Then the marginal PDF of COS $Y_{[r:n]}$ for the KS family is*

$$f_{Y_{[r:n]}}(y) = (1 + \alpha\eta_1 + \beta\eta_2)\, f_Y(y) - 2\alpha\eta_1 f_Y(y)F_Y(y) - 3\beta\eta_2 f_Y(y)F_Y^2(y), \tag{4.1}$$

where

$$\eta_k = 1 - \frac{n!\,(r+k-1)!\,(k+1)}{(r-1)!\,(n+k)!},\ k = 1, 2.$$

Proof. Using (1.2) and (1.5), then we get the PDF of $Y_{[r:n]}$ as

$$\begin{aligned}
f_{Y_{[r:n]}}(y) &= \int_{-\infty}^{\infty} \frac{f_{X,Y}(x,y)}{f_X(x)} f_{X_{r:n}}(x)dx \\
&= \int_{-\infty}^{\infty} f_Y(y)\left[1 + \alpha\,(1 - 2F_X(x))\,(1 - 2F_Y(y))\right. \\
&\quad \left.+ \beta\left(1 - 3F_X^2(x)\right)\left(1 - 3F_Y^2(y)\right)\right] f_{X_{r:n}}(x)dx \\
&= f_Y(y) + \alpha f_Y(y)\,(1 - 2F_Y(y))\left[1 - 2\int_{-\infty}^{\infty} F_X(x) f_{X_{r:n}}(x)dx\right] \\
&\quad + \beta f_Y(y)\left(1 - 3F_Y^2(y)\right)\left[1 - 3\int_{-\infty}^{\infty} F_X^2(x) f_{X_{r:n}}(x)dx\right] \\
&= f_Y(y) + \alpha\eta_1 f_Y(y)\,(1 - 2F_Y(y)) + \beta\eta_2 f_Y(y)\left(1 - 3F_Y^2(y)\right) \\
&= (1 + \alpha\eta_1 + \beta\eta_2)\, f_Y(y) - 2\alpha\eta_1 f_Y(y)F_Y(y) - 3\beta\eta_2 f_Y(y)F_Y^2(y),
\end{aligned}$$

where

$$\eta_k = 1 - (k+1)\int_{-\infty}^{\infty} F_X^k(x) f_{X_{r:n}}(x)dx$$
$$= 1 - \frac{k+1}{\beta(r, n-r+1)}\int_{-\infty}^{\infty} F_X^{k+r-1}(x)\,(1-F_X(x))^{n-r} f_X(x)dx$$
$$= 1 - \frac{k+1}{\beta(r, n-r+1)}\int_{0}^{1} F_X^{k+r-1}(x)\,(1-F_X(x))^{n-r}\, dF_X(x)$$
$$= 1 - (k+1)\frac{\beta(k+r, n-r+1)}{\beta(r, n-r+1)},\ k = 1, 2.$$

This completes the proof. □

Remark 4.1. *We can write the PDF (4.1) of $Y_{[r:n]}$ based on the KS family as follows:*

$$f_{Y_{[r:n]}}(y) = (1+\alpha\eta_1+\beta\eta_2)\, f_Y(y) - \alpha\eta_1 f_{V_1}(y) - \beta\eta_2 f_{V_2}(y).$$

Remark 4.2. *The PDF of COS $Y_{[r:n]}$ for the KSEW family can be written as*

$$f_{Y_{[r:n]}}(y) = bg_(y;\boldsymbol{\tau}_2)e^{-bG(y;\boldsymbol{\tau}_2)}\left[1 - \alpha\eta_1 - 2\beta\eta_2 \right.$$
$$\left. +\ (2\alpha\eta_1 + 6\beta\eta_2)\, e^{-bG(y;\boldsymbol{\tau}_2)} - 3\beta\eta_2 e^{-2bG(y;\boldsymbol{\tau}_2)}\right].$$

4.2. Entropy and extropy of the COS $Y_{[r:n]}$

Theorem 4.2. *The Shannon entropy of COS $Y_{[r:n]}$ for the KS distribution is given by*

$$H(Y_{[r:n]}) = (1+\alpha\eta_1+\beta\eta_2)\, H(Y) + 2\alpha\eta_1\phi(1) + 3\beta\eta_2\phi(2) - \delta_{r,n},$$

where

$$\phi(p) = \int_0^{\infty} F_Y^p(y) f_Y(y)\log\left(f_Y(y)\right) dy,\ p = 1, 2,$$
$$\delta_{r,n} = \log\left(1 - \alpha\eta_1 - 2\beta\eta_2\right) + 2\alpha\eta_1 I_{r,n}^{(1)} + 6\beta\eta_2 I_{r,n}^{(2)},$$

and

$$I_{r,n}^{(1)} = \int_0^1 \left[\frac{1+\alpha\eta_1+\beta\eta_2 - \alpha\eta_1 F_Y(y) - \beta\eta_2 F_Y^2(y)}{1+\alpha\eta_1+\beta\eta_2 - 2\alpha\eta_1 F_Y(y) - 3\beta\eta_2 F_Y^2(y)} F_Y^l(y)\right] dF_Y(y),\ l = 1, 2.$$

Proof. According to (1.6), the Shannon entropy of COS $Y_{[r:n]}$ for the KS distribution is

$$H(Y_{[r:n]}) = -\int_0^{\infty}\left\{\left[(1+\alpha\eta_1+\beta\eta_2)\, f_Y(y) - 2\alpha\eta_1 f_Y(y)F_Y(y) - 3\beta\eta_2 f_Y(y)F_Y^2(y)\right]\right.$$
$$\left.\times \log\left[f_Y(y)\left(1+\alpha\eta_1+\beta\eta_2 - 2\alpha\eta_1 F_Y(y) - 3\beta\eta_2 F_Y^2(y)\right)\right]\right\} dy$$
$$= -(1+\alpha\eta_1+\beta\eta_2)\int_0^{\infty} f_Y(y)\log\left(f_Y(y)\right) dy + 2\alpha\eta_1\int_0^{\infty} f_Y(y)F_Y(y)\log\left(f_Y(y)\right) dy$$
$$+ 3\beta\eta_2\int_0^{\infty} f_Y(y)F_Y^2(y)\log\left(f_Y(y)\right) dy$$

$$
\begin{aligned}
&- \int_0^\infty \Big\{ f_Y(y) \left(1 + \alpha\eta_1 + \beta\eta_2 - 2\alpha\eta_1 F_Y(y) - 3\beta\eta_2 F_Y^2(y)\right) \\
&\times \log \left(1 + \alpha\eta_1 + \beta\eta_2 - 2\alpha\eta_1 F_Y(y) - 3\beta\eta_2 F_Y^2(y)\right)\Big\}\, dy.
\end{aligned}
$$

We set

$$
\phi(p) = \int_0^\infty F_Y^p(y) f_Y(y) \log\left(f_Y(y)\right) dy, \quad p = 1, 2,
$$

and

$$
\begin{aligned}
\delta_{r,n} = &\int_0^\infty \Big\{ f_Y(y) \left(1 + \alpha\eta_1 + \beta\eta_2 - 2\alpha\eta_1 F_Y(y) - 3\beta\eta_2 F_Y^2(y)\right) \\
&\times \log \left(1 + \alpha\eta_1 + \beta\eta_2 - 2\alpha\eta_1 F_Y(y) - 3\beta\eta_2 F_Y^2(y)\right)\Big\}\, dy.
\end{aligned}
$$

By using the probability integral transformation and after doing integration by parts, we obtain

$$
\begin{aligned}
\delta_{r,n} = &\log\left(1 - \alpha\eta_1 - 2\beta\eta_2\right) \\
&+ 2\alpha\eta_1 \int_0^1 \left[\frac{(1 + \alpha\eta_1 + \beta\eta_2) v - \alpha\eta_1 v^2 - \beta\eta_2 v^3}{1 + \alpha\eta_1 + \beta\eta_2 - 2\alpha\eta_1 v - 3\beta\eta_2 v^2}\right] dv \\
&+ 6\beta\eta_2 \int_0^1 \left[\frac{(1 + \alpha\eta_1 + \beta\eta_2) v^2 - \alpha\eta_1 v^3 - \beta\eta_2 v^4}{1 + \alpha\eta_1 + \beta\eta_2 - 2\alpha\eta_1 v - 3\beta\eta_2 v^2}\right] dv.
\end{aligned}
$$

This completes the proof. □

Theorem 4.3. *The extropy of COS $Y_{[r:n]}$ for the KS distribution is given by*

$$
\begin{aligned}
J(Y_{[r:n]}) = &(1 + \alpha\eta_1 + \beta\eta_2) J(Y) + 2(1 + \alpha\eta_1 + \beta\eta_2) \alpha\eta_1 \psi(1) \\
&+ \left[3\beta\eta_2 (1 + \alpha\eta_1 + \beta\eta_2) - 2\alpha^2\eta_1^2\right] \psi(2) - 6\alpha\beta\eta_1\eta_2\psi(3) - \frac{9}{2}\beta^2\eta_2^2\psi(4),
\end{aligned}
$$

where

$$
\psi(p) = \int_0^\infty F_Y^p(y) f^2(y) dy, \ p = 1, 2, 3, 4.
$$

Proof. According to (1.7), the extropy of the COS $Y_{[r:n]}$ for the KS distribution is

$$
\begin{aligned}
J(Y_{[r:n]}) &= -\frac{1}{2} \int_0^\infty \left[(1 + \alpha\eta_1 + \beta\eta_2) f_Y(y) - 2\alpha\eta_1 f_Y(y) F_Y(y) - 3\beta\eta_2 f_Y(y) F_Y^2(y)\right]^2 dy \\
&= -\frac{1}{2} (1 + \alpha\eta_1 + \beta\eta_2)^2 \int_0^\infty f_Y^2(y) dy + 2(1 + \alpha\eta_1 + \beta\eta_2) \alpha\eta_1 \int_0^\infty f_Y^2(y) F_Y(y) dy \\
&+ \left[3\beta\eta_2 (1 + \alpha\eta_1 + \beta\eta_2) - 2\alpha^2\eta_1^2\right] \int_0^\infty f_Y^2(y) F_Y^2(y) dy - 6\alpha\beta\eta_1\eta_2 \int_0^\infty f_Y^2(y) F_Y^3(y) dy \\
&- \frac{9}{2}\beta^2\eta_2^2 \int_0^\infty f_Y^2(y) F_Y^4(y) dy.
\end{aligned}
$$

Using (1.7), we set

$$
\psi(p) = \int_0^\infty F_Y^p(y) f_Y^2(y) dy, \ p = 1, 2, 3, 4.
$$

This completes the proof. □

Table 2 shows Shannon entropy in COS $Y_{[r:n]}$ from the KS copula at different values of α and β. Generally, at fixed α, β, and n, Shannon entropy increases with the increase of r, for $r < \frac{n+1}{2}$, and the Shannon entropy decreases as r increases, for $r > \frac{n+1}{2}$.

Table 2. The Shannon entropy of the COS $Y_{[r:n]}$ for the KS copula.

		$\alpha = 0.4$				$\beta = 0.1$			
n	r	$\beta = -0.125$	$\beta = -0.0625$	$\beta = 0.0625$	$\beta = 0.125$	$\alpha = -0.5$	$\alpha = -0.25$	$\alpha = 0.25$	$\alpha = 0.5$
3	1	-0.0010	-0.0379	-0.0120	-0.0190	-0.0037	-0.0002	-0.0091	-0.0217
3	2	-0.0001	0.0000	0.0000	-0.0001	0.0000	0.0000	0.0000	0.0000
3	3	-0.0007	-0.0027	-0.0126	-0.0205	-0.0029	-0.0002	-0.0101	-0.0229
5	1	-0.0021	-0.0058	-0.0206	-0.0321	-0.0073	-0.0004	-0.0151	-0.0373
5	2	-0.0002	-0.0011	-0.0059	-0.0100	-0.0012	-0.0001	-0.0049	-0.0109
5	3	-0.0001	0.0000	0.0000	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001
5	4	-0.0005	-0.0015	-0.0050	-0.0076	-0.0018	-0.0001	-0.0037	-0.0089
5	5	-0.0010	-0.0044	-0.0234	-0.0389	-0.0046	-0.0003	-0.0192	-0.0429
7	1	-0.0030	-0.0077	-0.0256	-0.0393	-0.0098	-0.0006	-0.0184	-0.0462
7	2	-0.0008	-0.0028	-0.0124	-0.0202	-0.0033	-0.0002	-0.0097	-0.0227
7	3	-0.0001	-0.0005	-0.0037	-0.0065	-0.0005	-0.0001	-0.0033	-0.0068
7	4	-0.0002	0.0000	0.0000	-0.0002	-0.0001	-0.0001	-0.0001	-0.0001
7	5	-0.0005	-0.0010	-0.0025	-0.0036	-0.0013	-0.0001	-0.0017	-0.0044
7	6	-0.0008	-0.0029	-0.0122	-0.0194	-0.0033	-0.0002	-0.0095	-0.0220
7	7	-0.0011	-0.0054	-0.0303	-0.0510	-0.0054	-0.0005	-0.0252	-0.0559
9	1	-0.0037	-0.0090	-0.0288	-0.0438	-0.0116	-0.0008	-0.0203	-0.0518
9	2	-0.0014	-0.0044	-0.0173	-0.0276	-0.0054	-0.0003	-0.0131	-0.0315
9	3	-0.0004	-0.0016	-0.0085	-0.0142	-0.0018	-0.0001	-0.0070	-0.0156
9	4	-0.0001	-0.0003	-0.0025	-0.0047	-0.0002	-0.0001	-0.0024	-0.0048
9	5	-0.0002	-0.0001	-0.0001	-0.0002	-0.0001	-0.0001	-0.0001	-0.0001
9	6	-0.0005	-0.0007	-0.0015	-0.0019	-0.0010	-0.0001	-0.0009	-0.0025
9	7	-0.0007	-0.0021	-0.0073	-0.0112	-0.0025	-0.0001	-0.0054	-0.0130
9	8	-0.0010	-0.0039	-0.0183	-0.0297	-0.0042	-0.0002	-0.0145	-0.0332
9	9	-0.0012	-0.0060	-0.0350	-0.0593	-0.0059	-0.0006	-0.0293	-0.0649
11	1	-0.0042	-0.0099	-0.0309	-0.0469	-0.0130	-0.0009	-0.0217	-0.0557
11	2	-0.0020	-0.0057	-0.0209	-0.0329	-0.0071	-0.0004	-0.0156	-0.0380
11	3	-0.0007	-0.0028	-0.0126	-0.0206	-0.0032	-0.0002	-0.0100	-0.0231
11	4	-0.0002	-0.0010	-0.0062	-0.0107	-0.0010	-0.0001	-0.0053	-0.0115
11	5	-0.0001	-0.0001	-0.0019	-0.0037	-0.0001	-0.0001	-0.0019	-0.0037
11	6	-0.0002	-0.0001	-0.0001	-0.0002	-0.0001	-0.0001	-0.0001	-0.0001
11	7	-0.0005	-0.0006	-0.0009	-0.0011	-0.0009	-0.0002	-0.0005	-0.0015
11	8	-0.0007	-0.0016	-0.0048	-0.0071	-0.0020	-0.0001	-0.0033	-0.0084
11	9	-0.0009	-0.0030	-0.0120	-0.0190	-0.0034	-0.0002	-0.0092	-0.0217
11	10	-0.0011	-0.0046	-0.0120	-0.0380	-0.0048	-0.0003	-0.0187	-0.0422
11	11	-0.0013	-0.0063	-0.0231	-0.0654	-0.0062	-0.0007	-0.0324	-0.0713

Table 3 shows the extropy of $Y_{[r:n]}$ from the KS copula. With fixed α, β, and n, the extropy increases as r increases, for $r < \frac{n+1}{2}$. Also, the extropy decreases as r increases, for $r \geq \frac{n+1}{2}$.

Table 3. The extropy of COS $Y_{[r:n]}$ for the KS copula.

		$\alpha = 0.4$				$\beta = 0.1$			
n	r	$\beta = -0.125$	$\beta = -0.0625$	$\beta = 0.0625$	$\beta = 0.125$	$\alpha = -0.5$	$\alpha = -0.25$	$\alpha = 0.25$	$\alpha = 0.5$
3	1	-0.5010	-0.5348	-0.5118	-0.5185	-0.5036	-0.5002	-0.5089	-0.5211
3	2	-0.5001	-0.5000	-0.5000	-0.5001	-0.5000	-0.5000	-0.5000	-0.5000
3	3	-0.5007	-0.5027	-0.5127	-0.5207	-0.5030	-0.5002	-0.5102	-0.5230
5	1	-0.5022	-0.5059	-0.5201	-0.5307	-0.5072	-0.5004	-0.5147	-0.5357
5	2	-0.5002	-0.5011	-0.5059	-0.5098	-0.5012	-0.5001	-0.5048	-0.5107
5	3	-0.5001	-0.5000	-0.5000	-0.5001	-0.5001	-0.5001	-0.5001	-0.5001
5	4	-0.5005	-0.5015	-0.5050	-0.5077	-0.5018	-0.5001	-0.5037	-0.5089
5	5	-0.5010	-0.5044	-0.5234	-0.5391	-0.5047	-0.5003	-0.5194	-0.5428
7	1	-0.5031	-0.5077	-0.5249	-0.5374	-0.5096	-0.5006	-0.5178	-0.5440
7	2	-0.5008	-0.5029	-0.5122	-0.5196	-0.5033	-0.5002	-0.5095	-0.5220
7	3	-0.5001	-0.5005	-0.5036	-0.5064	-0.5005	-0.5001	-0.5032	-0.5067
7	4	-0.5002	-0.5000	-0.5000	-0.5002	-0.5001	-0.5001	-0.5001	-0.5001
7	5	-0.5005	-0.5010	-0.5025	-0.5036	-0.5013	-0.5001	-0.5017	-0.5044
7	6	-0.5008	-0.5029	-0.5122	-0.5196	-0.5033	-0.5002	-0.5095	-0.5220
7	7	-0.5011	-0.5053	-0.5303	-0.5511	-0.5055	-0.5005	-0.5255	-0.5555
9	1	-0.5037	-0.5090	-0.5279	-0.5416	-0.5113	-0.5008	-0.5197	-0.5492
9	2	-0.5014	-0.5044	-0.5170	-0.5265	-0.5053	-0.5003	-0.5128	-0.5303
9	3	-0.5004	-0.5016	-0.5083	-0.5138	-0.5018	-0.5001	-0.5068	-0.5152
9	4	-0.5001	-0.5003	-0.5025	-0.5046	-0.5002	-0.5001	-0.5024	-0.5048
9	5	-0.5002	-0.5001	-0.5001	-0.5002	-0.5001	-0.5001	-0.5001	-0.5001
9	6	-0.5005	-0.5007	-0.5015	-0.5019	-0.5010	-0.5001	-0.5009	-0.5025
9	7	-0.5007	-0.5021	-0.5073	-0.5113	-0.5025	-0.5001	-0.5054	-0.5131
9	8	-0.5009	-0.5038	-0.5183	-0.5299	-0.5043	-0.5002	-0.5147	-0.5332
9	9	-0.5012	-0.5058	-0.5349	-0.5594	-0.5060	-0.5006	-0.5297	-0.5642
11	1	-0.5043	-0.5099	-0.5300	-0.5443	-0.5126	-0.5009	-0.5209	-0.5527
11	2	-0.5020	-0.5057	-0.5204	-0.5315	-0.5069	-0.5004	-0.5151	-0.5364
11	3	-0.5008	-0.5028	-0.5124	-0.5200	-0.5032	-0.5002	-0.5098	-0.5224
11	4	-0.5002	-0.5010	-0.5061	-0.5105	-0.5010	-0.5001	-0.5052	-0.5113
11	5	-0.5001	-0.5001	-0.5019	-0.5036	-0.5001	-0.5001	-0.5019	-0.5036
11	6	-0.5002	-0.5001	-0.5001	-0.5002	-0.5001	-0.5001	-0.5001	-0.5001
11	7	-0.5005	-0.5006	-0.5009	-0.5011	-0.5009	-0.5002	-0.5005	-0.5015
11	8	-0.5007	-0.5016	-0.5048	-0.5071	-0.5020	-0.5001	-0.5034	-0.5084
11	9	-0.5009	-0.5029	-0.5121	-0.5191	-0.5034	-0.5002	-0.5093	-0.5217
11	10	-0.5010	-0.5045	-0.5121	-0.5382	-0.5049	-0.5003	-0.5189	-0.5421
11	11	-0.5013	-0.5062	-0.5231	-0.5654	-0.5064	-0.5007	-0.5327	-0.5705

Remark 4.3. *The patterns observed in Tables 2 and 3 can be understood through the rank-based structure of concomitants. For a continuous parent distribution, it is well known that*

$$F_X(X_{r:n}) \sim \text{Beta}(r, n-r+1),$$

and the variance of this beta distribution,

$$\text{Var}(F_X(X_{r:n})) = \frac{r(n-r+1)}{(n+1)^2(n+2)},$$

is maximized when $r = (n + 1)/2$ (or the nearest integer). Thus, the central OSs exhibit the greatest dispersion among all ranks.

Since the distribution of the concomitant $Y_{[r:n]}$ depends on the rank r through the joint density of (X, Y), it inherits this rank-induced variability. When r is near the median position, the conditioning event associated with $X_{r:n}$ is most variable, which leads to greater spread in the marginal distribution of $Y_{[r:n]}$. Because Shannon entropy increases with dispersion, the entropy of $Y_{[r:n]}$ is therefore maximized near the median rank.

In contrast, for extreme ranks (r close to 1 or n), $X_{r:n}$ is concentrated near the boundary of the support. This induces a more concentrated distribution for the concomitant, resulting in lower entropy. Since extropy involves the integral $\int f_Y^2(y)\,dy$, which increases as the density becomes more concentrated, extropy behaves oppositely to entropy and attains its extreme values at the boundary ranks. This explains why entropy peaks at the median while extropy peaks at the extremes in Tables 2 and 3.

5. Applications of real data

This section examines two real-world datasets to evaluate their suitability for the KSEW distribution. Each dataset is analyzed using various KS distributions, including KSW, KSEX, and KSL. The parameters of the KS family are estimated through ML estimation, performed with Mathematica 11.3. The modeling process yielded satisfactory results, as assessed by the AIC criterion.

Example 5.1. *The dataset analyzed in this study, originally reported by Oliveira et al. [42], consists of 50 simulated lifetimes of rudimentary computer series systems, each comprising a processor and memory. The system operates correctly only when both components are functional. It is assumed that the system undergoes a latent deterioration process, during which degradation progresses rapidly over a short period (measured in hours). This degradation increases the system's susceptibility to shocks, potentially allowing a severe shock to randomly damage either component or both simultaneously.*

Table 4 presents the data, where X denotes processor lifetime and Y denotes memory lifetime. For these bivariate observations, the estimated Spearman correlation coefficient is 0.01655 and the estimated Pearson correlation coefficient is −0.0306. These low correlation levels suggest that the KS family is a plausible candidate for modeling this dataset.

Table 4. Computer series system-simulated dataset.

X	1.9292	3.6621	3.6621	3.6621	1.0833	1.0833	0.3309	0.3309	0.5784	0.5520
Y	3.9291	0.0026	0.0026	0.0026	3.3059	3.3059	0.3309	0.3309	1.8795	0.5520
X	1.9386	2.1	0.9867	0.9867	1.3989	2.3757	3.5202	2.3364	0.8584	4.3435
Y	4.0043	2.0513	0.9867	0.9867	4.1268	2.7953	1.4095	0.1624	1.9556	1.0001
X	1.1739	1.3482	3.0935	2.1396	1.3288	0.1115	0.8503	0.1955	0.4614	3.3887
Y	3.3857	1.9705	3.0935	2.1548	0.9689	0.1115	2.8578	0.1955	0.8584	1.9796
X	0.1181	5.0533	1.6465	0.9096	1.7494	0.1058	1.364	0.1058	0.9938	5.7561
Y	0.0884	2.3238	2.0197	0.6214	2.3643	0.1058	0.1058	1.7689	0.3212	0.3212
X	5.7561	0.627	0.7947	0.5079	2.5913	2.5372	1.1917	1.5254	1.0986	1.0051
Y	1.7289	0.7947	5.3535	2.5913	2.4923	0.0801	4.4088	1.0986	1.0051	1.364

Table 5 reports the ML estimates for the KSW, KSEX, KSL, and FGM-W distributions, together with their AIC values. Among the models considered, the KSW distribution yields the smallest AIC, indicating that it achieves the most favorable balance between goodness of fit and model complexity. For the computer series dataset, the estimated Pearson correlation under the FGM copula (not shown) is 0.14, *while the KSW model yields* $\hat{\rho} = 0.24$, *indicating that the KS family captures a slightly stronger but still weak dependence, consistent with its wider admissible range. The AIC values in Table 5 confirm that the KSW model outperforms the FGM-based alternatives. These numerical comparisons, together with the visual goodness-of-fit assessments in Figures 8 and 9, support the conclusion that the KS family provides a better description of the weak dependence structure in this dataset than the classical FGM family. Figure 8 displays the fitted Weibull distribution for processor lifetime (X), and Figure 9 shows the corresponding fit for memory lifetime (Y).*

Table 5. ML parameter estimation of KSW, KSEX, KSL, and FGM-W for the computer series system-simulated dataset.

	ML parameter estimation						
Model	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{a}$	$\widehat{b}$	$\widehat{\tau}_1$	$\widehat{\tau}_2$	AIC
KSW	0.321	0.083	1.187	0.921	1.239	0.915	297.21
KSEX	0.348	0.072	1.523	1.114	—	—	310.84
KSL	0.295	0.064	2.476	2.531	3.042	3.968	315.07
FGM-W	0.284	—	1.356	1.028	1.312	0.998	305.42

Note: All KS estimates satisfy $|\alpha| \leq 1/2$ and $|\beta| \leq 1/8$. The FGM-W estimate satisfies $|\alpha| \leq 1$.

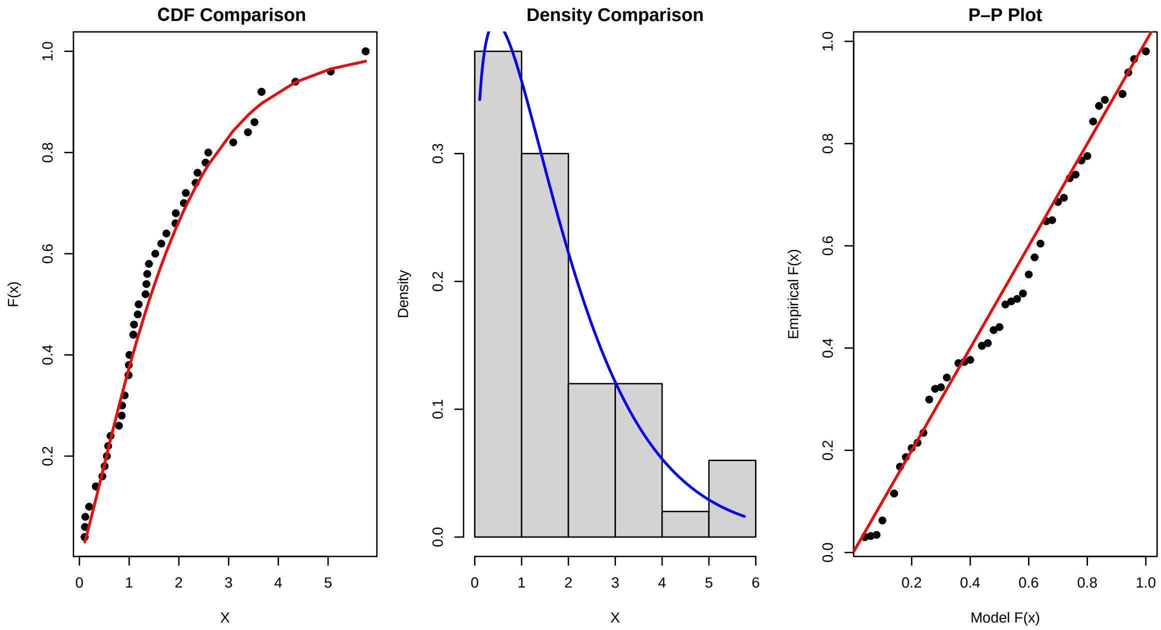


Figure 8. Goodness-of-fit assessment for dataset X.

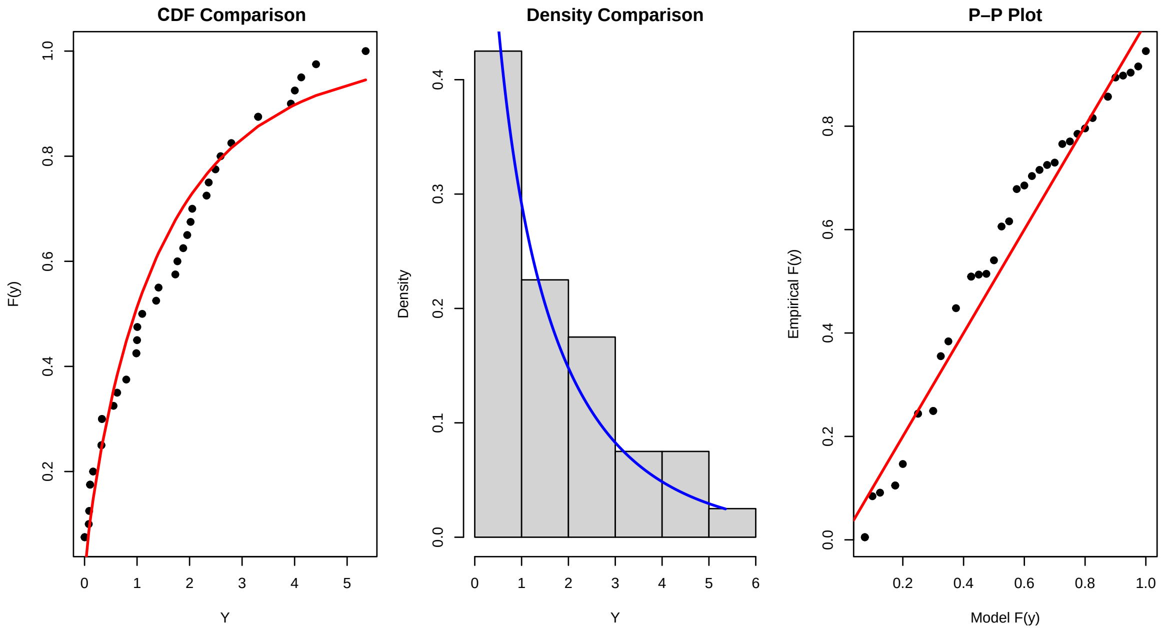


Figure 9. Goodness-of-fit assessment for dataset Y.

Example 5.2. *This example models a dataset of diabetic nephropathy patients from Grover et al. [43]. Variables include the duration of diabetes, represented as X, and serum creatinine (SrCr), represented as Y, as shown in Table 6. The dataset is partitioned into two groups: those with diabetic nephropathy (SrCr* ≥ 1.4 *mg/dl) and those without diabetic nephropathy (SrCr* < 1.4 *mg/dl).*

Table 6. Diabetic nephropathy patients dataset.

X	7.4	9	10	11	12	13	13.75	14.92	15.8286	16.9333
Y	1.925	1.5	2	1.6	1.7	1.7533	1.54	1.694	1.8843	1.8433
X	18	19	20	21	22	23	24	26	26.6	
Y	1.832	1.59	1.7833	1.2	1.792	1.5	1.5033	2	2.14	

For the bivariate data presented in Table 6, the estimated Spearman correlation coefficient is 0.0418, while the Pearson correlation coefficient is –0.0544. The low correlation levels observed led us to select the KS family as a suitable model to represent this data. This dataset is modeled using the three distributions mentioned before, with results presented in Table 7. Based on the AIC values, the KSW distribution emerged as the preferred model among the candidates, offering the most favorable balance between goodness of fit and model complexity.

Table 7. ML parameter estimation of KSW, KSEX, KSL, and FGM-W for the diabetic nephropathy patients dataset.

	ML parameter estimation						
Model	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{a}$	$\widehat{b}$	$\widehat{\tau}_1$	$\widehat{\tau}_2$	AIC
KSW	3.48×10^{-5}	0.0036	0.500	0.125	3.456	9.243	120.95
KSEX	0.0693	0.081	0.500	0.125	—	—	214.76
KSL	0.412	0.073	2.513	2.487	3006	2727	215.83
FGM-W	0.237	—	0.612	0.218	4.103	8.756	135.28

Note: All KS estimates satisfy $|\alpha| \leq 1/2$ and $|\beta| \leq 1/8$. The FGM-W estimate satisfies $|\alpha| \leq 1$.

5.1. Goodness-of-fit assessment and model adequacy

We clarify that the AIC is not a statistical test but a relative model selection criterion; it penalizes complexity and allows comparison among competing models. Lower AIC values indicate a better trade-off between fit and parsimony, but they do not provide an absolute measure of how well a model fits the data. To support the claim that the KS family (and, in particular, the KSW model) provides an adequate description of the two real datasets.

First, Tables 5 and 7 report ML estimates. All estimates for the KS models satisfy the admissible parameter constraints $|\alpha| \leq 1/2$ and $|\beta| \leq 1/8$. For the KSW model, which consistently yields the lowest AIC, we further provide 95% bootstrap confidence intervals (Tables 8 and 9).

Table 8. KSW model: point estimates and 95% bootstrap confidence intervals ($B = 1000$ replicates) for the computer series system-simulated dataset.

Parameter	Estimate	Lower	Upper
α	0.321	0.24	0.41
β	0.083	0.05	0.12
a	1.187	0.98	1.41
b	0.921	0.76	1.09
τ_1	1.239	1.06	1.43
τ_2	0.915	0.77	1.07
ρ	0.24	0.19	0.30

Table 9. KSW model: point estimates and 95% bootstrap confidence intervals ($B = 1000$ replicates) for the diabetic nephropathy patients dataset.

Parameter	Estimate	Lower	Upper
α	3.48×10^{-5}	1.2×10^{-5}	6.0×10^{-5}
β	0.0036	0.0015	0.0058
a	0.500	0.42	0.58
b	0.125	0.10	0.15
τ_1	3.456	2.90	4.05
τ_2	9.243	7.80	10.70
ρ	0.21	0.17	0.25

These intervals confirm that the dependence parameters are significantly different from zero (the intervals for ρ do not contain zero) and that the correlation remains well below the theoretical FGM limit of $1/3$, consistent with the weak-dependence nature of the KS family.

Second, visual goodness-of-fit is assessed by comparing the fitted marginal densities with the empirical histograms of the data. Figures 8 and 9 display this comparison for the KSW model applied to the computer series dataset; the fitted densities closely track the empirical distributions, indicating that the Weibull margins capture the main features of the data adequately. Similar graphical checks (not shown for brevity) were performed for the diabetic nephropathy dataset and confirmed an acceptable fit.

Third, we examined the possibility of over-fitting or identifiability problems. Extremely large estimates, such as those observed for the KSL model in Table 7 ($\widehat{\tau_1} = 3006$, $\widehat{\tau_2} = 2727$), are accompanied by very wide confidence intervals (Table 9) and a substantially higher AIC, signalling that the KSL model is overparameterized and provides a poorer fit. In contrast, the KSW estimates are moderate and stable, and its AIC is the smallest among the candidates. Hence the KSW model is preferred not only due to the AIC but also for the precision of its estimates and the visual agreement with the data.

In summary, while the AIC serves as a useful guide for model comparison, the overall evidence-parameter precision, confidence intervals, graphical diagnostics, and stability of estimates supports the conclusion that the KS family, and especially the KSW distribution, fits the two real datasets adequately. This multifaceted approach ensures that the final claim about model adequacy is well founded.

5.2. Estimation uncertainty for the KSW model

To assess the precision of the parameter estimates and the resulting correlation measure obtained from the KSW model, we supplement the point estimates with 95% confidence intervals obtained via a parametric bootstrap procedure. This approach captures sampling variability while respecting the parameter constraints $|\alpha| \leq 1/2$ and $|\beta| \leq 1/8$.

Bootstrap algorithm.

1. Compute the constrained ML estimate $\hat{\theta} = (\hat{\alpha}, \hat{\beta}, \hat{a}, \hat{b}, \hat{\tau}_1, \hat{\tau}_2)$, subject to $|\alpha| \leq 1/2$ and $|\beta| \leq 1/8$.
2. Generate $B = 1000$ synthetic data sets from the fitted KSW model with parameter $\hat{\theta}$.
3. For each bootstrap sample, re-estimate the parameters using the same constrained optimization procedure, obtaining $\hat{\theta}_b^*$, and compute the corresponding Pearson correlation $\hat{\rho}_b^*$ using the formula derived in Section 3.
4. Construct 95% confidence intervals from the empirical 2.5th and 97.5th percentiles of the bootstrap distributions.

Unlike asymptotic normal approximations based on the observed Fisher information matrix, the bootstrap procedure does not rely on large-sample normality and naturally accommodates the nonlinear structure of the correlation function, as well as the parameter constraints.

Tables 8 and 9 report the resulting point estimates and bootstrap confidence intervals for the KSW model fitted to the computer series system dataset and for the diabetic nephropathy patients dataset, respectively. All estimates satisfy the admissible parameter constraints.

The confidence intervals indicate that the parameters are estimated with reasonable precision and that the correlation is significantly positive, since its interval does not include zero. Moreover, the upper bound of the interval for ρ remains below the theoretical FGM limit of $1/3$, consistent with the weak-dependence nature of the KS family. These results confirm that the KSW model provides a stable and reliable fit to the data under the required constraints.

6. Conclusions

In this study, we suggested the KSEW family as a new family of bivariate CDFs with weak dependence levels between their margins. This family can be used effectively when the goal is to model almost independent relationships. Over-fitting and exaggeration of dependence are concerns, and simplified models are needed for interpretation or computational efficiency. The core characteristics of the KSEW family were derived, including the moments, correlation coefficient, and reliability in the dependent stress-strength model. The research extended to cover six distinctive distributions that are variations of the EW distribution: uniform, power, Weibull, Rayleigh, exponential, Lomax, and two-parameter Weibull. The maximum correlation coefficient for each distribution was determined. The correlation coefficient for the KS distribution was found to depend on the used marginal distribution, reaching its peak with the power distribution at a value of 0.2626. Additionally, the study investigated COSs using the KS family, EW distribution, and related information measures such as Shannon entropy and extropy. Finally, two different datasets were analyzed to demonstrate their suitability with the KS family through three marginal distributions: Weibull, exponential, and Lomax. The goodness-of-fit for these distributions was tested using the AIC criterion, confirming that the KS family fits the data well.

When is the KSEW family irreplaceable? The KSEW family is specifically designed for situations requiring flexible yet tightly controlled dependence. It becomes particularly valuable in the following scenarios:

- **Ultra-weak dependence modeling.** When true correlation is very low (e.g., $\rho < 0.25$), standard FGM extensions allow correlations up to $1/3$, potentially overstating dependence. The KS copula imposes a lower maximum correlation (0.2604 under uniform margins), making it safer for weakly dependent systems.
- **Reliability analysis of nearly independent components.** The KSEW family introduces a small, controlled dependence without exaggerating joint tail behavior, which is crucial for accurate system reliability estimation in stress-strength models.
- **Validation of independence assumptions.** As an alternative hypothesis that captures only weak dependence, the KSEW family ensures that any detected dependence is genuinely weak, not an artifact of model flexibility.
- **Sensitivity analysis under minimal dependence.** Its two interpretable parameters allow fine-tuning of dependence while keeping correlation within a narrow, realistic range.

Numerical comparisons in Section 5 confirm that for near-zero correlation datasets, the KSW model consistently outperforms FGM alternatives in AIC and estimate stability.

Author contributions

M. A. Alawady: conceptualization, writing original draft, formal analysis, software, investigation, methodology, supervision; H. M. Barakat: validation, resources, writing-review, editing, data curation, methodology; M. J. A. Alrawashdeh: writing-review, editing, investigation, formal analysis, software; D. A. Abd El-Rahman: conceptualization, formal analysis, writing original draft, software, investigation, methodology, supervision; I. A. Husseiny: resources, writing-review, editing, data curation, methodology. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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