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Research article

Relaxed contractions in suprametric spaces: A unified framework with applications to nonlinear differential models

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Abstract: This paper develops a relaxed fixed-point framework for nonlinear operators acting on suprametric spaces. A new class of control functions Θ^R is introduced, allowing strictly increasing but possibly discontinuous behaviors that go beyond the classical ϑ-contraction structure. Within this setting, several relaxed $\vartheta_{\mathcal{R}}$-type contractive conditions are formulated through max-based suprametric functionals and on complete suprametric spaces. These conditions guarantee the existence and uniqueness of fixed-points under a suitable jump requirement on the control function. The theory is supported by explicit examples showing how discontinuities and nonlinear growth patterns influence convergence. Finally, two differential models, namely a second-order particle motion problem and a fourth-order beam equation, are used to demonstrate that their associated integral operators admit unique solutions within the proposed relaxed suprametric framework.

Keywords: suprametric space; relaxed ϑ-contraction; fixed-point theory; nonlinear operators; boundary value problems
Mathematics Subject Classification: 47H10, 54H25

1. Introduction and preliminaries

Fixed-point theory has been a central tool in nonlinear analysis since the formulation of the classical contraction principle [1]. Over time, numerous extensions of this principle have been developed to accommodate structures beyond the standard metric setting, including modular metrics, b-metrics, and partial metrics. In particular, several suprametric generalizations have been introduced to model distance structures dominated by maximum-type behavior rather than additive geometry. Initial developments in suprametric spaces and their applications were presented in [2–4]. Related structural and analytical investigations, including extensions of suprametric frameworks and applications to nonlinear dynamic models, appear in [5, 6]. These developments have been motivated by the need to analyze nonlinear problems where classical additive geometry is replaced by max-dominated or irregular behaviors, particularly in models governed by the largest local deviation rather than an accumulated effect. In this direction, suprametric spaces provide a natural analytical environment, offering a structure in which the contractive mechanism must be tailored to the maximum type inequality inherent in the space.

Parallel to such geometric generalizations, a wide range of generalized contractions have been introduced to capture nonlinear operators' behavior that cannot be described by traditional Lipschitz-type conditions. Among these, ϑ-contractions and their relaxed variants have attracted significant attention due to their flexibility in replacing the fixed contraction constant with a monotone control function. The foundational formulation of this approach was introduced by Jleli and Samet in [7], while related developments in generalized metric frameworks can be found in [8, 9]. Later studies established several variants of ϑ-type contractions and Suzuki-type formulations, providing new fixed-point results in complete metric spaces [10–12]. This approach allows the contractive condition to depend on nonlinear transformations of the distance, thereby accommodating mappings whose local behavior is not uniformly controlled.

Further nonlinear contraction schemes have also been explored through different control mechanisms. In particular, Marija-type ϑ-contractions and related relaxed contraction principles have been investigated in [13, 14]. These frameworks broaden the contractive theory by allowing max-type, integral-type, or hybrid control structures governing the operators' behavior. A common feature of such approaches is that the contraction is controlled through an auxiliary function rather than a fixed numerical constant, enabling the treatment of mappings that may exhibit jumps, oscillatory patterns, or other non smooth dynamics.

Motivated by these advances, the present work develops a new relaxed fixed-point framework adapted specifically to suprametric spaces by introducing the class Θ^R of control functions. This family includes strictly increasing but possibly discontinuous functions, extending far beyond the classical ϑ-types and allowing contractive mechanisms that are compatible with the max-dominated nature of suprametrics. The resulting theory yields the existence and uniqueness of fixed-points for a broad class of nonlinear operators, supported by illustrative examples and applications to differential models.

The present work is designed to support and extend the existing ϑ-type fixed-point theory rather than refute it. Compared with the classical Θ-contraction framework of Jleli–Samet [7], developed in complete metric spaces under structural conditions such as (ϑ_1)–(ϑ_3), we work on complete suprametric spaces $(\complement, d_s)$ and introduce a relaxed admissible family Θ_R in which the control function ϑ_R is only required to be nondecreasing; hence, it may be discontinuous, together with a quantified left-jump condition. This relaxation enlarges the admissible control functions and includes, as particular cases, several subclasses studied in [10–12]. It therefore provides a framework suitable for the analysis of suprametric models that arise naturally beyond the classical triangle inequality setting.

Moreover, our main results recover earlier theorems as special cases: When d_s reduces to a metric d and ϑ_R is chosen from the classical class Θ, Theorem 2.6 becomes a standard ϑ-contraction principle, and Theorem 2.10 contains the max-type condition. Table 1 summarizes these relationships.

Table 1. Comparison with representative ϑ-type fixed-point frameworks.

Reference	Space	Control class	Contractive quantity
Jleli–Samet [7]	Complete Metric $(\complement, d)$	Θ (e.g., (ϑ_1)–(ϑ_3))	$\vartheta(d(\eth\varkappa,\eth y)) \leq [\vartheta(d(\varkappa,y))]^k$ (basic ϑ-contraction)
Ahmad et al. [10]	Complete Metric $(\complement, d)$	Relaxed variants (e.g., dropping (ϑ_3))	Generalized ϑ-contractions in metric setting
Liu et al. [11], Hussain et al. [12]	Complete Metric $(\complement, d)$	Modified ϑ-classes (Suzuki-type and additional axioms)	Maximum-type or Suzuki-type functionals in a metric setting
Cvetković et al. [13]	Complete Metric $(\complement, d)$	Monotone ϑ + jump control	$\vartheta(d(\eth\varkappa,\eth y)) \leq [\vartheta(\max\{d(\varkappa,y), d(\varkappa,\eth\varkappa), d(y,\eth y)\})]^k$
This paper (Thm 2.6)	Complete suprametric $(\complement, d_s)$	Θ_R (nondecreasing, possibly discontinuous) + jump control	$\vartheta_R(d_s(\eth\varkappa,\eth y)) \leq [\vartheta_R(d_s(\varkappa,y))]^k$
This paper (Thm 2.10)	Complete suprametric $(\complement, d_s)$	Θ_R + jump control	$\vartheta_R(d_s(\eth\varkappa,\eth y)) \leq [\vartheta_R(\mathcal{M}_2(\varkappa,y))]^k$ (max-type including cross-terms)

Before proceeding to the main results, we recall the basic notions of suprametric spaces required in this work. We begin with the following definition.

Definition 1.1. Let $d_s : \complement \times \complement \to [0,\infty)$ be a mapping defined on a nonempty set $\complement$ that satisfies

(d_{s1}) $d_s(\varkappa,y) = 0 \Leftrightarrow \varkappa = y$,

(d_{s2}) $d_s(\varkappa,y) = d_s(y,\varkappa)$,

(d_{s3}) $d_s(\varkappa,z) \leq d_s\left(\varkappa,y\right) + d_s\left(y,z\right) + \curlyvee d_s\left(\varkappa,y\right) d_s\left(y,z\right)$,

for all $\varkappa, y \in \complement$, where $\curlyvee \geq 0$ is a constant, then the pair $(\complement, d_s)$ is called a suprametric space.

Every metric is a suprametric; however, several approaches exist to create a suprametric from a metric and, in general, omit the triangle inequality.

Example 1.2. Let $(\complement, d)$ be a metric space and let η, μ be two positive real numbers. Define the mappings $d_s^{\eta}, d_s^{\mu} : \complement \times \complement \to [0,\infty)$ by

- $d_s^{\eta}\left(\varkappa,y\right) = d\left(\varkappa,y\right)\left(d\left(\varkappa,y\right) + \eta\right)$,
- $d_s^{\mu}\left(\varkappa,y\right) = \mu\left(e^{d(\varkappa,y)} - 1\right)$

for all $\varkappa, y \in \complement$. Then d_s^{η} and d_s^{μ} are suprametrics with the constants $\curlyvee = \frac{2}{\eta}$ and $\curlyvee = \frac{1}{\mu}$, respectively.

Moreover, let $\complement = \mathbb{R}$ and $d\left(\varkappa,y\right) = |\varkappa - y|$ for all $\varkappa, y \in \complement$. If we define $d := d_s^{\eta}$ with $\eta = 1$ (respectively, $d := d_s^{\mu}$ with $\mu = 1$), then the resulting mapping d is not necessarily a metric since

$$d\,(0,1) + d\,(1,2) < d\,(0,2)\,.$$

Furthermore, in this case, d is not suprametric with $\curlyvee = \frac{1}{3}$; that is, if a mapping is suprametric with the constant $\curlyvee$, then it is also suprametric with any constant $\curlyvee' > \curlyvee$.

Definition 1.3. Let $(\complement, d_s)$ be a suprametric space.

(*i*) A sequence $\{\varkappa_n\}$ in $\complement$ is said to converge to a point $\varkappa \in \complement$ if, for every $\varepsilon > 0$, $N_\varepsilon \in \mathbb{N}$ exists such that

$$d_s(\varkappa_n, \varkappa) < \varepsilon \qquad \text{for all } n \geq N_\varepsilon.$$

In this case, we write $\lim_{n\to\infty} \varkappa_n = \varkappa$.

(*ii*) A sequence $\{\varkappa_n\}$ in $\complement$ is called a Cauchy sequence if, for every $\varepsilon > 0$, $N_\varepsilon \in \mathbb{N}$ exists such that

$$d_s(\varkappa_n, \varkappa_m) < \varepsilon \qquad \text{for all } m, n \geq N_\varepsilon.$$

(*iii*) The suprametric space $(\complement, d_s)$ is said to be complete if every Cauchy sequence in $\complement$ converges to some point of $\complement$.

(*iv*) If the suprametric d_s is continuous, then every convergent sequence in $\complement$ admits a unique limit.

First, a new class of Θ functions was defined by M. Jleli and B. Samet [7] as follows.

Definition 1.4. Let Θ be a set of functions $\vartheta : (0, \infty) \mapsto (1, \infty)$ such that

(ϑ_1) $\varkappa \leq y$ implies that $\vartheta(\varkappa) \leq \vartheta(y)$ the means that ϑ is nondecreasing

(ϑ_2) for all $(\varkappa_n) \subseteq (0, \infty)$

$$\lim_{n\to\infty} \vartheta(\varkappa_n) = 1 \Leftrightarrow \lim_{n\to\infty} \varkappa_n = 0,$$

(ϑ_3) $l \in (0, \infty]$ and $k \in (0, 1)$ exists, thus causing

$$\lim_{\varkappa\to 0} \frac{\vartheta(\varkappa) - 1}{\varkappa^k} = l.$$

The existence and uniqueness of fixed-points for the class of ϑ-contractions were examined within the framework of a generalized metric space as defined by Branciari [8]. In [9], a novel class Θ was enunciated, defined in the following.

Definition 1.5. Θ' denotes a class of functions $\vartheta : (0, \infty) \mapsto (1, \infty)$ satisfying all the conditions of Definition 1.4 along with the following additional condition:

(ϑ_4) ϑ is continuous.

In a related direction, the authors in [13] introduced another variant of ϑ-contractions characterized through a broader nonlinear contractive condition, thereby extending the classical framework. It is clear that the associated function class Θ' is a proper subset of Θ. Subsequently, Ahmad et al. [10] modified the standard definition of a ϑ-contraction and established several important fixed-point results in complete metric spaces. Their approach omits the condition (ϑ_3), reflecting a deliberate choice to simplify the structural requirements while still retaining substantial mathematical strength; the resulting family of admissible functions is denoted Θ^*. Since the classes generated under (ϑ_3) and (ϑ_4) do not coincide, the families Θ and Θ^* are not identical. We substantiate this observation with an illustrative example.

Example 1.6. We provide examples illustrating that the conditions (ϑ_3) and (ϑ_4) are independent and need not hold simultaneously. First consider the function $\vartheta(\varkappa) = e^{e^{\varkappa}-1}$ for $\varkappa \in (0, \infty)$. This function is continuous for all $\varkappa > 0$ and satisfies

$$\lim_{\varkappa \to 0^+} \frac{\vartheta(\varkappa) - 1}{\varkappa^k} = 0 \qquad \text{for every } k \in (0, 1),$$

showing that (ϑ_3) fails, although continuity is preserved. Next, for $k \in [\frac{1}{2}, 1)$, the function

$$\vartheta(\varkappa) = 1 + \sqrt{\varkappa}\,(1 + \lfloor \varkappa \rfloor), \qquad \varkappa > 0,$$

is not continuous anywhere on $(0, \infty)$; nevertheless, Condition (ϑ_4) remains valid. In both of the abovementioned cases, the properties (ϑ_1) and (ϑ_2) continue to hold. Thus, the classes Θ and Θ^* do not coincide, although their intersection is nonempty. Indeed, the function $\vartheta(\varkappa) = e^{\sqrt{\varkappa}}$ for $\varkappa > 0$ belongs to $\Theta \cap \Theta^*$.

Liu et al. [11] introduced an alternate way to deal with the class of $\vartheta \in \tilde{\Theta}$ capabilities by introducing an identical method with (ϑ_2)

$$(\vartheta_2^*) \inf_{\varkappa > 0} \vartheta(\varkappa) = 1.$$

The authors derived some fixed-point results for a class of ϑ^*-contractions via generalized Suzuki-ϑ contractions, where ϑ implies (ϑ_1), (ϑ_2^*), and (ϑ_4). The authors added another condition in [12] to $(\vartheta_1) - (\vartheta_3)$ as follows:

$$(\vartheta_5)\ \ \vartheta(\varkappa + y) \leq \vartheta(\varkappa)\vartheta(y).$$

By Θ^+, we define the family of all maps given as $\vartheta :]0, \infty[\to]1, \infty[$ preserving $(\vartheta_1) - (\vartheta_3)$ and (ϑ_5).

Definition 1.7. Let $\vartheta_{\mathcal{R}} : (0, \infty) \to (1, \infty)$ be a non decreasing function. The collection of all such mappings will be denoted by $\Theta_{\mathcal{R}}$. This newly introduced class forms a superset of the previously established families Θ, Θ^+, Θ^*, Θ', and related variants. We emphasize that the functions in $\Theta_{\mathcal{R}}$ are not a priori specified at 0. This can be resolved in a natural way by defining $\vartheta_{\mathcal{R}}(0) = 1$, a convention that does not alter the contractive condition and is fully compatible, when needed, with the structural requirements (ϑ_1)-(ϑ_4).

Theorem 1.8. *[13] Let $(\complement, d)$ be a complete metric space and $\supset : \complement \to \complement$ be an arbitrary self-map. Assume that there is a monotone (nondecreasing) function $\vartheta : (0, \infty) \to (1, \infty)$ together with a constant $k \in (0, 1)$ for which the implication*

$$\supset\varkappa \neq \supset y \quad \Longrightarrow \quad \vartheta\Big(d(\supset\varkappa, \supset y)\Big) \leq \Big[\vartheta\Big(\mathcal{N}(\varkappa, y)\Big)\Big]^k$$

is valid for every pair $\varkappa, y \in \complement$, where $\mathcal{N}(\varkappa, y) = \max\{d(\varkappa, y),\ d(\varkappa, \supset\varkappa),\ d(y, \supset y)\}$.

Moreover, whenever ϑ is discontinuous at a point $\varkappa > 0$, the size of the left-hand jump of ϑ does not exceed the quantity $\vartheta(\varkappa) - (\vartheta(\varkappa))^k$; equivalently, we have

$$\vartheta(\varkappa) - \lim_{y \to \varkappa^-} \vartheta(y) > (\vartheta(\varkappa))^k.$$

Under these assumptions, the mapping $\supset$ possesses a unique fixed point in the space $\complement$.

We now advance to our main result, where the relaxed ϑ-contractive structure is embedded into the suprametric framework, enabling a sharper and more flexible fixed-point analysis.

2. Modified results

In this section, we introduce simple contractions as a class and provide various illustrations of the proposed framework through numerical computations and graphical presentations. We prove the fixed-point results by combining the relaxed contraction class with different contractions.

Definition 2.1. A set Θ^R is a collection of all non decreasing functions $\vartheta_{\mathcal{R}} : (0, \infty) \mapsto (1, \infty)$, which satisfy only the condition (ϑ_1) with $\vartheta_{\mathcal{R}}(0) = 1$.

To illustrate Definition 2.1, we present several concrete functions as examples that belong to the set Θ^R. In particular, Examples 2.2, 2.3, 2.4, and 2.5 demonstrate the functions $\vartheta_{\mathcal{R}} : (0, \infty) \to (1, \infty)$ that are strictly increasing and satisfy condition (ϑ_1) but do not satisfy Conditions (ϑ_2) and (ϑ_3). These examples serve to highlight the type of functions captured by Θ^R and provide explicit constructions supporting the definition.

Example 2.2. Define $\vartheta_{\mathcal{R}} : (0, \infty) \mapsto (1, \infty)$ such that

$$\vartheta_{\mathcal{R}}(t) = \begin{cases} 1 + t^3, & \text{if } t \in (0, 1), \\ 2 + e^t, & \text{if } t \in [1, \infty), \end{cases}$$

where $\vartheta_{\mathcal{R}}$ is discontinuous at $t = 1$, as shown in Figure 1.

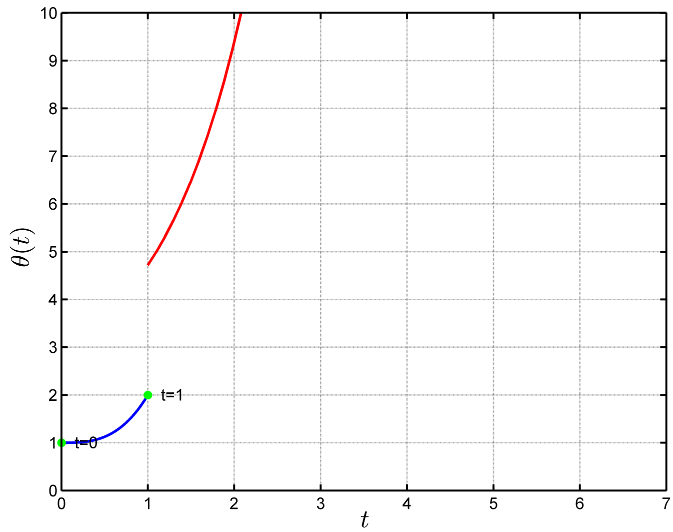

Figure 1. Discontinuity of $\vartheta_{\mathcal{R}}$ of Example 2.2.

For (ϑ_1): The given function is non decreasing over $(0, \infty)$. On $(0, 1)$, $\vartheta_{\mathcal{R}}(t) = 1 + t^3$ is increasing since its derivative $3t^2 > 0$. On $[1, \infty)$, $\vartheta_{\mathcal{R}}(t) = 2 + e^t$ is increasing since its derivative $e^t > 0$. At $t = 1$, the left-hand limit is $1 + 1^3 = 2$ and the right-hand value is $2 + e^1 > 2$, so the function remains non decreasing. Thus, (ϑ_1) is satisfied.

For (ϑ_2): For $t \to \infty$, $\vartheta_{\mathcal{R}}(t) = 2 + e^t \to \infty$. Since the limit is not 1, this function does not qualify for (ϑ_2).

For (ϑ_3): For $t \to 0^+$, $\vartheta_{\mathcal{R}}(t) = 1 + t^3$, so

$$\lim_{t\to 0^+} \frac{\vartheta_{\mathcal{R}}(t) - 1}{t^k} = \lim_{t\to 0^+} \frac{1 + t^3 - 1}{t^k} = \lim_{t\to 0^+} t^{3-k},$$

where $0 < k < 1$ and $3-k > 2$, this limit converges to $0 \notin (0, \infty]$, so (ϑ_3) fails to qualify. Consequently, the defined $\vartheta_{\mathcal{R}}(t)$ function satisfies Condition (ϑ_1) but not Conditions (ϑ_2) and (ϑ_3), so $\vartheta_{\mathcal{R}} \in \Theta^R$.

Example 2.3. Define $\vartheta_{\mathcal{R}} : (0, \infty) \mapsto (1, \infty)$ such that

$$\vartheta_{\mathcal{R}}(t) = \begin{cases} 1 + t^\alpha, & \text{if } t \in (0, 1), \\ 5 + \ln(t), & \text{if } t \in [1, \infty), \end{cases}$$

where $\alpha > 1$ and $\vartheta_{\mathcal{R}}$ is discontinuous at $t = 1$, as shown in Figure 2.

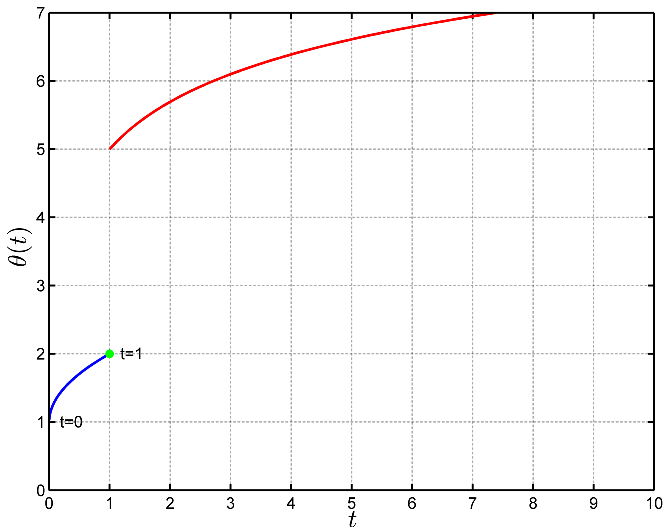

Figure 2. Discontinuity of $\vartheta_{\mathcal{R}}$ of Example 2.3.

For (ϑ_1): The given function is non decreasing over $(0, \infty)$. On $(0, 1)$, $\vartheta_{\mathcal{R}}(t) = 1 + t^\alpha$ is increasing, since its derivative $\alpha t^{\alpha-1} > 0$. On $[1, \infty)$, $\vartheta_{\mathcal{R}}(t) = 5 + \ln(t)$ is increasing since its derivative $1/t > 0$. At $t = 1$, the left-hand limit is 2 and the right-hand value is $5 + \ln(1) = 5$, so the function jumps from 2 to 5, but remains non decreasing. Thus, (ϑ_1) is satisfied.

For (ϑ_2): For $t \to \infty$, $\vartheta_{\mathcal{R}}(t) = 5 + \ln(t) \to \infty$. Since the limit is not 1, this function does not qualify for (ϑ_2).

For (ϑ_3): For $t \to 0^+$, $\vartheta_{\mathcal{R}}(t) = 1 + t^\alpha$, so

$$\lim_{t\to 0^+} \frac{\vartheta_{\mathcal{R}}(t) - 1}{t^k} = \lim_{t\to 0^+} \frac{1 + t^\alpha - 1}{t^k} = \lim_{t\to 0^+} t^{\alpha-k}.$$

where $0 < k < 1$ and $\alpha - k > \alpha - 1 > 0$. Since $\alpha > 1$, this limit converges to $0 \notin (0, \infty]$, so (ϑ_3) fails to qualify. Consequently, the defined $\vartheta_{\mathcal{R}}(t)$ function satisfies Condition (ϑ_1) but not Conditions (ϑ_2) and (ϑ_3), so $\vartheta_{\mathcal{R}} \in \Theta^R$.

Example 2.4. Define $\vartheta_{\mathcal{R}} : (0, \infty) \mapsto (1, \infty)$ such that

$$\vartheta_{\mathcal{R}}(t) = \begin{cases} 1 + t^3, & \text{if } t \in (0, 1), \\ 3 + \ln(t), & \text{if } t \in [1, \infty), \end{cases}$$

where $\vartheta_{\mathcal{R}}$ is discontinuous at $t = 1$, as shown in Figure 3.

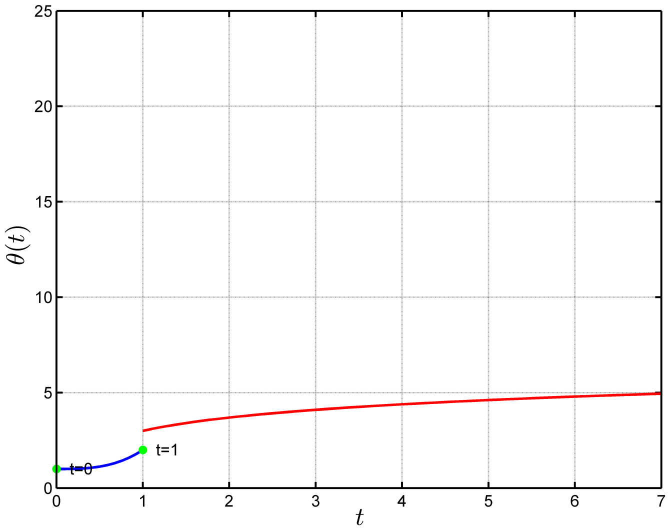

Figure 3. Discontinuity of $\vartheta_{\mathcal{R}}$ of Example 2.4.

For (ϑ_1): The given function is non decreasing over $(0, \infty)$. On $(0, 1)$, $\vartheta_{\mathcal{R}}(t) = 1 + t^3$ is increasing, since its derivative $3t^2 > 0$. On $[1, \infty)$, $\vartheta_{\mathcal{R}}(t) = 3 + \ln(t)$ is increasing since its derivative $1/t > 0$. At $t = 1$, the left-hand limit is $1 + 1^3 = 2$ and the right-hand value is $3 + \ln(1) = 3$, so the function jumps from 2 to 3, but remains non decreasing. Thus, (ϑ_1) is satisfied.

For (ϑ_2): For $t \to \infty$, $\vartheta_{\mathcal{R}}(t) = 3 + \ln(t) \to \infty$. Since the limit is not 1, this function does not qualify for (ϑ_2).

For (ϑ_3): For $t \to 0^+$, $\vartheta_{\mathcal{R}}(t) = 1 + t^3$, so:

$$\lim_{t \to 0^+} \frac{\vartheta_{\mathcal{R}}(t) - 1}{t^k} = \lim_{t \to 0^+} \frac{1 + t^3 - 1}{t^k} = \lim_{t \to 0^+} t^{3-k}$$

where $0 < k < 1$ and $3 - k > 2$, this limit converges to $0 \notin (0, \infty]$, so (ϑ_3) fails to qualify. Consequently, the defined $\vartheta_{\mathcal{R}}(t)$ function satisfies Condition (ϑ_1) but not Conditions (ϑ_2) and (ϑ_3), so $\vartheta_{\mathcal{R}} \in \Theta^R$.

Example 2.5. Define $\vartheta_{\mathcal{R}} : (0, \infty) \mapsto (1, \infty)$ such that

$$\vartheta_{\mathcal{R}}(t) = \begin{cases} 1 + \sqrt{t^3}, & \text{if } t \in (0, 1), \\ 3 + \ln(t), & \text{if } t \in [1, \infty), \end{cases}$$

where $\vartheta_{\mathcal{R}}$ is discontinuous at $t = 1$, as shown in Figure 4.

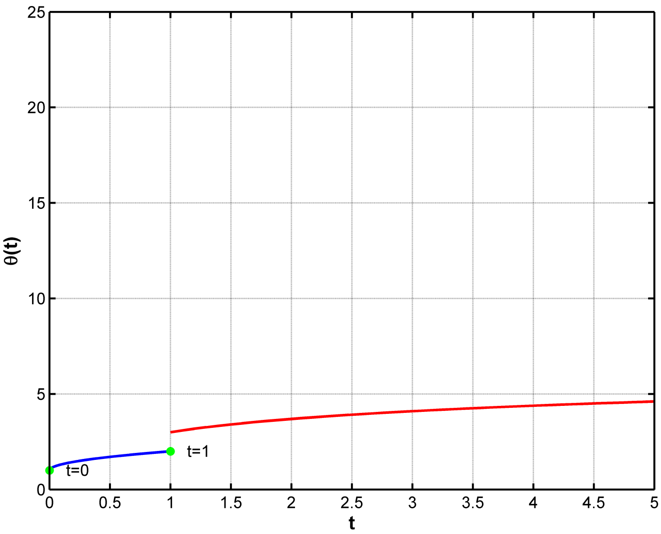

Figure 4. Discontinuity of $\vartheta_{\mathcal{R}}$ of Example 2.5.

For (ϑ_1): The given function is non decreasing over $(0, \infty)$. On $(0, 1)$, $\vartheta_{\mathcal{R}}(t) = 1 + \sqrt{t^3}$ is increasing, since its derivative $\frac{3}{2}\sqrt{t} > 0$. On $[1, \infty)$, $\vartheta_{\mathcal{R}}(t) = 3 + \ln(t)$ is increasing since its derivative $1/t > 0$. At $t = 1$, the left-hand limit is $1 + \sqrt{1} = 2$ and the right-hand value is $3 + \ln(1) = 3$, so the function jumps from 2 to 3, but remains non decreasing. Thus, (ϑ_1) is satisfied.

For (ϑ_2): For $t \to \infty$, $\vartheta_{\mathcal{R}}(t) = 3 + \ln(t) \to \infty$. Since the limit is not 1, this function does not qualify for (ϑ_2).

For (ϑ_3): For $t \to 0^+$, $\vartheta_{\mathcal{R}}(t) = 1 + \sqrt{t^3}$, so:

$$\lim_{t\to 0^+} \frac{\vartheta_{\mathcal{R}}(t) - 1}{t^k} = \lim_{t\to 0^+} \frac{1 + \sqrt{t^3} - 1}{t^k} = \lim_{t\to 0^+} t^{\frac{3}{2}-k},$$

where $0 < k < 1$ and $\frac{3}{2} - k > 0$, this limit converges to $0 \notin (0, \infty]$, so (ϑ_3) fails to qualify. Consequently, the defined $\vartheta_{\mathcal{R}}(t)$ function satisfies condition (ϑ_1) but not conditions (ϑ_2) and (ϑ_3), so $\vartheta_{\mathcal{R}} \in \Theta^R$.

Theorem 2.6. *Let* $\eth : \complement \to \complement$ *be a mapping on a complete suprametric space* $(\complement, d_s)$. *Assume that there is a nondecreasing function* $\vartheta_{\mathcal{R}} : (0, \infty) \to (1, \infty)$ *and a constant* $k \in (0, 1)$ *such that*

$$\eth\varkappa \neq \eth y \implies \vartheta_{\mathcal{R}}(d_s(\eth\varkappa, \eth y)) \leq (\vartheta_{\mathcal{R}}(d_s(\varkappa, y)))^k. \tag{2.1}$$

Moreover, at every discontinuity point $t > 0$ *of* $\vartheta_{\mathcal{R}}$, *the left-jump satisfies*

$$\vartheta_{\mathcal{R}}(t) - \lim_{s\to t^-} \vartheta_{\mathcal{R}}(s) \leq \vartheta_{\mathcal{R}}(t) - (\vartheta_{\mathcal{R}}(t))^k.$$

Then $\eth$ *admits a unique fixed-point in* $\complement$.

Proof. Construct a sequence $(\varkappa_n) \subseteq \complement$ by picking an arbitrary $\varkappa_0 \in \complement$ in such a way that $\varkappa_n = \eth\varkappa_{n-1}$ holds for all $n \in \mathbb{N}$. In the event that $\varkappa_n = \varkappa_{n-1}$ for some $n \in \mathbb{N}$, then $\varkappa_{n-1}$ is a fixed-point of $\eth$. Alternatively, suppose that for all $n \in \mathbb{N}$, the expression $\varkappa_n \neq \varkappa_{n-1}$ holds, so we will evaluate the components of the Cauchy sequence $(\varkappa_n)$ by utilizing (2.1), in such a way that

$$\vartheta_{\mathcal{R}}(d_s(\varkappa_n, \varkappa_{n+1})) = \vartheta_{\mathcal{R}}(d_s(\eth\varkappa_{n-1}, \eth\varkappa_n)) \leq (\vartheta_{\mathcal{R}}(d_s(\varkappa_{n-1}, \varkappa_n)))^k.$$

Proceeding similarly for each $n \in \mathbb{N}$, we obtain

$$\vartheta_{\mathcal{R}}\left(d_s\left(\varkappa_n, \varkappa_{n+1}\right)\right) \leq \left(\vartheta_{\mathcal{R}}\left(d_s\left(\varkappa_0, \varkappa_1\right)\right)\right)^{k^n}.$$

Taking the limit $n \to \infty$, we get

$$1 \leq \lim_{n\to\infty} \vartheta_{\mathcal{R}}\left(d_s\left(\varkappa_n, \varkappa_{n+1}\right)\right) \leq \lim_{n\to\infty} \left(\vartheta_{\mathcal{R}}\left(d_s\left(\varkappa_0, \varkappa_1\right)\right)\right)^{k^n} = 1.$$

Moreover

$$\begin{aligned} \vartheta_{\mathcal{R}}\left(d_s\left(\varkappa_n, \varkappa_{n+1}\right)\right) &\leq \left(\vartheta_{\mathcal{R}}\left(d_s(\varkappa_{n-1}, \varkappa_n)\right)\right)^k \\ &< \left(\vartheta_{\mathcal{R}}\left(d_s(\varkappa_{n-1}, \varkappa_n)\right)\right). \end{aligned}$$

Thus, for any $n \in \mathbb{N}$, we arrive at

$$d_s\left(\varkappa_n, \varkappa_{n+1}\right) < d_s\left(\varkappa_{n-1}, \varkappa_n\right).$$

As a sequence, $(d_s\left(\varkappa_{n-1}, \varkappa_n\right))$ is a monotone decreasing sequence. This implies that its limit exists and $a = \inf_{n\in\mathbb{N}} d_s\left(\varkappa_{n-1}, \varkappa_n\right) = \lim_{n\to\infty} d_s\left(\varkappa_{n-1}, \varkappa_n\right)$.

Taking $a > 0$, we obtain

$$\vartheta_{\mathcal{R}}(a) \leq \lim_{n\to\infty} \vartheta_{\mathcal{R}}\left(d_s\left(\varkappa_n, \varkappa_{n+1}\right)\right) = 1,$$

which leads to

$$\lim_{n\to\infty} d_s\left(\varkappa_n, \varkappa_{n+1}\right) = 0. \tag{2.2}$$

Suppose, on the contrary, that $(\varkappa_n)$ is not a Cauchy sequence. Moreover, note that the function $\vartheta_{\mathcal{R}}$ is monotonic, meaning that its set of discontinuities is countable. As a result, $\varepsilon > 0$ exists such that this value does not correspond to a discontinuity of $\vartheta_{\mathcal{R}}$, and there are strictly increasing sequences $(n_i), (m_i) \subseteq \mathbb{N}$, where $m_i > n_i$, for any $i \in \mathbb{N}$ and

$$d_s\left(\varkappa_{n_i}, \varkappa_{m_i}\right) \geq \varepsilon \text{ and } d_s\left(\varkappa_{n_i}, \varkappa_{m_i-1}\right) < \varepsilon, \tag{2.3}$$

where n_i is the minimal value that emphasizes the careful selection of indices in the sequences to ensure that the subsequences can be effectively analyzed in such a way that

$$n_i = \min\left\{i \leq j \mid d_s\left(\varkappa_j, \varkappa_m\right) \geq \varepsilon \wedge m > j\right\},$$

and

$$m_i = \min\left\{n_i < n_i \mid d_s\left(\varkappa_{n_i}, \varkappa_j\right) \geq \varepsilon\right\}.$$

Furthermore,

$$\begin{aligned} \varepsilon \leq d_s\left(\varkappa_{n_i}, \varkappa_{m_i}\right) &\leq d_s\left(\varkappa_{n_i}, \varkappa_{m_i-1}\right) + d_s\left(\varkappa_{m_i-1}, \varkappa_{m_i}\right) + \vee d_s\left(\varkappa_{n_i}, \varkappa_{m_i-1}\right) d_s\left(\varkappa_{m_i-1}, \varkappa_{m_i}\right) \\ &\leq \varepsilon + d_s\left(\varkappa_{m_i-1}, \varkappa_{m_i}\right) + \vee\varepsilon d_s\left(\varkappa_{m_i-1}, \varkappa_{m_i}\right), \end{aligned}$$

letting the limit $i \to \infty$, the terms $d_s\left(\varkappa_{n_i}, \varkappa_{n_i-1}\right)$ and $d_s\left(\varkappa_{m_i-1}, \varkappa_{m_i}\right)$ approaches 0. In the above, we get

$$\lim_{i\to\infty} d_s\left(\varkappa_{n_i}, \varkappa_{m_i}\right) = \varepsilon. \tag{2.4}$$

In addition, from $(d_{s}3)$, we have

$$\begin{aligned} d_s(\varkappa_{n_i},\varkappa_{m_i}) &\leq d_s(\varkappa_{n_i},\varkappa_{n_i-1}) + d_s(\varkappa_{n_i-1},\varkappa_{m_i}) + \curlyvee d_s(\varkappa_{n_i},\varkappa_{n_i-1})\, d_s(\varkappa_{n_i-1},\varkappa_{m_i}) \\ &\leq d_s(\varkappa_{n_i},\varkappa_{n_i-1}) + d_s(\varkappa_{n_i-1},\varkappa_{m_i-1}) + d_s(\varkappa_{m_i-1},\varkappa_{m_i}) + \curlyvee d_s(\varkappa_{n_i-1},\varkappa_{m_i-1})\, d_s(\varkappa_{m_i-1},\varkappa_{m_i}) \\ &\quad + \curlyvee\, d_s(\varkappa_{n_i},\varkappa_{n_i-1})\, d_s(\varkappa_{n_i-1},\varkappa_{m_i}) \\ &\leq d_s(\varkappa_{n_i},\varkappa_{n_i-1}) + d_s(\varkappa_{n_i-1},\varkappa_{m_i-1}) + d_s(\varkappa_{m_i-1},\varkappa_{m_i}) + \curlyvee d_s(\varkappa_{n_i-1},\varkappa_{m_i-1})\, d_s(\varkappa_{m_i-1},\varkappa_{m_i}) \\ &\quad + \curlyvee\, d_s(\varkappa_{n_i},\varkappa_{n_i-1})\left(d_s(\varkappa_{n_i-1},\varkappa_{n_i}) + d_s(\varkappa_{n_i},\varkappa_{m_i}) + \curlyvee d_s(\varkappa_{n_i-1},\varkappa_{n_i})\, d_s(\varkappa_{n_i},\varkappa_{m_i})\right). \end{aligned}$$

Equivalently

$$\begin{pmatrix} \left(1 - \curlyvee d_s(\varkappa_{n_i},\varkappa_{n_i-1}) - \curlyvee^2 d_s(\varkappa_{n_i},\varkappa_{n_i-1})\right) d_s(\varkappa_{n_i},\varkappa_{m_i}) \\ -d_s(\varkappa_{n_i},\varkappa_{n_i-1}) - d_s(\varkappa_{m_i-1},\varkappa_{m_i}) - \curlyvee d_s(\varkappa_{n_i-1},\varkappa_{m_i})^2 \end{pmatrix} (1 + \curlyvee d_s(\varkappa_{m_i-1},\varkappa_{m_i}))^{-1} \leq d_s(\varkappa_{n_i-1},\varkappa_{m_i-1})$$

and

$$\begin{aligned} d_s(\varkappa_{n_i-1},\varkappa_{m_i-1}) &\leq d_s(\varkappa_{n_i-1},\varkappa_{n_i}) + d_s(\varkappa_{n_i},\varkappa_{m_i-1}) + \curlyvee d_s(\varkappa_{n_i-1},\varkappa_{n_i})\, d_s(\varkappa_{n_i},\varkappa_{m_i-1}) \\ &\leq d_s(\varkappa_{n_i-1},\varkappa_{n_i}) + d_s(\varkappa_{n_i},\varkappa_{m_i}) + d_s(\varkappa_{m_i},\varkappa_{m_i-1}) + \curlyvee d_s(\varkappa_{n_i},\varkappa_{m_i})\, d_s(\varkappa_{m_i},\varkappa_{m_i-1}) \\ &\quad + \curlyvee\, d_s(\varkappa_{n_i-1},\varkappa_{n_i})\, d_s(\varkappa_{n_i},\varkappa_{m_i-1}) \\ &\leq d_s(\varkappa_{n_i-1},\varkappa_{n_i}) + d_s(\varkappa_{n_i},\varkappa_{m_i}) + d_s(\varkappa_{m_i},\varkappa_{m_i-1}) + \curlyvee d_s(\varkappa_{n_i},\varkappa_{m_i})\, d_s(\varkappa_{m_i},\varkappa_{m_i-1}) \\ &\quad + \curlyvee\, d_s(\varkappa_{n_i-1},\varkappa_{n_i})\left(d_s(\varkappa_{n_i},\varkappa_{n_i-1}) + d_s(\varkappa_{n_i-1},\varkappa_{m_i-1}) + \curlyvee d_s(\varkappa_{n_i},\varkappa_{n_i-1})\, d_s(\varkappa_{n_i-1},\varkappa_{m_i-1})\right). \end{aligned}$$

Equivalently

$$\begin{aligned} &\left(1 - \curlyvee d_s(\varkappa_{n_i},\varkappa_{n_i-1}) - \curlyvee^2 d_s(\varkappa_{n_i},\varkappa_{n_i-1})\right) d_s(\varkappa_{n_i-1},\varkappa_{m_i-1}) \\ &\leq d_s(\varkappa_{n_i-1},\varkappa_{n_i}) + d_s(\varkappa_{m_i},\varkappa_{m_i-1}) + \curlyvee d_s(\varkappa_{n_i-1},\varkappa_{n_i}) + (1 + d_s(\varkappa_{m_i},\varkappa_{m_i-1}))\, d_s(\varkappa_{n_i},\varkappa_{m_i}). \end{aligned}$$

Therefore, letting $i \to \infty$ in the inequalities above, and from (2.2) and (2.4), we conclude that

$$\lim_{i\to\infty} d_s(\varkappa_{n_i-1},\varkappa_{m_i-1}) = \varepsilon. \tag{2.5}$$

Putting $\varkappa = \varkappa_{n_i}$ and $y = \varkappa_{m_i}$ in (2.1), we get

$$\vartheta_{\mathcal{R}}(\varepsilon) \leq \vartheta_{\mathcal{R}}(d_s(\varkappa_{n_i},\varkappa_{m_i})) \leq (\vartheta_{\mathcal{R}}(d_s(\varkappa_{n_i-1},\varkappa_{m_i-1})))^k.$$

Hence, taking (2.4) and (2.5) into account, we have

$$\begin{aligned}\vartheta_{\mathcal{R}}(\varepsilon) &\leq \lim_{i\to\infty} \vartheta_{\mathcal{R}}\left(d_s\left(\varkappa_{n_i}, \varkappa_{m_i}\right)\right) \\ &\leq \lim_{i\to\infty} \left(\vartheta_{\mathcal{R}}\left(d_s\left(\varkappa_{n_i-1}, \varkappa_{m_i-1}\right)\right)\right)^k\end{aligned}$$

as $i \to \infty$, we get $\vartheta_{\mathcal{R}}(\varepsilon) \leq (\vartheta_{\mathcal{R}}(\varepsilon))^k$, which is a contradiction. Thus, $(\varkappa_n)$ is a Cauchy sequence. As a result, there is some $\varkappa^* \in \complement$ in such a way that $\lim_{n\to\infty} \varkappa_n = \varkappa^*$ and

$$\vartheta_{\mathcal{R}}(d_s\left(\eth\varkappa^*, \eth\varkappa_n\right)) \leq \vartheta_{\mathcal{R}}(d_s\left(\varkappa^*, \varkappa_n\right))^k.$$

As $\lim_{n\to\infty} d_s\left(\eth\varkappa^*, \varkappa_{n+1}\right) = d_s\left(\eth\varkappa^*, \varkappa^*\right)$, from the estimation of

$$\vartheta_{\mathcal{R}}\left(d_s\left(\eth\varkappa^*, \varkappa_{n+1}\right)\right) \leq \left(\vartheta_{\mathcal{R}}\left(d_s\left(\varkappa^*, \eth\varkappa^*\right)\right)\right)^k,$$

we obtain the contradiction, since it must be

$$\lim_{n\to\infty} \vartheta_{\mathcal{R}}\left(d_s\left(\eth\varkappa^*, \varkappa_n\right)\right) > \left(\vartheta_{\mathcal{R}}\left(d_s\left(\varkappa^*, \eth\varkappa^*\right)\right)\right)^k,$$

endorsing that $\varkappa^*$ is a fixed-point of the mapping $\eth$. If $\eth\varsigma = \varsigma$ and $\varsigma \neq \varkappa^*$, then

$$\begin{aligned}\lim_{n\to\infty} \vartheta_{\mathcal{R}}\left(d_s\left(\varkappa^*, \varsigma\right)\right) &= \left(\vartheta_{\mathcal{R}}\left(d_s\left(\eth\varkappa^*, \eth\varsigma\right)\right)\right) \\ &\leq \left(\vartheta_{\mathcal{R}}\left(d_s\left(\varkappa^*, \varsigma\right)\right)\right)^k,\end{aligned}$$

and $\varkappa^*$ is a unique fixed-point of the mapping $\eth$. □

Example 2.7. Given the complete suprametric space $(\complement, d_s)$, where $\complement = [0, \infty)$ and

$$d_s(\varkappa, y) = \left|\varkappa - y\right| \left(\left|\varkappa - y\right| + \curlyvee\right)$$

with $\curlyvee = 2$. Define a mapping such that $\eth\varkappa = \frac{\varkappa}{2}$.

We then show that $d_s(\eth\varkappa, \eth y) = \frac{1}{4}|\varkappa - y|^2 + |\varkappa - y|$ and $d_s(\varkappa, y) = |\varkappa - y|^2 + 2|\varkappa - y|$. Now, by using the contraction discussed in Eq (2.1) and utilizing the $\vartheta_{\mathcal{R}}$ mentioned in Example 2.2 as follows, we have

$$\vartheta_{\mathcal{R}}(t) = \begin{cases} 1 + t^3, & \text{if } t \in (0, 1), \\ 2 + e^t, & \text{if } t \in [1, \infty), \end{cases}$$

which only satisfies ϑ_1 but not ϑ_2 and ϑ_3. This choice of ϑ_R is consistent with Condition (2.1), as illustrated in Figures 5 and 6.

Case (i): For $t \in (0, 1)$ and $\eth\varkappa \neq \eth y$, we calculate

$$1 + \left[\frac{1}{4}|\varkappa - y|^2 + |\varkappa - y|\right]^3 \leq \left[1 + \left(|\varkappa - y|^2 + 2|\varkappa - y|\right)^3\right]^k.$$

Figure 5 demonstrates that the inequality above is valid for all $\varkappa, y \in (0, 1)$.

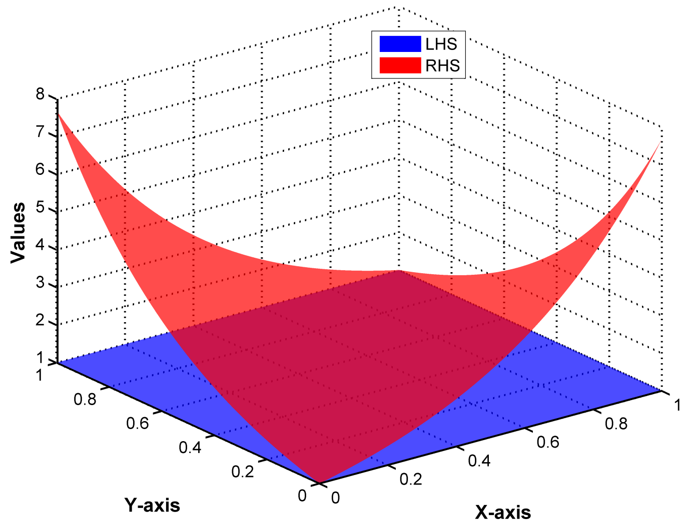

Figure 5. Case (i) of Example 2.7.

Case (ii): For $t \in [1, \infty)$ and $\partial\varkappa \neq \partial y$, we obtain

$$2 + \exp^{\left[\frac{1}{4}|\varkappa - y|^2 + |\varkappa - y|\right]} \leq \left[2 + \exp^{|\varkappa - y|^2 + 2|\varkappa - y|}\right]^k.$$

From Figure 6, we see that the inequality above is satisfied for all $\varkappa, y \in [1, \infty)$.

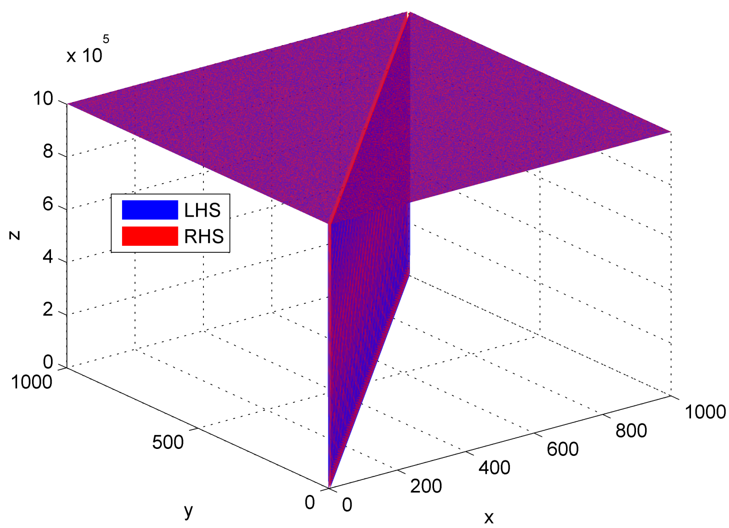

Figure 6. Case (ii) of Example 2.7.

For discontinuous $\vartheta_{\mathcal{R}}$, we require the following left-jump condition at each point of the discontinuity t:

$$\vartheta_{\mathcal{R}}(t) - \lim_{y \to t-} \vartheta_{\mathcal{R}}(y) > (\vartheta_{\mathcal{R}}(t))^k.$$

Since $\vartheta_{\mathcal{R}}$ exhibits a discontinuity at $t = 1$, we obtain the following result:

$$\begin{aligned} \vartheta_{\mathcal{R}}(1) - \lim_{y \to t-} \vartheta_{\mathcal{R}}(1) &> (\vartheta_{\mathcal{R}}(1))^k, \\ (2 + 2.718) - 2 &> (4.718)^k, \\ 2.718 &> (4.718)^k. \end{aligned}$$

Given $k = 0.64$, all the necessary conditions for Theorem 2.6 are satisfied, and 0 is the unique fixed-point of the mapping $\eth$.

Theorem 2.8. *Let $\eth : \complement \to \complement$ be a mapping on a complete suprametric space $(\complement, d_s)$. Assume that a nondecreasing function $\vartheta_{\mathcal{R}} : (0, \infty) \to (1, \infty)$ and a constant $k \in (0, 1)$ exists such that*

$$\eth\varkappa \neq \eth y \implies \vartheta_{\mathcal{R}}(d_s(\eth\varkappa, \eth y)) \leq (\vartheta_{\mathcal{R}}(\mathcal{M}_1(\varkappa, y)))^k, \tag{2.6}$$

where

$$\mathcal{M}_1(\varkappa, y) = \max\left\{ d_s(\varkappa, y),\ \frac{d_s(\varkappa, \eth\varkappa) + d_s(y, \eth y)}{2},\ \frac{d_s(\varkappa, \eth y) + d_s(y, \eth\varkappa)}{2} \right\}.$$

Moreover, at every discontinuity $t > 0$ of $\vartheta_{\mathcal{R}}$, the left-jump satisfies

$$\vartheta_{\mathcal{R}}(t) - \lim_{s \to t^-} \vartheta_{\mathcal{R}}(s) > (\vartheta_{\mathcal{R}}(t))^k.$$

Then $\eth$ admits a unique fixed-point in $\complement$.

Proof. Construct a sequence $(\varkappa_n) \subseteq \complement$ by picking an arbitrary $\varkappa_0 \in \complement$ in such a way that $\varkappa_n = \eth\varkappa_{n-1}$ holds for all $n \in \mathbb{N}$. In the event that $\varkappa_n = \varkappa_{n-1}$ for some $n \in \mathbb{N}$, then $\varkappa_{n-1}$ is a fixed-point of $\eth$. Alternatively, suppose that for all $n \in \mathbb{N}$, the expression $\varkappa_n \neq \varkappa_{n-1}$ holds, so we will evaluate the components of the Cauchy sequence $(\varkappa_n)$ by utilizing (2.6), in such a way that

$$\vartheta_{\mathcal{R}}(d_s(\varkappa_n, \varkappa_{n+1})) = \vartheta_{\mathcal{R}}(d_s(\eth\varkappa_{n-1}, \eth\varkappa_n)) \leq (\vartheta_{\mathcal{R}}(\mathcal{M}_1(\varkappa_{n-1}, \varkappa_n)))^k,$$

where

$$\begin{aligned} \mathcal{M}_1(\varkappa_{n-1}, \varkappa_n) &= \max\{d_s(\varkappa_{n-1}, \varkappa_n), \frac{d_s(\varkappa_{n-1}, \eth\varkappa_{n-1}) + d_s(\varkappa_n, \eth\varkappa_n)}{2}, \frac{d_s(\varkappa_{n-1}, \eth\varkappa_n) + d_s(\varkappa_n, \eth\varkappa_{n-1})}{2}\} \\ &= \max\{d_s(\varkappa_{n-1}, \varkappa_n), \frac{d_s(\varkappa_{n-1}, \varkappa_n) + d_s(\varkappa_n, \varkappa_{n+1})}{2}, \frac{d_s(\varkappa_{n-1}, \varkappa_{n+1})}{2}\}. \end{aligned}$$

If $\max\{d_s(\varkappa_{n-1}, \varkappa_n), \frac{d_s(\varkappa_{n-1}, \varkappa_n) + d_s(\varkappa_n, \varkappa_{n+1})}{2}, \frac{d_s(\varkappa_{n-1}, \varkappa_{n+1})}{2}\} \neq d_s(\varkappa_{n-1}, \varkappa_n)$ for some $n \in \mathbb{N}$, we will lead to a contradiction. Therefore, we conclude that

$$\vartheta_{\mathcal{R}}(d_s(\varkappa_n, \varkappa_{n+1})) \leq (\vartheta_{\mathcal{R}}(d_s(\varkappa_{n-1}, \varkappa_n)))^k.$$

By proceeding with similar process for each $n \in \mathbb{N}$, we obtain

$$\vartheta_{\mathcal{R}}(d_s(\varkappa_n, \varkappa_{n+1})) \leq (\vartheta_{\mathcal{R}}(d_s(\varkappa_0, \varkappa_1)))^{k^n}.$$

Taking the limit $n \to \infty$, we get

$$1 \leq \lim_{n \to \infty} \vartheta_{\mathcal{R}}(d_s(\varkappa_n, \varkappa_{n+1})) \leq \lim_{n \to \infty} (\vartheta_{\mathcal{R}}(d_s(\varkappa_0, \varkappa_1)))^{k^n} = 1.$$

Moreover, we have

$$\begin{aligned}\vartheta_{\mathcal{R}}\left(d_s\left(\varkappa_n,\varkappa_{n+1}\right)\right) &\leq \left(\vartheta_{\mathcal{R}}\left(d_s(\varkappa_{n-1},\varkappa_n)\right)\right)^k \\ &< \left(\vartheta_{\mathcal{R}}\left(d_s(\varkappa_{n-1},\varkappa_n)\right)\right).\end{aligned}$$

Thus, for any $n \in \mathbb{N}$, we arrive at

$$d_s\left(\varkappa_n,\varkappa_{n+1}\right) < d_s\left(\varkappa_{n-1},\varkappa_n\right),$$

and so $(d_s\left(\varkappa_{n-1},\varkappa_n\right))$ is a monotone decreasing sequence. This amounts to saying that its limit exists and $a = \inf_{n\in\mathbb{N}} d_s\left(\varkappa_{n-1},\varkappa_n\right) = \lim_{n\to\infty} d_s\left(\varkappa_{n-1},\varkappa_n\right)$.

Taking $a > 0$, we have

$$\vartheta_{\mathcal{R}}(a) \leq \lim_{n\to\infty} \vartheta_{\mathcal{R}}\left(d_s\left(\varkappa_n,\varkappa_{n+1}\right)\right) = 1,$$

which leads to

$$\lim_{n\to\infty} d_s\left(\varkappa_n,\varkappa_{n+1}\right) = 0. \tag{2.7}$$

On the contrary, assume that $(\varkappa_n)$ is not a Cauchy sequence. Moreover, note that the function $\vartheta_{\mathcal{R}}$ is monotonic, which means that its set of discontinuities is countable. As a result, $\varepsilon > 0$ exists such that this value does not correspond to a discontinuity of $\vartheta_{\mathcal{R}}$, and there are strictly increasing sequences $(n_i), (m_i) \subseteq \mathbb{N}$, where $m_i > n_i$, for any $i \in \mathbb{N}$ and

$$d_s\left(\varkappa_{n_i},\varkappa_{m_i}\right) \geq \varepsilon \text{ and } d_s\left(\varkappa_{n_i},\varkappa_{m_i-1}\right) < \varepsilon, \tag{2.8}$$

where n_i is the minimal value that emphasizes the careful selection of indices in the sequences to ensure that the subsequences can be effectively analyzed in such a way that

$$n_i = \min\left\{i \leq j \mid d_s\left(\varkappa_j,\varkappa_m\right) \geq \varepsilon \wedge m > j\right\},$$

and

$$m_i = \min\left\{n_i < n_i \mid d_s\left(\varkappa_{n_i},\varkappa_j\right) \geq \varepsilon\right\}.$$

Furthermore,

$$\begin{aligned}\varepsilon \leq d_s\left(\varkappa_{n_i},\varkappa_{m_i}\right) &\leq d_s\left(\varkappa_{n_i},\varkappa_{m_i-1}\right) + d_s\left(\varkappa_{m_i-1},\varkappa_{m_i}\right) + \vee d_s\left(\varkappa_{n_i},\varkappa_{m_i-1}\right) d_s\left(\varkappa_{m_i-1},\varkappa_{m_i}\right) \\ &\leq \varepsilon + d_s\left(\varkappa_{m_i-1},\varkappa_{m_i}\right) + \vee\varepsilon d_s\left(\varkappa_{m_i-1},\varkappa_{m_i}\right).\end{aligned}$$

Taking the limit $i \to \infty$, the terms $d_s\left(\varkappa_{n_i},\varkappa_{n_i-1}\right)$ and $d_s\left(\varkappa_{m_i-1},\varkappa_{m_i}\right)$ approach 0 above, we get

$$\lim_{i\to\infty} d_s\left(\varkappa_{n_i},\varkappa_{m_i}\right) = \varepsilon. \tag{2.9}$$

In addition, from (d_{s3}), we have

$$\begin{aligned}d_s\left(\varkappa_{n_i},\varkappa_{m_i}\right) &\leq d_s\left(\varkappa_{n_i},\varkappa_{n_i-1}\right) + d_s\left(\varkappa_{n_i-1},\varkappa_{m_i}\right) + \vee d_s\left(\varkappa_{n_i},\varkappa_{n_i-1}\right) d_s\left(\varkappa_{n_i-1},\varkappa_{m_i}\right) \\ &\leq d_s\left(\varkappa_{n_i},\varkappa_{n_i-1}\right) + d_s\left(\varkappa_{n_i-1},\varkappa_{m_i-1}\right) + d_s\left(\varkappa_{m_i-1},\varkappa_{m_i}\right) + \vee d_s\left(\varkappa_{n_i-1},\varkappa_{m_i-1}\right) d_s\left(\varkappa_{m_i-1},\varkappa_{m_i}\right)\end{aligned}$$

$$+\curlyvee\, d_s(\varkappa_{n_i},\varkappa_{n_i-1})\, d_s(\varkappa_{n_i-1},\varkappa_{m_i})$$

$$\leq d_s(\varkappa_{n_i},\varkappa_{n_i-1}) + d_s(\varkappa_{n_i-1},\varkappa_{m_i-1}) + d_s(\varkappa_{m_i-1},\varkappa_{m_i}) + \curlyvee d_s(\varkappa_{n_i-1},\varkappa_{m_i-1})\, d_s(\varkappa_{m_i-1},\varkappa_{m_i})$$

$$+\curlyvee\, d_s(\varkappa_{n_i},\varkappa_{n_i-1})\left(d_s(\varkappa_{n_i-1},\varkappa_{n_i}) + d_s(\varkappa_{n_i},\varkappa_{m_i}) + \curlyvee d_s(\varkappa_{n_i-1},\varkappa_{n_i})\, d_s(\varkappa_{n_i},\varkappa_{m_i})\right).$$

Equivalently

$$\begin{pmatrix} \left(1-\curlyvee d_s(\varkappa_{n_i},\varkappa_{n_i-1}) - \curlyvee^2 d_s(\varkappa_{n_i},\varkappa_{n_i-1})\right) d_s(\varkappa_{n_i},\varkappa_{m_i}) \\ -d_s(\varkappa_{n_i},\varkappa_{n_i-1}) - d_s(\varkappa_{m_i-1},\varkappa_{m_i}) - \curlyvee d_s(\varkappa_{n_i-1},\varkappa_{m_i})^2 \end{pmatrix} \left(1+\curlyvee d_s(\varkappa_{m_i-1},\varkappa_{m_i})\right)^{-1} \leq d_s(\varkappa_{n_i-1},\varkappa_{m_i-1})$$

and

$$\begin{aligned} d_s(\varkappa_{n_i-1},\varkappa_{m_i-1}) &\leq d_s(\varkappa_{n_i-1},\varkappa_{n_i}) + d_s(\varkappa_{n_i},\varkappa_{m_i-1}) + \curlyvee d_s(\varkappa_{n_i-1},\varkappa_{n_i})\, d_s(\varkappa_{n_i},\varkappa_{m_i-1}) \\ &\leq d_s(\varkappa_{n_i-1},\varkappa_{n_i}) + d_s(\varkappa_{n_i},\varkappa_{m_i}) + d_s(\varkappa_{m_i},\varkappa_{m_i-1}) + \curlyvee d_s(\varkappa_{n_i},\varkappa_{m_i})\, d_s(\varkappa_{m_i},\varkappa_{m_i-1}) \\ &\quad + \curlyvee\, d_s(\varkappa_{n_i-1},\varkappa_{n_i})\, d_s(\varkappa_{n_i},\varkappa_{m_i-1}) \\ &\leq d_s(\varkappa_{n_i-1},\varkappa_{n_i}) + d_s(\varkappa_{n_i},\varkappa_{m_i}) + d_s(\varkappa_{m_i},\varkappa_{m_i-1}) + \curlyvee d_s(\varkappa_{n_i},\varkappa_{m_i})\, d_s(\varkappa_{m_i},\varkappa_{m_i-1}) \\ &\quad + \curlyvee\, d_s(\varkappa_{n_i-1},\varkappa_{n_i})\left(d_s(\varkappa_{n_i},\varkappa_{n_i-1}) + d_s(\varkappa_{n_i-1},\varkappa_{m_i-1}) + \curlyvee d_s(\varkappa_{n_i},\varkappa_{n_i-1})\, d_s(\varkappa_{n_i-1},\varkappa_{m_i-1})\right). \end{aligned}$$

Equivalently

$$\begin{aligned} &\left(1-\curlyvee d_s(\varkappa_{n_i},\varkappa_{n_i-1}) - \curlyvee^2 d_s(\varkappa_{n_i},\varkappa_{n_i-1})\right) d_s(\varkappa_{n_i-1},\varkappa_{m_i-1}) \\ &\leq d_s(\varkappa_{n_i-1},\varkappa_{n_i}) + d_s(\varkappa_{m_i},\varkappa_{m_i-1}) + \curlyvee d_s(\varkappa_{n_i-1},\varkappa_{n_i}) + \left(1+d_s(\varkappa_{m_i},\varkappa_{m_i-1})\right) d_s(\varkappa_{n_i},\varkappa_{m_i}). \end{aligned}$$

Therefore, letting $i \to \infty$ in the inequalities above and from (2.7) and (2.9), we conclude that

$$\lim_{i\to\infty} d_s(\varkappa_{n_i-1},\varkappa_{m_i-1}) = \varepsilon. \tag{2.10}$$

Likewise, we can see that

$$\lim_{i\to\infty} d_s(\varkappa_{n_i-1},\varkappa_{m_i}) = \lim_{i\to\infty} d_s(\varkappa_{m_i-1},\varkappa_{n_i}) = \varepsilon. \tag{2.11}$$

Putting $\varkappa = \varkappa_{n_i}$ and $y = \varkappa_{m_i}$ in (2.6), we get

$$\vartheta_{\mathcal{R}}(\varepsilon) \leq \vartheta_{\mathcal{R}}(d_s(\varkappa_{n_i},\varkappa_{m_i})) \leq (\vartheta_{\mathcal{R}}(\mathcal{M}_1(\varkappa_{n_i-1},\varkappa_{m_i-1})))^k,$$

where

$$\mathcal{M}_1(\varkappa_{n_i-1},\varkappa_{m_i-1}) = \max\left\{ \begin{array}{l} d_s(\varkappa_{n_i-1},\varkappa_{m_i-1}),\ \frac{d_s(\varkappa_{n_i-1},\varkappa_{n_i})+d_s(\varkappa_{m_i-1},\varkappa_{m_i})}{2}, \\ \frac{d_s(\varkappa_{n_i-1},\varkappa_{m_i})+d_s(\varkappa_{m_i-1},\varkappa_{n_i})}{2} \end{array} \right\}.$$

Hence, taking (2.10) and (2.11) into account, we find that $\lim_{i\to\infty}\mathcal{M}_1\left(\varkappa_{n_i-1},\varkappa_{m_i-1}\right)=\varepsilon$, starting from a particular point $i_0\in\mathbb{N}$. Consequently, we have

$$\begin{aligned}\vartheta_{\mathcal{R}}(\varepsilon) &\leq \lim_{i\to\infty}\vartheta_{\mathcal{R}}\left(d_s\left(\varkappa_{n_i},\varkappa_{m_i}\right)\right)\\ &\leq \lim_{i\to\infty}\left(\vartheta_{\mathcal{R}}\left(\mathcal{M}_1\left(\varkappa_{n_i-1},\varkappa_{m_i-1}\right)\right)\right)^k\end{aligned}$$

as $i\to\infty$, we get $\vartheta_{\mathcal{R}}(\varepsilon)\leq(\vartheta_{\mathcal{R}}(\varepsilon))^k$, which is a contradiction. So $(\varkappa_n)$ is a Cauchy sequence. As a result, there is some $\varkappa^*\in\complement$ in such a way that $\lim_{n\to\infty}\varkappa_n=\varkappa^*$ and

$$\vartheta_{\mathcal{R}}(d_s\left(\eth\varkappa^*,\varkappa_{n+1}\right))\leq\vartheta_{\mathcal{R}}(\mathcal{M}_1\left(\varkappa^*,\varkappa_n\right))^k,$$

where

$$\mathcal{M}_1\left(\varkappa^*,\varkappa_n\right)=\max\left\{d_s\left(\varkappa^*,\varkappa_n\right),\frac{d_s\left(\varkappa^*,\eth\varkappa^*\right)+d_s\left(\varkappa_n,\varkappa_{n+1}\right)}{2},\frac{d_s\left(\varkappa^*,\eth\varkappa_n\right)+d_s\left(\varkappa_n,\eth\varkappa^*\right)}{2}\right\}.$$

Letting $n\to\infty$ leads to the conclusion $\mathcal{M}_1\left(\varkappa^*,\varkappa_n\right)=d_s\left(\varkappa^*,\varkappa_n\right)$ starting from some $n_0\in\mathbb{N}$. As $\lim_{n\to\infty}d_s\left(\eth\varkappa^*,\varkappa_{n+1}\right)=d_s\left(\eth\varkappa^*,\varkappa^*\right)$, from the estimation of

$$\vartheta_{\mathcal{R}}\left(d_s\left(\eth\varkappa^*,\varkappa_{n+1}\right)\right)\leq\left(\vartheta_{\mathcal{R}}\left(d_s\left(\varkappa^*,\eth\varkappa^*\right)\right)\right)^k,$$

we obtain the contradiction, since it must be

$$\lim_{n\to\infty}\vartheta_{\mathcal{R}}\left(d_s\left(\eth\varkappa^*,\varkappa_n\right)\right)>\left(\vartheta_{\mathcal{R}}\left(d_s\left(\varkappa^*,\eth\varkappa^*\right)\right)\right)^k,$$

showing that $\varkappa^*$ is a fixed-point of the mapping $\eth$. If $\eth\varsigma=\varsigma$ and $\varsigma\neq\varkappa^*$, then

$$\begin{aligned}\lim_{n\to\infty}\vartheta_{\mathcal{R}}\left(d_s\left(\varkappa^*,\varsigma\right)\right) &= \left(\vartheta_{\mathcal{R}}\left(d_s\left(\eth\varkappa^*,\eth\varsigma\right)\right)\right)\\ &\leq \left(\vartheta_{\mathcal{R}}\left(\mathcal{M}_1\left(\varkappa^*,\varsigma\right)\right)\right)^k,\end{aligned}$$

where $\mathcal{M}_1\left(\varkappa^*,\varsigma\right)=d_s\left(\varkappa^*,\varsigma\right)$ and $\varkappa^*$ is a unique fixed-point of the mapping $\eth$. □

Example 2.9. Given the complete suprametric space $(\complement,d_s)$, where $\complement=[0,\infty)$ and

$$d_s(\varkappa,y)=\left|\varkappa-y\right|\left(\left|\varkappa-y\right|+\curlyvee\right),$$

with $\curlyvee=2$. Define a mapping such that $\eth\varkappa=\frac{\varkappa+1}{10}$. From here, we get

$$d_s(\eth\varkappa,\eth y)=\frac{1}{100}\left(|\varkappa-y|^2+20|\varkappa-y|\right),\qquad d_s(\varkappa,y)=|\varkappa-y|^2+2|\varkappa-y|.$$

$$\frac{d_s(\varkappa,\eth\varkappa)+d_s(y,\eth y)}{2}=\frac{1}{200}\Big(|9\varkappa-1|(|9\varkappa-1|+20)+|9y-1|(|9y-1|+20)\Big).$$

$$\frac{d_s(\varkappa,\eth y)+d_s(y,\eth\varkappa)}{2}=\frac{1}{200}\Big(|10\varkappa-y-1|(|10\varkappa-y-1|+20)+|10y-\varkappa-1|(|10y-\varkappa-1|+20)\Big).$$

Now we use the contraction discussed in Eq (2.6) and utilizing the $\vartheta_{\mathcal{R}}$ mentioned in Example 2.2 as follows

$$\vartheta_{\mathcal{R}}(t) = \begin{cases} 1 + t^3, & \text{if } t \in (0, 1), \\ 2 + e^t, & \text{if } t \in [1, \infty), \end{cases}$$

which only satisfies ϑ_1 but not ϑ_2 and ϑ_3. We now have to choose the maximum for both cases

$$\begin{aligned} \mathcal{M}_1(\varkappa, y) &= \max\Big\{ d_s(\varkappa, y), \ \frac{d_s(\varkappa, \partial\varkappa) + d_s(y, \partial y)}{2}, \ \frac{d_s(\varkappa, \partial y) + d_s(y, \partial\varkappa)}{2} \Big\} \\ &= \max\Big\{|\varkappa - y|^2 + 2|\varkappa - y|, \ \frac{1}{200}\Big(|9\varkappa - 1|(|9\varkappa - 1| + 20) + |9y - 1|(|9y - 1| + 20)\Big), \\ &\quad \frac{1}{200}\Big(|10\varkappa - y - 1|(|10\varkappa - y - 1| + 20) + |10y - \varkappa - 1|(|10y - \varkappa - 1| + 20)\Big)\Big\}. \end{aligned}$$

From the inequality above, we see that $\mathcal{M}_1(\varkappa, y) = |\varkappa - y|^2 + 2|\varkappa - y|$. This expression will be used in the following two cases.
Case (i): For $t \in (0, 1)$ and $\partial\varkappa \neq \partial y$, we calculate

$$1 + \left[\frac{1}{100}\left(|\varkappa - y|^2 + 20|\varkappa - y|\right)\right]^3 \leq \left[1 + \left(|\varkappa - y|^2 + 2|\varkappa - y|\right)^3\right]^k.$$

Figure 7 demonstrates that the inequality above is valid for all $\varkappa, y \in (0, 1)$.

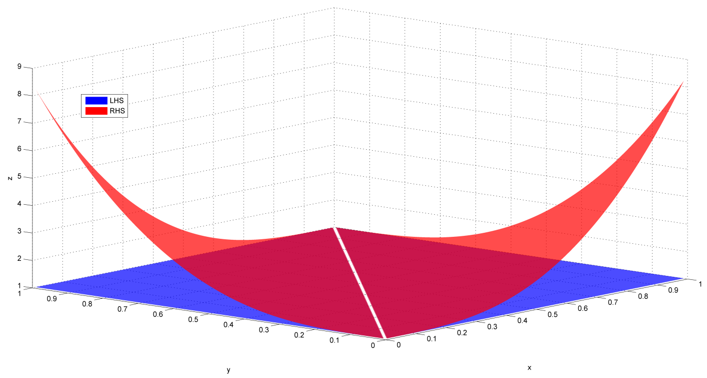

Figure 7. Case (i) of Example 2.9.

Case (ii): For $t \in [1, \infty)$ and $\partial\varkappa \neq \partial y$, we obtain

$$2 + \exp^{\left[\frac{1}{100}\left(|\varkappa - y|^2 + 20|\varkappa - y|\right)\right]} \leq \left[2 + \exp^{|\varkappa - y|^2 + 2|\varkappa - y|}\right]^k.$$

From Figure 8, we see that the inequality above is satisfied for all $\varkappa, y \in [1, \infty)$.

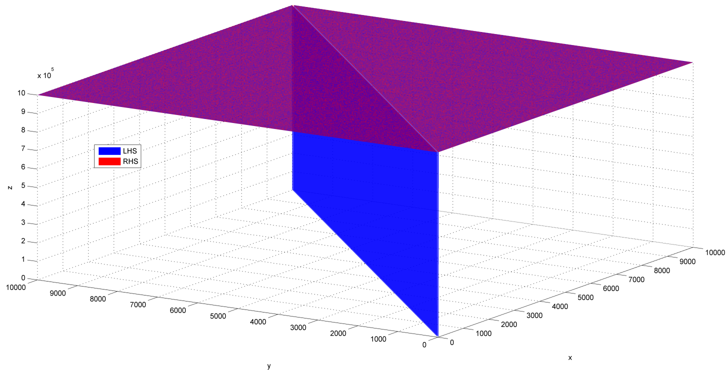

Figure 8. Case (ii) of Example 2.9.

For a discontinuous $\vartheta_{\mathcal{R}}$, we require the following left-jump condition at each point of the discontinuity t

$$\vartheta_{\mathcal{R}}(t) - \lim_{y \to t-} \vartheta_{\mathcal{R}}(y) > (\vartheta_{\mathcal{R}}(t))^k.$$

Since $\vartheta_{\mathcal{R}}$ exhibits a discontinuity at $t = 1$, we obtain the following result:

$$\begin{aligned} \vartheta_{\mathcal{R}}(1) - \lim_{y \to t-} \vartheta_{\mathcal{R}}(1) &> (\vartheta_{\mathcal{R}}(1))^k, \\ (2 + 2.718) - 2 &> (4.718)^k, \\ 2.718 &> (4.718)^k. \end{aligned}$$

Given $k = 0.64$, all the necessary conditions for Theorem 2.8 are fulfilled, and $\frac{1}{9}$ is the unique fixed-point of the mapping $\eth$.

Theorem 2.10. *Let* $\eth : \mathbb{C} \to \mathbb{C}$ *be a mapping on a complete suprametric space* $(\mathbb{C}, d_s)$*, and let* $\vartheta_{\mathcal{R}}$ *be a nondecreasing function with a constant* $k \in (0, 1)$ *such that*

$$\eth\varkappa \neq \eth y \implies \vartheta_{\mathcal{R}}(d_s(\eth\varkappa, \eth y)) \leq (\vartheta_{\mathcal{R}}(\mathcal{M}_2(\varkappa, y)))^k, \tag{2.12}$$

where

$$\mathcal{M}_2(\varkappa, y) = \max\left\{ d_s(\varkappa, y),\ d_s(\varkappa, \eth\varkappa),\ d_s(y, \eth y),\ d_s(\varkappa, \eth y),\ d_s(y, \eth\varkappa) \right\}.$$

For each discontinuity $t > 0$ *of the function* $\vartheta_{\mathcal{R}}$*, the left-hand jump satisfies*

$$\vartheta_{\mathcal{R}}(t) - \lim_{s \to t^-} \vartheta_{\mathcal{R}}(s) > (\vartheta_{\mathcal{R}}(t))^k.$$

Then $\eth$ *admits a unique fixed-point in* $\mathbb{C}$*.*

Proof. Construct a sequence $(\varkappa_n) \subseteq \mathbb{C}$ by picking an arbitrary $\varkappa_0 \in \mathbb{C}$ in such a way that $\varkappa_n = \eth\varkappa_{n-1}$ holds for all $n \in \mathbb{N}$. In the event that $\varkappa_n = \varkappa_{n-1}$ for some $n \in \mathbb{N}$, then $\varkappa_{n-1}$ is a fixed-point of $\eth$. Alternatively, suppose that for all $n \in \mathbb{N}$, the expression $\varkappa_n \neq \varkappa_{n-1}$ holds, so we will evaluate the components of the Cauchy sequence $(\varkappa_n)$ by utilizing (2.12), in such a way that

$$\vartheta_{\mathcal{R}}(d_s(\varkappa_n, \varkappa_{n+1})) = \vartheta_{\mathcal{R}}(d_s(\eth\varkappa_{n-1}, \eth\varkappa_n)) \leq (\vartheta_{\mathcal{R}}(\mathcal{M}_2(\varkappa_{n-1}, \varkappa_n)))^k,$$

where

$$\mathcal{M}_2(\varkappa_{n-1},\varkappa_n) = \max\{d_s(\varkappa_{n-1},\varkappa_n), d_s(\varkappa_{n-1},\eth\varkappa_{n-1}), d_s(\varkappa_n,\eth\varkappa_n), d_s(\varkappa_{n-1},\eth\varkappa_n), d_s(\varkappa_n,\eth\varkappa_{n-1})\}$$

$$= \max\{d_s(\varkappa_{n-1},\varkappa_n), d_s(\varkappa_n,\varkappa_{n+1}), d_s(\varkappa_{n-1},\varkappa_{n+1})\}.$$

If $\max\{d_s(\varkappa_{n-1},\varkappa_n), d_s(\varkappa_n,\varkappa_{n+1}), d_s(\varkappa_{n-1},\varkappa_{n+1})\} \neq d_s(\varkappa_{n-1},\varkappa_n)$ for some $n \in \mathbb{N}$, this leads to a contradiction. We therefore conclude that

$$\vartheta_{\mathcal{R}}(d_s(\varkappa_n,\varkappa_{n+1})) \leq (\vartheta_{\mathcal{R}}(d_s(\varkappa_{n-1},\varkappa_n)))^k.$$

Proceeding with a similar process for each $n \in \mathbb{N}$, we gain

$$\vartheta_{\mathcal{R}}(d_s(\varkappa_n,\varkappa_{n+1})) \leq (\vartheta_{\mathcal{R}}(d_s(\varkappa_0,\varkappa_1)))^{k^n}.$$

Taking limit $n \to \infty$, we get

$$1 \leq \lim_{n\to\infty} \vartheta_{\mathcal{R}}(d_s(\varkappa_n,\varkappa_{n+1})) \leq \lim_{n\to\infty} (\vartheta_{\mathcal{R}}(d_s(\varkappa_0,\varkappa_1)))^{k^n} = 1.$$

Moreover,

$$\begin{aligned}\vartheta_{\mathcal{R}}(d_s(\varkappa_n,\varkappa_{n+1})) &\leq (\vartheta_{\mathcal{R}}(d_s(\varkappa_{n-1},\varkappa_n)))^k \\ &< (\vartheta_{\mathcal{R}}(d_s(\varkappa_{n-1},\varkappa_n))).\end{aligned}$$

Thus, for any $n \in \mathbb{N}$, we arrive at

$$d_s(\varkappa_n,\varkappa_{n+1}) < d_s(\varkappa_{n-1},\varkappa_n),$$

and $(d_s(\varkappa_{n-1},\varkappa_n))$ is a monotone decreasing sequence. This amounts to saying that its limit exists and $a = \inf_{n\in\mathbb{N}} d_s(\varkappa_{n-1},\varkappa_n) = \lim_{n\to\infty} d_s(\varkappa_{n-1},\varkappa_n)$.

Taking $a > 0$, we obtain

$$\vartheta_{\mathcal{R}}(a) \leq \lim_{n\to\infty} \vartheta_{\mathcal{R}}(d_s(\varkappa_n,\varkappa_{n+1})) = 1,$$

which means that

$$\lim_{n\to\infty} d_s(\varkappa_n,\varkappa_{n+1}) = 0. \tag{2.13}$$

On the contrary, assume that $(\varkappa_n)$ is not a Cauchy sequence. Moreover, note that the function $\vartheta_{\mathcal{R}}$ is monotonic, meaning that its set of discontinuities is countable. As a result, $\varepsilon > 0$ exists such that this value does not correspond to a discontinuity of $\vartheta_{\mathcal{R}}$, and there are strictly increasing sequences $(n_i), (m_i) \subseteq \mathbb{N}$, where $m_i > n_i$, for any $i \in \mathbb{N}$ and

$$d_s(\varkappa_{n_i},\varkappa_{m_i}) \geq \varepsilon \text{ and } d_s(\varkappa_{n_i},\varkappa_{m_i-1}) < \varepsilon, \tag{2.14}$$

where n_i is the minimal value that emphasizes the careful selection of indices in the sequences to ensure that the subsequences can be effectively analyzed in such a way that

$$n_i = \min\left\{i \leq j \mid d_s\left(\varkappa_j,\varkappa_m\right) \geq \varepsilon \wedge m > j\right\},$$

and

$$m_i = \min\left\{n_i < n_i \mid d_s\left(\varkappa_{n_i}, \varkappa_j\right) \geq \varepsilon\right\}.$$

Furthermore,

$$\begin{aligned} \varepsilon \leq d_s\left(\varkappa_{n_i}, \varkappa_{m_i}\right) &\leq d_s\left(\varkappa_{n_i}, \varkappa_{m_i-1}\right) + d_s\left(\varkappa_{m_i-1}, \varkappa_{m_i}\right) + \vee d_s\left(\varkappa_{n_i}, \varkappa_{m_i-1}\right) d_s\left(\varkappa_{m_i-1}, \varkappa_{m_i}\right) \\ &\leq \varepsilon + d_s\left(\varkappa_{m_i-1}, \varkappa_{m_i}\right) + \vee\varepsilon d_s\left(\varkappa_{m_i-1}, \varkappa_{m_i}\right), \end{aligned}$$

by letting the limit $i \to \infty$, the terms $d_s\left(\varkappa_{n_i}, \varkappa_{n_i-1}\right)$ and $d_s\left(\varkappa_{m_i-1}, \varkappa_{m_i}\right)$ approach 0 in the above, and we get

$$\lim_{i\to\infty} d_s\left(\varkappa_{n_i}, \varkappa_{m_i}\right) = \varepsilon. \tag{2.15}$$

In addition, from (d_{s3}), we have

$$\begin{aligned} d_s\left(\varkappa_{n_i}, \varkappa_{m_i}\right) &\leq d_s\left(\varkappa_{n_i}, \varkappa_{n_i-1}\right) + d_s\left(\varkappa_{n_i-1}, \varkappa_{m_i}\right) + \vee d_s\left(\varkappa_{n_i}, \varkappa_{n_i-1}\right) d_s\left(\varkappa_{n_i-1}, \varkappa_{m_i}\right) \\ &\leq d_s\left(\varkappa_{n_i}, \varkappa_{n_i-1}\right) + d_s\left(\varkappa_{n_i-1}, \varkappa_{m_i-1}\right) + d_s\left(\varkappa_{m_i-1}, \varkappa_{m_i}\right) + \vee d_s\left(\varkappa_{n_i-1}, \varkappa_{m_i-1}\right) d_s\left(\varkappa_{m_i-1}, \varkappa_{m_i}\right) \\ &\quad + \vee\, d_s\left(\varkappa_{n_i}, \varkappa_{n_i-1}\right) d_s\left(\varkappa_{n_i-1}, \varkappa_{m_i}\right) \\ &\leq d_s\left(\varkappa_{n_i}, \varkappa_{n_i-1}\right) + d_s\left(\varkappa_{n_i-1}, \varkappa_{m_i-1}\right) + d_s\left(\varkappa_{m_i-1}, \varkappa_{m_i}\right) + \vee d_s\left(\varkappa_{n_i-1}, \varkappa_{m_i-1}\right) d_s\left(\varkappa_{m_i-1}, \varkappa_{m_i}\right) \\ &\quad + \vee\, d_s\left(\varkappa_{n_i}, \varkappa_{n_i-1}\right)\left(d_s\left(\varkappa_{n_i-1}, \varkappa_{n_i}\right) + d_s\left(\varkappa_{n_i}, \varkappa_{m_i}\right) + \vee d_s\left(\varkappa_{n_i-1}, \varkappa_{n_i}\right) d_s\left(\varkappa_{n_i}, \varkappa_{m_i}\right)\right). \end{aligned}$$

Equivalently

$$\left(\begin{array}{l} \left(1 - \vee d_s\left(\varkappa_{n_i}, \varkappa_{n_i-1}\right) - \vee^2 d_s\left(\varkappa_{n_i}, \varkappa_{n_i-1}\right)\right) d_s\left(\varkappa_{n_i}, \varkappa_{m_i}\right) \\ -d_s\left(\varkappa_{n_i}, \varkappa_{n_i-1}\right) - d_s\left(\varkappa_{m_i-1}, \varkappa_{m_i}\right) - \vee d_s(\varkappa_{n_i-1}, \varkappa_{m_i})^2 \end{array}\right)\left(1 + \vee d_s\left(\varkappa_{m_i-1}, \varkappa_{m_i}\right)\right)^{-1} \leq d_s\left(\varkappa_{n_i-1}, \varkappa_{m_i-1}\right)$$

and

$$\begin{aligned} d_s\left(\varkappa_{n_i-1}, \varkappa_{m_i-1}\right) &\leq d_s\left(\varkappa_{n_i-1}, \varkappa_{n_i}\right) + d_s\left(\varkappa_{n_i}, \varkappa_{m_i-1}\right) + \vee d_s\left(\varkappa_{n_i-1}, \varkappa_{n_i}\right) d_s\left(\varkappa_{n_i}, \varkappa_{m_i-1}\right) \\ &\leq d_s\left(\varkappa_{n_i-1}, \varkappa_{n_i}\right) + d_s\left(\varkappa_{n_i}, \varkappa_{m_i}\right) + d_s\left(\varkappa_{m_i}, \varkappa_{m_i-1}\right) + \vee d_s\left(\varkappa_{n_i}, \varkappa_{m_i}\right) d_s\left(\varkappa_{m_i}, \varkappa_{m_i-1}\right) \\ &\quad + \vee\, d_s\left(\varkappa_{n_i-1}, \varkappa_{n_i}\right) d_s\left(\varkappa_{n_i}, \varkappa_{m_i-1}\right) \\ &\leq d_s\left(\varkappa_{n_i-1}, \varkappa_{n_i}\right) + d_s\left(\varkappa_{n_i}, \varkappa_{m_i}\right) + d_s\left(\varkappa_{m_i}, \varkappa_{m_i-1}\right) + \vee d_s\left(\varkappa_{n_i}, \varkappa_{m_i}\right) d_s\left(\varkappa_{m_i}, \varkappa_{m_i-1}\right) \\ &\quad + \vee\, d_s\left(\varkappa_{n_i-1}, \varkappa_{n_i}\right)\left(d_s\left(\varkappa_{n_i}, \varkappa_{n_i-1}\right) + d_s\left(\varkappa_{n_i-1}, \varkappa_{m_i-1}\right) + \vee d_s\left(\varkappa_{n_i}, \varkappa_{n_i-1}\right) d_s\left(\varkappa_{n_i-1}, \varkappa_{m_i-1}\right)\right). \end{aligned}$$

Equivalently

$$\left(1 - \curlyvee d_s\left(\varkappa_{n_i}, \varkappa_{n_i-1}\right) - \curlyvee^2 d_s\left(\varkappa_{n_i}, \varkappa_{n_i-1}\right)\right) d_s\left(\varkappa_{n_i-1}, \varkappa_{m_i-1}\right)$$

$$\leq d_s\left(\varkappa_{n_i-1}, \varkappa_{n_i}\right) + d_s\left(\varkappa_{m_i}, \varkappa_{m_i-1}\right) + \curlyvee d_s\left(\varkappa_{n_i-1}, \varkappa_{n_i}\right) + \left(1 + d_s\left(\varkappa_{m_i}, \varkappa_{m_i-1}\right)\right) d_s\left(\varkappa_{n_i}, \varkappa_{m_i}\right).$$

Therefore, letting $i \to \infty$ in the inequalities above, from (2.13) and (2.15), we conclude that

$$\lim_{i\to\infty} d_s\left(\varkappa_{n_i-1}, \varkappa_{m_i-1}\right) = \varepsilon. \tag{2.16}$$

Likewise, we can obtain

$$\lim_{i\to\infty} d_s\left(\varkappa_{n_i-1}, \varkappa_{m_i}\right) = \lim_{i\to\infty} d_s\left(\varkappa_{m_i-1}, \varkappa_{n_i}\right) = \varepsilon. \tag{2.17}$$

Putting $\varkappa = \varkappa_{n_i}$ and $y = \varkappa_{m_i}$ in (2.12), we get

$$\vartheta_{\mathcal{R}}(\varepsilon) \leq \vartheta_{\mathcal{R}}(d_s\left(\varkappa_{n_i}, \varkappa_{m_i}\right)) \leq (\vartheta_{\mathcal{R}}(\mathcal{M}_2\left(\varkappa_{n_i-1}, \varkappa_{m_i-1}\right)))^k,$$

where

$$\mathcal{M}_2\left(\varkappa_{n_i-1}, \varkappa_{m_i-1}\right) = \max\left\{d_s\left(\varkappa_{n_i-1}, \varkappa_{m_i-1}\right), d_s\left(\varkappa_{n_i-1}, \varkappa_{n_i}\right) d_s\left(\varkappa_{m_i-1}, \varkappa_{m_i}\right), d_s\left(\varkappa_{n_i-1}, \varkappa_{m_i}\right), d_s\left(\varkappa_{m_i-1}, \varkappa_{n_i}\right)\right\}.$$

Hence, taking (2.16) and (2.17) into account, we find that $\lim_{i\to\infty} \mathcal{M}_2\left(\varkappa_{n_i-1}, \varkappa_{m_i-1}\right) = \varepsilon$, starting from a particular point $i_0 \in \mathbb{N}$. Consequently,

$$\begin{aligned}\vartheta_{\mathcal{R}}(\varepsilon) &\leq \lim_{i\to\infty} \vartheta_{\mathcal{R}}\left(d_s\left(\varkappa_{n_i}, \varkappa_{m_i}\right)\right)\\ &\leq \lim_{i\to\infty} \left(\vartheta_{\mathcal{R}}\left(\mathcal{M}_2\left(\varkappa_{n_i-1}, \varkappa_{m_i-1}\right)\right)\right)^k,\end{aligned}$$

as $i \to \infty$, and we get $\vartheta_{\mathcal{R}}(\varepsilon) \leq (\vartheta_{\mathcal{R}}(\varepsilon))^k$, which is a contradiction. So $(\varkappa_n)$ is a Cauchy sequence. As a result, there is some $\varkappa^* \in \complement$ in such a way that $\lim_{n\to\infty} \varkappa_n = \varkappa^*$ and

$$\vartheta_{\mathcal{R}}(d_s\left(\eth\varkappa^*, \varkappa_{n+1}\right)) \leq \vartheta_{\mathcal{R}}(\mathcal{M}_2\left(\varkappa^*, \varkappa_n\right))^k,$$

where

$$\mathcal{M}_2\left(\varkappa^*, \varkappa_n\right) = \max\left\{d_s\left(\varkappa^*, \varkappa_n\right), d_s\left(\varkappa^*, \eth\varkappa^*\right), \mathrm{d}\left(\varkappa_n, \varkappa_{n+1}\right), d_s\left(\varkappa^*, \varkappa_{n+1}\right), d_s\left(\varkappa_n, \eth\varkappa^*\right)\right\}.$$

Letting $n \to \infty$ leads to the conclusion that $\mathcal{M}_2\left(\varkappa^*, \varkappa_n\right) = d_s\left(\varkappa^*, \varkappa_n\right)$ starting from some $n_0 \in \mathbb{N}$. As $\lim_{n\to\infty} d_s\left(\eth\varkappa^*, \varkappa_{n+1}\right) = d_s\left(\eth\varkappa^*, \varkappa^*\right)$, from the estimation of

$$\vartheta_{\mathcal{R}}\left(d_s\left(\eth\varkappa^*, \varkappa_{n+1}\right)\right) \leq \left(\vartheta_{\mathcal{R}}\left(d_s\left(\varkappa^*, \eth\varkappa^*\right)\right)\right)^k,$$

we obtain a contradiction, since it must be

$$\lim_{n\to\infty} \vartheta_{\mathcal{R}}\left(d_s\left(\eth\varkappa^*, \varkappa_n\right)\right) > \left(\vartheta_{\mathcal{R}}\left(d_s\left(\varkappa^*, \eth\varkappa^*\right)\right)\right)^k,$$

indicating that $\varkappa^*$ is a fixed-point of the mapping $\eth$. If $\eth\varsigma = \varsigma$ and $\varsigma \neq \varkappa^*$, then

$$\begin{aligned}\lim_{n\to\infty} \vartheta_{\mathcal{R}}\left(d_s\left(\varkappa^*, \varsigma\right)\right) &= \left(\vartheta_{\mathcal{R}}\left(d_s\left(\eth\varkappa^*, \eth\varsigma\right)\right)\right)\\ &\leq \left(\vartheta_{\mathcal{R}}\left(\mathcal{M}_2\left(\varkappa^*, \varsigma\right)\right)\right)^k,\end{aligned}$$

where $\mathcal{M}_2\left(\varkappa^*, \varsigma\right) = d_s\left(\varkappa^*, \varsigma\right)$ and $\varkappa^*$ is a unique fixed-point of the mapping $\eth$. □

Remark 1. If d_s is a metric d in particular, a suprametric with $\curlyvee = 0$ and $\vartheta_R = \vartheta \in \Theta$, then Theorem 2.6 reduces to the classical ϑ-contraction principle in complete metric spaces and, for instance, choosing $\vartheta(t) = e^t$ yields the Banach contraction form $d(\eth\varkappa, \eth y) \leq k\, d(\varkappa, y)$. Furthermore, in a metric space, let $N(\varkappa, y) = \max\{d(\varkappa, y), d(\varkappa, \eth\varkappa), d(y, \eth y)\}$ as in [13]. Since $N(\varkappa, y) \leq \mathcal{M}_2(\varkappa, y)$ and ϑ is nondecreasing, any mapping satisfying the contractive condition of [13] also satisfies our condition in Theorem 2.10. Hence Theorem 2.10 contains [13, Theorem 3.5] as a particular case.

Example 2.11. Given the complete suprametric space $(\complement, d_s)$, where $\complement = [0, \infty)$ and

$$d_s(\varkappa, y) = \left|\varkappa - y\right| \left(\left|\varkappa - y\right| + \curlyvee\right)$$

with $\curlyvee = 2$. Define a mapping such that

$$\eth\varkappa = \frac{2\varkappa + 1}{20}.$$

From here, we get

$$d_s(\eth\varkappa, \eth y) = \frac{1}{100}\left(|\varkappa - y|^2 + 20|\varkappa - y|\right), \qquad d_s(\varkappa, y) = |\varkappa - y|^2 + 2|\varkappa - y|,$$

and

$$d_s(\varkappa, \eth\varkappa) = \frac{1}{400}\Big(|18\varkappa - 1|(|18\varkappa - 1| + 40)\Big), \qquad d_s(y, \eth y) = \frac{1}{400}\Big(|18y - 1|(|18y - 1| + 40)\Big),$$

$$d_s(\varkappa, \eth y) = \frac{1}{400}\Big(|20\varkappa-2y-1|(|20\varkappa-2y-1|+40)\Big), \qquad d_s(y, \eth\varkappa) = \frac{1}{400}\Big(|20y-2\varkappa-1|(|20y-2\varkappa-1|+40)\Big).$$

Now by using the contraction discussed in Eq (2.12) and utilizing the $\vartheta_{\mathcal{R}}$ mentioned in Example 2.2 as follows:

$$\vartheta_{\mathcal{R}}(t) = \begin{cases} 1 + t^3, & \text{if } t \in (0, 1), \\ 2 + e^t, & \text{if } t \in [1, \infty), \end{cases}$$

which only satisfies ϑ_1 but not ϑ_2 and ϑ_3. We now have to choose the maximum for both cases

$$\begin{aligned} \mathcal{M}_2(\varkappa, y) &= \max\Big\{ d_s(\varkappa, y),\ d_s(\varkappa, \eth\varkappa),\ d_s(y, \eth y),\ d_s(\varkappa, \eth y),\ d_s(y, \eth\varkappa)\Big\} \\ &= \max\Big\{|\varkappa - y|^2 + 2|\varkappa - y|,\ \frac{1}{400}\Big(|18\varkappa - 1|(|18\varkappa - 1| + 40)\Big),\ \frac{1}{400}\Big(|18y - 1|(|18y - 1| + 40)\Big) \\ &\quad \frac{1}{400}\Big(|20\varkappa - 2y - 1|(|20\varkappa - 2y - 1| + 40)\Big),\ \frac{1}{400}\Big(|20y - 2\varkappa - 1|(|20y - 2\varkappa - 1| + 40)\Big)\Big\}. \end{aligned}$$

From the inequality above, we see that $\mathcal{M}_2(\varkappa, y) = |\varkappa - y|^2 + 2|\varkappa - y|$. This expression will be used in the following two cases.

Case (i): For $t \in (0, 1)$ and $\eth\varkappa \neq \eth y$, we calculate

$$1 + \left[\frac{1}{100}\left(|\varkappa - y|^2 + 20|\varkappa - y|\right)\right]^3 \leq \left[1 + \left(|\varkappa - y|^2 + 2|\varkappa - y|\right)^3\right]^k.$$

Figure 9 demonstrates that the inequality above is valid for all $\varkappa, y \in (0, 1)$.

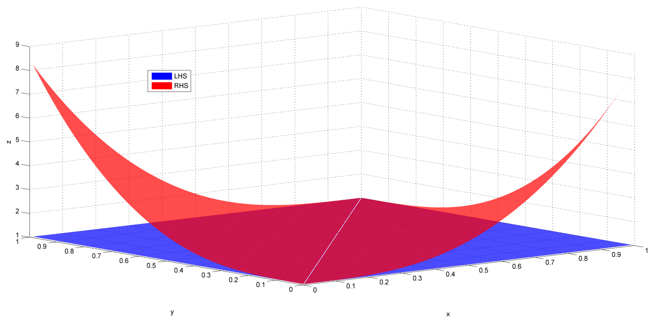

Figure 9. Case (i) of Example 2.11.

Case (ii): For $t \in [1, \infty)$ and $\eth\varkappa \neq \eth y$, we find that

$$2 + \exp^{\left[\frac{1}{100}\left(|\varkappa - y|^2 + 20|\varkappa - y|\right)\right]} \leq \left[2 + \exp^{|\varkappa - y|^2 + 2|\varkappa - y|}\right]^k.$$

From Figure 10, we see that the inequality above is satisfied for all $\varkappa, y \in [1, \infty)$.

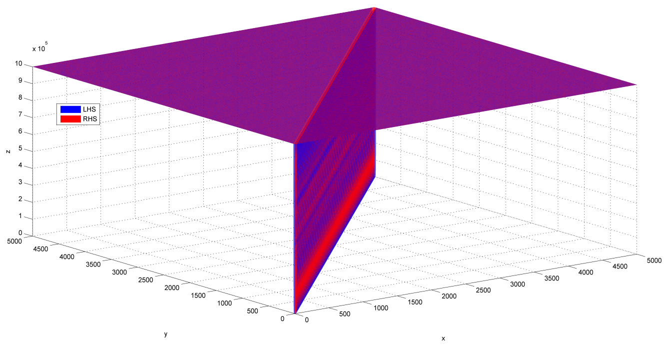

Figure 10. Case (ii) of Example 2.11.

For a discontinuous $\vartheta_{\mathcal{R}}$, we require the following left-jump condition at each point of the discontinuity t:

$$\vartheta_{\mathcal{R}}(t) - \lim_{y \to t-} \vartheta_{\mathcal{R}}(y) > (\vartheta_{\mathcal{R}}(t))^k.$$

Since $\vartheta_{\mathcal{R}}$ exhibits a discontinuity at $t = 1$, we obtain the following result:

$$\begin{aligned} \vartheta_{\mathcal{R}}(1) - \lim_{y \to t-} \vartheta_{\mathcal{R}}(1) &> (\vartheta_{\mathcal{R}}(1))^k, \\ (2 + 2.718) - 2 &> (4.718)^k, \\ 2.718 &> (4.718)^k. \end{aligned}$$

Given $k = 0.64$, all the necessary conditions for Theorem 2.10 are fulfilled and $\frac{1}{18}$ is the unique fixed-point of the mapping $\eth$.

3. Applications of the proposed fixed-point results

Let $\mathbb{C} = C[[0,1],\mathbb{R}]$ be a set of all real valued continuous functions and let $d_s : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$ be defined as

$$d_s(\varkappa, y) = \sup_{t\in[0,1]} \left|\varkappa(t) - y(t)\right|,$$

then for all $\varkappa, y \in \mathbb{C}$ and $t \in [0,1]$. Clearly, $(\mathbb{C}, d_s)$ is a complete suprametric space.

In what follows, we connect the abstract fixed-point framework developed in Section 2 to concrete differential models. For each problem, we first transform the differential equation into an equivalent integral equation by using the appropriate Green kernel, which then defines a self-mapping $\eth : \mathbb{C} \to \mathbb{C}$ on the complete suprametric space $(\mathbb{C}, d_s)$. Consequently, the existence and uniqueness of the solutions reduce to the existence of a fixed-point of $\eth$. In Subsection 3.1, we treat a second-order initial value problem related to the motion of a particle, and in Subsection 3.2, we address a fourth-order two-point beam boundary value problem; in both cases, we verify a relaxed $\vartheta_{\mathcal{R}}$-contractive condition with jump control and, where needed, a maximum-type functional so that Theorem 2.6 or Theorem 2.10 applies.

3.1. Initial value problem related to the motion of a particle

Problem: A particle of mass m is at rest at $\varkappa = 0$, $t = 0$. A force f starts acting on it in the $\varkappa$ direction such that its velocity jumps from 0 to 1 immediately after $t = 0$. Find the position of the particle at time t.

Now, we will assume an equation of motion that is a second-order differential equation as follows:

$$\begin{cases} m\frac{d^2\varkappa}{dt^2} = f(t, \varkappa(t)), \\ \varkappa(0) = 0, \ \varkappa'(0) = 1, \end{cases} \tag{3.1}$$

for all $t \in [0,1]$ and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

The initial value problem (3.1) is a general mathematical model that describes the motion of a particle of mass m under the influence of an external time and position-dependent force $f(t,\varkappa)$. Indeed, Eq (3.1) follows directly from Newton's second law, $m\varkappa'' = F$, with $F = f(t,\varkappa)$ being a general nonlinear force field. Since the initial conditions provide the release of the particle from the origin with the unit of initial velocity, the trajectory is uniquely determined for $t > 0$. Since the right-hand side depends only on t and $\varkappa$, the system models both conservative and nonconservative forces that do not depend explicitly on velocity. Under standard assumptions, such as the continuity or local Lipschitz continuity of f, the existence and uniqueness of the solution to the initial value problem (3.1) follow directly from the classical Picard-Lindelöf theorem. This well-posedness corresponds to the physical fact that the future motion of a particle should be completely determined by its initial position and velocity. Depending on the structure of $f(t,\varkappa)$, the problem (3.1) covers a wide range of physical phenomena: Time-dependent forcing, nonlinear restoring forces, and parametric effects. For example, the choices $f(t,\varkappa) = -k\varkappa$ and $f(t,\varkappa) = -k\varkappa^3$ recover models for linear and nonlinear oscillators, respectively, while more general t-dependence may give rise to various phenomena, including external driving or parametric resonance. Basic studies of such initial value problems form a fundamental part of both classical mechanics and the modern qualitative theory of ordinary differential equations.

From (3.1), $\varkappa''(t) = \frac{1}{m} f(t, \varkappa(t))$. Integrating once from 0 to t and using $\varkappa'(0) = 1$, we get

$$\varkappa'(t) = 1 + \frac{1}{m} \int_0^t f(s, \varkappa(s))\, ds.$$

Integrating again and using $\varkappa(0) = 0$, then we have

$$\varkappa(t) = t + \frac{1}{m} \int_0^t (t - s)\, f(s, \varkappa(s))\, ds.$$

Thus, the Green function is

$$G(t, s) = \begin{cases} \dfrac{t-s}{m}, & 0 \le s \le t, \\ 0, & t < s \le 1, \end{cases}$$

and (3.1) is equivalent to

$$\varkappa(t) = t + \int_0^1 G(t, s)\, f(s, \varkappa(s))\, ds.$$

We now prove the existence of a solution of the second-order differential equation (3.1). It is worth noting that $\varkappa(t)$ is a solution of (3.1) if and only if $\varkappa \in \complement$ is a solution of the equivalent integral equation of (3.1) given by

$$\varkappa(t) = t + \int_0^1 G(t, s) f(s, \varkappa(s))\, ds, \quad t \in [0, 1]. \tag{3.2}$$

Theorem 3.1. *Let $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ be a function upholding the following condition for some constant $\lambda > 0$:*

$$\left| f(s, \varkappa(s)) - f(s, y(s)) \right| \le e^{-\lambda s} \left| \varkappa(s) - y(s) \right|, \qquad s \in [0, 1].$$

Then $\varkappa_0 \in \complement$ exists for all $t \in [0, 1]$, where $\eth : \complement \to \complement$, and the second-order differential equation (3.1) has a solution in $\complement$. Consequently, the second-order differential equation (3.1) *representing the equation of motion has a solution.*

Proof. Let $\varkappa_0 \in \complement$ be a solution of the integral equation

$$\varkappa(t) = t + \int_0^1 G(t, s) f(s, \varkappa(s))\, ds, \quad t \in [0, 1].$$

Since $\eth : \complement \to \complement$ is well defined, we have

$$\eth\varkappa(t) = t + \int_0^1 G(t, s) f(s, \varkappa(s))\, ds.$$

Now, consider

$$\begin{aligned} \left| \eth\varkappa(t) - \eth y(t) \right| &= \left| t + \int_0^1 G(t, s) f(s, \varkappa(s))\, ds - t - \int_0^1 G(t, s) f(s, y(s))\, ds \right| \\ &\le \int_0^1 |G(t, s)| \left| f(s, \varkappa(s)) - f(s, y(s)) \right| ds \end{aligned}$$

$$\leq \int_0^1 |G(t,s)|\, e^{-\lambda s} \left|\varkappa(s) - y(s)\right| ds$$
$$\leq \left(\sup_{s\in[0,1]} |\varkappa(s) - y(s)|\right) \int_0^1 |G(t,s)|\, e^{-\lambda s}\, ds.$$

Set

$$\alpha := \sup_{t\in[0,1]} \int_0^1 |G(t,s)|\, e^{-\lambda s}\, ds,$$

and assume that $\alpha \in (0,1)$. Therefore, we have

$$d_s(\partial\varkappa, \partial y) \leq \alpha\, d_s(\varkappa, y).$$

We now choose the admissible control function $\vartheta_{\mathcal{R}}(t) = e^t$ for $t > 0$. It then follows that

$$\vartheta_{\mathcal{R}}\big(d_s(\partial\varkappa, \partial y)\big) = e^{d_s(\partial\varkappa,\partial y)} \leq e^{\alpha\, d_s(\varkappa,y)} = \Big(\vartheta_{\mathcal{R}}(d_s(\varkappa, y))\Big)^{\alpha}.$$

If we take $k = \alpha \in (0,1)$, all the conditions of Theorem 2.6 are satisfied. Hence, ∂ has a fixed-point, and the integral equation (3.2) has a solution. □

3.1.1. Numerical illustration for the initial value problem (IVP) (3.1)

In order to complement the existence and uniqueness statement with a simple numerical profile, we consider a representative nonlinear force satisfying the hypothesis of the previous theorem. Initially, set $m = 1$ and choose

$$f(t,\varkappa) = e^{-t}\tanh(\varkappa), \qquad t \in [0,1],\ \varkappa \in \mathbb{R}.$$

Since $|\tanh a - \tanh b| \leq |a - b|$, we have

$$|f(t,\varkappa) - f(t,y)| = e^{-t}|\tanh\varkappa - \tanh y| \leq e^{-t}|\varkappa - y|,$$

which matches Condition (3.1) with $\lambda = 1$. We compute the solution of the IVP

$$\varkappa''(t) = e^{-t}\tanh(\varkappa(t)), \qquad \varkappa(0) = 0, \quad \varkappa'(0) = 1,$$

by a standard fourth-order Runge−Kutta method on $[0,1]$ with the step size $h = 10^{-4}$. Moreover, we also verify the convergence of the Picard-type iteration induced by the equivalent integral equation

$$\varkappa_{n+1}(t) = t + \int_0^t (t-s)\, e^{-s}\tanh(\varkappa_n(s))\, ds, \qquad \varkappa_0(t) = t,$$

where the integral is approximated by the composite trapezoidal rule on a uniform grid.

Table 2. Example of numerical values for the IVP (3.1) with $m = 1$ and $f(t,\varkappa) = e^{-t}\tanh(\varkappa)$.

t	$\varkappa(t)$	$\varkappa'(t)$
0.0	0.000000	1.000000
0.2	0.201205	1.017462
0.4	0.408694	1.060659
0.6	0.626397	1.117866
0.8	0.856168	1.180074
1.0	1.098335	1.241004

Table 3. Picard iteration errors $d_s(\varkappa_{n+1}, \varkappa_n) = \sup_{t\in[0,1]} |\varkappa_{n+1}(t) - \varkappa_n(t)|$ for the IVP simulation.

Iteration n	$d_s(\varkappa_{n+1}, \varkappa_n)$
0	9.618539×10^{-2}
1	2.126607×10^{-3}
2	2.304112×10^{-5}
3	1.486672×10^{-7}
4	6.317611×10^{-10}
5	1.897815×10^{-12}

These numerical results provide an explicit solution profile and illustrate the rapid convergence of the fixed-point iteration corresponding to the operator $\supset$ defined by (3.2).

3.2. Application to a fourth-order two-point boundary value problem

Fourth-order nonlinear boundary value problems (BVPs) are central to the qualitative theory of differential equations for a wide range of practical models. Issues like these are inherent in the analysis of deflection of elastic beams, in which the Euler-Bernoulli theory results in a fourth-order differential operator, which characterizes the bending of a beam when a load is applied to it. It is found that allowing the nonlinearity f to be a function of both the displacement and the slope $\varkappa$ gives a general framework that is able to be used to capture nonlinear material's behavior, shear-dependent reactions, as well as non-classical loadings commonly found in modern engineering applications, such as micro and nano-structural mechanics, post-buckling analysis, and beam-foundation interactions. Mathematically, fourth-order BVPs, which have nonlinear dependence on the derivatives of lower orders, for instance $\varkappa(t)$, have significant analytical problems. The presence of the term $\varkappa'(t)$, in contrast to variational problems where f is the gradient of a potential, generally precludes an underlying variational formulation and hence rules out the direct use of classical critical point theory. This encourages the creation of alternative methods, such as fixed-point theory in cones, Leray-Schauder degrees, and upper-lower solution methods. A large body of literature is dedicated to establishing conditions that ensure the existence, multiplicity, positivity, and qualitative properties of solutions to such fourth-order problems.

A fourth-order two-point boundary value problem describes the deformations of the elastic beam as per the controls at the ends of the beam. Due to the nature and characterization of the equilibrium state, these higher order beam equations have significant applications pertaining to mechanics and engineering related to bridges, ships, automobiles, buildings, and aircraft. Our aim is to illustrate sufficient conditions for the existence of the solution of the two point fourth-order boundary value problem portraying the deformations of an elastic beam, which play a significant role in the problems related to mechanics and engineering.

Consider the following two-point, fourth-order boundary value problem:

$$\begin{cases} \varkappa''''(t) = f(t, \varkappa(t), \varkappa'(t)), \\ \varkappa(0) = \varkappa'(0) = \varkappa''(1) = \varkappa'''(1) = 0, \end{cases} \tag{3.3}$$

where $f : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function. Integrating the differential equation in (3.3) four times and applying the boundary conditions successively, one obtains an equivalent integral representation of the solution. Consequently, Problem (3.3) admits a solution $\varkappa \in \complement = C[[0,1],\mathbb{R}]$ of the form

$$\varkappa(t) = \int_0^1 \dot{\mathsf{G}}(t,s)\, f(s,\varkappa(s),\varkappa'(s))\, ds, \quad t \in [0,1], \tag{3.4}$$

where the Green kernel is explicitly given by

$$\dot{\mathsf{G}}(t,s) = \frac{1}{6}\begin{cases} s^2(3t-s), & 0 \le s \le t, \\ t^2(3s-t), & t \le s \le 1. \end{cases} \tag{3.5}$$

The next theorem provides us a sufficient condition for the existence and uniqueness of the solution of Problem (3.3).

Theorem 3.2. *Suppose that the following condition holds: for each $\varkappa, y \in \complement$ and for all $s \in [0,1]$, we have*

$$\left| f(s,\varkappa(s),\varkappa'(s)) - f(s,y(s),y'(s)) \right| \le L \left| \varkappa(s) - y(s) \right|,$$

where $L > 0$ is a constant. Let

$$\beta := L \sup_{t\in[0,1]} \int_0^1 \left| \dot{\mathsf{G}}(t,s) \right| ds,$$

and assume that $\beta \in (0,1)$. Then the existence of a solution for the integral equation (3.4) *provides a solution to the problem describing the deformation of an elastic beam* (3.3).

Proof. Define an operator $\eth : \complement \to \complement$, given by

$$\eth\varkappa(t) = \int_0^1 \dot{\mathsf{G}}(t,s) f(s,\varkappa(s),\varkappa'(s))\, ds,$$

with $\eth$ being well-defined for all $\varkappa, y \in \complement$, and we have

$$\begin{aligned} \left|\eth\varkappa(t) - \eth y(t)\right| &= \left| \int_0^1 \dot{\mathsf{G}}(t,s) f(s,\varkappa(s),\varkappa'(s))\, ds - \int_0^1 \dot{\mathsf{G}}(t,s) f(s,y(s),y'(s))\, ds \right| \\ &\le \left| f(s,\varkappa(s),\varkappa'(s)) - f(s,y(s),y'(s)) \right| \left\{ \int_0^1 \left|\dot{\mathsf{G}}(t,s)\right| ds \right\} \\ &\le \left|\varkappa(t) - y(t)\right| \left\{ L \int_0^1 \left|\dot{\mathsf{G}}(t,s)\right| ds \right\}. \end{aligned}$$

Therefore, we arrive at

$$\left|\eth\varkappa(t) - \eth y(t)\right| \le \beta \left|\varkappa(t) - y(t)\right|.$$

We now choose the admissible control function $\vartheta_{\mathcal{R}}(t) = e^t$ for $t > 0$ and we obtain

$$\vartheta_{\mathcal{R}}(d_s(\eth\varkappa,\eth y)) = e^{d_s(\eth\varkappa,\eth y)} \le e^{\beta\, d_s(\varkappa,y)} = \left(\vartheta_{\mathcal{R}}(d_s(\varkappa,y))\right)^{\beta}.$$

Since $d_s(\varkappa,y) \le \mathcal{M}_2(\varkappa,y)$, we conclude that

$$\vartheta_{\mathcal{R}}(d_s(\eth\varkappa,\eth y)) \le (\vartheta_{\mathcal{R}}(\mathcal{M}_2(\varkappa,y)))^{\beta}.$$

In particular, for $k = \beta \in (0,1)$, we obtain the required relaxed $\vartheta_{\mathcal{R}}$-contraction where

$$\mathcal{M}_2(\varkappa(t),y(t)) = \max\left\{ d_s(\varkappa(t),y(t)),\ d_s(\varkappa(t),\eth\varkappa(t)),\ d_s(y(t),\eth y(t)),\ d_s(\varkappa(t),\eth y(t)),\ d_s(y(t),\eth\varkappa(t)) \right\}.$$

Hence, all the conditions of Theorem 2.10 are fulfilled, and we conclude that the fourth-order boundary value problem (3.4) describing the deformation of an elastic beam has a unique solution in $\complement$. □

3.2.1. Numerical illustration for the beam boundary value problem (BVP) (3.3)

We also provide a simple numerical demonstration for the fourth-order beam model. Consider a representative continuous load of the form

$$f(t, \varkappa, u) = t(1 - t) + 0.3\sin(\varkappa).$$

Note that f does not explicitly depend on u in this test case. Since $|\sin a - \sin b| \le |a - b|$, we obtain

$$|f(t, \varkappa, u) - f(t, y, v)| = 0.3|\sin \varkappa - \sin y| \le 0.3|\varkappa - y|,$$

so the Lipschitz-type condition in Theorem 3.2 holds with $L = 0.3$. We compute the numerical solution of the equivalent integral equation (3.4) by Picard iteration as follows:

$$\varkappa_{n+1}(t) = \int_0^1 \dot{\mathsf{G}}(t, s)\, f(s, \varkappa_n(s), \varkappa_n'(s))\, ds, \qquad \varkappa_0(t) \equiv 0,$$

where the integral is approximated by the composite trapezoidal rule on a uniform grid. Here, $N = 1200$ subintervals are used. Table 4 reports sampled deflection values, and Table 5 shows the decay of $d_s(\varkappa_{n+1}, \varkappa_n)$.

Table 4. Sample of numerical deflection values for the beam BVP (3.3) with $f(t, \varkappa, u) = t(1 - t) + 0.3\sin(\varkappa)$.

t	$\varkappa(t)$
0.0	0.000000
0.2	0.001479
0.4	0.005077
0.6	0.009748
0.8	0.014807
1.0	0.019941

Table 5. Picard iteration errors $d_s(\varkappa_{n+1}, \varkappa_n)$ for the BVP integral formulation (3.4).

Iteration n	$d_s(\varkappa_{n+1}, \varkappa_n)$
0	1.944443×10^{-2}
1	4.842737×10^{-4}
2	1.175840×10^{-5}
3	2.853142×10^{-7}
4	6.922952×10^{-9}
5	1.679806×10^{-10}

Therefore, beyond guaranteeing the existence and uniqueness, the computations above provide explicit approximate solution values and confirm the contraction-driven convergence behavior predicted by our theoretical framework.

4. Conclusions and future directions

In this work, we developed a relaxed fixed-point framework on suprametric spaces by introducing the new control class Θ^R and establishing existence and uniqueness theorems for nonlinear operators governed by max-dominated geometry. By connecting these analytical results with concrete differential models, specifically, a second-order IVP and a fourth-order BVP, we demonstrated that relaxed $\vartheta_{\mathcal{R}}$-contractions provide an effective tool for treating operators that fall outside the scope of classical contraction theory. The flexibility of Θ^R, together with the structural properties of suprametrics, allows discontinuous or highly irregular control mechanisms, thereby extending and unifying several existing fixed-point schemes in the literature.

Our findings show that suprametric structures, when combined with relaxed control functions, offer a robust analytical bridge between maximum type contractive mappings and nonlinear differential systems. The illustrative examples confirm that the behavior of solutions is strongly influenced by the nonlinear control encoded by $\vartheta_{\mathcal{R}}$, supporting the theoretical framework established in the paper. Consequently, the results presented here contribute both to the theoretical development of generalized contraction principles and to their applicability in real world models where dominance by local maxima or abrupt changes plays a central role.

Several promising research directions naturally arise from the present framework:

- Extending the relaxed $\vartheta_{\mathcal{R}}$-contraction approach to multivalued, random, or hybrid operators defined on suprametric spaces.
- Investigating suprametric analogues of higher-order nonlinear differential systems, nonlocal boundary value problems, or integro- differential models.
- Developing numerical algorithms driven by the relaxed control structure, such as adaptive Mann-Ishikawa or viscosity-type iterations, for applications where max-type dynamics dominate.
- Exploring connections with control theory, elastic and viscoelastic modeling, and stability analysis in mechanical or engineering systems governed by discontinuous forces.
- Applying suprametric fixed-point techniques in data-driven modeling, including neural networks or learning architectures where max-based or abrupt transition dynamics naturally arise.

The theory introduced in this paper opens several new avenues for unifying suprametric geometry, relaxed contraction principles, and the analysis of nonlinear systems. This provides a fertile ground for further developments both in foundational fixed-point theory and in applied areas such as engineering, mechanics, and computational mathematics.

Author contributions

A. Büyükkaya: Conceptualization, methodology, software, validation, formal analysis, investigation, resources; E. Girgin: Conceptualization, methodology, software, validation, formal analysis, investigation, resources, data curation, writing-original draft preparation, writing-review and editing, visualization; H. Ahmad: Methodology, formal analysis, investigation, writing-review and editing, visualization; M. Younis: Data curation, writing-original draft preparation, writing-review and editing, visualization, M. Öztürk: Data curation, writing-original draft preparation, writing-review and editing, visualization, supervision, project administration, funding acquisition. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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