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Research article

Some fixed points in complete extended b-metric spaces for single-valued and multi-valued mappings

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Abstract: In this paper, I prove the existence and uniqueness of fixed points in complete extended b-metric spaces for Kannan and Chatterjea contractions in a single-valued setting. We also prove the existence of fixed points in complete extended b-metric spaces for the existence of fixed points in complete extended b-metric spaces for Nadler and Kannan-type contractions in a multi-valued setting.

Keywords: fixed point; b-metric space; Nadler theorem; multivalued mapping; Kannan contraction; Chatriga contraction
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1. Introduction

Fixed-point theory represents a central and well-established domain in mathematical analysis, owing to its fundamental role in the study of the existence, uniqueness, and stability of solutions to a wide class of mathematical models. In particular, fixed-point techniques have proved to be an indispensable analytical tool for transforming differential and integral equations into operator equations whose solvability can be addressed within appropriate functional frameworks.

This approach has been successfully applied to various classes of problems, including impulsive differential equations and boundary value problems, where contraction-type fixed-point principles are frequently used to guarantee the solvability and convergence of iterative schemes [1, 2]. Such applications have reinforced the relevance of fixed-point methods in nonlinear analysis and motivated the development of increasingly general fixed-point results. Moreover, recent advances have extended fixed-point theory to more general metric frameworks, highlighting the significance of generalized contractive conditions and completeness assumptions in ensuring uniqueness and convergence of fixed-points in b-metric and related spaces [3, 4].

In recent years, considerable attention has been directed toward extending classical fixed-point theory beyond standard metric spaces (see, e.g., [5]. In this context, generalized metric structures and weakened contractive conditions have emerged as effective tools for broadening the applicability of fixed-point techniques. In particular, b-metric spaces have attracted interest as a natural generalization of metric spaces, allowing greater flexibility in the underlying distance structure while preserving the essential convergence properties. Moreover, related structures such as a b-metric spaces endowed with additional relations have also been considered in this direction (see [6]).

Within this framework, Koleva and Zlatanov [7] investigated fixed-point problems for Chatterjea-type contractions in complete b-metric spaces. They established existence and uniqueness results for the fixed-points of single-valued mappings and analyzed the convergence behavior of the associated iterative sequences. More recently, Hong [8] studied generalized non-expansive mappings of the Hardy–Rogers type in complete b-metric spaces and proved common fixed-point theorems for pairs of mappings, thereby extending and unifying several classical fixed-point principles.

In addition, recent studies have addressed both single-valued and multi-valued mappings under generalized contractive conditions and related approaches (see [9–12]), further enriching the theory.

Consider a mapping Γ acting on a set V. A point γ in V is termed a fixed-point of Γ if $\Gamma\gamma = \gamma$. In 1922, Banach [13] established the Banach's contraction principle, a renowned theorem that pertains to mappings that contract in a complete metric space. Later, in 1968, Kannan [14] presented an existence and uniqueness result for fixed points of Kannan contractions defined on metric spaces (V, δ) satisfying the inequality

$$\delta(\Gamma\gamma, \Gamma\xi) \leqslant \rho\left[\delta(\gamma, \Gamma\gamma) + \delta(\xi, \Gamma\xi)\right], \text{for } 0 < \rho < \frac{1}{2},$$

for each $\gamma, \xi \in V$.

Similarly, in 1972, Chatterjea [15] proved the existence and uniqueness of fixed-points for Chatterjea contractions, characterized by the inequality

$$\delta(\Gamma\gamma, \Gamma\xi) \leqslant \rho\left[\delta(\gamma, \Gamma\xi) + \delta(\xi, \Gamma\gamma)\right], \text{ with } 0 < \rho < \frac{1}{2},$$

for each $\gamma, \xi \in V$. Since then, many mathematicians have explored fixed-point theorems in various metric spaces.

As part of this ongoing generalization, Bakhtin introduced the notion of a b-metric space as an extension of the classical metric space concept [16]. This structure relaxes the triangular inequality through the introduction of a control parameter, thereby providing a broader setting for fixed-point analysis.

Definition 1. *[16] Let V be a nonempty set and let $s \geq 1$ be a real number. A function $\delta : V \times V \to [0, \infty)$ is called a b-metric if, for all $\gamma, \xi, \zeta \in V$, the following conditions are satisfied:*

(i) $\delta(\gamma, \xi) = 0$ *if and only if* $\gamma = \xi$*;*
(ii) $\delta(\gamma, \xi) = \delta(\xi, \gamma)$*;*
(iii) $\delta(\gamma, \xi) \leq s(\delta(\gamma, \zeta) + \delta(\zeta, \xi))$.

The pair (V, δ) is called a b-metric space.

The following example, which appears in the literature on generalized distance functions and is reported in Bourbaki's treatise on general topology, also illustrates the concept of a b-metric space.

Example 1. *[16] Let $V = L_p[0,1]$ be the space of all real functions $\gamma(t), t \in [0,1]$ such that $\int_0^1 |\gamma(t)|^p dt < \infty$ with $0 < p < 1$. Define $\delta : V \times V \to [0,\infty]$ as:*

$$\delta(\gamma,\xi) = \left(\int_0^1 |\gamma(t) - \xi(t)|^p dt\right)^{\frac{1}{p}}.$$

Then δ is b-metric space with $s = 2^{\frac{1}{p}}$.

Remark 1. *In contrast to the case $0 < p < 1$, when $p > 1$ the distance defined in the example above satisfies the classical triangle inequality as a consequence of Minkowski's inequality. Consequently, the associated space is a metric space in the usual sense, and the generalized b-metric framework is no longer applicable.*

In 1993, Czerwik [17] presented Banach's fixed-point theorem adapted for b-metric spaces. Furthermore, in 2013, Kir and Kiziltunc [18] demonstrated Kannan's fixed-point theorem within the framework of b-metric spaces and also established the Chatterjea fixed-point theorem in b-metric spaces.

In 2017, Kamran et al., [19] introduced the concept of an extended b-metric space as follows:

Definition 2. *[19] Let V be a non-empty set and $\theta : V \times V \to [1,\infty)$. A mapping $\delta_\theta : V \times V \to [0,\infty)$ is an extended b-metric if for, all $\gamma, \xi, \zeta \in V$, it satisfies:*
$(\delta_\theta 1)$ $\delta_\theta(\gamma,\xi) = 0$ iff $\gamma = \xi$;
$(\delta_\theta 2)$ $\delta_\theta(\gamma,\xi) = \delta_\theta(\xi,\gamma)$;
$(\delta_\theta 3)$ $\delta_\theta(\gamma,\zeta) \le \theta(\gamma,\zeta)\left[\delta_\theta(\gamma,\xi) + \delta_\theta(\xi,\zeta)\right]$.
The pair (V,δ_θ) is called an extended b-metric space.

Remark 2. *If $1 \le \theta(\gamma,\xi) = s$ for $\gamma,\xi \in V$, then the extended b-metric space reduces to a b-metric space. Therefore, a b-metric space is a generalization of metric space, and an extended b-metric space extends the concept of a b-metric space.*

The following example is adapted from Huang [20] and is presented here to illustrate the concept of an extended b-metric space.

Example 2. *Let $V = \mathbb{R}$. Define two functions $\delta_\theta : V \times V \to [0,+\infty)$ and $\theta : V \times V \to [1,+\infty)$ as follows:*

$$\delta_\theta(\gamma,\xi) = \begin{cases} \frac{|\gamma|+|\xi|}{1+|\gamma|+|\xi|}, & \gamma,\xi \in V, \gamma \neq \xi, \\ 0, & \gamma = \xi, \end{cases}$$

and $\theta(\gamma,\xi) = 1 + |\gamma| + |\xi|$, for all $\gamma,\xi \in V$. Then (V,δ_θ) is an extended b-metric space.

Indeed, $(\delta_\theta 1)$ and $(\delta_\theta 2)$ in Definition (2) are satisfied. To prove $(\delta_\theta 3)$ in Definition (2), let $\gamma,\xi,\zeta \in V$.

(i) It is clear that if $\gamma = \xi$, then $(\delta_\theta 3)$ holds.

(ii) If $\gamma \neq \xi$ and $\gamma = \zeta$, then

$$\theta(\gamma,\xi)\left[\delta_\theta(\gamma,\zeta) + \delta_\theta(\zeta,\xi)\right] = (1 + |\gamma| + |\xi|)\left(0 + \frac{|\zeta| + |\xi|}{1 + |\zeta| + |\xi|}\right),$$

$$= (1 + |\gamma| + |\xi|) \cdot \frac{|\zeta| + |\xi|}{1 + |\zeta| + |\xi|},$$
$$\geq \frac{|\zeta| + |\xi|}{1 + |\zeta| + |\xi|},$$
$$= \delta_\theta(\zeta, \xi).$$

(iii) Similarly to (ii), we can prove if $\gamma \neq \xi$ and $\xi = \zeta$.

(iv) If $\gamma \neq \xi, \xi \neq \zeta$, and $\gamma \neq \zeta$, by the fact that $f(r) = \frac{r}{1+r}$ is nondecreasing on $[0, +\infty)$ and $|\gamma| + |\xi| \leq |\gamma| + |\zeta| + |\xi|$, it then follows that

$$\theta(\gamma, \xi)\left[\delta_\theta(\gamma, \zeta) + \delta_\theta(\zeta, \xi)\right] = (1 + |\gamma| + |\xi|)\left(\frac{|\gamma| + |\zeta|}{1 + |\gamma| + |\zeta|} + \frac{|\zeta| + |\xi|}{1 + |\zeta| + |\xi|}\right),$$
$$\geq \frac{|\gamma| + |\xi|}{1 + |\gamma| + |\xi|} = \delta_\theta(\gamma, \xi).$$

It follows that $(\delta_\theta 3)$ holds. Hence, (V, δ_θ) is an extended b-metric space.

Definition 3. *[19] Let (V, δ_θ) be an extended b-metric space. Then we have the following:*

1) *A sequence $\{\gamma_n\}$ in V is said to converge to $\gamma \in V$ if, for every $\epsilon > 0$, $N_\epsilon \in \mathbb{N}$ exists such that $\delta_\theta(\gamma_n, \gamma) < \epsilon$ for all $n \geq N_\epsilon$.*
2) *A sequence $\{\gamma_n\}$ in V is said to be Cauchy if, for every $\epsilon > 0$, $N_\epsilon \in \mathbb{N}$ exists such that $\delta_\theta(\gamma_n, \gamma_m) < \epsilon$ for all $n, m \geq N_\epsilon$.*
3) *An extended b-metric space (V, δ_θ) is complete if every Cauchy sequence in V is convergent.*

In addition, Kamran et al., [19] proved the Banach's contraction principle in complete extended b-metric spaces. In Section 2, we will prove the existence and uniqueness of a fixed-point on extended b-metric spaces for the Kannan and Chatterijea contractions.

Before entering into the study of multi-valued mappings, let us revisit the concept of distance between two sets. In 1905, Pompeiu [21] introduced the distance between two closed and bounded sets, $\mathcal{A}$ and $\mathcal{B}$, in the complex plane as $\Delta(\mathcal{A}, \mathcal{B}) = \max\{\delta(\alpha, \mathcal{B}) : \alpha \in \mathcal{A}\}$ where $\delta(\alpha, \mathcal{B}) = \min\{\delta(\alpha, \beta) : \beta \in \mathcal{B}\}$ and $\delta(\alpha, \beta)$, which represents the Euclidean distance between points α and β. However, this definition fails to satisfy the metric condition due to its lack of symmetry. Consequently, Pompeiu [21] proposed a metric distance between two closed and bounded sets, $\mathcal{A}$ and $\mathcal{B}$, in the complex plane as $P(\mathcal{A}, \mathcal{B}) = \Delta(\mathcal{A}, \mathcal{B}) + \Delta(\mathcal{B}, \mathcal{A})$. In 1914, Hausdorff [22] formalized this concept in a more general metric space setting as shown below.

Definition 4. *[22] Let (V, δ) be a metric space. Let $CB(V)$ enote the family of all nonempty closed and bounded subsets of V.*

For $\mathcal{A}, \mathcal{B} \in CB(V)$ and $\alpha \in \mathcal{A}$, define

$$\Delta(\alpha, \mathcal{B}) = \inf\{\delta(\alpha, \beta) : \beta \in \mathcal{B}\}.$$

The Hausdorff metric $H : CB(V) \times CB(V) \to \mathbb{R}^+$ is given by

$$H(\mathcal{A}, \mathcal{B}) = \max\left\{\sup_{\alpha \in \mathcal{A}} \Delta(\alpha, \mathcal{B}), \sup_{\beta \in \mathcal{B}} \Delta(\beta, \mathcal{A})\right\}.$$

If (V, δ) is complete, then $(CB(V), H)$ is also complete and is called the Hausdorff metric space.

Multi-valued mappings extend the concept of single-valued mappings by associating each point in the domain with a set of values rather than a single value. This concept has found wide application across various mathematical fields, including optimization, control theory, and differential equations, as well as in the analysis of dynamic systems and partial differential equations. Classical contributions by renowned mathematicians such as N. Bourbaki, S. Banach, and S. Mazur [16, 23, 24] significantly advanced the understanding of multi-valued mappings through their work on fixed-point theorems and set-valued analysis. The original and systematic formulation of the definition adopted in this work is attributed to Nadler (in 1969) [25], which leads to the following definition.

Definition 5. *[25] Let V be a nonempty set. Γ is said to be a multi-valued mapping on V if Γ is a mapping from V to the power set of V denoted by $\Gamma : V \to 2^V$. Furthermore, an element $\gamma \in V$ is said to be a fixed-point of a multi-valued mapping $\Gamma : V \to 2^V$ if $\gamma \in \Gamma(V)$.*

In 1969, Nadler extended the Banach contraction principle and proved the existence of fixed-point for the multi-valued mapping $\Gamma : V \to CB(V)$ which satisfies

$$H(\Gamma\gamma, \Gamma\xi) \leqslant \alpha\delta(\gamma, \xi), \quad \alpha \in [0, 1).$$

In [26], Khine remarked that every multi-valued mapping is a continuous mapping.

In Section 3, we will prove the existence of fixed points for multi-valued mappings in extended b-metric spaces which satisfy Nadler and Kannan type contractions.

2. Kannan and Chatterjea contractions in complete extended b-metric spaces

In this section, we establish some fixed-point theorems on extended b-metric spaces with Kannan and Chatterijea contractions.

Theorem 1. *Let (V, δ_θ) be a complete extended b-metric space where δ_θ is a continuous function. Let $\Gamma : V \longrightarrow V$, satisfying*

$$\delta_\theta(\Gamma\gamma, \Gamma\xi) \leq \rho\left[\delta_\theta(\gamma, \Gamma\gamma) + \delta_\theta(\xi, \Gamma\xi)\right], \quad 0 < \rho < \frac{1}{2}, \tag{2.1}$$

such that for each $\gamma_0 \in V$, $\lim_{n,m\to\infty} \theta(\gamma_n, \gamma_m) < \frac{1-\rho}{\rho}$, where $\gamma_n = \Gamma^n\gamma_0$, $n \in \mathbb{N}$, and $\theta(\gamma, \xi) < \frac{1}{\rho}$ for each $\gamma, \xi \in V$. Then Γ has a unique fixed point.

Proof. Let $\gamma_0 \in V$. Define an iteration sequence by $\gamma_1 = \Gamma\gamma_0, \gamma_2 = \Gamma\gamma_1 = \Gamma^2\gamma_0, \ldots \gamma_n = \Gamma\gamma_{n-1} = \Gamma^n_{\gamma_0}$.

By applying Inequality (2.1), then for $n \in \mathbb{N}$, we get the following:

$$\begin{aligned}\delta_\theta(\gamma_n, \gamma_{n+1}) &= \delta_\theta(\Gamma\gamma_{n-1}, \Gamma\gamma_n)\\ &\leq \rho\delta_\theta(\gamma_{n-1}, \Gamma\gamma_{n-1}) + \rho\delta_\theta(\gamma_n, \Gamma\gamma_n)\\ &= \rho\delta_\theta(\gamma_{n-1}, \gamma_n) + \rho\delta_\theta(\gamma_n, \gamma_{n+1}).\end{aligned}$$

Therefore,

$$\delta_\theta(\gamma_n, \gamma_{n+1}) \leq \frac{\rho}{1-\rho}\delta_\theta(\gamma_{n-1}, \gamma_n). \tag{2.2}$$

By using Inequality (2.2) we get the following:

$$\delta_\theta(\gamma_n, \gamma_{n+1}) \leq \left(\frac{\rho}{1-\rho}\right)^n \delta_\theta(\gamma_0, \gamma_1). \tag{2.3}$$

By using the extended b-metric condition ($\delta_\theta 3$) and Inequality (2.3), for each $n, m \in \mathbb{N}$ and $m \geq n$ we then have:

$$\begin{aligned}
\delta_\theta(\gamma_n, \gamma_m) &\leqslant \theta(\gamma_n, \gamma_m)\left[\delta_\theta(\gamma_n, \gamma_{n+1}) + \delta_\theta(\gamma_{n+1}, \gamma_m)\right] \\
&\leqslant \theta(\gamma_n, \gamma_m)\delta_\theta(\gamma_n, \gamma_{n+1}) + \theta(\gamma_n, \gamma_m)\theta(\gamma_{n+1}, \gamma_m)\left[\delta_\theta(\gamma_{n+1}, \gamma_{n+2}) + \delta_\theta(\gamma_{n+2}, \gamma_m)\right] \\
&\leqslant \theta(\gamma_n, \gamma_m)\delta_\theta(\gamma_n, \gamma_{n+1}) + \theta(\gamma_n, \gamma_m)\theta(\gamma_{n+1}, \gamma_m)\delta_\theta(\gamma_{n+1}, \gamma_{n+2}) \\
&\quad + \theta(\gamma_n, \gamma_m)\theta(\gamma_{n+1}, \gamma_m)\delta_\theta(\gamma_{n+2}, \gamma_m) \\
&\vdots \\
&\leqslant \theta(\gamma_n, \gamma_m)\delta_\theta(\gamma_n, \gamma_{n+1}) + \theta(\gamma_n, \gamma_m)\theta(\gamma_{n+1}, \gamma_m)\delta_\theta(\gamma_{n+1}, \gamma_{n+2}) \\
&\quad + \cdots + \theta(\gamma_n, \gamma_m)\theta(\gamma_{n+1}, \gamma_m)\cdots\theta(\gamma_{m-2}, \gamma_m)\delta_\theta(\gamma_{m-2}, \gamma_{m-1}) \\
&\quad + \theta(\gamma_n, \gamma_m)\theta(\gamma_{n+1}, \gamma_m)\cdots\theta(\gamma_{m-2}, \gamma_m)\delta_\theta(\gamma_{m-1}, \gamma_m) \\
&\leqslant \theta(\gamma_n, \gamma_m)\left(\frac{\rho}{1-\rho}\right)^n \delta_\theta(\gamma_0, \gamma_1) + \theta(\gamma_n, \gamma_m)\theta(\gamma_{n+1}, \gamma_m)\left(\frac{\rho}{1-\rho}\right)^{n+1} \delta_\theta(\gamma_0, \gamma_1) \\
&\quad + \cdots + \theta(\gamma_n, \gamma_m)\theta(\gamma_{n+1}, \gamma_m)\cdots\theta(\gamma_{m-2}, \gamma_m)\left(\frac{\rho}{1-\rho}\right)^{m-2} \delta_\theta(\gamma_0, \gamma_1) \\
&\quad + \theta(\gamma_n, \gamma_m)\theta(\gamma_{n+1}, \gamma_m)\cdots\theta(\gamma_{m-2}, \gamma_m)\left(\frac{\rho}{1-\rho}\right)^{m-1} \delta_\theta(\gamma_0, \gamma_1) \\
&\leqslant \delta_\theta(\gamma_0, \gamma_1)\left[\prod_{i=1}^{n}\theta(\gamma_i, \gamma_m)\left(\frac{\rho}{1-\rho}\right)^n + \prod_{i=1}^{n+1}\theta(\gamma_i, \gamma_m)\left(\frac{\rho}{1-\rho}\right)^{n+1}\right. \\
&\qquad \left. + \cdots + \prod_{i=1}^{m-2}\theta(\gamma_i, \gamma_m)\left(\frac{\rho}{1-\rho}\right)^{m-2} + \prod_{i=1}^{m-1}\theta(\gamma_i, \gamma_m)\left(\frac{\rho}{1-\rho}\right)^{m-1}\right].
\end{aligned}$$

In this case

$$\delta_\theta(\gamma_n, \gamma_m) \leqslant \delta_\theta(\gamma_0, \gamma_1)\left[\sum_{j=1}^{m-1}\left(\frac{\rho}{1-\rho}\right)^j \prod_{i=1}^{j}\theta(\gamma_i, \gamma_m) - \sum_{j=1}^{n-1}\left(\frac{\rho}{1-\rho}\right)\prod_{i=1}^{j}\theta(\gamma_i, \gamma_m)\right].$$

By taking $S_n = \sum_{j=1}^{n}\left(\frac{\rho}{1-\rho}\right)^j \prod_{i=1}^{j}\theta(\gamma_i, \gamma_m)$ and using the ratio test, we get

$$\sum_{j=1}^{\infty}\left(\frac{\rho}{1-\rho}\right)^j \prod_{i=1}^{j}\theta(\gamma_i, \gamma_m) \text{ is convergent.}$$

Therefore,

$$\delta_\theta(\gamma_n, \gamma_m) \longrightarrow 0 \text{ as } n, m \to \infty.$$

It is follows that $(\gamma_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. By the completeness of V, $(\gamma_n)_{n\in\mathbb{N}}$ is convergent, so $\gamma \in V$ exists such that γ_n converge to γ, i.e., $\lim_{n\to\infty}\delta_\theta(\gamma_n, \gamma) = 0$.

We now prove γ is the fixed point of Γ by using the extended b-metric condition ($\delta_\theta 3$) and Inequality (2.1), we have

$$\begin{aligned}
\delta_\theta(\gamma,\Gamma\gamma) &\leqslant \theta(\gamma,\Gamma\gamma)\left[\delta_\theta(\gamma,\gamma_n)+\gamma_\theta(\gamma_n,\Gamma\gamma)\right]\\
&\leqslant \theta(\gamma,\Gamma\gamma)\,\delta_\theta(\gamma,\gamma_n)+\theta(\gamma,\Gamma\gamma)\,\delta_\theta(\Gamma\gamma_{n-1},\Gamma\gamma)\\
&\leqslant \theta(\gamma,\Gamma\gamma)\,\delta_\theta(\gamma,\gamma_n)+\rho\theta(\gamma,\Gamma\gamma)\left[\delta_\theta(\gamma_{n-1},\Gamma\gamma_{n-1})+\delta_\theta(\gamma,\Gamma\gamma)\right]\\
\left[1-\rho\theta(\gamma,\Gamma\gamma)\right]\delta_\theta(\gamma,\Gamma\gamma) &\leqslant \theta(\gamma,\Gamma\gamma)\left[\delta_\theta(\gamma,\gamma_n)+\rho\delta_\theta(\gamma_{n-1},\gamma_n)\right].
\end{aligned}$$

Therefore

$$\delta_\theta(\gamma,\Gamma\gamma) \leqslant \frac{1}{1-\rho\theta(\gamma,\Gamma\gamma)}\theta(\gamma,\Gamma\gamma)\left[\delta_\theta(\gamma,\gamma_n)+\rho\delta_\theta(\gamma_{n-1},\gamma_n)\right].$$

For $n \to \infty$, we get:

$$\delta_\theta(\delta,\Gamma\delta) = 0 \Longleftrightarrow \gamma = \Gamma\gamma.$$

This implies that γ is a fixed point of the mapping Γ.

To prove the uniqueness, let ξ be another fixed point, i.e., $\xi = \Gamma\xi$

$$\begin{aligned}
\delta_\theta(\gamma,\xi) &= \delta_\theta(\Gamma\gamma,\Gamma\xi)\\
&\leqslant \rho\delta_\theta(\gamma,\Gamma\gamma)+\rho\delta_\theta(\xi,\Gamma\xi)\\
&\leqslant \lambda\delta_\theta(\gamma,\gamma)+\rho\delta_\theta(\xi,\xi).
\end{aligned}$$

Thus,

$$\delta_\theta(\gamma,\xi) \leqslant 0, \text{ which implies } \delta_\theta(\gamma,\xi) = 0.$$

Therefore

$$\gamma = \xi.$$

□

The following example is designed to verify that the assumptions of Theorem 1 are satisfied. and to highlight the feasibility of the proposed conditions.

Example 3. *Let $V = \mathbb{Z}$. Define a mapping $\delta_\theta : V \times V \to [0,+\infty)$ as*

$$\delta_\theta(x,y) = |\xi-\mu|, \quad \textit{for all } \xi,\mu \in V.$$

Moreover, define the control function $\theta : V \times V \to \mathbb{Z}^+$ as

$$\theta(\xi,\mu) = |\xi|+|\mu|+1, \quad \textit{for all } \xi,\mu \in V.$$

We now verify that (V,δ_θ) satisfies the conditions of an extended b-metric space. Clearly, $\delta_\theta(\xi,\mu) = 0$ if and only if $\xi = \mu$, and $\delta_\theta(\xi,\mu) = \delta_\theta(\mu,\xi)$ for all $\xi,\mu \in V$. Hence, Conditions ($\delta_\theta 1$) and ($\delta_\theta 2$) of Definition 2 are satisfied.

It remains to verify Condition ($\delta_\theta 3$). Let $\xi,\mu,\zeta \in V$. Since

$$\delta_\theta(\xi,\mu) = |\xi-\mu| \leq |\xi-\zeta|+|\zeta-\mu|,$$

and $\theta(\xi,\mu) = |\xi| + |\mu| + 1 \geq 1$, *it follows that*

$$\delta_\theta(\xi,\mu) \leq \theta(\xi,\mu)(\delta_\theta(\xi,\zeta) + \delta_\theta(\zeta,\mu)).$$

Therefore, (V,δ_θ) *is an extended b-metric space.*

Next, define a mapping $\Gamma : V \to V$ *as*

$$\Gamma(\xi) = \begin{cases} 0, & \text{if } \xi \in \{0,1,-1\}, \\ 1, & \text{if } \xi \in A = \mathbb{Z}^- \setminus \{-1\}, \\ -1 & \text{if } \xi \in B = \mathbb{Z}^+ \setminus \{1\}. \end{cases}$$

We now verify that the contractive condition stated in Theorem 1 is satisfied. Clearly, Inequality (2.1) holds when $\xi = \mu$*. Hence, we consider the case* $\xi \neq \mu$*.*

We distinguish the proof into the following cases:

Case 1. *If* $\xi,\mu \in \mathbb{Z}^+$*, then Inequality (2.1) can be verified directly.*

Case 2. *If* $\xi,\mu \in \mathbb{Z}^-$*, a similar argument shows that Inequality (2.1) holds.*

Case 3. *If* $\xi,\mu \in \{0,-1,1\}$*, Inequality (2.1) is satisfied by direct computation.*

Case 4. *Let* $\xi = 0$ *and* $\mu \in B$*. Then Inequality (2.1) holds, since*

$$1 = \delta_\theta(0,-1) = \delta_\theta(\Gamma\xi,\Gamma\mu) < \frac{1}{2}\Big[\delta_\theta(\xi,\Gamma\xi) + \delta_\theta(\mu,\Gamma\mu)\Big] = \frac{1}{2}[0 + |1+\mu|],$$

which is true for all $\mu \in \mathbb{Z}^+ \setminus \{1\}$*.*

The next case can be treated analogously.

Case 5. *Let* $\xi = 0$ *and* $\mu \in A$*. Then Inequality (2.1) holds, since*

$$2 = \delta_\theta(1,-1) = \delta_\theta(\Gamma\xi,\Gamma\mu) < \frac{1}{2}\Big[\delta_\theta(\xi,\Gamma\xi) + \delta_\theta(\mu,\Gamma\mu)\Big] = \frac{1}{2}[|\xi-1| + |\mu+1|],$$

which is true for all $\mu \in \mathbb{Z}^- \setminus \{-1\}$*.*

Case 6. *Let* $\xi \in A$ *and* $\mu \in B$ *or vice versa. Then Inequality (2.1) holds, since*

$$1 = \delta_\theta(0,-1) = \delta_\theta(\Gamma\xi,\Gamma\mu) < \frac{1}{2}\Big[\delta_\theta(\xi,\Gamma\xi) + \delta_\theta(\mu,\Gamma\mu)\Big] = \frac{1}{2}[0 + |1+\mu|],$$

which is true for all $\mu \in \mathbb{Z}^+ \setminus \{1\}$ *and* $\mu \in \mathbb{Z}^- \setminus \{-1\}$*.*

Combining all the cases above, we conclude that Inequality (2.1) is satisfied throughout the complete extended b-metric space $(\mathbb{Z},\delta_\theta)$*. Hence, the mapping admits a uniqUe fixed point, which is given by* $\xi = 0$*.*

In 2013, Kir and Kiziltunc proved the existence and uniqueness of a fixed point for Kannan contractions in b-metric spaces [18]. This result appears as a special case of our previous theorem, which provides a more general setting. Specifically, by taking $\theta(\gamma,\xi) = s$, where $s \geq 1$, our result reduces to that of Kir and Kiziltunc. We therefore obtain the following corollary as a direct consequence.

Corollary 1. *Let (V,δ) be a complete b-metric space with a constant $s \geq 1$, where δ is a continuous function. Let $\Gamma : V \to V$ satisfy the following condition:*

$$\delta(\Gamma\gamma,\Gamma\xi) \leq \rho\left[\delta(\gamma,\Gamma\gamma)+\delta(\xi,\Gamma\xi)\right], \quad \text{where } 0<\rho<\frac{1}{2}, \text{ and } s<\frac{1-\rho}{\rho}.$$

Then Γ has a unique fixed point in V, and for any initial point $\gamma_0 \in V$, the iterative sequence $\gamma_{n+1} = \Gamma(\gamma_n)$ converges to this fixed point.

Proof. By choosing $\theta(\gamma,\xi) = s$, for every $\gamma,\xi \in V$, where s is a consistent function, the space (V,δ_θ) reduces to a b-metric space. Moreover, we have

$$\lim_{n,m\to\infty}\theta(\gamma_n,\gamma_m) = \lim_{n,m\to\infty} s \leq \frac{1-\rho}{\rho},$$

where the sequence (γ_n) is defined as $\gamma_n = \Gamma^n(\gamma_0)$ for some $\gamma_0 \in V$.

Since

$$\delta_\theta(\gamma,\xi) = s \leq \frac{1-\rho}{\rho} < \frac{1}{\rho} \quad \text{for all } \gamma,\xi \in V,$$

it follows that the contractive inequality required in Theorem 1 is satisfied. Consequently, by Theorem 1, the mapping Γ admits a unique fixed point in V. □

From the previously proven theorem, if we substitute $\theta(\gamma,\xi) = 1$ for every $\gamma,\xi \in V$, we obtain the following result, which was originally established by Kannan in 1968 in [14]. This result is as follows.

Corollary 2. *Let (V,δ) be a complete metric space, and let $\Gamma : V \to V$ satisfy the following condition:*

$$\delta(\Gamma\gamma,\Gamma\xi) \leq \rho\left[\delta(\gamma,\Gamma\gamma)+\delta(\xi,\Gamma\xi)\right], \quad \text{for all } \gamma,\xi \in V,$$

where $0<\rho<\frac{1}{2}$. Then Γ has a unique fixed point in V, and for any initial point $\gamma_0 \in V$, the iterative sequence $\gamma_{n+1} = \Gamma(\gamma_n)$ converges to this fixed point.

This result is a special case of the previously stated corollary, where $s = 1$, and b-metric spaces reduce to standard metric spaces.

In the next theorem, we will prove the fixed points in extended b-metric spaces for a mapping $\Gamma : V \to V$ which satisfies Chatterijea contraction inequality.

Theorem 2. *Let (V,δ_θ) be a complete extended b-metric space where δ_θ is a continuous function. Let $\Gamma : V \longrightarrow V$ satisfying*

$$\delta_\theta(\Gamma\gamma,\Gamma\xi) \leq \rho\left[\delta_\theta(\gamma,\Gamma\xi)+\delta_\theta(\xi,\Gamma\gamma)\right], \quad 0<\rho<\frac{1}{2}, \tag{2.4}$$

such that for each $\gamma_0 \in V$ $\lim_{n,m\to\infty}\theta(\gamma_n,\gamma_m) < \frac{1-\rho}{\rho}$, $\lim_{n\to\infty}\theta(\gamma_n,\gamma_{n+2}) < \frac{1-\rho}{\rho}$, where $\gamma_n = \Gamma^n\gamma_0$, $n\in\mathbb{N}$ and $\theta(\gamma,\xi) < \frac{1}{\rho}$ for each $\gamma,\xi \in V$.

Then Γ has a unique fixed point.

Proof. Let $\gamma_0 \in V$ be arbitrary. Define $\gamma_1 = \Gamma\gamma_0, \gamma_2 = \Gamma\gamma_1 = \Gamma^2\gamma_0, \ldots \gamma_n = \Gamma\gamma_{n-1} = \Gamma^n_{\gamma_0}$.
For $n \in \mathbb{N}$, we have

$$\begin{aligned}
\delta_\theta(\gamma_n, \gamma_{n+1}) &= \delta_\theta(\Gamma\gamma_{n-1}, \Gamma\gamma_n) \\
&\leqslant \rho\delta_\theta(\gamma_{n-1}, \Gamma\gamma_n) + \rho\delta_\theta(\gamma_n, \Gamma\gamma_{n-1}) \\
&\leqslant \rho\delta_\theta(\gamma_{n-1}, \gamma_{n+1}) + \rho\delta_\theta(\gamma_n, \gamma_n) \\
&\leqslant \rho\delta_\theta(\gamma_{n-1}, \gamma_{n+1}) \\
&\leqslant \rho\theta(\gamma_{n-1}, \gamma_{n+1})\left[\delta_\theta(\gamma_{n-1}, \gamma_n) + \delta_\theta(\gamma_n, \gamma_{n+1})\right],
\end{aligned}$$

$$\text{so, } \left[1 - \rho\theta(\gamma_{n-1}, \gamma_{n+1})\right]\delta_\theta(\gamma_n, \gamma_{n+1}) \leqslant \rho\theta(\gamma_{n-1}, \gamma_{n+1})\,\delta_\theta(\gamma_{n-1}, \gamma_n).$$

Therefore

$$\delta_\theta(\gamma_n, \gamma_{n+1}) \leqslant \frac{\rho\theta(\gamma_{n-1}, \gamma_{n+1})}{1 - \rho\theta(\gamma_{n-1}, \gamma_{n+1})}\delta_\theta(\gamma_{n-1}, \gamma_n). \tag{2.5}$$

By using (2.5), we get

$$\delta_\theta(\gamma_n, \gamma_{n+1}) \leqslant \frac{\rho^n\theta(\gamma_{n-1}, \gamma_{n+1})\,\theta(\gamma_{n-2}, \gamma_n)\ldots\theta(\gamma_0, \gamma_2)}{(1 - \rho\theta(\gamma_{n-1}, \gamma_{n+1}))\cdot(1 - \rho\theta(\gamma_{n-2}, \gamma_n))\ldots(1 - \rho\theta(\gamma_0, \gamma_2))}\delta_\theta(\gamma_0, \gamma_1). \tag{2.6}$$

To prove that $(\gamma_n)_{n\in\mathbb{N}}$ is Cauchy, let $n, m \in \mathbb{N}, \quad m \geq n$.

$$\begin{aligned}
\delta_\theta(\gamma_n, \gamma_m) &\leqslant \theta(\gamma_n, \gamma_m)\,\delta_\theta(\gamma_n, \gamma_{n+1}) + \theta(\gamma_n, \gamma_m)\,\delta_\theta(\gamma_{n+1}, \gamma_m) \\
&\leqslant \theta(\gamma_n, \gamma_m)\,\delta_\theta(\gamma_n, \gamma_{n+1}) + \theta(\gamma_n, \gamma_m)\,\theta(\gamma_{n+1}, \gamma_m)\left[\delta_\theta(\gamma_{n+1}, \gamma_{n+2}) + \delta_\theta(\gamma_{n+2}, \gamma_m)\right] \\
&\leqslant \theta(\gamma_n, \gamma_m)\,\delta_\theta(\gamma_n, \gamma_{n+1}) + \theta(\gamma_n, \gamma_m)\,\theta(\gamma_{n+1}, \gamma_m)\,\delta_\theta(\gamma_{n+1}, \gamma_{n+2}) + \theta(\gamma_n, \gamma_m)\,\theta(\gamma_{n+1}, \gamma_m)\,\delta_\theta(\gamma_{n+2}, \gamma_m).
\end{aligned}$$

After the nth time and by using inequality (2.6), we get:

$$\begin{aligned}
\delta_\theta(\gamma_n, \gamma_m) \leqslant{}& \theta(\gamma_n, \gamma_m)\,\delta_\theta(\gamma_n, \gamma_{n+1}) + \theta(\gamma_n, \gamma_m)\,\theta(\gamma_{n+1}, \gamma_m)\,\delta_\theta(\gamma_{n+1}, \gamma_{n+2}) + \\
& \cdots + \theta(\gamma_n, \gamma_m)\,\theta(\gamma_{n+1}, \gamma_m)\cdots\theta(\gamma_{m-2}, \gamma_m)\,\delta_\theta(\gamma_{m-2}, \gamma_{m-1}) + \\
& \theta(\gamma_n, \gamma_m)\,\theta(\gamma_{n+1}, \gamma_m)\cdots\theta(\gamma_{m-2}, \gamma_m)\,\delta_\theta(\gamma_{m-1}, \gamma_m). \\
& \leqslant \theta(\gamma_n, \gamma_m)\,\rho^n\prod_{i=1}^{n}\frac{\theta(\gamma_{i-1}, \gamma_{i+1})}{1 - \rho\theta(\gamma_{i-1}, \gamma_{i+1})}\delta_\theta(\gamma_0, \gamma_1) + \\
& \theta(\gamma_n, \gamma_m)\,\theta(\gamma_{n+1}, \gamma_m)\,\rho^{n+1}\prod_{i=1}^{n+1}\frac{\theta(\gamma_{i-1}, \gamma_{i+1})}{1 - \rho\theta(\gamma_{i-1}, \gamma_{i+1})}\delta_\theta(\gamma_0, \gamma_1) + \\
& \cdots + \theta(\gamma_n, \gamma_m)\cdots\theta(\gamma_{m-2}, \gamma_m)\,\rho^{m-2}\prod_{i=1}^{m-2}\frac{\theta(\gamma_{i-1}, \gamma_{i+1})}{1 - \rho\theta(\gamma_{i-1}, \gamma_{i+1})}\delta_\theta(\gamma_0, \gamma_1) + \\
& \theta(\gamma_n, \gamma_m)\cdots\theta(\gamma_{m-2}, \gamma_m)\,\rho^{m-1}\prod_{i=1}^{m-1}\frac{\theta(\gamma_{i-1}, \gamma_{i+1})}{1 - \rho\theta(\gamma_{i-1}, \gamma_{i+1})}\delta_\theta(\gamma_0, \gamma_1) \\
& \leqslant \theta(\gamma_1, \gamma_m)\,\theta(\gamma_2, \gamma_m)\cdots\theta(\gamma_n, \gamma_m)\,\rho^n\prod_{i=1}^{n}\frac{\theta(\gamma_{i-1}, \gamma_{i+1})}{1 - \rho\theta(\gamma_{i-1}, \gamma_{i+1})}\delta_\theta(\gamma_0, \gamma_1) + \\
& \theta(\gamma_1, \gamma_m)\,\theta(\gamma_2, \gamma_m)\cdots\theta(\gamma_{n+1}, \gamma_m)\,\rho^n\prod_{i=1}^{n+1}\frac{\theta(\gamma_{i-1}, \gamma_{i+1})}{1 - \rho\theta(\gamma_{i-1}, \delta_{i+1})}\delta_\theta(\gamma_0, \gamma_1) +
\end{aligned}$$

$$\cdots + \theta(\gamma_1,\gamma_m)\,\theta(\gamma_2,\gamma_m)\cdots\theta(\gamma_{m-2},\gamma_m)\,\rho^{m-2}\prod_{i=1}^{m-2}\frac{\theta(\gamma_{i-1},\gamma_{i+1})}{1-\rho\theta(\gamma_{i-1},\gamma_{i+1})}\delta_\theta(\gamma_0,\gamma_1) +$$

$$\theta(\gamma_1,\gamma_m)\,\theta(\gamma_2,\gamma_m)\cdots\theta(\gamma_{m-1},\gamma_m)\,\lambda^{m-1}\prod_{i=1}^{m-1}\frac{\theta(\gamma_{i-1},\gamma_{i+1})}{1-\rho\theta(\gamma_{i-1},\gamma_{i+1})}\delta_\theta(\gamma_0,\gamma_1)$$

$$\leqslant \delta_\theta(\gamma_0,\gamma_1)\left[\rho^n\prod_{i=1}^{n}\frac{\theta(\gamma_i,\gamma_m)\,\theta(\gamma_{i-1},\gamma_{i+1})}{1-\rho\theta(\gamma_{i-1},\gamma_{i+1})} + \cdots + \rho^{m-1}\prod_{i=1}^{m-1}\frac{\theta(\gamma_i,\gamma_m)\,\theta(\gamma_{i-1},\gamma_{i+1})}{1-\rho\theta(\gamma_{i-1},\gamma_{i+1})}\right].$$

Take $S_n = \sum_{j=1}^{n}\lambda\prod_{i=1}^{j}\frac{\theta(\gamma_i,\gamma_m)\,\theta(\gamma_{i-1},\gamma_{i+1})}{1-\lambda\theta(\gamma_{i-1},\gamma_{i+1})}$.

Since $\lim_{n,m\to\infty}\theta(\gamma_n,\gamma_m) \leqslant \frac{1-\rho}{\rho}$ and $\lim_{n\to\infty}\theta(\gamma_n,\gamma_{n+2}) < \frac{1-\rho}{\rho}$, and by the Ratio test, $\delta_\theta(\gamma_n,\gamma_m)$ converges to zero as $n,m\to\infty$.

By completeness, $(\gamma_n)_{n\in\mathbb{N}}$ is convergent, say to γ.

We noe show that x is a fixed point.

$$\begin{aligned}
\delta_\theta(\gamma,\Gamma\gamma) &\leqslant \theta(\gamma,\Gamma\gamma)\left[\delta_\theta(\gamma,\gamma_n) + \delta_\theta(\gamma_n,\Gamma\gamma)\right]\\
&\leqslant \theta(\gamma,\Gamma\gamma)\,\delta_\theta(\gamma,\gamma_n) + \theta(\gamma,\Gamma\gamma)\,\delta_\theta(\gamma_n,\Gamma\gamma)\\
&\leqslant \theta(\gamma,\Gamma\gamma)\,\delta_\theta(\gamma,\gamma_n) + \theta(\gamma,\Gamma\gamma)\,\delta_\theta(\Gamma\gamma_{n-1},\Gamma\gamma)\\
&\leqslant \theta(\gamma,\Gamma\gamma)\,\delta_\theta(\gamma,\gamma_n) + \rho\theta(\gamma,\Gamma\gamma)\left[\delta_\theta(\gamma_{n-1},\Gamma\gamma) + \delta_\theta(\gamma,\Gamma\gamma_{n-1})\right]\\
&\leqslant \theta(\gamma,\Gamma\gamma)\,\delta_\theta(\gamma,\gamma_n) + \rho\theta(\gamma,\Gamma\gamma)\left[\delta_\theta(\gamma_{n-1},\Gamma\gamma) + \delta_\theta(\gamma,\gamma_n)\right]\\
&\leqslant \theta(\gamma,\Gamma\gamma)\,\delta_\theta(\gamma,\gamma_n) + \rho\theta(\gamma,\Gamma\gamma)\left[\delta_\theta(\gamma,\gamma_n) + \delta_\theta(\gamma_{n-1},\Gamma\gamma)\right]\\
&\leqslant \theta(\gamma,\Gamma\gamma)\,\delta_\theta(\gamma,\gamma_n) + \rho\theta(\gamma,\Gamma\gamma)\,\delta_\theta(\gamma,\gamma_n) + \rho\theta(\gamma,\Gamma\gamma)\left[\delta_\theta(\Gamma\gamma_{n-2},\Gamma\gamma)\right]\\
&\leqslant [1+\rho]\,\theta(\gamma,\Gamma\gamma)\,\delta_\theta(\gamma,\gamma_n) + \rho^2\theta(\gamma,\Gamma\gamma)\left[\delta_\theta(\gamma_{n-2},\Gamma\gamma) + \delta_\theta(\gamma,\Gamma\gamma_{n-2})\right]\\
&\leqslant [1+\rho]\,\theta(\gamma,\Gamma\gamma)\,\delta_\theta(\gamma,\gamma_n) + \rho^2\theta(\gamma,\Gamma\gamma)\left[\delta_\theta(\gamma_{n-2},\Gamma\gamma) + \delta_\theta(x\gamma,\gamma_{n-1})\right]\\
&\leqslant [1+\rho]\,\theta(\gamma,\Gamma\gamma)\,\delta_\theta(\gamma,\gamma_n) + \rho^2\theta(\gamma,\Gamma\gamma)\,\delta_\theta(\gamma,\gamma_{n-1}) + \rho^2\theta(\gamma,\Gamma\gamma)\left[\delta_\theta(\gamma_{n-2},\Gamma\gamma)\right]\\
&\;\vdots\\
&\leqslant [1+\rho]\,\theta(\gamma,\Gamma\gamma)\,\delta_\theta(\gamma,\gamma_n) + \rho^2\theta(\gamma,\Gamma\gamma)\,\delta_\theta(\gamma,\gamma_{n-1}) +\\
&\quad\cdots + \rho^{n-1}\theta(\gamma,\Gamma\gamma)\,\delta_\theta(\gamma,\gamma_2) + \rho^n\theta(\gamma,\Gamma\gamma)\,\delta_\theta(\gamma,\gamma_1) + \rho^n\theta(\gamma,\Gamma\gamma)\delta_\theta(\gamma_0,\Gamma\gamma),
\end{aligned}$$

as $n\to\infty$ and because $\lambda < \frac{1}{2}$, we get, $\delta_\theta(\gamma,\Gamma\gamma) = 0$ i.e, $\gamma = \Gamma\gamma$.

For the purpose of proving uniqueness, suppose that ξ is another fixed point, i.e., $\xi = \Gamma\xi$.

$$\begin{aligned}
\delta_\theta(\gamma,\xi) &= \delta_\theta(\Gamma\gamma,\Gamma\xi)\\
&\leqslant \rho\delta_\theta(\gamma,\Gamma\xi) + \rho\delta_\theta(\xi,\Gamma\gamma)\\
&\leqslant \rho\delta_\theta(\gamma,\xi) + \rho\delta_\theta(\xi,\gamma),\\
\text{then}(1-2\rho)\delta_\theta(\gamma,\xi) &\leqslant 0.
\end{aligned}$$

Accordingly, $\delta(\gamma,\xi) = 0$, which means that $\gamma = \xi$. □

The aim of the next example is to illustrate the applicability of Theorem 2 by showing that the contraction condition is satisfied in the underlying extended metric space. As a result, Theorem 2 guarantees the existence of a unique fixed point.

Example 4. *Let $V = \mathbb{Z}$. Define a mapping $\delta_\theta : V \times V \to [0, +\infty)$ as*

$$\delta_\theta(x, y) = |\xi - \mu|, \quad \textit{for all } \xi, \mu \in V.$$

Moreover, define the control function $\theta : V \times V \to \mathbb{Z}^+$ as

$$\theta(\xi, \mu) = |\xi| + |\mu| + 1, \quad \textit{for all } \xi, \mu \in V.$$

In Example 3, we considered the same subset and showed that the defined distance satisfies all the conditions of an extended b-metric space.

Now, we define a mapping $\Gamma : V \to V$ by $\Gamma\xi = 2$ for all $\xi \in \mathbb{Z}$

For any $\xi, \mu \in \mathbb{Z}$, it holds that

$$0 = \delta_\theta(\Gamma\xi, \Gamma\mu) < \rho\left[\delta_\theta(\xi, \Gamma\mu) + \delta_\theta(\mu, \Gamma\xi)\right].$$

Hence, the Inequality (2.4) holds, and, therefore, the mapping possesses a unique fixed point, which is $\xi = 2$.

In 2013, Kir and Kiziltunc proved the existence and uniqueness of a fixed point for hatterijea contractions on b-metric spaces [18]. This result appears as a special case of our previous theorem, which provides a more general setting. Specifically, by taking $\theta(\gamma, \xi) = s$, where $s \geq 1$, our result reduces to that of Kir and Kiziltunc. We therefore obtain the following corollary as a direct consequence.

Corollary 3. *Let (V, δ) be a complete b-metric space with $s \geq 1$, where δ is a continuous mapping. Let $\Gamma : V \longrightarrow V$ satisfying*

$$\delta(\Gamma\gamma, \Gamma\xi) \leq \rho\left[\delta(\gamma, \Gamma\xi) + \delta(\xi, \Gamma\gamma)\right], \quad 0 < \rho < \frac{1}{2}.$$

such that for each $s < \frac{1-\rho}{\rho}$, T has a unique fixed point.

Proof. By choosing $\theta(\gamma, \xi) = s$, for every $\gamma, \xi \in V$, where s is a consistent function, the space (V, δ_θ) reduces to a b-metric space. Moreover, we have

$$\lim_{n,m\to\infty} \theta(\gamma_n, \gamma_m) = \lim_{n,m\to\infty} s \leq \frac{1-\rho}{\rho},$$

where the sequence (γ_n) is defined as $\gamma_n = \Gamma^n(\gamma_0)$ for some $\gamma_0 \in V$.

Since

$$\delta_\theta(\gamma, \xi) = s \leq \frac{1-\rho}{\rho} < \frac{1}{\rho} \quad \text{for all } \gamma, \xi \in V,$$

it follows that the contractive inequality required in Theorem 2 is satisfied. Consequently, by Theorem 2, the mapping Γ admits a unique fixed point in V. □

From the previously proven theorem, if we substitute $\theta(\gamma, \xi) = 1$ for every $\gamma, \xi \in V$, we obtain the following result, which was originally established by Chatterjea in 1972 [15]. This result is as follows.

Corollary 4. *Let (V, δ) be a complete metric space. Let $\Gamma : V \longrightarrow V$ satisfying*

$$\delta(\Gamma\gamma, \Gamma\xi) \leq \rho\left[\delta(\gamma, \Gamma\xi) + \delta(\xi, \Gamma\gamma)\right], \quad 0 < \rho < \frac{1}{2}.$$

Then, Γ has a unique fixed point.

By choosing $s = 1$, the b-metric framework reduces to the classical metric setting. Hence, the result follows immediately from the previously stated corollary, since all the imposed conditions are fulfilled by the corresponding constant control function.

3. Nadlar and Kannan type contractions in complete extended b-metric spaces for multi-valued mappings

In this section, we will prove the existence of fixed points for multi-valued mapping in extended b-metric spaces which satisfy Nadler- and Kannan-type contractions.

The following lemma has an important role in the proof of the main theorem.

Lemma 1. *[25] Let (V, δ) be a metric space, and $\mathcal{A}, \mathcal{B} \in CB(V)$. Then for each $\alpha \in \mathcal{A}$ and $\epsilon > 0$, $\beta \in \mathcal{B}$ exists such that*

$$\delta(\alpha, \beta) \leq H(\mathcal{A}, \mathcal{B}) + \epsilon.$$

In the next theorem, we will prove the existence of a fixed point in extended b-metric spaces for a mapping $\Gamma : V \to CB(V)$, which satisfies the Nadler contraction.

Theorem 3. *Let (V, δ_θ) be a complete extension b-metric space and let $\Gamma : V \longrightarrow CB(V)$ be multi-valued mapping from V into the set of all closed bounded subsets of V, $CB(V)$, satisfying the following:*

$$H(\Gamma\gamma, \Gamma\xi) \leqslant \kappa\delta_\theta(\gamma, \xi), \quad \kappa \in (0, 1). \tag{3.1}$$

Such that for each $\gamma_0 \in V \quad \lim_{n,m\to\infty} \theta(\gamma_n, \gamma_m) < \frac{1}{\kappa}$ where $\gamma_n = \Gamma^n\gamma_0, n \in \mathbb{N}$ and $\theta(\gamma, \xi) < \frac{1}{\kappa}$ for each $\gamma, \xi \in V$. Then Γ has a fixed point.

Proof. Let γ_0 be an arbitrary element in V. Define an iteration sequence by $\gamma_1 = \Gamma\gamma_0, \gamma_2 = \Gamma\gamma_1 = \Gamma^2 x_0, \ldots \gamma_n = \Gamma\gamma_{n-1} = \Gamma\gamma_0{}^n$.

Then $\gamma_2 \in \Gamma\gamma_1$ exists such that $\delta_\theta(\gamma_1, \gamma_2) \leqslant H(\Gamma\gamma_0, \Gamma\gamma_1) + \kappa$,
and thus$\gamma_3 \in \Gamma\gamma_2$ exists such that $\delta_\theta(\gamma_2, \gamma_3) \leqslant H(\Gamma\gamma_1, \Gamma\gamma_2) + \kappa^2$,
$\vdots$
which implies that $\gamma_{n+1} \in \Gamma\gamma_n$ exists such that $\delta_\theta(\gamma_n, \gamma_{n+1}) \leqslant H(\Gamma\gamma_{n-1}, \Gamma\gamma_n) + \kappa^n$.

For $n \in \mathbb{N}$, we have:

$$\delta_\theta(\gamma_{n+1}, \gamma_n) \leqslant H(\Gamma\gamma_{n-1}, \Gamma\gamma_n) + \kappa^n$$

$$
\begin{aligned}
&\leqslant \kappa\delta_\theta(\gamma_{n-1},\gamma_n)+\kappa^n\\
&\leqslant \kappa\left[H(\Gamma\gamma_{n-2},\Gamma\gamma_{n-1})+\kappa^{n-1}\right]+\kappa^n\\
&\leqslant \kappa\left[\kappa\delta_\theta(\gamma_{n-2},\gamma_{n-1})+\kappa^{n-1}\right]+\kappa^n\\
&\leqslant \kappa^2\delta_\theta(\gamma_{n-2},\gamma_{n-1})+2\kappa^n\\
&\vdots\\
&\leqslant \kappa^n\delta_\theta(\gamma_0,\gamma_1)+n\kappa^n,
\end{aligned}
$$

Since $\kappa \leqslant 1$, κ^n and $n\kappa^n$ convergent to zero as n approaches to infinity. Therefore, we have

$$\lim_{n\to\infty}\delta_\theta(\gamma_n,\gamma_{n+1})=0. \tag{3.2}$$

For $n, m \in \mathbb{N}$ where $m > n$, we have

$$
\begin{aligned}
\delta_\theta(\gamma_n,\gamma_m) &\leqslant \theta(\gamma_n,\gamma_m)\left[\delta_\theta(\gamma_n,\gamma_{n+1})+\delta_\theta(\gamma_{n+1},\gamma_m)\right]\\
&\leqslant \theta(\gamma_n,\gamma_m)\cdot\delta_\theta(\gamma_n,\gamma_{n+1})+\theta(\gamma_n,\gamma_m)\,\delta_\theta(\gamma_{n+1},\gamma_m)\\
&\leqslant \theta(\gamma_n,\gamma_m)\,\delta_\theta(\gamma_n,\gamma_{n+1})+\theta(\gamma_n,\gamma_m)\,\theta(\gamma_{n+1},\gamma_m)\left[\delta_\theta(\gamma_{n+1},\gamma_{n+2})+\delta_\theta(\gamma_{n+2},\gamma_m)\right]\\
&\leqslant \theta(\gamma_n,\gamma_m)\,\delta_\theta(\gamma_n,\gamma_{n+1})+\theta(\gamma_n,\gamma_m)\,\theta(\gamma_{n+1},\gamma_m)\,\delta_\theta(\gamma_{n+1},\gamma_{n+2})\\
&+\theta(\gamma_n,\gamma_m)\,\theta(\gamma_{n+1},\gamma_m)\,\delta_\theta(\gamma_{n+2},\gamma_m)\\
&\vdots\\
&\leqslant \theta(\gamma_n,\gamma_m)\,\delta_\theta(\gamma_n,\gamma_{n+1})+\theta(\gamma_n,\gamma_m)\,\theta(\gamma_{n+1},\gamma_m)\,\delta_\theta(\gamma_{n+1},\gamma_{n+2})+\\
&\cdots+\theta(\gamma_n,\gamma_m)\,\theta(\gamma_{n+1},\gamma_m)\cdots\theta(\gamma_{m-2},\gamma_m)\,\delta_\theta(\gamma_{m-1},\gamma_m)\\
&\leqslant \theta(\gamma_1,\gamma_m)\,\theta(\gamma_2,\gamma_m)\cdots\theta(\gamma_n,\gamma_m)\,\delta_\theta(\gamma_n,\gamma_{n+1})+\\
&\theta(\gamma_1,\gamma_m)\cdots\theta(\gamma_n,\gamma_m)\,\theta(\gamma_{n+1},\gamma_m)\,\delta_\theta(\gamma_{n+1},\gamma_{n+2})+\cdots+\\
&\theta(\gamma_1,\gamma_m)\cdots\theta(\gamma_n,\gamma_m)\ldots\theta(\gamma_{m-2},\gamma_m)\,\theta(\gamma_{m-1},\gamma_m)\,\delta_\theta(\gamma_{m-1},\gamma_m).
\end{aligned}
$$

$$
\begin{aligned}
\delta_\theta(\gamma_n,\gamma_m) &\leqslant \left[\kappa^n\delta_\theta(\gamma_0,\gamma_1)+n\kappa^n\right]\cdot\prod_{i=1}^{n}\theta(\gamma_i,\gamma_m)+\left[\kappa^{n+1}\delta_\theta(\gamma_0,\gamma_1)+(n+1)\kappa^{n+1}\right]\prod_{i=1}^{n+1}\theta(\gamma_i,\gamma_m)\\
&+\cdots+\left[\kappa^{m-1}\delta(\gamma_0,\gamma_1)+(m-1)\kappa^{m-1}\right]\cdot\prod_{i=1}^{m-1}\theta(\gamma_i,\gamma_m)\\
&\leqslant \delta_\theta(\gamma_0,\gamma_1)\left[\kappa^n\prod_{i=1}^{n}\theta(\gamma_i,\gamma_m)+\kappa^{n+1}\prod_{i=1}^{n+1}\theta(\gamma_i,\gamma_m)+\cdots+\kappa^{m-1}\prod_{i=1}^{m-1}\theta(\gamma_i,\gamma_m)\right]+\\
&n\kappa^n\cdot\prod_{i=1}^{n}\theta(\gamma_i,\gamma_m)+(n+1)\kappa^{n+1}\cdot\prod_{i=1}^{n+1}\theta(\gamma_i,\gamma_m)+\cdots+(m-1)\kappa^{m-1}\cdot\prod_{i=1}^{m-1}\theta(\gamma_i,\gamma_m)\\
&\leqslant \delta_\theta(\gamma_0,\gamma_1)\cdot\left[\sum_{j=n}^{m-1}\kappa^j\prod_{i=1}^{j}\theta(\gamma_i,\gamma_m)\right]+\sum_{j=n}^{m-1}j\kappa^j\prod_{i=1}^{j}\theta(\gamma_i,\gamma_m)\\
&\leqslant \delta_\theta(\gamma_0,\gamma_1)\left[\sum_{j=1}^{\infty}\kappa^j\prod_{i=1}^{j}\theta(\gamma_i,\gamma_m)\right]+\sum_{j=1}^{\infty}j\kappa^j\prod_{i=1}^{j}\theta(\gamma_i,\gamma_m).
\end{aligned}
$$

Applying the ratio test and using $\kappa \leq 1$, $\lim_{n,m\to\infty} \theta(\gamma_n, \gamma_m) \leq \frac{1}{\kappa}$, and then for $n, m \to \infty$, we get (γ_n) is a Cauchy sequence in V and, by the completeness of V, γ in V exists such that $\lim_{n\to\infty} \delta_\theta(\gamma_n, \gamma) = 0$.

Now, we will prove that Γ has a fixed point. Assume that $\gamma \notin \Gamma\gamma$.

$$\begin{aligned}
\delta_\theta(\gamma, \Gamma\gamma) = \inf_{\alpha\in\Gamma\gamma} \delta_\theta(\gamma, \alpha) &\leqslant \inf_{\alpha\in\Gamma\gamma} \left[\theta(\gamma, \alpha)\, \delta(\gamma, \gamma_{n+1}) + \theta(\gamma, \alpha)\, \delta_\theta(\gamma_{n+1}, \alpha)\right] \\
&\leqslant \delta_\theta(\gamma, \gamma_{n+1}) \inf_{\alpha\in\Gamma\gamma} \theta(\gamma, \alpha) + \inf_{\alpha\in\Gamma\gamma} \theta(\gamma, \alpha)\delta_\theta(\gamma_{n+1}, \alpha) \\
&\leqslant \delta_\theta(\gamma, \gamma_{n+1}) \inf_{\alpha\in\Gamma\gamma} \theta(\gamma, \alpha) + \inf_{\alpha\in\Gamma\gamma} \theta(\gamma, \alpha) \cdot \inf_{\alpha\in\Gamma\gamma} \delta_\theta(\gamma_{n+1}, \alpha) \\
&\leqslant \delta(\gamma, \gamma_{n+1}) \inf_{\alpha\in\Gamma\gamma} \theta(\gamma, \alpha) + \inf_{\alpha\in\Gamma\gamma} \theta(\gamma, \alpha) H(\Gamma\gamma_n, \Gamma\gamma) \\
&\leqslant \delta_\theta(\gamma, \gamma_{n+1}) \inf_{\alpha\in\Gamma\gamma} \theta(\gamma, \alpha) + \inf_{\alpha\in\Gamma\gamma} \theta(\gamma, \alpha) .\kappa\delta_\theta(\gamma_n, \gamma).
\end{aligned}$$

Hence

$$\delta_\theta(\gamma, \Gamma\gamma) \longrightarrow 0 \quad \text{as } n \to \infty,$$

which implies that $\gamma \in \Gamma\gamma$. Therefore, γ is a fixed point of Γ □

The following example is motivated by the standard multivalued contraction framework introduced by Nadler [25] and is constructed to illustrate the applicability of Theorem 3.

Example 5. *Let $V = [0, \infty)$ and define the function $\delta_\theta : V \times V \to [0, \infty)$ by*

$$\delta_\theta(\xi, \mu) = |\xi - \mu|,$$

where the control function $\theta : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ is given by

$$\theta(\xi, \mu) = |\xi| + |\mu| + 1.$$

Define the multi-valued mapping $\Gamma : V \to CB(V)$ by

$$\Gamma(\xi) = \left[\frac{\xi}{4}, \frac{\xi}{2}\right], \quad \xi \in V.$$

It is straightforward to verify that (V, δ_θ) forms an extended b-metric space, particularly since the control function θ satisfies $\theta(\xi, \mu) \geq 1$ for all $\xi, \mu \in V$.

For $\xi \in V$ and $\mu \in V$, define the distance from ξ to $\Gamma(\mu)$ as

$$\delta_\theta(\xi, \Gamma\mu) = \inf_{\zeta\in\Gamma(\mu)} |\xi - \zeta| = \begin{cases} 0, & \text{if } \xi \in \Gamma\mu, \\ |\xi - \frac{\mu}{4}|, & \text{if } \xi < \frac{\mu}{4}, \\ |\xi - \frac{\mu}{2}|, & \text{if } \xi > \frac{\mu}{2}. \end{cases}$$

Similarly, define the distance from μ to $\Gamma(\xi)$ as

$$\delta_\theta(\mu, \Gamma\xi) = \inf_{\zeta\in\Gamma(\xi)} |\mu - \zeta| = \begin{cases} 0, & \text{if } \mu \in \Gamma\xi, \\ |\mu - \frac{\xi}{4}|, & \text{if } \mu < \frac{\xi}{4}, \\ |\mu - \frac{\xi}{2}|, & \text{if } \mu > \frac{\xi}{2}. \end{cases}$$

The Hausdorff distance between $\Gamma(\xi)$ *and* $\Gamma(\mu)$ *is then defined as*

$$\begin{aligned}
H(\Gamma\xi, \Gamma\mu) &= \max\left\{\sup_{\xi\in\Gamma(\xi)} \delta_\theta(\xi, \Gamma\mu), \sup_{\mu\in\Gamma(\mu)} \delta_\theta(\mu, \Gamma\xi)\right\} \\
&= \max\left\{\left|\frac{\xi}{4} - \frac{\mu}{4}\right|, \left|\frac{\xi}{2} - \frac{\mu}{2}\right|\right\} \\
&= \frac{1}{2}\left|\xi - \mu\right|.
\end{aligned}$$

Thus, Condition (3.1) is satisfied. Consequently, Γ *admits a fixed point, which is* $\xi = 0$.

The previous theorem is a generalization of the Nadler fixed-point theorem, which was proved in [17] for metric spaces and b-metric spaces.

Corollary 5. *Let* (V, δ) *be a complete b-metric space with* $s \geq 1$, *where* δ *is a continuous function. Let* $\Gamma : V \longrightarrow CB(V)$, *satisfying*

$$H(\Gamma\gamma, \Gamma\xi) \leq \kappa\left[\delta(\gamma, \xi)\right], \quad 0 < \kappa < \frac{1}{s}.$$

Then Γ *has a fixed point.*

Proof. By taking $\theta(\gamma, \xi) = s$ with $s \geq 1$, the contractive condition is fulfilled; consequently, all the hypotheses of Theorem 3 are satisfied, and hence the mapping admits a unique fixed point. □

Furthermore, by taking $\theta(\gamma, \xi) = s$ with $s \geq 1$, Theorem (3) reduces to the classical Nadler fixed-point theorem for multi-valued contractions. Consequently, we obtain the following corollary.

Corollary 6. *Let* (V, δ) *be a complete metric space. Let* $\Gamma : V \longrightarrow CB(V)$, *satisfying*

$$H(\Gamma\gamma, \Gamma\xi) \leq \kappa\left[\delta(\gamma, \xi)\right], \quad 0 < \kappa < 1.$$

Then Γ *has a fixed point.*

Proof. The result follows directly from Theorem 3 by taking $\theta(\gamma, \xi) = s$ with $s \geq 1$. □

In the next theorem, we will prove the existence of a fixed point in extended b-metric spaces for a mapping $\Gamma : V \to CB(V)$ which satisfies Kannan-type contraction.

Theorem 4. *Let* (V, δ) *be a complete extended b-metric space and let* $\Gamma : V \longrightarrow CB(V)$, *be multi-valued mapping from V into the set of all closed bounded subsetS of V,* $CB(V)$, *such that*

$$H(\Gamma\gamma, \Gamma\gamma) \leqslant \kappa[\delta_\theta(\gamma, \Gamma\gamma) + \delta_\theta(\xi, \Gamma\xi)] \quad 0 < \kappa < \frac{1}{2}, \tag{3.3}$$

For $\gamma_0 \in V$ $\lim_{n,m\to\infty} \theta(\gamma_n, \gamma_m) < \frac{1-\kappa}{\kappa}$ *where* $\gamma_n = \Gamma^n\gamma_0, n, m \in \mathbb{N}$ *and* $\theta(\gamma, \xi) < \frac{1}{\kappa}$ *for each* $\gamma, \xi \in V$. *Then* Γ *has a fixed point.*

Proof. Let $\gamma_0 \in V$ and define $\rho = \frac{\kappa}{1-\kappa} < 1$. Then, $\gamma_1 \in \Gamma\gamma_0$ exists.
It follows that we have

$$\gamma_2 \in \Gamma\gamma_1 \text{ such that } \quad \delta_\theta(\gamma_1, \gamma_2) \leqslant H(\Gamma\gamma_0, \Gamma\gamma_1) + \rho,$$
$$\vdots$$
$$\text{Hence, } \gamma_n \in \Gamma\gamma_{n-1} \text{ exists such that } \quad \delta_\theta(\gamma_{n-1}, \gamma_n) \leqslant H(\Gamma\gamma_{n-2}, \Gamma\gamma_{n-1}) + \rho^{n-1}.$$

Therefore, , for $n \in \mathbb{N}$, we have

$$\begin{aligned}\delta_\theta(\gamma_n, \gamma_{n+1}) &\leqslant H(\Gamma\gamma_{n-1}, \Gamma\gamma_n) + \rho^n \\ &\leqslant \kappa\delta_\theta(\gamma_{n-1}, \Gamma\gamma_{n-1}) + \kappa\delta_\theta(\gamma_n, \Gamma\gamma_n) + \rho^n \\ &\leqslant \kappa\delta_\theta(\gamma_{n-1}, \gamma_n) + \kappa\delta_\theta(\gamma_n, \gamma_{n+1}) + \rho^n.\end{aligned}$$

Therefore

$$\begin{aligned}\delta_\theta(\gamma_n, \gamma_{n+1}) &\leqslant \frac{\kappa}{1-\kappa}\delta_\theta(\gamma_{n-1}, \gamma_n) + \frac{\rho^n}{1-\kappa} \\ &\leqslant \rho\delta_\theta(\gamma_{n-1}, \gamma_n) + \frac{\rho^n}{1-\kappa} \\ &\leqslant \rho\left[\rho\delta(\gamma_{n-2}, \gamma_{n-1}) + \frac{\rho^{n-1}}{1\kappa}\right] + \frac{\rho^n}{1-\kappa} \\ &\leqslant \rho^2\delta(\gamma_{n-2}, \gamma_{n-1}) + \frac{2\rho^n}{1-\kappa} \\ &\vdots \\ &\leqslant \rho^n\delta(x\gamma_0, \gamma_1) + \frac{n\rho^n}{1-\kappa}.\end{aligned}$$

Since $\rho < 1$, then ρ^n and $n\rho^n$ converge to zero as $n \to \infty$.
For $n, m \in \mathbb{N}$ and $m > n$, we get the following:

$$\begin{aligned}\delta_\theta(\gamma_n, \gamma_m) &\leqslant \theta(\gamma_n, \gamma_m)\left[\delta_\theta(\gamma_n, \gamma_{n+1}) + \delta_\theta(\gamma_{n+1}, \gamma_m)\right] \\ &\leqslant \theta(\gamma_n, \gamma_m)\delta_\theta(\gamma_n, \gamma_{n+1}) + \theta(\gamma_n, \gamma_m)\delta_\theta(\gamma_{n+1}, \gamma_m) \\ &\leqslant \theta(\gamma_n, \gamma_m)\delta_\theta(\gamma_n, \gamma_{n+1}) + \theta(\gamma_n, \gamma_m)\theta(\gamma_{n+1}, \gamma_m)\delta_\theta(\gamma_{n+1}.\gamma_{n+2}) + \\ &\cdots + \theta(\gamma_n, \gamma_m)\cdots\theta(\gamma_{m-1}, \gamma_m)\delta_\theta(\gamma_{m-2}, \gamma_m) \\ &\vdots \\ &\leqslant \rho^n\delta_\theta(\gamma_0, \gamma_1)\prod_{i=1}^{n}\theta(\gamma_i, \gamma_m) + \frac{n\rho^n}{1-\kappa}\prod_{i=1}^{n}\theta(\gamma_i, \gamma_m) + \cdots + \\ &\rho^{m-1}\delta_\theta(\gamma_0, \gamma_1)\cdot\prod_{i=1}^{m-1}\theta(\gamma_i, \gamma_m) + \frac{(m-1)\rho^{m-1}}{1-\kappa}\prod_{i=1}^{m-1}\theta(\gamma_i, \gamma_m).\end{aligned}$$

By the ratio test and since $\rho < 1$, $(\gamma_n)_{n\in\mathbb{N}}$ is therefore a Cauchy sequence and by the completeness of V, γ in V exists such that $\lim_{n\to\infty}\delta(\gamma_n, \gamma) = 0$.

Now, we prove that $\gamma \in \Gamma\gamma$. Assume that $\gamma \notin \Gamma\gamma$.

$$\begin{aligned}
\delta_\theta(\gamma, \Gamma\gamma) &= \inf_{\alpha\in\Gamma\gamma} \delta_\theta(\gamma, \alpha) \leqslant \inf_{\alpha\in\Gamma\gamma} \left[\theta(\gamma, \alpha)\left(\delta_\theta(\gamma, \gamma_{n+1}) + \delta_\theta(\gamma_{n+1}, \alpha)\right)\right] \\
&\leqslant \delta_\theta(\gamma, \gamma_{n+1}) \cdot \inf_{\alpha\in\Gamma\gamma} \theta(\gamma, \alpha) + \inf_{\alpha\in\Gamma\gamma} \theta(\gamma, \alpha) \cdot \delta_\theta(\gamma_{n+1}, \Gamma\gamma) \\
&\leqslant \delta_\theta(\gamma, \gamma_{n+1}) \inf_{\alpha\in\Gamma\gamma} \theta(\gamma, \alpha) + \inf_{\alpha\in\Gamma\gamma} \theta(\gamma, \alpha) \cdot H(\Gamma\gamma_n, \Gamma\gamma) \\
&\leqslant \delta_\theta(\gamma, \gamma_{n+1}) \inf_{\alpha\in\Gamma\gamma} \theta(\gamma, \alpha) + \inf_{\gamma\in\Gamma\gamma} \theta(\gamma, \alpha)\kappa\delta_\theta(\gamma_n, \Gamma\gamma_n) + \inf_{\alpha\in\Gamma\gamma} \theta(\gamma, \alpha) \cdot \kappa\delta_\theta(\gamma, \Gamma\gamma).
\end{aligned}$$

Therefore

$$\delta_\theta(\gamma, \Gamma\gamma,) \leqslant \frac{1}{1-\kappa} \inf_{\alpha\in\Gamma\gamma} \theta(\gamma, \alpha)\left[\delta_\theta(\gamma, \gamma_{n+1}) + \kappa\delta_\theta(\gamma_n, \gamma_{n+1})\right].$$

Consequently, $\delta(\gamma, \Gamma\gamma) = 0$ as $n \to \infty$, which means that $\gamma \in \Gamma\gamma$. □

The previous theorem extends several known fixed-point results in the literature. In particular, it generalizes the corresponding result established in [11] for mappings defined on complete b-metric spaces. As a consequence, we obtain the following corollary.

Corollary 7. *Let (V, δ) be a complete b-metric space with $s \geq 1$, where δ_θ is a continuous function. Let $\Gamma : V \longrightarrow CB(V)$, satisfying*

$$H(\Gamma\gamma, \Gamma\xi) \leq \kappa\left[\delta_\theta(\gamma, \Gamma\gamma) + \delta_\theta(\xi, \Gamma\xi)\right], \quad 0 < \kappa < \frac{1}{2},$$

such that for each $s < \frac{1-\kappa}{\kappa}$, T has a fixed point.

Proof. The result follows directly from Theorem 4 by taking $\theta(\gamma, \xi) = s$ for all $\gamma, \xi \in V$, where $s \geq 1$. Moreover, since $0 < \kappa < \frac{1}{2}$ and $s < \frac{1-\kappa}{\kappa}$, it follows that

$$\lim_{n,m\to\infty} \theta(\gamma_n, \gamma_m) = s < \frac{1-\kappa}{\kappa}, \text{ and } \theta(\gamma, \xi) = s < \frac{1}{\kappa} \text{ for all } \gamma, \xi \in V$$

Therefore, the contractive condition given in Eq 3.3 is satisfied. Consequently, by Theorem 4, the mapping Γ admits a fixed point. □

In 2015, Ume established the existence of fixed points for multivalued mappings in complete metric spaces under a Kannan-type contraction [12].

Corollary 8. *[12] Let (V, δ) be a complete metric space and let $\Gamma : V \longrightarrow CB(V)$, satisfying*

$$H(\Gamma\gamma, \Gamma\gamma) \leq \kappa\left[\delta_\theta(\gamma, \Gamma\gamma) + \delta_\theta(\xi, \Gamma\xi)\right], \quad 0 < \kappa < \frac{1}{2}.$$

Then, Γ has a fixed point.

Proof. By choosing $s = 1$ and because all the assumptions of Theorem 4 are satisfied, hence, Γ has a fixed point. □

4. Conclusions

In this paper, we established new fixed-point results in the framework of extended b-metric spaces for both single-valued and multi-valued mappings. For single-valued operators acting from V into V, generalized contractive conditions were introduced, leading to fixed point results that extend several classical contraction principles.

Moreover, in the multi-valued setting, suitable generalizations of Nadler-type contractions were formulated, together with generalized Kannan-type conditions. The obtained results demonstrate the flexibility of extended b-metric spaces in treating multi-valued operators under weak contractive assumptions and provide a basis for further developments in generalized fixed-point theory.

Use of Generative-AI tools declaration

The author declares she has not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares that there is no conflict of interest in this paper.

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