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Research article

Integral-type operators acting on a Hilbert-Bergman space of logarithmic weights on the unit ball

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Abstract: For a given holomorphic function g on the open unit ball in $\mathbb{C}^N$, we consider the following integral operators

$$T_g f(z) = \int_0^1 f(tz)\mathcal{R}g(tz)\frac{dt}{t} \quad \text{and} \quad I_g f(z) = \int_0^1 \mathcal{R}f(tz)g(tz)\frac{dt}{t}$$

on a Hilbert-Bergman space of logarithmic weights. By using an estimate for the norm of the Hilbert-Bergman space in terms of the radial derivative, we describe necessary and sufficient conditions for the boundedness of the operators. We also estimate the essential norm of these operators via the boundary behavior of some quantities that involve a symbol function g.

Keywords: Bounded operator; essential norm; holomorphic function; open unit ball
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1. Introduction

Let $\mathbb{N}$ be the set of positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{C}$ be the set of complex numbers, $\mathbb{C}^N$, $N \in \mathbb{N}$, be the N-dimensional complex vector space with the standard Hermitian inner product

$$\langle z, w\rangle = \sum_{j=1}^{N} z_j \overline{w}_j,$$

where $z = (z_1, \ldots, z_N)$ and $w = (w_1, \ldots, w_N)$, and the norm $|z| = \sqrt{\langle z, z\rangle}$, $\mathbb{B}_N = \mathbb{B}$ be the open unit ball in $\mathbb{C}^N$, that is, $\mathbb{B} = \{z \in \mathbb{C}^N : |z| < 1\}$, $\mathbb{B}_1 = \mathbb{D}$ (the open unit disk in $\mathbb{C}$), $\mathbb{S}$ be the unit sphere in $\mathbb{C}^N$, that is, the set $\{z \in \mathbb{C}^N : |z| = 1\}$, $d\sigma(\zeta)$ be the normalized surface measure on $\mathbb{S}$ (i.e., $\sigma(\mathbb{S}) = 1$), and $dV(z)$ be the normalized Lebesgue volume measure on $\mathbb{B}$ (i.e., $V(\mathbb{B}) = 1$). Positive and continuous functions on domains in $\mathbb{C}^N$ are called weight functions or weights [3], and frequently occur in definitions of various spaces of functions [1, 5, 17].

We denote multi-indices by α, that is, $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}_0^N$. If α is a multi-index, then $|\alpha| := \alpha_1 + \cdots + \alpha_N$ and $\alpha! := \alpha_1! \cdots \alpha_N!$.

Let $H(\mathbb{B})$ be the space of all holomorphic functions in the ball $\mathbb{B}$. It is known that such functions are analytic on $\mathbb{B}$ and that $f \in H(\mathbb{B})$ can be written in the form $f(z) = \sum_\alpha a_\alpha z^\alpha$, where α are multi-indices, $a_\alpha \in \mathbb{C}$, and $z^\alpha = z_1^{\alpha_1} \cdots z_N^{\alpha_N}$. For more information on the functions see, for instance, [8, 15].

Since the 1980s, composition operators on spaces of holomorphic functions over the unit ball, such as Hardy and Bergman spaces [11,15], have been extensively studied. Subsequently, as a generalization of these studies, weighted composition operators have been investigated by many authors. In parallel with the study of composition and weighted composition operators, integral-type operators on various domains have also attracted considerable attention (for instance, see [4, 18] for the operators on the unit polydisk, [6, 13] for the operators on the unit ball, [20, 23] for some of their extensions, as well as the related references therein). A central theme in these studies is the characterization of operator-theoretic properties of operators acting on function spaces in terms of function-theoretic properties of the symbols defining the operators. This research direction originated from the natural question of whether phenomena observed in spaces of analytic functions of one complex variable persist or are closely related in the several-variable setting. However, when passing to several complex variables, new difficulties arise that are absent in the one-variable theory. For example, there is no a unique natural way to generalize the one-dimensional differentiation operator $Df = f'$ in the several-variable setting because it is sometimes replaced by the radial derivative operator [10, 22, 26], or some other operators that contain partial derivatives. Consequently, it has been necessary to further develop the theory of holomorphic function spaces on the domains in $\mathbb{C}^N$ alongside the study of operators acting on them. Within this context, the analysis of linear operators on spaces of holomorphic functions in several variables has led to important insights into the structural properties of the underlying function spaces themselves. From this perspective, this line of research acquires intrinsic significance.

In the spirit of earlier work by many authors, in this paper, we focus on the study of integral-type operators defined as follows [9, 11]. For a given $g \in H(\mathbb{B})$, we put

$$T_g f(z) = \int_0^1 f(tz)\mathcal{R}g(tz)\frac{dt}{t} \quad \text{and} \quad I_g f(z) = \int_0^1 \mathcal{R}f(tz)g(tz)\frac{dt}{t}$$

for $z \in \mathbb{B}$ and $f \in H(\mathbb{B})$. Here, the radial derivative $\mathcal{R}f$ of $f \in H(\mathbb{B})$ is defined by

$$\mathcal{R}f(z) = \sum_{j=1}^{N} z_j D_j f(z),$$

where $D_j = \frac{\partial}{\partial z_j}$ $(1 \le j \le N)$. The purpose of this paper is to study these integral-type operators. These operators naturally arise as higher dimensional analogues of classical integral operators (e.g., of the Volterra operator or Cesàro operator, on analytic function spaces over the open unit disk $\mathbb{D}$; see [14]).

Let $\gamma \in (-1, +\infty)$ and δ be a non-positive number. We put

$$\omega_{\gamma,\delta}(z) = \left(\ln \frac{1}{|z|}\right)^{\gamma} \left[\ln\left(1 - \frac{1}{\ln |z|}\right)\right]^{\delta}.$$

We consider the following weighted Bergman space with respect to the weight function $\omega_{\gamma,\delta}$:

$$A^2_{\gamma,\delta}(\mathbb{B}) = \left\{ f \in H(\mathbb{B}) \ : \ \int_{\mathbb{B}} |f(z)|^2 \omega_{\gamma,\delta}(z) dV(z) < +\infty \right\}.$$

For $f,\ g \in A^2_{\gamma,\delta}(\mathbb{B})$, the scalar-valued function

$$\langle f, g\rangle_{\gamma,\delta} = \int_{\mathbb{B}} f(z)\overline{g(z)}\omega_{\gamma,\delta}(z) dV(z)$$

is a Hermitian inner product on $A^2_{\gamma,\delta}(\mathbb{B})$, and

$$\|f\|_{\gamma,\delta} = \sqrt{\langle f, f\rangle_{\gamma,\delta}}$$

defines the associated norm. It is not difficult to see that $A^2_{\gamma,\delta}(\mathbb{B})$ is a complete space, that is, a Hilbert space. When $\delta = 0$, the space $A^2_{\gamma,0}(\mathbb{B})$ is equivalent with the classical weighted Bergman space. For the operator T_g that acts on the classical weighted Bergman space, a complete characterization for the boundedness and compactness of the operator was given in [25]. For some generalizations of the results, see [12, 28]. For some generalizations of the operators T_g and I_g, see [19, 29, 30]. Therefore, our interest is in the case when $\delta < 0$.

In [21], Stević and Jiang investigated the following integral operator

$$f \mapsto \int_0^z f(\zeta) g(\zeta) d\zeta, \qquad z \in \mathbb{D},$$

on the space $A^2_{\gamma,\delta}(\mathbb{D})$, that is, on the unit disk $\mathbb{D}$. They obtained characterizations for the boundedness and compactness of this operator, as well as estimates for its essential norm. In the setting of the unit ball, the boundedness and compactness of a sum of product-type operators acting from the logarithmic weighted Bergman space into the weighted-type space were studied in [26].

Motivated by these studies, we aim to carry out an analysis of integral-type operators T_g and I_g on the space $A^2_{\gamma,\delta}(\mathbb{B})$. Our first goal is to give a characterization for the boundedness of T_g on $A^2_{\gamma,\delta}(\mathbb{B})$. We show that the boundedness of T_g is completely determined by a growth condition of the radial derivative function $\mathcal{R}g$. The second result is to establish an estimation for the essential norm of T_g. In general, the essential norm $\|T\|_e$ of a bounded operator T on a Hilbert space H is defined by

$$\|T\|_e = \inf\{\|T - K\|_{\text{op}} \ : \ K \text{ is compact on } H\}.$$

Here, $\|T\|_{\text{op}}$ denotes the operator norm of the bounded operator T. Since T is compact on H if and only if $\|T\|_e = 0$, it plays an important role in the study of the compactness of operators. By using suitable test functions and approximation arguments based on the homogeneous expansion of holomorphic functions, we have an estimation of the essential norm in terms of a boundary behavior of $\mathcal{R}g$. As

a consequence, we also obtain necessary and sufficient conditions for the compactness of T_g. These arguments crucially rely on estimating the norm of $A^2_{\gamma,\delta}(\mathbb{B})$ via integral expressions involving the radial derivative. Finally, we investigate the boundedness and the essential norm estimate of the operator I_g.

Throughout the paper, we denote $A \lesssim B$ if there is a positive constant C such that $A \le CB$. If both $A \lesssim B$ and $B \lesssim A$ hold, then one says that $A \asymp B$. We fix the parameters $\gamma \in (-1, +\infty)$ and $\delta \in (-\infty, 0)$ that define the space $A^2_{\gamma,\delta}(\mathbb{B})$.

2. The space $A^2_{\gamma,\delta}(\mathbb{B})$

Since the operator T_g satisfies $\mathcal{R}[T_g f](z) = f(z)\mathcal{R}g(z)$, it is useful to estimate the norm $\|f\|_{\gamma,\delta}$ in terms of the integral of the radial derivative $\mathcal{R}f$. First, we establish this norm estimate. As an application of it, we characterize the boundedness of T_g and estimate the essential norm of the operator in the next section.

Lemma 1. *For a fixed $r \in (0,1)$, if $z \in \mathbb{B}$ satisfies $r \le |z| < 1$, then*

$$\int_{|z|}^1 \omega_{\gamma,\delta}(t) t^{2N-1} \ln \frac{t}{|z|} dt \asymp \omega_{\gamma+2,\delta}(z). \tag{2.1}$$

Proof. The relation (2.1) is verified by the asymptotic relation $\ln 1/x \asymp 1 - x$ as $x \to 1$. □

The following lemma was proven in [21].

Lemma 2. *Let g be a nondecreasing and nonnegative function on $[0,1)$ and h be a positive function on $(0,1)$ which is continuous on $[0,1)$. Then, for each fixed $r_0 \in (0,1)$,*

$$\int_0^1 g(r)h(r)dr \asymp \int_{r_0}^1 g(r)h(r)dr.$$

Corollary 1. *For each $f \in H(\mathbb{B})$,*

$$\int_0^1 \omega_{\gamma,\delta}(r) r^{2N-1} dr \int_{\mathbb{S}} |f(r\zeta)|^2 d\sigma(\zeta) \asymp \int_{r_0}^1 \omega_{\gamma,\delta}(r) r^{2N-1} dr \int_{\mathbb{S}} |f(r\zeta)|^2 d\sigma(\zeta).$$

Proof. It is well known that the $\mathcal{M}$-subharmonicity of $|f|^2$ on $\mathbb{B}$ implies that the square integral mean

$$M_2(f,r)^2 := \int_{\mathbb{S}} |f(r\zeta)|^2 d\sigma(\zeta)$$

is an increasing function on r. Hence, this result is an immediate consequence of Lemma 2. □

The following lemma is a special case of Theorem 3.3 in [2] (see also Lemma 4 in [24]).

Lemma 3. *For any $f \in H(\mathbb{B})$ which is continuous on $\overline{\mathbb{B}}$,*

$$\int_{\mathbb{S}} |f(\zeta)|^2 d\sigma(\zeta) = |f(0)|^2 + \frac{2}{N} \int_{\mathbb{B}} \frac{|\mathcal{R}f(z)|^2}{|z|^{2N}} \ln \frac{1}{|z|} dV(z). \tag{2.2}$$

Theorem 1. *For $f \in H(\mathbb{B})$ and $r_0 \in (0,1)$, the following relations hold:*

$$\|f\|^2_{\gamma,\delta} \lesssim \frac{4}{r_0^{2(N-1)}} \int_{\mathbb{B}\setminus r_0\overline{\mathbb{B}}} \frac{|\mathcal{R}f(z)|^2}{|z|^2} \omega_{\gamma+2,\delta}(z) dV(z) + \left(\frac{2N\omega_{\gamma+2,\delta}(r_0)}{\ln \frac{1+r_0}{2r_0}} + \int_{\mathbb{B}\setminus r_0\overline{\mathbb{B}}} \omega_{\gamma,\delta}(z) dV(z) \right) \int_{\mathbb{S}} |f(r'\zeta)|^2 d\sigma(\zeta), \tag{2.3}$$

where $r' = \frac{1+r_0}{2}$, and

$$\|f\|^2_{\gamma,\delta} \gtrsim |f(0)|^2 \int_{\mathbb{B}\setminus r_0\overline{\mathbb{B}}} \omega_{\gamma,\delta}(z) dV(z) + 4 \int_{\mathbb{B}\setminus r_0\overline{\mathbb{B}}} \frac{|\mathcal{R}f(z)|^2}{|z|^2} \omega_{\gamma+2,\delta}(z) dV(z). \tag{2.4}$$

Proof. Let $f \in H(\mathbb{B})$ and $r_0 \in (0,1)$. By using the polar coordinates (see [15, p.13]) and (2.2), we obtain

$$\int_{r\mathbb{B}} \frac{|\mathcal{R}f(z)|^2}{|z|^{2N}} \ln \frac{r}{|z|} dV(z) = \frac{N}{2} \left(\int_{\mathbb{S}} |f(r\zeta)|^2 d\sigma(\zeta) - |f(0)|^2 \right) \tag{2.5}$$

for each $r \in (0,1)$.

Hence, from Corollary 1, it follows that

$$\|f\|^2_{\gamma,\delta} \asymp |f(0)|^2 \int_{\mathbb{B}\setminus r_0\overline{\mathbb{B}}} \omega_{\gamma,\delta}(w) dV(w) + 4 \int_{r_0}^1 \omega_{\gamma,\delta}(r) r^{2N-1} dr \int_{r\mathbb{B}} \frac{|\mathcal{R}f(w)|^2}{|w|^{2N}} \ln \frac{r}{|w|} dV(w). \tag{2.6}$$

By applying the polar coordinates to the inner integral in the second integral in relation (2.6), it follows that

$$\begin{aligned}
&\int_{r_0}^1 \omega_{\gamma,\delta}(r) r^{2N-1} dr \int_{r\mathbb{B}} \frac{|\mathcal{R}f(w)|^2}{|w|^{2N}} \ln \frac{r}{|w|} dV(w) \\
&= 2N \int_{r_0}^1 \omega_{\gamma,\delta}(r) r^{2N-1} dr \int_0^r \rho^{2N-1} d\rho \int_{\mathbb{S}} \frac{|\mathcal{R}f(\rho\eta)|^2}{\rho^{2N}} \ln \frac{r}{\rho} d\sigma(\eta) \\
&= 2N \int_0^{r_0} \int_{\mathbb{S}} \frac{|\mathcal{R}f(\rho\eta)|^2}{\rho^{2N}} d\sigma(\eta) \rho^{2N-1} d\rho \int_{r_0}^1 \omega_{\gamma,\delta}(r) \ln \frac{r}{\rho} r^{2N-1} dr \\
&+ 2N \int_{r_0}^1 \int_{\mathbb{S}} \frac{|\mathcal{R}f(\rho\eta)|^2}{\rho^{2N}} d\sigma(\eta) \rho^{2N-1} d\rho \int_{\rho}^1 \omega_{\gamma,\delta}(r) \ln \frac{r}{\rho} r^{2N-1} dr.
\end{aligned} \tag{2.7}$$

By Fubini's theorem, we have

$$\begin{aligned}
&2N \int_0^{r_0} \int_{\mathbb{S}} \frac{|\mathcal{R}f(\rho\eta)|^2}{\rho^{2N}} d\sigma(\eta) \rho^{2N-1} d\rho \int_{r_0}^1 \omega_{\gamma,\delta}(r) \ln \frac{r}{\rho} r^{2N-1} dr \\
&= \int_{r_0}^1 \omega_{\gamma,\delta}(r) r^{2N-1} dr \int_{r_0\overline{\mathbb{B}}} \frac{|\mathcal{R}f(z)|^2}{|z|^{2N}} \ln \frac{r}{|z|} dV(z) \\
&= \int_{r_0\overline{\mathbb{B}}} \frac{|\mathcal{R}f(z)|^2}{|z|^{2N}} dV(z) \int_{r_0}^1 \omega_{\gamma,\delta}(r) r^{2N-1} \ln \frac{r}{|z|} dr.
\end{aligned} \tag{2.8}$$

Lemma 1 gives

$$\int_{r_0}^{1}\omega_{\gamma,\delta}(r)r^{2N-1}\ln\frac{r}{|z|}dr=\int_{r_0}^{1}\omega_{\gamma,\delta}(r)r^{2N-1}\left(\ln\frac{r}{r_0}+\ln\frac{r_0}{|z|}\right)dr$$
$$\asymp\omega_{\gamma+2,\delta}(r_0)+\int_{r_0}^{1}\omega_{\gamma,\delta}(r)r^{2N-1}\ln\frac{r_0}{|z|}dr. \tag{2.9}$$

Lemma 3 shows

$$\begin{aligned}\omega_{\gamma+2,\delta}(r_0)\int_{r_0\overline{\mathbb{B}}}\frac{|\mathcal{R}f(z)|^2}{|z|^{2N}}dV(z)&=\omega_{\gamma+2,\delta}(r_0)\int_{r_0\overline{\mathbb{B}}}\frac{|\mathcal{R}f(z)|^2}{|z|^{2N}}\frac{\ln\frac{r'}{r_0}}{\ln\frac{r'}{r_0}}dV(z)\\&\le\frac{\omega_{\gamma+2,\delta}(r_0)}{\ln\frac{r'}{r_0}}\int_{r_0\overline{\mathbb{B}}}\frac{|\mathcal{R}f(z)|^2}{|z|^{2N}}\ln\frac{r'}{|z|}dV(z)\\&\le\frac{\omega_{\gamma+2,\delta}(r_0)}{\ln\frac{r'}{r_0}}\int_{r'\overline{\mathbb{B}}}\frac{|\mathcal{R}f(z)|^2}{|z|^{2N}}\ln\frac{r'}{|z|}dV(z)\\&=\frac{N\omega_{\gamma+2,\delta}(r_0)}{2\ln\frac{1+r_0}{2r_0}}\left(\int_{\mathbb{S}}|f(r'\zeta)|^2d\sigma(\zeta)-|f(0)|^2\right).\end{aligned} \tag{2.10}$$

The polar coordinates, Lemma 3, and the monotonicity of the integral means show

$$\begin{aligned}&\int_{r_0\overline{\mathbb{B}}}\frac{|\mathcal{R}f(z)|^2}{|z|^{2N}}dV(z)\int_{r_0}^{1}\omega_{\gamma,\delta}(r)r^{2N-1}\ln\frac{r_0}{|z|}dr\\&=\frac{\int_{\mathbb{B}\setminus r_0\overline{\mathbb{B}}}\omega_{\gamma,\delta}(z)dV(z)}{4}\left(\int_{\mathbb{S}}|f(r_0\zeta)|^2d\sigma(\zeta)-|f(0)|^2\right)\\&\le\frac{\int_{\mathbb{B}\setminus r_0\overline{\mathbb{B}}}\omega_{\gamma,\delta}(z)dV(z)}{4}\left(\int_{\mathbb{S}}|f(r'\zeta)|^2d\sigma(\zeta)-|f(0)|^2\right).\end{aligned} \tag{2.11}$$

Thus, from (2.8)–(2.11), it follows that

$$\begin{aligned}&2N\int_0^{r_0}\int_{\mathbb{S}}\frac{|\mathcal{R}f(\rho\eta)|^2}{\rho^{2N}}d\sigma(\eta)\rho^{2N-1}d\rho\int_{r_0}^{1}\omega_{\gamma,\delta}(r)\ln\frac{r}{\rho}r^{2N-1}dr\\&\lesssim\left(\frac{N\omega_{\gamma,\delta+2}(r_0)}{2\ln\frac{1+r_0}{2r_0}}+\frac{\int_{\mathbb{B}\setminus r_0\overline{\mathbb{B}}}\omega_{\gamma,\delta}(z)dV(z)}{4}\right)\left(\int_{\mathbb{S}}|f(r'\zeta)|^2d\sigma(\zeta)-|f(0)|^2\right).\end{aligned} \tag{2.12}$$

By employing Lemma 1, we have

$$\begin{aligned}&2N\int_{r_0}^{1}\int_{\mathbb{S}}\frac{|\mathcal{R}f(\rho\eta)|^2}{\rho^{2N}}d\sigma(\eta)\rho^{2N-1}d\rho\int_{\rho}^{1}\omega_{\gamma,\delta}(r)\ln\frac{r}{\rho}r^{2N-1}dr\\&\asymp 2N\int_{r_0}^{1}\rho^{2N-1}\omega_{\gamma+2,\delta}(\rho)d\rho\int_{\mathbb{S}}\frac{|\mathcal{R}f(\rho\eta)|^2}{\rho^{2N}}d\sigma(\eta)\\&=\int_{\mathbb{B}\setminus r_0\overline{\mathbb{B}}}\frac{|\mathcal{R}f(z)|^2}{|z|^{2N}}\omega_{\gamma+2,\delta}(z)dV(z)\le\frac{1}{{r_0}^{2(N-1)}}\int_{\mathbb{B}\setminus r_0\overline{\mathbb{B}}}\frac{|\mathcal{R}f(z)|^2}{|z|^2}\omega_{\gamma+2,\delta}(z)dV(z).\end{aligned} \tag{2.13}$$

By (2.6), (2.7), (2.12) and (2.13), we obtain the desired upper estimate (2.3) for $\|f\|^2_{\gamma,\delta}$.

Lemma 1, Fubini's theorem, and equations (2.6) and (2.7), also imply that

$$\begin{aligned}\|f\|^2_{\gamma,\delta} \gtrsim & |f(0)|^2 \int_{\mathbb{B}\setminus r_0\overline{\mathbb{B}}} \omega_{\gamma,\delta}(w)dV(w) \\ & + 8N \int_{r_0}^1 \rho^{2N-1}\omega_{\gamma+2,\delta}(\rho)d\rho \int_{\mathbb{S}} \frac{|\mathcal{R}f(\rho\eta)|^2}{\rho^{2N}} d\sigma(\eta) \\ \geq & |f(0)|^2 \int_{\mathbb{B}\setminus r_0\overline{\mathbb{B}}} \omega_{\gamma,\delta}(w)dV(w) + 4 \int_{\mathbb{B}\setminus r_0\overline{\mathbb{B}}} \frac{|\mathcal{R}f(z)|^2}{|z|^2}\omega_{\gamma+2,\delta}(z)dV(z),\end{aligned}$$

thus completing the proof. □

As an immediate consequence, we also obtain the following characterization for $A^2_{\gamma,\delta}(\mathbb{B})$.

Corollary 2. *For each $f \in H(\mathbb{B})$, f belongs to $A^2_{\gamma,\delta}(\mathbb{B})$ if and only if*

$$\int_{\mathbb{B}} |\mathcal{R}f(z)|^2\omega_{\gamma+2,\delta}(z)dV(z) < +\infty.$$

Proposition 1. *For each $f \in A^2_{\gamma,\delta}(\mathbb{B})$, it holds that*

$$|f(z)|^2 \lesssim \frac{1}{(1-|z|)^{N+1}\omega_{\gamma,\delta}(z)}\|f\|^2_{\gamma,\delta}$$

for $z \in \mathbb{B} \setminus \{0\}$.

Proof. Fix $z \in \mathbb{B} \setminus \{0\}$ and $0 < r < 1$, and put $B(z, r) = \varphi_z(r\mathbb{B})$ where φ_z is the ball automorphism that interchanges z and 0. By the subharmonicity of $|f|^2$, we have

$$|f(0)|^2 \leq \frac{1}{V(r\mathbb{B})} \int_{r\mathbb{B}} |f(w)|^2 dV(w).$$

Since $f(z) = f(\varphi_z(0))$, a change of variables shows

$$\begin{aligned}|f(z)|^2 &\leq \frac{1}{V(r\mathbb{B})} \int_{B(z,r)} |f(w)|^2 \frac{(1-|z|^2)^{N+1}}{|1-\langle z, w\rangle|^{2(N+1)}} dV(w) \\ &\leq \frac{2^{N+1}}{V(r\mathbb{B})(1-|z|)^{N+1}} \int_{B(z,r)} |f(w)|^2 dV(w).\end{aligned}$$

Thus, if we take $r = 1/2$, then we obtain

$$|f(z)|^2 \leq \frac{2^{3N+1}}{(1-|z|)^{N+1}} \int_{B(z,1/2)} |f(w)|^2 dV(w). \tag{2.14}$$

For each $w \in B(z, 1/2)$, put $\zeta = \varphi_z(w)$. Since φ_z is an involution, we see that $w = \varphi_z(\zeta)$ and $|\zeta| < 1/2$. Hence, we have

$$1 - |w|^2 = \frac{(1-|z|^2)(1-|\zeta|^2)}{|1-\langle \zeta, z\rangle|^2} \leq \frac{1+|\zeta|}{1-|\zeta|}(1-|z|^2) \leq 3(1-|z|^2).$$

Since $w \in B(z, 1/2)$ if and only if $z \in B(w, 1/2)$, we also have

$$1 - |z|^2 \leq 3(1 - |w|^2),$$

and so

$$\frac{1}{3}(1 - |z|^2) \leq 1 - |w|^2 \leq 3(1 - |z|^2).$$

This estimate shows that $\omega_{\gamma,\delta}(z) \asymp \omega_{\gamma,\delta}(w)$ for $w \in B(z, 1/2)$. Hence, we obtain

$$\omega_{\gamma,\delta}(z) \int_{B(z,1/2)} |f(w)|^2 dV(w) \asymp \int_{B(z,1/2)} |f(w)|^2 \omega_{\gamma,\delta}(w) dV(w) \leq \|f\|^2_{\gamma,\delta}. \tag{2.15}$$

By (2.14) and (2.15), we have

$$|f(z)|^2 \lesssim \frac{1}{(1 - |z|)^{N+1} \omega_{\gamma,\delta}(z)} \|f\|^2_{\gamma,\delta},$$

which finishes the proof. □

For the case when $z = 0$, the maximum modulus principle for f and Proposition 1, show that

$$|f(0)| \leq \max_{|z|=\frac{1}{2}} |f(z)| \lesssim \max_{|z|=\frac{1}{2}} \left(\frac{1}{(1 - |z|)^{N+1} \omega_{\gamma,\delta}(z)} \right)^{1/2} \|f\|_{\gamma,\delta}.$$

For a fixed $z \in \mathbb{B}$, Proposition 1 shows that the point evaluation $\Lambda_z : f \mapsto f(z)$ at z is a bounded linear functional on $A^2_{\gamma,\delta}(\mathbb{B})$. Hence, by the Riesz representation theorem [7], there is a unique kernel function $K_{\gamma,\delta,z} \in A^2_{\gamma,\delta}(\mathbb{B}) \setminus \{0\}$ such that

$$f(z) = \Lambda_z(f) = \int_{\mathbb{B}} f(w) \overline{K_{\gamma,\delta,z}(w)} \omega_{\gamma,\delta}(w) dV(w), \tag{2.16}$$

for $f \in A^2_{\gamma,\delta}(\mathbb{B})$ (see also [15, 16] for related representations).

Applying this reproducing formula to the kernel function $K_{\gamma,\delta,z}$ implies that

$$K_{\gamma,\delta,z}(z) = \|K_{\gamma,\delta,z}\|^2_{\gamma,\delta} > 0. \tag{2.17}$$

From (2.17), and using Proposition 1, we obtain

$$\|K_{\gamma,\delta,z}\|_{\gamma,\delta} \lesssim \left(\frac{1}{(1 - |z|)^{N+1} \omega_{\gamma,\delta}(z)} \right)^{1/2}$$

for each $z \in \mathbb{B}$.

Lemma 4. *$K_{\gamma,\delta,z}$ has the expansion:*

$$K_{\gamma,\delta,z}(w) = \sum_{k=0}^{+\infty} \frac{(N + k - 1)!}{2N!k!} \frac{\overline{\langle z, w \rangle}^k}{\int_0^1 r^{2(k+N)-1} \omega_{\gamma,\delta}(r) dr}, \qquad w \in \mathbb{B}.$$

Proof. For each multi-index $\alpha \in \mathbb{N}_0^N$, a straightforward calculation with the polar coordinates shows that

$$\|z^\alpha\|_{\gamma,\delta}^2 = \frac{2N!\alpha!}{(N+|\alpha|-1)!}\int_0^1 r^{2(|\alpha|+N)-1}\omega_{\gamma,\delta}(r)dr$$

and $\{z^\alpha/\|z^\alpha\|_{\gamma,\delta}\}$ forms a complete orthonormal set of $A^2_{\gamma,\delta}(\mathbb{B})$. Thus, we have

$$\begin{aligned}
K_{\gamma,\delta,z}(w) &= \sum_{k=0}^{\infty}\sum_{|\alpha|=k}\frac{\langle K_{\gamma,\delta,z}, w^\alpha\rangle_{\gamma,\delta}}{\|w^\alpha\|_{\gamma,\delta}^2}w^\alpha \\
&= \sum_{k=0}^{\infty}\sum_{|\alpha|=k}\frac{(N+|\alpha|-1)!}{2N!\alpha!}\frac{\overline{z^\alpha}w^\alpha}{\int_0^1 r^{2(|\alpha|+N)-1}\omega_{\gamma,\delta}(r)dr} \\
&= \sum_{k=0}^{\infty}\frac{(N+k-1)!}{2N!k!\int_0^1 r^{2(k+N)-1}\omega_{\gamma,\delta}(r)dr}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\overline{z^\alpha}w^\alpha \\
&= \sum_{k=0}^{\infty}\frac{(N+k-1)!}{2N!k!}\frac{\overline{\langle z,w\rangle}^k}{\int_0^1 r^{2(k+N)-1}\omega_{\gamma,\delta}(r)dr}.
\end{aligned}$$

This is a desired expansion formula for the kernel function $K_{\gamma,\delta,z}$. □

Lemma 5. *For each $a \in \mathbb{B}\setminus\{0\}$, put*

$$f_a(z) = \frac{(1-|a|^2)^{-\frac{\delta}{2}}}{(1-\langle z,a\rangle)^{\frac{\gamma-\delta+N+1}{2}}}\left[\ln\left(1-\frac{1}{\ln|a|}\right)\right]^{-\frac{\delta}{2}}, \qquad z\in\mathbb{B}. \tag{2.18}$$

Then, $\|f_a\|_{\gamma,\delta} \asymp 1$ for $a \in \mathbb{B}\setminus\{0\}$ which is sufficiently close to $\mathbb{S}$, and $f_a \to 0$ uniformly on compact subsets of $\mathbb{B}$ as $|a| \to 1^-$.

Proof. Since the real valued function

$$h(x) = (1-x)\ln\Big(1-\frac{1}{\ln x}\Big), \quad (0<x<1)$$

satisfies the relations

$$\lim_{x\to+0}h(x) = 0 \quad \text{and} \quad \lim_{x\to1^-}h(x) = 0,$$

we easily see that $f_a \to 0$ uniformly on compact subsets of $\mathbb{B}$ as $|a| \to 1^-$.

On the other hand, since

$$\omega_{\gamma,\delta}(a) \asymp (1-|a|)^\gamma\Big(\ln\Big(1-\frac{1}{\ln|a|}\Big)\Big)^\delta,$$

it follows from Proposition 1 that

$$\Big(\omega_{\gamma,\delta}(a)(1-|a|)^{N+1}\Big)^{-1} \asymp |f_a(a)|^2 \lesssim \Big(\omega_{\gamma,\delta}(a)(1-|a|)^{N+1}\Big)^{-1}\|f_a\|_{\gamma,\delta}^2,$$

and so $1 \lesssim \|f_a\|_{\gamma,\delta}$. The upper estimate $\|f_a\|_{\gamma,\delta} \lesssim 1$ can be found in [26, Theorem 3.4]. Hence, we finish the proof. □

Lemma 6. *The functions f_a defined in* (2.18) *converge to* 0 *as $|a| \to 1$ weakly in $A^2_{\gamma,\delta}(\mathbb{B})$.*

Proof. Take $h \in A^2_{\gamma,\delta}(\mathbb{B})$ and $\varepsilon > 0$. Since $\overline{h} \in L^2(\omega_{\gamma,\delta}dV)$, we can choose a continuous function g with compact support such that

$$\left[\int_{\mathbb{B}} \left|\overline{h(z)} - g(z)\right|^2 \omega_{\gamma,\delta}(z)dV(z)\right]^{1/2} < \varepsilon.$$

Hence, we obtain

$$\begin{aligned}|\langle f_a, h\rangle_{\gamma,\delta}| &\leq \int_{\mathbb{B}} |f_a(z)| \left|\overline{h(z)} - g(z)\right| \omega_{\gamma,\delta}(z)dV(z) + \int_{\mathbb{B}} |f_a(z)||g(z)|\omega_{\gamma,\delta}(z)dV(z)\\ &\leq \varepsilon\|f_a\|_{\gamma,\delta} + \int_{\mathbb{B}} |f_a(z)||g(z)|\omega_{\gamma,\delta}(z)dV(z).\end{aligned}$$

Since g has a compact support $\mathrm{supp}(g)$, $\{f_a\}$ is a bounded family in $A^2_{\gamma,\delta}(\mathbb{B})$, and $f_a \to 0$ as $|a| \to 1$ uniformly on $\mathrm{supp}(g)$, it holds that

$$\limsup_{|a|\to 1} |\langle f_a, h\rangle_{\gamma,\delta}| \lesssim \varepsilon. \tag{2.19}$$

Since ε is an arbitrary positive number, from (2.19), we get $\lim_{|a|\to 1}\langle f_a, h\rangle_{\gamma,\delta} = 0$, for each $h \in A^2_{\gamma,\delta}(\mathbb{B})$, from which, together with the Riesz representation theorem, we have $f_a \to 0$ weakly in $A^2_{\gamma,\delta}(\mathbb{B})$. □

3. Boundedness, essential norm and compactness of T_g

Theorem 2. *Let $g \in H(\mathbb{B})$. Then, T_g is bounded on $A^2_{\gamma,\delta}(\mathbb{B})$ if and only if g satisfies the condition*

$$\sup_{z\in\mathbb{B}}(1 - |z|)|\mathcal{R}g(z)| < +\infty. \tag{3.1}$$

Proof. First, assume that g satisfies (3.1) and fix $r_0 \in (1/2, 1)$ arbitrarily. In view of Theorem 1, we may prove that

$$\int_{\mathbb{B}\setminus r_0\overline{\mathbb{B}}} \frac{|\mathcal{R}[T_g f](z)|^2}{|z|^2}\omega_{\gamma+2,\delta}(z)dV(z)$$

and

$$\int_{\mathbb{S}} |T_g f(r'\zeta)|^2 d\sigma(\zeta)$$

are dominated by $\|f\|^2_{\gamma,\delta}$. Here, r' denotes $\frac{1+r_0}{2}$.

Since $\mathcal{R}[T_g f](z) = f(z)\mathcal{R}g(z)$, Corollary 1 gives

$$\begin{aligned}\int_{\mathbb{B}\setminus r_0\overline{\mathbb{B}}} \frac{|\mathcal{R}[T_g f](z)|^2}{|z|^2}\omega_{\gamma+2,\delta}(z)dV(z) &\asymp \int_{\mathbb{B}\setminus r_0\overline{\mathbb{B}}} \frac{|f(z)\mathcal{R}g(z)|^2}{|z|^2}(1 - |z|)^2\omega_{\gamma,\delta}(z)dV(z)\\ &\leq \left(\frac{\sup_{z\in\mathbb{B}}(1 - |z|)|\mathcal{R}g(z)|}{r_0}\right)^2 \int_{\mathbb{B}\setminus r_0\overline{\mathbb{B}}} |f(z)|^2\omega_{\gamma,\delta}(z)dV(z)\\ &\asymp \left(\frac{\sup_{z\in\mathbb{B}}(1 - |z|)|\mathcal{R}g(z)|}{r_0}\right)^2 \|f\|^2_{\gamma,\delta}.\end{aligned} \tag{3.2}$$

On the other hand, since $T_g f(0) = 0$ and

$$|g(0)|^2 \le \int_{\mathbb{S}} |g(r'\zeta)|^2 d\sigma(\zeta) < +\infty,$$

we can derive from (2.5) that

$$\begin{aligned}\int_{\mathbb{S}} |T_g f(r'\zeta)|^2 d\sigma(\zeta) &= \frac{2}{N} \int_{r'\mathbb{B}} \frac{|f(z)\mathcal{R}g(z)|^2}{|z|^{2N}} \ln \frac{r'}{|z|} dV(z) \\ &\le \max_{w \in r'\overline{\mathbb{B}}} |f(z)|^2 \left(\int_{\mathbb{S}} |g(r'\zeta)|^2 d\sigma(\zeta) - |g(0)|^2 \right).\end{aligned}$$

By combining this estimate with Proposition 1, we have

$$\int_{\mathbb{S}} |T_g f(r'\zeta)|^2 d\sigma(\zeta) \lesssim \max_{z \in r'\overline{\mathbb{B}}} \left(\frac{1}{(1-|z|)^{N+1} \omega_{\gamma,\delta}(z)} \right) \left(\int_{\mathbb{S}} |g(r'\zeta)|^2 d\sigma(\zeta) - |g(0)|^2 \right) \|f\|^2_{\gamma,\delta}. \tag{3.3}$$

From (2.3), (3.2) and (3.3), we have $\|T_g f\|^2_{\gamma,\delta} \lesssim \|f\|^2_{\gamma,\delta}$ for $f \in A^2_{\gamma,\delta}(\mathbb{B})$, and so $T_g : A^2_{\gamma,\delta}(\mathbb{B}) \to A^2_{\gamma,\delta}(\mathbb{B})$ is bounded.

Next, we assume that T_g is bounded on $A^2_{\gamma,\delta}(\mathbb{B})$. For a fixed $a \in \mathbb{B}$ we consider the function f_a in (2.18). Since the boundedness of the operator and Theorem 1 imply that $\mathcal{R}[T_g f_a] \in A^2_{\gamma+2,\delta}(\mathbb{B})$, an application of Proposition 1 to $\mathcal{R}[T_g f_a]$ shows that

$$\begin{aligned}(1-|a|)|\mathcal{R}g(a)||f_a(a)| &\lesssim \left(\frac{(1-|a|)^2}{(1-|a|)^{N+1} \omega_{\gamma+2,\delta}(a)} \right)^{1/2} \|\mathcal{R}[T_g f_a]\|_{\gamma+2,\delta} \\ &\lesssim \left(\frac{1}{(1-|a|)^{N+1} \omega_{\gamma,\delta}(a)} \right)^{1/2} \|\mathcal{R}[T_g f_a]\|_{\gamma+2,\delta}. \end{aligned} \tag{3.4}$$

From the proof of Lemma 5, we have

$$|f_a(a)| \asymp ((1-|a|)^{N+1} \omega_{\gamma,\delta}(a))^{-1/2}. \tag{3.5}$$

From Corollary 1 and (2.4), it follows that

$$\begin{aligned}\|\mathcal{R}[T_g f_a]\|^2_{\gamma+2,\delta} &\asymp \int_{\mathbb{B} \setminus r_0 \overline{\mathbb{B}}} |\mathcal{R}[T_g f_a](w)|^2 \omega_{\gamma+2,\delta}(w) dV(w) \\ &\le \int_{\mathbb{B} \setminus r_0 \overline{\mathbb{B}}} \frac{|\mathcal{R}[T_g f_a](w)|^2}{|w|^2} \omega_{\gamma+2,\delta}(w) dV(w) \\ &\lesssim \|T_g f_a\|^2_{\gamma,\delta} \end{aligned} \tag{3.6}$$

for a fixed $r_0 \in (1/2, 1)$. From (3.4)–(3.6), we have

$$(1-|a|)|\mathcal{R}g(a)| \lesssim \|\mathcal{R}[T_g f_a]\|_{\gamma+2,\delta} \lesssim \|T_g f_a\|_{\gamma,\delta} \lesssim \|T_g\|_{\text{op}}, \tag{3.7}$$

where $\|T_g\|_{\text{op}}$ denotes the operator norm of $T_g : A^2_{\gamma,\delta}(\mathbb{B}) \to A^2_{\gamma,\delta}(\mathbb{B})$. Since this estimate is uniform on $a \in \mathbb{B} \setminus \{0\}$, and obviously holds for $a = 0$, we get (3.1), accomplishing the proof. □

Next, we will estimate the essential norm $\|T_g\|_e$ of the operator T_g. In order to estimate it, we need some results on an approximation by the homogeneous polynomial for $f \in A^2_{\gamma,\delta}(\mathbb{B})$.

For $f \in H(\mathbb{B})$, let

$$f(z) = \sum_{k=0}^{+\infty} \sum_{|\alpha|=k} c_\alpha z^\alpha,$$

where $\alpha = (\alpha_1, \ldots, \alpha_N)$ is a multi-index, be its homogeneous expansion.

For each positive integer n, put

$$P_n f(z) = \sum_{k=0}^{n-1} \sum_{|\alpha|=k} c_\alpha z^\alpha \quad \text{and} \quad Q_n f(z) = \sum_{k=n}^{+\infty} \sum_{|\alpha|=k} c_\alpha z^\alpha.$$

Then, P_n and Q_n are bounded operators on $A^2_{\gamma,\delta}(\mathbb{B})$.

Proposition 2. *For $f \in A^2_{\gamma,\delta}(\mathbb{B})$, it holds that*

$$\lim_{n\to+\infty} \|P_n f - f\|_{\gamma,\delta} = 0.$$

Proof. For each $r \in (0,1)$ and $\zeta \in \mathbb{S}$, we denote $f_r(z) = f(rz)$ $(z \in \mathbb{B})$ and $f_\zeta(\lambda) = f(\lambda\zeta)$ $(\lambda \in \mathbb{C}, |\lambda| < 1)$. Since $(f_r)_\zeta$ belongs to the disk algebra, and so the Hardy space H^2, the boundedness of the Szegö projection on L^2 implies that

$$\int_0^{2\pi} |P_n[(f_r)_\zeta](e^{i\theta})|^2 \frac{d\theta}{2\pi} \le C \int_0^{2\pi} |(f_r)_\zeta(e^{i\theta})|^2 \frac{d\theta}{2\pi}$$

for some positive constant C which is independent of n (see [27, Proposition 1 and Corollary 3]). Applying integration by slices in this inequality gives

$$\int_{\mathbb{S}} |P_n f(r\zeta)|^2 d\sigma(\zeta) \le C \int_{\mathbb{S}} |f(r\zeta)|^2 d\sigma(\zeta).$$

Using polar coordinates, we also obtain that $\|P_n f\|^2_{\gamma,\delta} \le C\|f\|^2_{\gamma,\delta}$, that is, $\|P_n\|_{\mathrm{op}} \le C$ for any positive integer n. By [27, Proposition 1], we see that $\|P_n f - f\|_{\gamma,\delta} \to 0$ as $n \to +\infty$. □

Corollary 3. $\sup_{n\ge1} \|Q_n\|_{\mathrm{op}} < +\infty$.

Proof. By Proposition 2, $Q_n f \to 0$ in $A^2_{\gamma,\delta}(\mathbb{B})$ as $n \to +\infty$, for each $f \in A^2_{\gamma,\delta}(\mathbb{B})$. Thus, this results can be verified by the uniform boundedness principle. □

Theorem 3. *Let $g \in H(\mathbb{B})$. Suppose that $T_g : A^2_{\gamma,\delta}(\mathbb{B}) \to A^2_{\gamma,\delta}(\mathbb{B})$ is bounded. Then, the following asymptotic relation holds:*

$$\|T_g\|_e \asymp \limsup_{|z|\to1^-} (1-|z|)|\mathcal{R}g(z)|.$$

Hence, T_g is compact on $A^2_{\gamma,\delta}(\mathbb{B})$ if and only if

$$\lim_{|z|\to1^-} (1-|z|)|\mathcal{R}g(z)| = 0.$$

Proof. First, we prove the lower estimate

$$\|T_g\|_e \gtrsim \limsup_{|z|\to 1^-}(1-|z|)|\mathcal{R}g(z)|. \tag{3.8}$$

Let $K : A^2_{\gamma,\delta}(\mathbb{B}) \to A^2_{\gamma,\delta}(\mathbb{B})$ be an arbitrary compact operator. Let f_a be the functions in (2.18). Since $f_a \to 0$ weakly as $|a| \to 1^-$, we see that $\|Kf_a\|_{\gamma,\delta} \to 0$ as $|a| \to 1^-$ and

$$\|T_g - K\|_{\mathrm{op}} \geq \limsup_{|a|\to 1^-} \|T_g f_a\|_{\gamma,\delta}. \tag{3.9}$$

From (3.7), we have

$$(1-|a|)|\mathcal{R}g(a)| \lesssim \|T_g f_a\|_{\gamma,\delta}.$$

By combining this with (3.9), we have

$$\|T_g - K\|_{\mathrm{op}} \gtrsim \limsup_{|a|\to 1^-}(1-|a|)|\mathcal{R}g(a)|.$$

Since a compact operator $K : A^2_{\gamma,\delta}(\mathbb{B}) \to A^2_{\gamma,\delta}(\mathbb{B})$ is arbitrary, we obtain the lower estimate (3.8).

Next, we prove the upper estimate:

$$\|T_g\|_e \lesssim \limsup_{|z|\to 1^-}(1-|z|)|\mathcal{R}g(z)|. \tag{3.10}$$

Since $T_g = T_g(P_n + Q_n)$ and P_n is compact on $A^2_{\gamma,\delta}(\mathbb{B})$, we obtain

$$\|T_g\|_e \leq \liminf_{n\to+\infty} \|T_g Q_n\|_{\mathrm{op}}. \tag{3.11}$$

Take $f \in A^2_{\gamma,\delta}(\mathbb{B})$ with $\|f\|_{\gamma,\delta} \leq 1$ and fix $r_0 \in (0,1)$. By (2.3), we have that $\|T_g Q_n f\|^2_{\gamma,\delta}$ is dominated by the sum of

$$T_1(r_0,n,f) := \frac{4}{r_0^{2(N-1)}} \int_{\mathbb{B}\setminus r_0\overline{\mathbb{B}}} \frac{|\mathcal{R}[T_g Q_n f](z)|^2}{|z|^2} \omega_{\gamma+2,\delta}(z)dV(z)$$

and

$$T_2(r_0,n,f) := \left(\frac{2N\omega_{\gamma+2,\delta}(r_0)}{\ln\frac{1+r_0}{2r_0}} + \int_{\mathbb{B}\setminus r_0\overline{\mathbb{B}}} \omega_{\gamma,\delta}(z)dV(z) \right) \int_{\mathbb{S}} |T_g Q_n f(r'\zeta)|^2 d\sigma(\zeta),$$

where $r' = \frac{r_0+1}{2}$. Corollary 1 gives

$$\begin{aligned} T_1(r_0,n,f) &\asymp \frac{4}{r_0^{2(N-1)}} \int_{\mathbb{B}\setminus r_0\overline{\mathbb{B}}} \frac{|\mathcal{R}g(z)|^2 |Q_n f(z)|^2}{|z|^2} (1-|z|^2)^2 \omega_{\gamma,\delta}(z)dV(z) \\ &\leq \frac{4}{r_0^{2N}} \sup_{|z|>r_0}(1-|z|^2)^2|\mathcal{R}g(z)|^2 \int_{\mathbb{B}\setminus r_0\overline{\mathbb{B}}} |Q_n f(z)|^2 \omega_{\gamma,\delta}(z)dV(z) \\ &\asymp \frac{4}{r_0^{2N}} \sup_{|z|>r_0}(1-|z|^2)^2|\mathcal{R}g(z)|^2 \|Q_n f\|^2_{\gamma,\delta}. \end{aligned} \tag{3.12}$$

By noting $T_g Q_n f(0) = 0$, Lemma 3 gives

$$\int_{\mathbb{S}} |T_g Q_n f(r'\zeta)|^2 d\sigma(\zeta) = \frac{2}{N} \int_{\mathbb{B}} \frac{|\mathcal{R}[(T_g Q_n f)_{r'}](z)|^2}{|z|^{2N}} \ln\frac{1}{|z|} dV(z).$$

By the change of variables $w = r'z$, we obtain

$$\int_{\mathbb{B}} \frac{|\mathcal{R}[(T_g Q_n f)_{r'}](z)|^2}{|z|^{2N}} \ln \frac{1}{|z|} dV(z) = \int_{r'\mathbb{B}} \frac{|Q_n f(w)\mathcal{R}g(w)|^2}{|w|^{2N}} \ln \frac{r'}{|w|} dV(w).$$

Hence, by an application of Lemma 3 once again, we obtain

$$\begin{aligned}\int_{\mathbb{S}} |T_g Q_n f(r'\zeta)|^2 d\sigma(\zeta) &\leq \max_{|w|\leq r'} |Q_n f(w)|^2 \frac{2}{N} \int_{r'\mathbb{B}} \frac{|\mathcal{R}g(w)|^2}{|w|^{2N}} \ln \frac{r'}{|w|} dV(w) \\ &= \max_{|w|\leq r'} |Q_n f(w)|^2 \left(\int_{\mathbb{S}} |g(r'\zeta)|^2 d\sigma(\zeta) - |g(0)|^2 \right). \end{aligned} \tag{3.13}$$

Since the orthogonality of $\{z^\alpha : \alpha \in \mathbb{N}_0^N\}$ with respect to the weighted measure $\omega_{\gamma,\delta}(\cdot)dV$ indicates that Q_n is self-adjoint on $A^2_{\gamma,\delta}(\mathbb{B})$, the reproducing formula (2.16) shows

$$|Q_n f(w)| \leq \int_{\mathbb{B}} |f(\xi)||Q_n K_{\gamma,\delta,w}(\xi)|\omega_{\gamma,\delta}(\xi)dV(\xi). \tag{3.14}$$

Put $r'' = \frac{r'+1}{2}$ and $C(\gamma, \delta, r_0) = \max\{1, (1 - r'')^\gamma\}(\log \frac{e}{1-r''})^\delta$. Then,

$$\begin{aligned}\int_0^1 r^{2(|\alpha|+N)-1} \omega_{\gamma,\delta}(r)dr &\asymp \int_0^1 r^{2(|\alpha|+N)-1}(1-r)^\gamma \left(\log \frac{e}{1-r}\right)^\delta dr \\ &\geq C(\gamma, \delta, r_0) \int_0^{r''} r^{2(|\alpha|+N)-1} dr \\ &= \frac{C(\gamma, \delta, r_0)}{2(|\alpha|+N)} r''^{2(|\alpha|+N)} \\ &= \frac{C(\gamma, \delta, r_0)}{2(|\alpha|+N)} \left(\frac{3+r_0}{4}\right)^{2(|\alpha|+N)}. \end{aligned}$$

Combining this with Lemma 4 gives

$$|Q_n K_{\gamma,\delta,w}(\xi)| \lesssim \sum_{k=n}^{+\infty} \frac{(N+k)!}{N!k!} \left(\frac{4}{3+r_0}\right)^{2(k+N)} \left(\frac{1+r_0}{2}\right)^k \tag{3.15}$$

for any $w \in r'\overline{\mathbb{B}}$.

From (3.14) and (3.15), we have

$$\begin{aligned}|Q_n f(w)| &\leq \sum_{k=n}^{+\infty} \frac{(N+k)!}{N!k!} \left(\frac{4}{3+r_0}\right)^{2(k+N)} \left(\frac{1+r_0}{2}\right)^k \int_{\mathbb{B}} |f(\xi)|\omega_{\gamma,\delta}(\xi)dV(\xi) \\ &\leq \sum_{k=n}^{+\infty} \frac{(N+k)!}{N!k!} \left(\frac{4}{3+r_0}\right)^{2(k+N)} \left(\frac{1+r_0}{2}\right)^k \|f\|_{\gamma,\delta} \left[\int_{\mathbb{B}} \omega_{\gamma,\delta}(\xi)dV(\xi)\right]^{\frac{1}{2}} \end{aligned} \tag{3.16}$$

for any $w \in r'\overline{\mathbb{B}}$.

From (3.13), (3.16), and the fact $\frac{16(1+r_0)}{2(3+r_0)^2} < 1$, we obtain

$$\lim_{n\to+\infty} \sup_{\|f\|_{\gamma,\delta}\leq 1} T_2(r_0, n, f) = 0 \tag{3.17}$$

for any $r_0 \in (0,1)$. Combining (3.11), (3.12), (3.17), and Corollary 3, it follows that

$$\|T_g\|_e \lesssim \sup_{n\geq 1}\|Q_n\|_{\mathrm{op}} \lim_{r_0\to 1^-}\sup_{|z|>r_0}(1-|z|^2)|\mathcal{R}g(z)|,$$

and so the upper estimate (3.10) holds. □

4. The operator I_g

As another application of Theorem 1, we can obtain characterizations for the boundedness and the compactness of the operator $I_g : A^2_{\gamma,\delta}(\mathbb{B}) \to A^2_{\gamma,\delta}(\mathbb{B})$. Since we have that $\mathcal{R}[I_g f](z) = \mathcal{R}f(z)g(z)$, most of the arguments in the previous section work for the operator I_g.

Theorem 4. *Let $g \in H(\mathbb{B})$. Then, I_g is bounded on $A^2_{\gamma,\delta}(\mathbb{B})$ if and only if g is a bounded holomorphic function on $\mathbb{B}$.*

Proof. First, suppose that g is bounded on $\mathbb{B}$. For a fixed $r_0 \in (1/2,1)$, put $r' = \frac{1+r_0}{2}$. Then, inequality (2.3) gives

$$\begin{aligned}\|I_g f\|^2_{\gamma,\delta} \lesssim \frac{4}{r_0^{2(N-1)}}\int_{\mathbb{B}\setminus r_0\overline{\mathbb{B}}}\frac{|\mathcal{R}f(z)|^2|g(z)|^2}{|z|^2}\omega_{\gamma+2,\delta}(z)dV(z)\\ +\left(\frac{2N\omega_{\gamma+2,\delta}(r_0)}{\ln\frac{1+r_0}{2r_0}}+\int_{\mathbb{B}\setminus r_0\overline{\mathbb{B}}}\omega_{\gamma,\delta}(z)dV(z)\right)\int_{\mathbb{S}}|I_g f(r'\zeta)|^2 d\sigma(\zeta).\end{aligned}$$

From inequality (2.4), it follows that

$$4\int_{\mathbb{B}\setminus r_0\overline{\mathbb{B}}}\frac{|\mathcal{R}f(z)|^2|g(z)|^2}{|z|^2}\omega_{\gamma+2,\delta}(z)dV(z) \lesssim \|g\|^2_\infty\|f\|^2_{\gamma,\delta}.$$

By applying Lemma 3 twice, we obtain

$$\begin{aligned}\int_{\mathbb{S}}|I_g f(r'\zeta)|^2 d\sigma(\zeta) &= \frac{2}{N}\int_{r'\mathbb{B}}\frac{|\mathcal{R}f(z)g(z)|^2}{|z|^{2N}}\ln\frac{r'}{|z|}dV(z)\\ &\lesssim \|g\|^2_\infty\int_{\mathbb{S}}|f(r'\zeta)|^2 d\sigma(\zeta).\end{aligned}$$

By Proposition 1, we have

$$\int_{\mathbb{S}}|f(r'\zeta)|^2 d\sigma(\zeta) \lesssim \frac{1}{(1-r')^{N+1}\omega_{\gamma,\delta}(r')}\|f\|^2_{\gamma,\delta}.$$

These inequalities imply that $\|I_g f\|_{\gamma,\delta} \lesssim \|g\|_\infty\|f\|_{\gamma,\delta}$ for all $f \in A^2_{\gamma,\delta}(\mathbb{B})$. Thus, I_g is bounded on $A^2_{\gamma,\delta}(\mathbb{B})$.

Conversely, we assume that I_g is bounded on $A^2_{\gamma,\delta}(\mathbb{B})$. By Corollary 2, we see that $\mathcal{R}[I_g f]$ belongs to $A^2_{\gamma+2,\delta}(\mathbb{B})$ for all $f \in A^2_{\gamma,\delta}(\mathbb{B})$. Hence, Proposition 1 shows that

$$|\mathcal{R}f(z)g(z)| = |\mathcal{R}[I_g f](z)| \leq \left(\frac{1}{(1-|z|)^{N+1}\omega_{\gamma+2,\delta}(z)}\right)^{1/2}\|\mathcal{R}[I_g f]\|_{\gamma+2,\delta} \tag{4.1}$$

for $f \in A^2_{\gamma,\delta}(\mathbb{B})$ and $z \in \mathbb{B}$.

As in (3.6) in Theorem 2, we also obtain that

$$\|\mathcal{R}[I_g f]\|_{\gamma+2,\delta} \lesssim \|I_g f\|_{\gamma,\delta} \leq \|I_g\|_{\mathrm{op}} \|f\|_{\gamma,\delta} \tag{4.2}$$

for any $f \in A^2_{\gamma,\delta}(\mathbb{B})$. Now, we take $r \in (0,1)$ and consider the functions f_a in (2.18) for $a \in \mathbb{B}$ with $r < |a| < 1$. Since $\|f_a\|_{\gamma,\delta} \lesssim 1$ and

$$|\mathcal{R} f_a(a)| \asymp \frac{\gamma - \delta + N + 1}{2} \frac{|a|^2}{1-|a|^2} \left(\frac{1}{(1-|a|)^{N+1} \omega_{\gamma,\delta}(a)} \right)^{1/2}, \tag{4.3}$$

applying (4.1) and (4.2) to f_a, yields

$$|g(a)| \lesssim \frac{2}{(\gamma - \delta + N + 1) r^2} \|I_g\|_{\mathrm{op}}$$

for any $a \in \mathbb{B}$ with $r < |a| < 1$. This implies that g is bounded on $\mathbb{B}$. □

As already suggested, the following theorem can be proven by suitably modifications of the proof of Theorem 3. Therefore, we only describe the necessary ones.

Theorem 5. *Suppose that $g \in H(\mathbb{B})$ induces a bounded operator $I_g : A^2_{\gamma,\delta}(\mathbb{B}) \to A^2_{\gamma,\delta}(\mathbb{B})$. Then,*

$$\|I_g\|_e \asymp \limsup_{|z| \to 1^-} |g(z)|.$$

Thus, I_g is compact on $A^2_{\gamma,\delta}(\mathbb{B})$ if and only if $|g(z)| \to 0$ as $|z| \to 1^-$.

Proof. Instead of (3.9), we have

$$\|I_g - K\|_{\mathrm{op}} \geq \limsup_{|a| \to 1^-} \|I_g f_a\|_{\gamma,\delta}$$

for any compact operator K on $A^2_{\gamma,\delta}(\mathbb{B})$ and the test functions f_a in Lemma 5. By (4.1)-(4.3), we obtain

$$\|I_g f_a\|_{\gamma,\delta} \gtrsim \frac{\gamma - \delta + N + 1}{2} |a| |g(a)|$$

for $a \in \mathbb{B}$ sufficiently close to $\mathbb{S}$. Hence, we have that $\|I_g\|_e \gtrsim \limsup_{|z| \to 1^-} |g(z)|$.

For the upper estimate, we take $f \in A^2_{\gamma,\delta}(\mathbb{B})$ with $\|f\|_{\gamma,\delta} \leq 1$ and fix $r_0 \in (0,1)$. Additionally, we put

$$I_1(r_0, n, f) := \frac{4}{r_0^{2(N-1)}} \int_{\mathbb{B} \setminus r_0 \overline{\mathbb{B}}} \frac{|\mathcal{R}[I_g Q_n f](z)|^2}{|z|^2} \omega_{\gamma+2,\delta}(z) dV(z)$$

and

$$I_2(r_0, n, f) := \left(\frac{2N \omega_{\gamma+2,\delta}(r_0)}{\ln \frac{1+r_0}{2r_0}} + \int_{\mathbb{B} \setminus r_0 \overline{\mathbb{B}}} \omega_{\gamma,\delta}(z) dV(z) \right) \int_{\mathbb{S}} |I_g Q_n f(r'\zeta)|^2 d\sigma(\zeta),$$

where $r' = \frac{r_0+1}{2}$.

As in (3.12), we have

$$I_1(r_0, n, f) \lesssim \sup_{|z|>r_0} |g(z)|^2 \sup_{n\geq 1} \|Q_n\|_{\text{op}}^2. \tag{4.4}$$

On the one hand, by using the same argument as that used to derive (3.13), we obtain

$$\begin{aligned}\int_{\mathbb{S}} |I_g Q_n f(r'\zeta)|^2 d\sigma(\zeta) &\leq \max_{|w|\leq r'} |g(w)|^2 \frac{2}{N} \int_{r'\mathbb{B}} \frac{|\mathcal{R}[Q_n f](w)|^2}{|w|^{2N}} \ln \frac{r'}{|w|} dV(w) \\ &\leq \max_{|w|\leq r'} |g(w)|^2 \int_{\mathbb{S}} |Q_n f(r'\zeta)|^2 d\sigma(\zeta) \\ &\leq \max_{|w|\leq r'} |g(w)|^2 \cdot \max_{|w|=r'} |Q_n f(w)|^2.\end{aligned}$$

Estimate (3.16) implies that

$$\sup_{\|f\|_{\gamma,\delta}\leq 1} \max_{|w|=r'} |Q_n f(w)|^2 \to 0$$

as $n \to +\infty$, and so

$$\lim_{n\to+\infty} \sup_{\|f\|_{\gamma,\delta}\leq 1} I_2(r_0, n, f) = 0 \tag{4.5}$$

for any $r_0 \in (0, 1)$. From (4.4) and (4.5), we obtain the upper estimate $\|I_g\|_e \leq \limsup_{|z|\to 1^-} |g(z)|$, completing the proof. □

5. Conclusions

Here, we consider two integral operators on a Hilbert-Bergman space of logarithmic type weights on the open unit ball in the Euclidean complex vector space $\mathbb{C}^N$. We give some necessary and sufficient conditions for the boundedness of the operators, and also estimate the essential norm of these operators via the boundary behavior of the involving symbol function, considerably improving some results in the literature. To do this, we use several auxiliary results, estimates and formulas that could be useful in related studies. The methods, ideas, and results, presented here, as well as their modifications, could be employed in other investigations on the topic.

Author contributions

S. Stević: Conceptualization, Investigation, Writing-review and editing; S.-I. Ueki: Investigation, Writing-review and editing. S. Stević proposed the investigation. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

This work does not have any conflict of interest.

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