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Research article

Asymptotic expansions of powered order statistics for generalized Maxwell distribution

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Abstract: Let $M_{n,r}$ denote the rth largest-order statistics of a sequence of independent random variables with common generalized Maxwell distribution with positive parameter k. This paper mainly considers the higher-order asymptotic expansions of the distribution of normalized powered order statistics $|M_{n,r}|^t$ and the corresponding convergence rates under different norming constants. The results show that when $t = 2k$ and the optimal norming constants are selected, and the convergence speed of the distribution of $|M_{n,r}|^t$ toward its extreme limit is proportional to $1/(\log n)^2$, and for the case of $t \neq 2k$, the convergence rate is proportional to $1/\log n$. The main results are confirmed by some numerical analysis.

Keywords: asymptotic expansion; convergence rate; generalized Maxwell distribution; powered order statistic
Mathematics Subject Classification: Primary 60F15; Secondary 60G70

1. Introduction

The generalized Maxwell distribution (GMD for short) was first proposed by Vodǎ [1]. It represents a parameter family of probability distribution, comprising the classic Maxwell distribution as its special case. The probability density function (pdf) of GMD(k) with shape parameter $k \in (0, +\infty)$ is given by

$$f_k(x) = \frac{k}{2^{1/2k}\sigma^{2+1/k}\Gamma(1+1/2k)} x^{2k} \exp\left(-\frac{x^{2k}}{2\sigma^2}\right), \; x \in (0, +\infty), \tag{1.1}$$

where σ is a positive constant and $\Gamma(\cdot)$ represents the Gamma function. Let $F_k(x)$ stand for its cumulative distribution function (cdf). Vodǎ [1] discussed the statistical properties of GMD(k). For the generalization and application of Maxwell distribution family, see [2, 3] and literature therein.

In recent years, the asymptotic behaviors concerning the normalized powered extreme of certain samples have attracted the attention of scholars. Hall [4] established the asymptotic distribution behavior of powered extremes from normal samples. Zhou and Ling [5] derived the higher-order expansions for distributions and densities of powered extremes for normal samples. Further, Xiong and Peng [6] and Lu and Peng [7] extended Hall's result to the skew-normal case and general error distribution case, respectively. Hashorva et al. [8] considered asymptotic expansions for the distribution of the maximum in bivariate normal triangular arrays. For additional work on the extreme value distribution, moment and density of given distributions, and corresponding speeds of convergence, we recommend that readers refer to [9–11].

Motivated by the work of [12], the aim of this paper is to study the asymptotic expansions and speeds of convergence of powered order statistics from generalized Maxwell distribution. Let $\{X_n, n\geq 1\}$ be an independent random sequence with the common cdf F obeying GMD(k) with its pdf defined by (1.1). For $r \in \mathbb{N}$, let $M_{n,r}$ represent the rth largest order statistics of $\{X_k,\ 1 \leq k \leq n\}$ and $M_n = M_{n,1} = \bigvee_{i=1}^{n} X_i$.

We all know that the norming constants play a very important role in estimating the convergence rate of the normalized maximum value, see [12, 13]. For the generalized Maxwell distribution, it is necessary to study how to find proper norming constants $c_n \in (0,\ \infty)$ and $d_n \in \mathbb{R}$ to make

$$\mathbf{P}\left(|M_{n,r}|^t \leq c_n x + d_n\right) \to \Lambda_r(x),\ x \in \mathbb{R}, \tag{1.2}$$

hold, as $n \to \infty$. Here, $\Lambda_r(x) = \Lambda(x)\sum_{i=0}^{r-1} e^{-ix}/i!$, with $\Lambda(x) = \exp\{-e^{-x}\}$ and $t \in (0,\ \infty)$ is the power index. Further, we will study the higher-order asymptotic expansions of power order statistics $|M_{n,r}|^t$ and the speeds of convergence of (1.2) under different norming constants conditions.

For GMD(k), Huang and Chen [14] defined

$$\begin{aligned}
\alpha_n =& \frac{\sigma^{\frac{1}{k}}}{k(2\log n)^{1-\frac{1}{2k}}},\\
\beta_n =& (2\log n)^{\frac{1}{2k}}\sigma^{\frac{1}{k}} + \frac{\sigma^{\frac{1}{k}}\left[\log\log n - 2k\log\Gamma\left(1+\frac{1}{2k}\right)\right]}{2k^2(2\log n)^{1-\frac{1}{2k}}} + \frac{1-2k}{32k^4}\frac{(\log\log n)^2}{(\log n)^{2-\frac{1}{2k}}},
\end{aligned} \tag{1.3}$$

and gained

$$\mathbf{P}\left(M_n \leq \alpha_n x + \beta_n\right) \to \Lambda(x),\ \text{as } n \to \infty.$$

Moreover, Huang et al. [15] proved that the speed of convergence of normalized maximum M_n to its extreme value limit $\Lambda(x)$ is of order $O(1/\log n)$ with optimal norming constants $a_n = k^{-1}\sigma^2 b_n^{1-2k}$ and b_n is the solution of the following equation:

$$2^{\frac{1}{2k}}\sigma^{\frac{1}{k}}\Gamma\left(1+\frac{1}{2k}\right)b_n^{-1}\exp\left(\frac{b_n^{2k}}{2\sigma^2}\right) = n,\ \text{or}\ \frac{\sigma^2}{k}nf_k(b_n) = b_n^{2k-1}. \tag{1.4}$$

Let $1 - F_t(x) = P(|X|^t \leq x),\ x \in (0,\ +\infty)$, stand for the cdf of $|X|^t$. One can easily check that $1 - F_t(x) = 1 - F_k(x^{1/t})$. Therefore, by Lemma 3.1 of [16], we get

$$
\begin{aligned}
-F_t(x) =& f_k(x^{\frac{1}{t}})\frac{\sigma^2}{k}x^{\frac{1-2k}{t}}\Big(1+\frac{\sigma^2}{k}x^{-\frac{2k}{t}}+\frac{(1-2k)\sigma^4}{k^2}x^{-\frac{4k}{t}}+\frac{(1-2k)(1-4k)\sigma^6}{k^3}x^{-\frac{6k}{t}}+O(x^{-\frac{8k}{t}})\Big) \qquad (1.5)\\
=&\frac{1}{2^{1/2k}\sigma^{1/k}\Gamma(1+1/2k)}\Big(1+\frac{\sigma^2}{k}x^{-\frac{2k}{t}}+\frac{(1-2k)\sigma^4}{k^2}x^{-\frac{4k}{t}}\\
&+\frac{(1-2k)(1-4k)\sigma^6}{k^3}x^{-\frac{6k}{t}}+O(x^{-\frac{8k}{t}})\Big)\exp\left(\frac{1}{t}\log x-\frac{x^{\frac{2k}{t}}}{2\sigma^2}\right)\\
=&\frac{\exp(-1/2\sigma^2)}{2^{1/2k}\sigma^{1/k}\Gamma(1+1/2k)}\Big(1+\frac{\sigma^2}{k}x^{-\frac{2k}{t}}+\frac{(1-2k)\sigma^4}{k^2}x^{-\frac{4k}{t}}\\
&+\frac{(1-2k)(1-4k)\sigma^6}{k^3}x^{-\frac{6k}{t}}+O(x^{-\frac{8k}{t}})\Big)\exp\left(-\int_1^x\frac{g(y)}{\tilde{f}(y)}dy\right), \qquad (1.6)
\end{aligned}
$$

where $g(y) = 1 - k^{-1}\sigma^2 y^{-2k/t} \to 1$, and $\tilde{f}(y) = k^{-1}\sigma^2 t y^{1-2k/t}$ with $\tilde{f}'(y) \to 0$ as $y \to \infty$. Consequently, by (1.4) and (1.5), we get $d_n = b_n^t$. From (1.6), it implies that $c_n = \tilde{f}(d_n) = k^{-1}\sigma^2 t b_n^{t-2k}$, i.e.,

$$c_n = k^{-1}\sigma^2 t b_n^{t-2k},\ d_n = b_n^t. \qquad (1.7)$$

The norming constants can also be defined by

$$\alpha_n^* = t\alpha_n\beta_n^{t-1},\ \beta_n^* = \beta_n^t. \qquad (1.8)$$

With regard to normal distribution, Hall [12] proved that when we choose the optimal norming constants, the best speed of convergence of normalized $|M_{n,r}|^2$ is proportional to $1/(\log n)^2$. Inspired by Hall's ideal, next we will explore the choice of norming constants when $t = 2k$, and the following conclusions will show that the corresponding convergence speed under this kind of normalizing has also been improved.

In the special case of $t = 2k$, let $F(x) = P(|X|^{2k} \le x)$ express the cdf of $|X|^{2k}$, which implies that $1 - F(x) = 1 - F_k(x^{1/2k})$. Here, we assume $1 + \sigma^2/k \neq 0$. By Lemma 3.1 of [16], we get

$$
\begin{aligned}
1-F_k(x)=&\frac{1}{2^{\frac{1}{2k}}\sigma^{\frac{1}{k}}\Gamma(1+\frac{1}{2k})}x^{\frac{1}{2k}}\exp\left(-\frac{x}{2\sigma^2}\right)\left(1+\frac{\sigma^2}{k}x^{-1}+\frac{(1-2k)\sigma^4}{k^2}x^{-2}+\frac{(1-2k)(1-4k)\sigma^6}{k^3}x^{-3}+O(x^{-4})\right)\\
\overset{(a)}{=}&\frac{1}{2^{\frac{1}{2k}}\sigma^{\frac{1}{k}}\Gamma(1+\frac{1}{2k})}\left(1+\frac{(1-2k)\sigma^4}{k^2}x^{-2}-\frac{4(1-2k)\sigma^6}{k^2}x^{-3}+O(x^{-4})\right)\left(1+\frac{\sigma^2}{k}x^{-1}\right)x^{\frac{1}{2k}}\exp\left(-\frac{x}{2\sigma^2}\right)\\
=&C\left(1+\frac{(1-2k)\sigma^4}{k^2}x^{-2}-\frac{4(1-2k)\sigma^6}{k^2}x^{-3}+O(x^{-4})\right)\exp\left(-\int_1^x\frac{\tilde{g}(y)}{\hat{f}(y)}dy\right),
\end{aligned}
$$

where (a) follows from the Taylor's expansion

$$
\begin{aligned}
(1+x)^a =& 1+ax+(a(a-1)/2)x^2+(a(a-1)(a-2)/6)x^3+O(x^4),\ x\to 0,\ a\in\mathbb{R},\\
C =& \frac{(1+\frac{\sigma^2}{k})\exp(-\frac{1}{2\sigma^2})}{2^{\frac{1}{2k}}\sigma^{\frac{1}{k}}\Gamma(1+\frac{1}{2k})},\\
\tilde{g}(y) =& 1-\frac{1-2k}{k^2}\sigma^4 y^{-2}\to 1,\ \text{as } y\to\infty,
\end{aligned}
$$

$$\hat{f}(y) = 2\sigma^2\left(1+\frac{\sigma^2}{k}y^{-1}\right),\ \text{with } \hat{f}'(y) = -\frac{2\sigma^4}{k}y^{-2} \to 0,\ \text{as } y\to\infty.$$

Now, analogous to [12], we make the choice of

$$\begin{aligned} d_n^* &= b_n^{2k} + \frac{2\sigma^4}{k}b_n^{-2k}, \\ c_n^* &= \hat{f}(d_n^*) = \hat{f}(b_n^{2k}) = 2\sigma^2 + \frac{2\sigma^4}{k}b_n^{-2k}, \end{aligned} \tag{1.9}$$

with b_n determined by (1.4).

The contents of this paper are constructed as follows: The main results are presented in Section 2. Section 3 provides some numerical analysis. Some auxiliary lemmas and proofs of the main results are given in Section 4. Section 5 summarizes the entire paper and provides a future outlook.

2. Main results

In this section, we give the main results. In the whole article, let $\Lambda_r(x) = \Lambda(x)\sum_{i=0}^{r-1} e^{-ix}/i!$ for $r\in\mathbb{N}$ and $\Lambda_r(x) = 0$ for $r\in(-\infty,\ 0]$.

Higher-order expansions of the distribution of the powered order statistics and associating optimal speeds of convergence in this limit law are established in the following theorems.

Theorem 2.1. *Let $\{X_1,\ X_2,\cdots,X_n,\cdots\}$ be an independent random variable sequence with common cdf GMD(k) with $k\in(0,\ \infty)$, and let $M_{n,r}$ stand for the rth largest order statistics of $\{X_1,\ X_2,\cdots,X_n\}$. Then, for $k\in(0,+\infty)$ and $t=2k$, with norming constants given by (1.8), we have*

$$\begin{aligned} &(\log\log n)\left[\frac{\log n}{(\log\log n)^2}(\mathbf{P}(|M_{n,r}|^t \le \alpha_n^* x+\beta_n^*)-\Lambda_r(x)) - \frac{(2k-1)e^{-rx}}{16k^3(r-1)!}\Lambda(x)\right] \\ &\to \frac{(2k-1)[x-\log\Gamma(1+\frac{1}{2k})]-1}{4k^2}\frac{e^{-rx}}{(r-1)!}\Lambda(x). \end{aligned} \tag{2.1}$$

The following theorem reveals that, when $t = 2k$, the distributional speed of convergence can be bettered in optimal norming constants c_n^* and d_n^*.

Theorem 2.2. *Under the conditions of Theorem 2.1, we have*
(i) for $k\in(0,+\infty)$ and $t\in(0,\ 2k)\bigcup(2k,\ +\infty)$, with norming constants defined by (1.7), we have

$$\begin{aligned} &b_n^{2k}\left[b_n^{2k}\left(\mathbf{P}\left(|M_{n,r}|^t \le c_n x + d_n\right) - \Lambda_r(x)\right) - \Lambda(x)P_k(x)\frac{e^{-(r-1)x}}{(r-1)!}\right] \\ &\to \left[Q_k(x) - \frac{(r-1)e^x-1}{2}P_k^2(x)\right]\frac{e^{-(r-1)x}}{(r-1)!}\Lambda(x), \end{aligned} \tag{2.2}$$

as $n\to\infty$, where

$$P_k(x) = \left(\frac{2k-t}{2k}x^2 - \frac{1}{k}x - \frac{1}{k}\right)\sigma^2 e^{-x}, \tag{2.3}$$

and

$$Q_k(x) = -\Big(\frac{(2k-t)^2}{8k^2}x^4 - \frac{(2k-t)[2(2k-t)+3]}{6k^2}x^3 + \frac{1-2k}{2k^2}x^2 + \frac{1-2k}{k^2}x + \frac{1-2k}{k^2}\Big)\sigma^4 e^{-x}. \tag{2.4}$$

(ii) for $k \in (0, +\infty)$ and $t = 2k$, with norming constants determined by (1.9),

$$b_n^{2k}\left[b_n^{4k}\left(\mathbf{P}\left(|M_{n,r}|^t \le c_n^* x + d_n^*\right) - \Lambda_r(x)\right) - \Lambda(x)S_k(x)\frac{e^{-(r-1)x}}{(r-1)!}\right] \to \Lambda(x)T_k(x)\frac{e^{-(r-1)x}}{(r-1)!}, \tag{2.5}$$

where

$$S_k(x) = \frac{2kx^2 + (4k-2)x + 4k - 3}{2k^2}\sigma^4 e^{-x}, \tag{2.6}$$

and

$$T_k(x) = -\frac{8kx^3 + (24k^2 - 12k + 12)x^2 + (48k^2 - 60k)x + 48k^2 - 60k + 14}{6k^3}\sigma^6 e^{-x}. \tag{2.7}$$

Remark 2.1. *Theorems 2.1 and 2.2 show that the asymptotic distribution of normalized powered order statistics $|M_{n,r}|^t$ from generalized Maxwell distribution is $\Lambda_r(x)$. That is,*
(i) for $k \in (0, +\infty)$ and $t \in (0, +\infty)$, with norming constants determined by (1.8), we have

$$\mathbf{P}\left(|M_{n,r}|^t \le \alpha_n^* x + \beta_n^*\right) \to \Lambda_r(x),\ x \in \mathbb{R}, \tag{2.8}$$

as $n \to \infty$.
(ii) for $k \in (0, +\infty)$ and $t \in (0,\ 2k) \bigcup (2k,\ \infty)$, with norming constants determined by (1.7), we have

$$\mathbf{P}\left(|M_{n,r}|^t \le c_n x + d_n\right) \to \Lambda_r(x),\ x \in \mathbb{R}, \tag{2.9}$$

as $n \to \infty$.
(iii) for $k \in (0, +\infty)$ and $t = 2k$, with norming constants determined by (1.9) , we have

$$\mathbf{P}\left(|M_{n,r}|^t \le c_n^* x + d_n^*\right) \to \Lambda_r(x),\ x \in \mathbb{R}, \tag{2.10}$$

as $n \to \infty$.

Remark 2.2. *In the case of $t \in (0,\ 2k) \bigcup (2k,\ +\infty)$ as well as the norming constants c_n and d_n defined by (1.7), Theorem 2.2(i) implies that the optimal speed of convergence of $(|M_{n,r}|^t - d_n)/c_n$ to $\Lambda_r(x)$ is only $1/\log n$ due to $b_n^{2k} \sim 2\sigma^2 \log n$ by (1.4). However, for the case of $t = 2k$ and the norming constants c_n^* and d_n^* given by (1.9), Theorem 2.2(ii) shows that the normalized $2k$ power of GMD(k) order statistics $(|M_{n,r}|^{2k} - d_n^*)/c_n^*$ tends to its limit at a speed of $1/(\log n)^2$.*

3. Numerical analysis

In this part, we conduct some numerical studies that illustrate the preciseness of asymptotic expansions of the cdf of $M_{n,r}$. Let $L_i(x)$, $i = 1,\ 2,\ 3$, separately denote the first-order, second-order, and third-order asymptotics of the cdf of $M_{n,r}$. By utilizing Theorem 2.2, we have

$$L_1(x) = \Lambda_r(x),$$

$$L_2(x) = \Lambda_r(x) + \Lambda(x) S_k(x) \frac{e^{-(r-1)x}}{(r-1)!} b_n^{-4k},$$
$$L_3(x) = \Lambda_r(x) + \Lambda(x) \left\{ S_k(x) b_n^{-4k} + T_k(x) b_n^{-6k} \right\} \frac{e^{-(r-1)x}}{(r-1)!},$$

where $S_k(x)$ and $T_k(x)$ are determined by Theorem 2.2. Here, $\Lambda_r(x) = \Lambda(x) \sum_{i=0}^{r-1} e^{-ix}/i!$, with $\Lambda(x) = \exp\{-e^{-x}\}$, and the norming constant b_n is provided by (1.4).

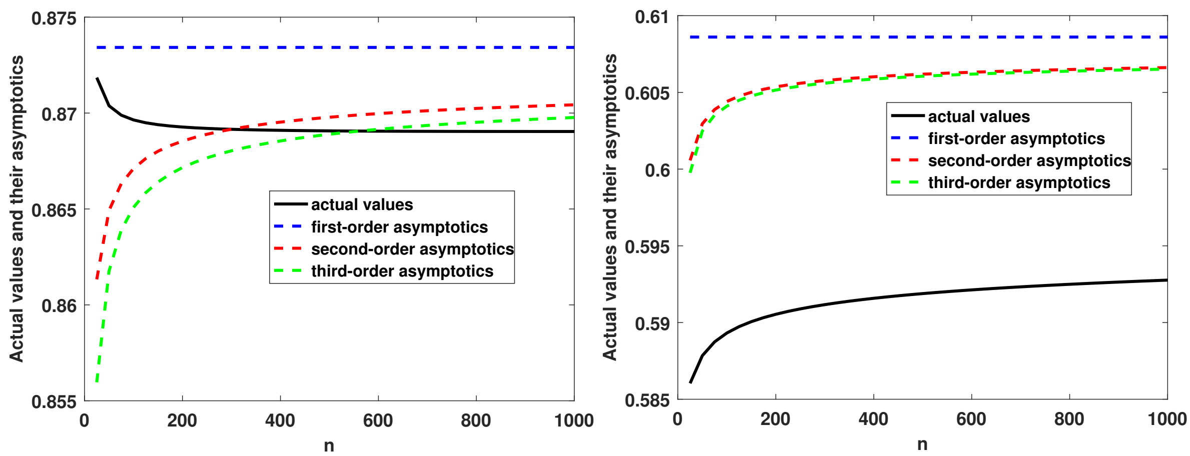
In order to compare the preciseness of actual values with its associating asymptotics, for given $x \in (0, \infty)$, let

$$\Delta_i(x) = \left| \mathbf{P}\left(|M_{n,r}|^{2k} \le c_n^* x + d_n^* \right) - L_i(x) \right|$$

stand for the absolute errors of the cdf of $M_{n,r}$, where $i = 1,\ 2,\ 3$, and the norming constants c_n^* and d_n^* are defined by (1.9). In the numerical analysis, we only consider the case of $r = 1$, then $\Lambda_1(x) = \Lambda(x)$ and $M_{n,1} = M_n = \max\{X_i,\ 1 \le i \le n\}$. In the whole experiment, let parameter $\sigma = 1$.

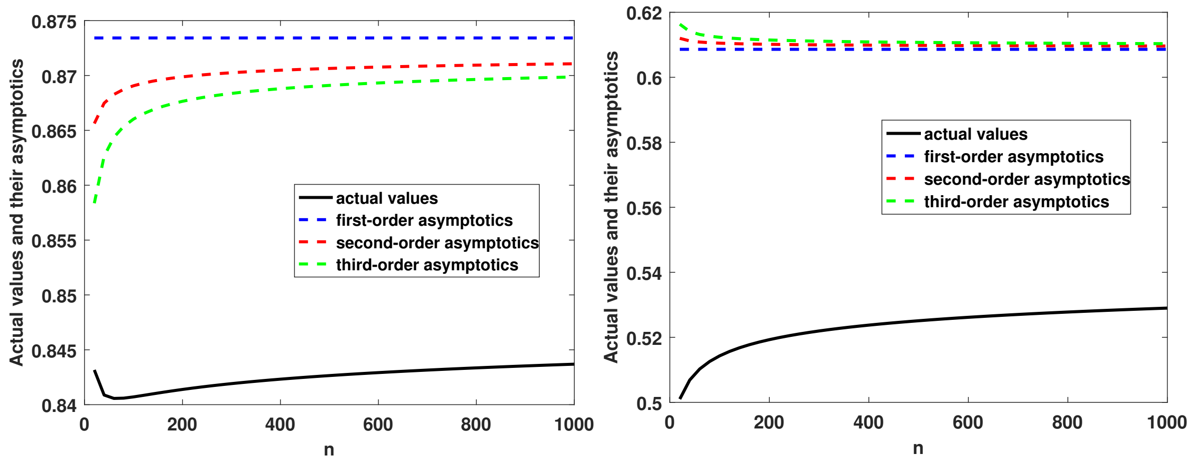
First of all, fixing $x = 2$ and $x = 0.7$, for n changing from 25 to 1000 in steps of 25, we calculate the absolute errors of the distribution function of $M_{n,r}^2$. The numerical results of $\Delta_i(x),\ i = 1,\ 2, 3$, are listed in Tables 1 and 2. The results show that as n increases, the accuracy of the three kinds of expansions of distribution function from $M_{n,r}^2$ is continuously improving.

In order to more intuitively show the accuracy of all asymptotic expansions when changing with n, for $k = 1$, we plot the true value and the corresponding asymptotic values of the distribution function of $M_{n,r}^2$ when x is fixed. In Figure 1, we compare all asymptotic values with the true value, where (a) is the case of $x = 2$, and (b) is the case of $x = 0.7$. Observing Figure 1, we can see that (i) except for some cases, as n increases, all asymptotic values are constantly approaching the true value, and (ii) except for some values, the third-order approximation is closer to the true value. Figures 2–4 represent the cases of $k = 0.5,\ 1.5,\ 0.25$, respectively. By observing Figures 1–4, we can find that when $k = 1.5$, all asymptotic values approach the true value best.



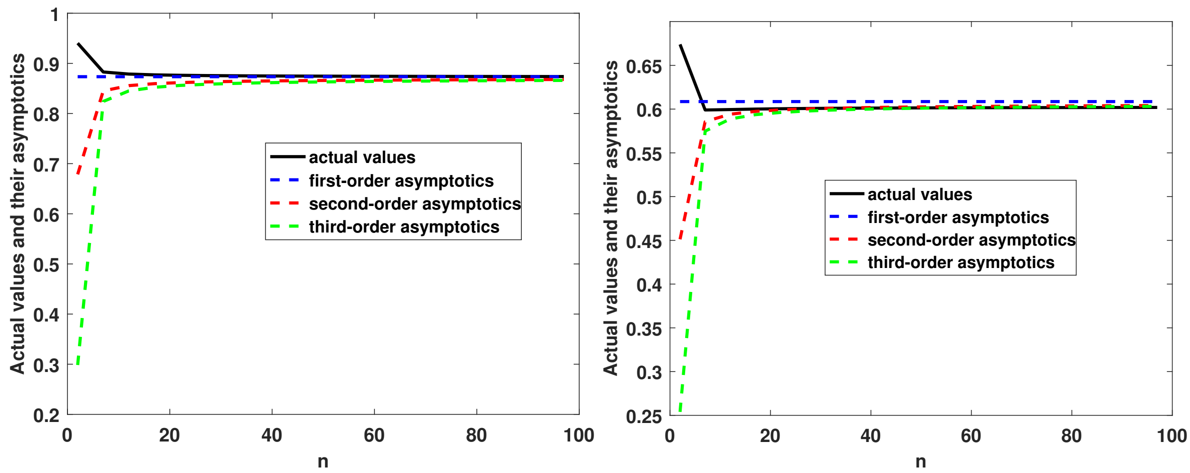
(a) $x = 2$ (b) $x = 0.7$

Figure 1. Actual values and its corresponding asymptotics of the cdf of $M_{n,r}^2$ with $k = 1$.



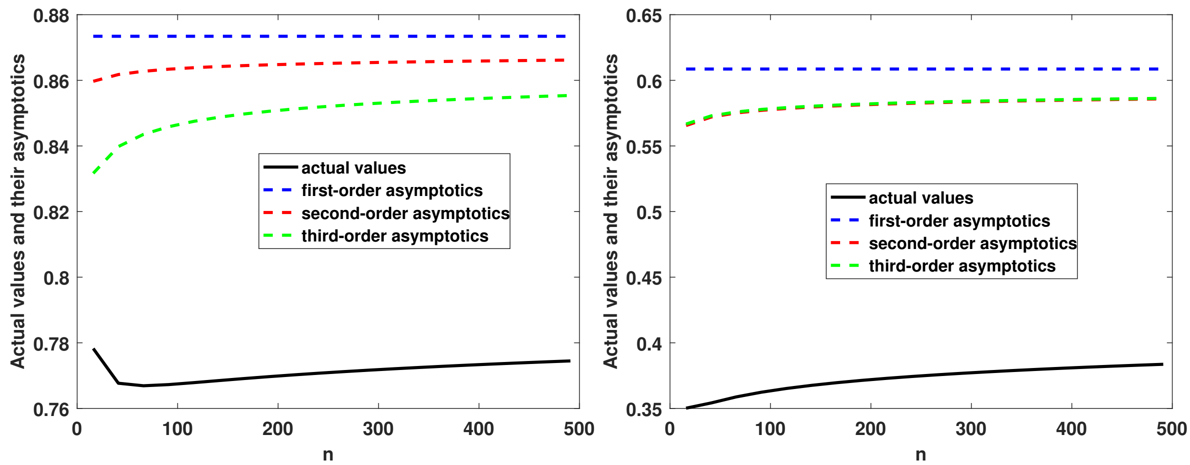
(a) $x = 2$ (b) $x = 0.7$

Figure 2. Actual values and its associating approximations of the cdf of $M_{n,r}$ with $k = 0.5$.



(a) $x = 2$ (b) $x = 0.7$

Figure 3. Actual values and its associating approximations of the cdf of $M_{n,r}^3$ with $k = 1.5$.



(a) $x = 2$ (b) $x = 0.7$

Figure 4. Actual values and its corresponding asymptotics of the cdf of $M_{n,r}^{1/2}$ with $k = 0.25$.

Table 1. Absolute errors between actual values and their asymptotics of the cdf at $x = 2$.

n	$\Delta_1(2)$	$\Delta_2(2)$	$\Delta_3(2)$
25	0.00157444605	0.0107865072	0.0163342704
50	0.00304304688	0.00553206093	0.00873758727
75	0.00353299047	0.00360462011	0.00603889288
100	0.00378014977	0.00255242804	0.00458670822
125	0.00392908314	0.00187205639	0.0036557076
150	0.0040282894	0.00138823105	0.00299747004
175	0.00409877316	0.0010225076	0.00250198332
200	0.00415114984	0.000734033845	0.00211238957
225	0.00419137546	0.000499239403	0.00179607412
250	0.00422305509	0.00030347675	0.00153286296
275	0.00424850042	0.000137115137	0.00130954212
300	0.00426926261	0.00000646493254	0.00111705827
325	0.00428642147	0.000131979245	0.000948979333
350	0.0043007518	0.000242888177	0.000800596648
375	0.00431282388	0.000341794702	0.000668375571
400	0.00432306665	0.000430698642	0.000549605433
425	0.00433180884	0.000511164663	0.00044216884
450	0.0043393065	0.000584436528	0.000344385102
475	0.00434576197	0.000651516785	0.000254901256
500	0.00435133716	0.000713223581	0.0001726145
525	0.00435616311	0.000770231969	0.0000966158965
550	0.00436034695	0.000823104465	0.0000261487797
575	0.00436397708	0.000872314003	0.0000394224404
600	0.00436712706	0.000918261432	0.000100636027
625	0.00436985853	0.000961289014	0.000157951219
650	0.00437222353	0.00100169096	0.000211762505
675	0.00437426627	0.00103972175	0.000262410875
700	0.00437602451	0.00107560276	0.000310192775
725	0.0043775307	0.00110952759	0.000355367299
750	0.00437881286	0.00114166634	0.000398162013
775	0.00437989527	0.00117216922	0.000438777714
800	0.0043807991	0.00120116935	0.00047739233
825	0.00438154286	0.00122878526	0.000514164161
850	0.0043821428	0.00125512281	0.000549234567
875	0.00438261323	0.00128027692	0.00058273023
900	0.00438296682	0.00130433294	0.000614765043
925	0.00438321477	0.00132736789	0.000645441729
950	0.00438336706	0.00134945146	0.000674853205
975	0.0043834326	0.00137064687	0.000703083747
1000	0.00438341933	0.00139101164	0.000730210001

Table 2. Absolute errors between practical values and its asymptotics of the cdf at $x = 0.7$.

n	$\Delta_1(0.7)$	$\Delta_2(0.7)$	$\Delta_3(0.7)$
25	0.022567308	0.0143501952	0.0135172198
50	0.0207645845	0.0150641643	0.0145828667
75	0.0198652076	0.0151203833	0.0147548865
100	0.0192883808	0.0150787131	0.0147732737
125	0.0188713153	0.0150149285	0.01474712
150	0.0185480499	0.0149473439	0.0147057228
175	0.0182858843	0.0148814429	0.0146593052
200	0.0180664068	0.0148189141	0.0146119593
225	0.017878298	0.0147601476	0.0145654329
250	0.0177141351	0.0147050611	0.0145204735
275	0.0175688038	0.0146534059	0.0144773704
300	0.0174386403	0.0146048872	0.0144361945
325	0.0173209352	0.0145592129	0.0143969111
350	0.0172136322	0.0145161118	0.0143594366
375	0.0171151353	0.0144753394	0.0143236662
400	0.017024183	0.0144366781	0.0142894892
425	0.0169397611	0.0143999357	0.0142567963
450	0.0168610435	0.0143649424	0.0142254833
475	0.0167873483	0.0143315483	0.0141954531
500	0.0167181066	0.0142996208	0.0141666156
525	0.0166528391	0.0142690423	0.0141388884
550	0.0165911384	0.0142397081	0.014112196
575	0.0165326556	0.0142115248	0.0140864693
600	0.0164770894	0.0141844088	0.0140616455
625	0.0164241782	0.014158285	0.0140376669
650	0.0163736934	0.0141330858	0.014014481
675	0.0163254341	0.01410875	0.0139920398
700	0.0162792229	0.0140852224	0.013970299
725	0.0162349023	0.0140624525	0.0139492182
750	0.016192332	0.0140403945	0.0139287602
775	0.0161513864	0.0140190066	0.0139088907
800	0.0161119528	0.0139982504	0.0138895781
825	0.0160739295	0.0139780908	0.0138707931
850	0.0160372245	0.0139584952	0.0138525088
875	0.0160017545	0.013939434	0.0138347
900	0.0159674434	0.0139208794	0.0138173434
925	0.0159342221	0.0139028061	0.0138004175
950	0.0159020272	0.0138851903	0.0137839019
975	0.0158708006	0.0138680101	0.013767778
1000	0.015840489	0.013851245	0.0137520282

4. Proofs

In order to facilitate the main results, several auxiliary lemmas are needed.

Lemma 4.1. *Let $F_k(x)$ stand for the cdf of GMD(k) with $k \in (0, +\infty)$ and $t = 2k$. For large n, we get*

$$\begin{aligned} &1 - F_k\left((\alpha_n^* x + \beta_n^*)^{\frac{1}{t}}\right) \\ =&n^{-1}e^{-x}\left(1 - \frac{(2k-1)(\log\log n)^2}{16k^3 \log n} - \frac{\{(2k-1)[x - \log\Gamma(1+\frac{1}{2k})] - 1\}\log\log n}{4k^2 \log n} + o\left(\frac{\log\log n}{\log n}\right)\right), \end{aligned} \tag{4.1}$$

where the norming constants α_n^* and β_n^* are defined by (1.8).

Proof. Notice for fixed $x \in (0, \infty)$ and large n that $\alpha_n^* x + \beta_n^* > 0$. Put below $z_{n,t}(x) = (\alpha_n^* x + \beta_n^*)^{1/t}$. Making use of the Taylor's expansion below:

$$(1+x)^a = 1 + ax + (a(a-1)/2)x^2 + (a(a-1)(a-2)/6)x^3 + O(x^4), \ x \to 0, \ a \in \mathbb{R},$$

we get

$$\begin{aligned} \beta_n^{2k} =& 2\sigma^2 \log n \left(1 + \frac{\log\log n - 2k\log\Gamma(1+\frac{1}{2k})}{4k^2 \log n}\right)^{2k} \\ =& 2\sigma^2 \log n \Bigg[1 + \frac{\log\log n - 2k\log\Gamma(1+\frac{1}{2k})}{2k \log n} \\ &+ \frac{(2k-1)[(\log\log n)^2 - 4k(\log\log n)(\log\Gamma(1+\frac{1}{2k})) + 4k^2(\log\Gamma(1+\frac{1}{2k}))^2]}{16k^3(\log n)^2} + o((\log n)^{-2})\Bigg], \end{aligned}$$

and

$$\begin{aligned} 1 + 2k\alpha_n\beta_n^{-1}x =& 1 + \frac{x}{\log n}\left(1 + \frac{\log\log n - 2k\log\Gamma(1+\frac{1}{2k})}{4k^2 \log n}\right)^{-1} \\ =& 1 + \frac{x}{\log n} - \frac{[\log\log n - 2k\log\Gamma(1+\frac{1}{2k})]x}{4k^2(\log n)^2} + o\left(\frac{\log\log n}{(\log n)^2}\right), \end{aligned} \tag{4.2}$$

which implies

$$\begin{aligned} (z_{n,t}(x))^{2k} =& \beta_n^{2k}(1 + 2k\alpha_n\beta_n^{-1}x) = 2\sigma^2 \log n \Bigg[1 + \frac{2kx + \log\log n - 2k\log\Gamma(1+\frac{1}{2k})}{2k\log n} \\ &+ \frac{(2k-1)(\log\log n)^2 + 4k(2k-1)[x - \log\Gamma(1+\frac{1}{2k})]\log\log n}{16k^3(\log n)^2} + o\left(\frac{\log\log n}{(\log n)^2}\right)\Bigg]. \end{aligned} \tag{4.3}$$

Employing (4.3), we get

$$\begin{aligned} &\frac{1}{2^{\frac{1}{2k}}\sigma^{\frac{1}{k}}\Gamma(1+\frac{1}{2k})} \exp\left(-\frac{(z_{n,t}(x))^{2k}}{2\sigma^2}\right) \\ =& \frac{n^{-1}e^{-x}}{2^{\frac{1}{2k}}\sigma^{\frac{1}{k}}}(\log n)^{-\frac{1}{2k}} \times \exp\left(-\frac{(2k-1)(\log\log n)^2 + 4k(2k-1)[x - \log\Gamma(1+\frac{1}{2k})]\log\log n}{16k^3 \log n} + o\left(\frac{(\log\log n)^2}{\log n}\right)\right) \end{aligned}$$

$$\overset{(a)}{=}\frac{n^{-1}e^{-x}}{2^{\frac{1}{2k}}\sigma^{\frac{1}{k}}}(\log n)^{-\frac{1}{2k}}\left(1-\frac{(2k-1)(\log\log n)^2+4k(2k-1)[x-\log\Gamma(1+\frac{1}{2k})]\log\log n}{16k^3\log n}+o\left(\frac{(\log\log n)^2}{\log n}\right)\right), \quad (4.4)$$

where (a) is due to the Taylor's expansion $e^x = 1 + x + x^2/2 + x^3/6 + o(x^3)$ as $x \to 0$.

By utilizing (4.2), we get

$$(1 + 2k\alpha_n\beta_n^{-1}x)^{\frac{1}{2k}} = \left(1 + \frac{x}{\log n} - \frac{[\log\log n - 2k\log\Gamma(1 + \frac{1}{2k})]x}{4k^2(\log n)^2} + o\left(\frac{\log\log n}{(\log n)^2}\right)\right)^{\frac{1}{2k}}$$

$$= 1 + \frac{x}{2k\log n} - \frac{[\log\log n - 2k\log\Gamma(1 + \frac{1}{2k}) - k(1 - 2k)x]x}{8k^3(\log n)^2} + o\left(\frac{\log\log n}{(\log n)^2}\right). \quad (4.5)$$

Through a combination of (1.3) and (4.5), we get

$$z_{n,t}(x) = \beta_n(1 + 2k\alpha_n\beta_n^{-1}x)^{\frac{1}{2k}}$$

$$=\sigma^{\frac{1}{k}}(2\log n)^{\frac{1}{2k}}\left(1+\frac{2k[x-2k\log\Gamma(1+\frac{1}{2k})]+\log\log n}{4k^2\log n}+\frac{1-2k}{32k^4}\frac{(\log\log n)^2}{(\log n)^2}+o\left(\frac{(\log\log n)^2}{(\log n)^2}\right)\right). \quad (4.6)$$

Additionally, it follows from (4.6) that

$$1 + \frac{\sigma^2}{k}(z_{n,t}(x))^{-2k} + \frac{(1 - 2k)\sigma^4}{k^2}(z_{n,t}(x))^{-4k} = 1 + \frac{2}{k}(\log n)^{-1} - \frac{\log\log n}{4k^2(\log n)^2} + o\left(\frac{\log\log n}{(\log n)^2}\right). \quad (4.7)$$

Therefore, combining with Lemma 3.1 of [16] and (4.4), (4.6), and (4.7), we obtain (4.1). □

Lemma 4.2. *Let $F_k(x)$ represent the cdf of GMD(x) with parameter $k \in (0, \infty)$, then*
(i) when $k \in (0, \infty)$ and $t \neq 2k$, with norming constants c_n and d_n provided by (1.7), we have

$$1 - F_k\left((c_nx + d_n)^{\frac{1}{t}}\right) = n^{-1}e^{-x}\left[1 - P_k(x)e^xb_n^{-2k} - Q_k(x)e^xb_n^{-4k} + o(b_n^{-4k})\right], \quad (4.8)$$

where $P_k(x)$ and $Q_k(x)$ are respectively determined by (2.3) and (2.4).
(ii) when $k \in (0, \infty)$ and $t = 2k$, with norming constants c_n^* and d_n^* given by (1.9), we have

$$1 - F_k\left((c_n^*x + d_n^*)^{\frac{1}{t}}\right) = n^{-1}e^{-x}\left[1 - S_k(x)e^xb_n^{-4k} - T_k(x)e^xb_n^{-6k} + o(b_n^{-6k})\right], \quad (4.9)$$

where $S_k(x)$ and $T_k(x)$ are respectively determined by (2.6) and (2.7).

Proof. For given $k \in (0,\infty)$ and large enough n, we see $c_nx + d_n > 0$. Let $z_{n,t}(x) = (c_nx + d_n)^{1/t}$ as follows:
(i) For $t \neq 2k$. Applying the definition of norming constants c_n and d_n provided by (1.7), it implies that

$$z_{n,t}(x) = (c_nx + d_n)^{\frac{1}{t}} = b_n\left(1 + \frac{\sigma^2}{k}tb_n^{-2k}x\right)^{\frac{1}{t}} = b_n\left(1 + \frac{\sigma^2}{k}b_n^{-2k}x + \frac{1-t}{2}\frac{\sigma^4}{k^2}b_n^{-4k}x^2 + o(b_n^{-4k})\right). \quad (4.10)$$

Similar to (4.10), we get

$$(z_{n,t}(x))^{2k} = b_n^{2k}\left(1 + \frac{\sigma^2}{k}tb_n^{-2k}x\right)^{\frac{2k}{t}}$$

$$=b_n^{2k}\left(1+2\sigma^2 b_n^{-2k}x+\frac{2k-t}{k}\sigma^4 b_n^{-4k}x^2+\frac{(2k-t)(2k-2t)}{3k^2}\sigma^6 b_n^{-6k}x^3+o(b_n^{-6k})\right). \tag{4.11}$$

By (4.11), we get

$$\begin{aligned}
&\exp\left(-\frac{(z_{n,t}(x))^{2k}}{2\sigma^2}\right)\\
&=\exp\left(-\frac{b_n^{2k}}{2\sigma^2}\right)e^{-x}\exp\left(-\frac{2k-t}{2k}\sigma^2 b_n^{-2k}x^2-\frac{(2k-t)(k-t)}{3k^2}\sigma^4 b_n^{-4k}x^3+o(b_n^{-4k})\right)\\
&=\exp\left(-\frac{b_n^{2k}}{2\sigma^2}\right)e^{-x}\left(1-\frac{2k-t}{2k}\sigma^2 b_n^{-2k}x^2-\frac{(2k-t)\sigma^4x^3[8(k-t)-3(2k-t)x]b_n^{-4k}}{24k^2}+o(b_n^{-4k})\right).
\end{aligned} \tag{4.12}$$

Combining with (1.4) and (4.12), it leads to

$$\begin{aligned}
&\frac{1}{2^{\frac{1}{2k}}\sigma^{\frac{1}{k}}\Gamma(1+\frac{1}{2k})}\exp\left(-\frac{(z_{n,t}(x))^{2k}}{2\sigma^2}\right)\\
=&n^{-1}e^{-x}b_n^{-1}\left(1-\frac{2k-t}{2k}\sigma^2 b_n^{-2k}x^2-\frac{(2k-t)\sigma^4x^3[8(k-t)-3(2k-t)x]b_n^{-4k}}{24k^2}+o(b_n^{-4k})\right).
\end{aligned} \tag{4.13}$$

Besides, by exploiting (4.10), we get

$$1+\frac{\sigma^2}{k}(z_{n,t}(x))^{-2k}+\frac{(1-2k)\sigma^4}{k^2}(z_{n,t}(x))^{-4k}+o(b_n^{-4k})=1+\frac{\sigma^2}{k}b_n^{-2k}+\frac{(1-2k-2kx)\sigma^4}{k^2}b_n^{-4k}+o(b_n^{-4k}). \tag{4.14}$$

Accordingly, combining with Lemma 3.1 of [16] and (4.10), (4.13), and (4.14), (4.8) is derived.
(ii) For $t = 2k$. Then, by norming constants c_n^* and d_n^* defined by (1.9), we get

$$\begin{aligned}
z_{n,t}(x)=&(c_n^*x+d_n^*)^{\frac{1}{t}}=b_n\left(1+2\sigma^2 t b_n^{-2k}x+\frac{2\sigma^4(x+1)}{k}b_n^{-4k}\right)^{\frac{1}{2k}}\\
=&b_n\left(1+\frac{\sigma^2}{k}xb_n^{-2k}+\frac{(1-2k)x^2+2x+2}{2k^2}\sigma^4 b_n^{-4k}\right.\\
&\left.+\frac{(1-2k)(1-4k)x^3+6(1-2k)x^2+6(1-2k)x}{6k^3}\sigma^6 b_n^{-6k}+o(b_n^{-6k})\right),
\end{aligned} \tag{4.15}$$

where b_n satisfies Eq (1.4). From (4.15), it is easy to see

$$(z_{n,t}(x))^{2k}=b_n^{2k}+2\sigma^2 x+\frac{2\sigma^4(x+1)}{k}b_n^{-2k},$$

which, combined with (1.4), deduces that

$$\begin{aligned}
&\frac{1}{2^{\frac{1}{2k}}\sigma^{\frac{1}{k}}\Gamma(1+\frac{1}{2k})}\exp\left(-\frac{(z_{n,t}(x))^{2k}}{2\sigma^2}\right)\\
=&n^{-1}e^{-x}b_n^{-1}\left(1-\frac{(x+1)\sigma^2}{k}b_n^{-2k}+\frac{(x+1)^2\sigma^4}{2k^2}b_n^{-4k}-\frac{(x+1)^3\sigma^6}{6k^3}b_n^{-6k}+o(b_n^{-6k})\right).
\end{aligned} \tag{4.16}$$

Furthermore, it follows from (4.15) that

$$\begin{aligned}
&1+\frac{\sigma^2}{k}(z_{n,t}(x))^{-2k}+\frac{(1-2k)\sigma^4}{k^2}(z_{n,t}(x))^{-4k}+\frac{(1-2k)(1-4k)\sigma^6}{k^3}(z_{n,t}(x))^{-6k}+o(b_n^{-6k})\\
=&1+\frac{\sigma^2}{k}b_n^{-2k}-\frac{(2kx-1+2k)\sigma^4}{k^2}b_n^{-4k}+\frac{4k^2x^2-2k(3-4k)x+8k^2-8k+1}{k^3}\sigma^6 b_n^{-6k}+o(b_n^{-6k}). \quad (4.17)
\end{aligned}$$

Thereby, the claim follows by (4.15)–(4.17) together with Lemma 3.1 of [16]. □

Lemma 4.3. *[7] Let $\{X_1,\ X_2,\cdots,X_n,\cdots\}$ be an independent random sequence following common cdf GMD(k) with $k\in(0,\ \infty)$ and let $M_{n,r}$ indicate the rth largest order statistics of $\{X_i,\ 1\le i\le n\}$. Suppose that there is a constant $z_{n,t}(x)>0$ such that $n(1-F_k(z_{n,t}(x)))\to e^{-x}$, then we get*

$$P\left(|M_{n,r}|^t\le(z_{n,t}(x))^t\right)-\Lambda_r(x)=\Lambda(x)\left\{1-\frac{r-1-e^{-x}}{2}(1-\delta_{n,k}(x))\right\}(1-\delta_{n,k}(x))\frac{e^{-rx}}{(r-1)!}+O(n^{-1}), \quad (4.18)$$

with $\delta_{n,k}(x)=ne^x(1-F_k(z_{n,t}(x)))$.

Proof of Theorem 2.1. With norming constants α_n^* and β_n^* provided by (1.8), set $z_{n,t}(x)=(\alpha_n^*x+\beta_n^*)^{1/t}$. Observe that $\delta_{n,k}(x)=ne^x[1-F_k(z_{n,t}(x))]$, and by (4.1), then we get

$$\delta_{n,k}(x)=1-\frac{(2k-1)(\log\log n)^2}{16k^3\log n}-\frac{\{(2k-1)[x-\log\Gamma(1+\frac{1}{2k})]-1\}\log\log n}{4k^2\log n}+o\left(\frac{\log\log n}{\log n}\right). \quad (4.19)$$

By combining (4.19) and (4.18), we have

$$\begin{aligned}
&P\left(|M_{n,r}|^t\le\alpha_n^*x+\beta_n^*\right)-\Lambda_r(x)=\Lambda(x)\left\{1-\frac{r-1-e^{-x}}{2}(1-\delta_{n,k}(x))\right\}(1-\delta_{n,k}(x))\frac{e^{-rx}}{(r-1)!}+O(n^{-1})\\
=&\Lambda(x)\left(\frac{(2k-1)(\log\log n)^2}{16k^3\log n}+\frac{\{(2k-1)[x-\log\Gamma(1+\frac{1}{2k})]-1\}\log\log n}{4k^2\log n}+o\left(\frac{\log\log n}{\log n}\right)\right)\\
&\times\left[1-\frac{r-1-e^{-x}}{2}\left(\frac{(2k-1)(\log\log n)^2}{16k^3\log n}+\frac{\{(2k-1)[x-\log\Gamma(1+\frac{1}{2k})]-1\}\log\log n}{4k^2\log n}\right.\right.\\
&\left.\left.+o\left(\frac{\log\log n}{\log n}\right)\right)\right]\times\frac{e^{-rx}}{(r-1)!}+O(n^{-1})\\
=&\Lambda(x)\left\{\frac{(2k-1)(\log\log n)^2}{16k^3\log n}+\frac{\{(2k-1)[x-\log\Gamma(1+\frac{1}{2k})]-1\}\log\log n}{4k^2\log n}\right\}\times\frac{e^{-(r-1)x}}{(r-1)!}+o\left(\frac{\log\log n}{\log n}\right).
\end{aligned}$$

Hence,

$$\left(\frac{\log n}{(\log\log n)^2}\right)\left(\mathbf{P}\left(|M_{n,r}|^t\le\alpha_n^*x+\beta_n^*\right)-\Lambda_r(x)\right)\to\frac{(2k-1)e^{-rx}}{16k^3(r-1)!}\Lambda(x),$$

and

$$\begin{aligned}
&(\log\log n)\left[\left(\frac{\log n}{(\log\log n)^2}\right)\left(\mathbf{P}\left(|M_{n,r}|^t\le\alpha_n^*x+\beta_n^*\right)-\Lambda_r(x)\right)-\frac{(2k-1)e^{-rx}}{16k^3(r-1)!}\Lambda(x)\right]\\
&\to\frac{(2k-1)[x-\log\Gamma(1+\frac{1}{2k})]-1}{4k^2}\frac{e^{-rx}}{(r-1)!}\Lambda(x),
\end{aligned}$$

as $n\to\infty$, which is (2.1). □

Proof of Theorem 2.2. (i). When $k \in (0, +\infty)$ and $t \in (0, 2k) \bigcup (2k, +\infty)$, with norming constants c_n and d_n determined by (1.7), set $z_{n,t}(x) = (c_n x + d_n)^{1/t}$. It follows from (4.8) that

$$\delta_{n,k}(x) = 1 - P_k(x)e^x b_n^{-2k} - Q_k(x)e^x b_n^{-4k} + o(b_n^{-4k}), \tag{4.20}$$

where $\delta_{n,k}(x) = ne^x(1 - F_k(z_{n,t}(x)))$, $P_k(x)$ and $Q_k(x)$ are separately defined by (2.3) and (2.4). Combining with Lemma 4.3 and (4.20), we get

$$\begin{aligned}
&P\left(|M_{n,r}|^t \le c_n x + d_n\right) - \Lambda_r(x)\\
=&\Lambda(x)\left\{1 - \frac{r-1-e^{-x}}{2}(1-\delta_{n,k}(x))\right\}(1-\delta_{n,k}(x))\frac{e^{-rx}}{(r-1)!} + O(n^{-1})\\
=&\Lambda(x)\left(P_k(x)e^x b_n^{-2k} + Q_k(x)e^x b_n^{-4k} + o(b_n^{-4k})\right)\\
&\times\left[1-\frac{r-1-e^{-x}}{2}(P_k(x)e^x b_n^{-2k}+Q_k(x)e^x b_n^{-4k}+o(b_n^{-4k}))\right]\frac{e^{-rx}}{(r-1)!}+O(n^{-1})\\
=&\Lambda(x)\left\{P_k(x)b_n^{-2k}+\left[Q_k(x)-\frac{(r-1)e^x-1}{2}P_k^2(x)\right]b_n^{-4k}\right\}\frac{e^{-(r-1)x}}{(-1)}+o(b_n^{-4k}).
\end{aligned}$$

Then, as $n \to \infty$,

$$b_n^{2k}\left\{\mathbf{P}\left(|M_{n,r}|^t \le c_n x + d_n\right) - \Lambda_r(x)\right\} \to \Lambda(x)P_k(x)\frac{e^{-(r-1)x}}{(r-1)!},$$

and

$$\begin{aligned}
&b_n^{2k}\left[b_n^{2k}\left(\mathbf{P}\left(|M_{n,r}|^t \le c_n x + d_n\right) - \Lambda_r(x)\right) - \Lambda(x)P_k(x)\frac{e^{-(r-1)x}}{(r-1)!}\right]\\
\to&\left[Q_k(x) - \frac{(r-1)e^x - 1}{2}P_k^2(x)\right]\frac{e^{-(r-1)x}}{(r-1)!}\Lambda(x),
\end{aligned}$$

which is (2.2).

(ii). For $k \in (0, \infty)$ and $t = 2k$, with norming constants c_n^* and d_n^* defined by (1.9), set $z_{n,t}(x) = (c_n^* x + d_n^*)^{1/t}$. By (4.9), we get

$$\delta_{n,k}(x) = 1 - S_k(x)e^x b_n^{-4k} - T_k(x)e^x b_n^{-6k} + o(b_n^{-6k}), \tag{4.21}$$

because of $\delta_{n,k}(x) = ne^x(1 - F_k(z_{n,t}(x)))$, where $S_k(x)$ and $T_k(x)$ are separately defined by (2.6) and (2.7).

Combining with Lemma 4.3 and (4.21), we get

$$\begin{aligned}
&P\left(|M_{n,r}|^t \le c_n^* x + d_n^*\right) - \Lambda_r(x)\\
=&\Lambda(x)\left\{1 - \frac{r-1-e^{-x}}{2}(1-\delta_{n,k}(x))\right\}(1-\delta_{n,k}(x))\frac{e^{-rx}}{(r-1)!} + O(n^{-1})\\
=&\Lambda(x)\left(S_k(x)e^x b_n^{-4k} + T_k(x)e^x b_n^{-6k} + o(b_n^{-6k})\right)\\
&\times\left[1-\frac{r-1-e^{-x}}{2}(S_k(x)e^x b_n^{-4k}+T_k(x)e^x b_n^{-6k}+o(b_n^{-6k}))\right]\frac{e^{-rx}}{(r-1)!}+O(n^{-1})
\end{aligned}$$

$$=\Lambda(x)\left\{S_k(x)b_n^{-4k}+T_k(x)b_n^{-6k}\right\}\frac{e^{-(r-1)x}}{(r-1)!}+o(b_n^{-6k}). \tag{4.22}$$

Consequently, it follows from (4.22) that as $n\to\infty$,

$$b_n^{4k}\left(\mathbf{P}\left(|M_{n,r}|^t\le c_n^*x+d_n^*\right)-\Lambda_r(x)\right)\to\Lambda(x)S_k(x)\frac{e^{-(r-1)x}}{(r-1)!},$$

and

$$b_n^{2k}\left[b_n^{4k}\left(\mathbf{P}\left(|M_{n,r}|^t\le c_n^*x+d_n^*\right)-\Lambda_r(x)\right)-\Lambda(x)S_k(x)\frac{e^{-(r-1)x}}{(r-1)!}\right]\to\Lambda(x)T_k(x)\frac{e^{-(r-1)x}}{(r-1)!},$$

which is (2.5). □

5. Conclusions

This paper establishes higher-order asymptotic expansions and convergence rates for powered order statistics drawn from the generalized Maxwell distribution. In the future, we will consider the density of higher-order asymptotic expansions of the order statistics of the generalized Maxwell distribution family of powers and the uniform convergence rate under linear normalizing.

Author contributions

Jianwen Huang: Writing -original draft, Investigation. Xinling Liu: Writing -review & editing, Supervision. Jinping Jia: Validation. Runke Wang: Visualization.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflict of interest.

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