AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(3): 8832–8841.
DOI:10.3934/math.2026363
Received: 23 December 2025
Revised: 14 February 2026
Accepted: 10 March 2026
Published: 31 March 2026

Research article

Weak solutions for two-dimensional magnetohydrodynamics equations with partial dissipation and magnetic diffusion

Shaoliang Yuan*

School of Big Data and Artificial Intelligence, Fujian Polytechnic Normal University, Fuzhou 350300, Fujian, China

* **Correspondence:** Email: 13640840@qq.com.

Abstract: In this article, we study the existence of weak solutions to two-dimensional incompressible magnetohydrodynamics (MHD) equations. For the case with only magnetic diffusion, the global existence of solutions in $L^\infty(0,\infty;H^1)$ was established by Kozono, whereas the existence of $L^\infty(0,\infty;H^1)$ solutions for the case with only dissipation is totally unknown. The purpose here is to consider the mixed case, that is, a system with partial dissipation and partial magnetic diffusion. We show that there exists a unique local solution with initial data in H^1 space.

Keywords: MHD equations; local existence; uniqueness; partial dissipation and magnetic diffusion
Mathematics Subject Classification: 35Q35, 35D30

1. Introduction

In this article, we study the Cauchy problem of two-dimensional incompressible magnetohydrodynamics (MHD) equations:

$$\boldsymbol{u}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{u} - \nu_1\partial_{xx}\boldsymbol{u} - \nu_2\partial_{yy}\boldsymbol{u} + \nabla p = \boldsymbol{B}\cdot\nabla\boldsymbol{B}, \tag{1.1a}$$

$$\boldsymbol{B}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{B} - \mu_1\partial_{xx}\boldsymbol{B} - \mu_2\partial_{yy}\boldsymbol{B} = \boldsymbol{B}\cdot\nabla\boldsymbol{u}, \tag{1.1b}$$

$$\nabla\cdot\boldsymbol{u} = \nabla\cdot\boldsymbol{B} = 0, \tag{1.1c}$$

$$(\boldsymbol{u},\boldsymbol{B})|_{t=0} = (\boldsymbol{u}_0,\boldsymbol{B}_0), \tag{1.1d}$$

where $\boldsymbol{u} = (u_1,u_2)$ denotes the velocity field, $\boldsymbol{B} = (B_1,B_2)$ is the magnetic field, and p stands for the scalar pressure. The nonnegative parameters ν_1, ν_2 are the dissipation coefficients, and μ_1, μ_2 are the magnetic diffusion coefficients.

Over past years, the existence, uniqueness, and regularity of incompressible MHD equations have been extensively studied by many mathematicians. For the case $\nu_1 = \nu_2 > 0$ and $\mu_1 = \mu_2 > 0$, Duvaut and Lions [1] proved that system (1.1) has a local and unique solution with $\boldsymbol{u}_0$ and $\boldsymbol{B}_0$ in $H^s(\mathbb{R}^2)(s \geq 2)$, and the solution is global provided that $\boldsymbol{u}_0$ and $\boldsymbol{B}_0$ are small enough. Later on, the smallness condition on the initial data was eliminated by Sermange and Teman in [2]. When $\nu_1 = \nu_2 = 0$ and $\mu_1 = \mu_2 > 0$, Kozono [3] proved the global existence of solutions in bounded domains with initial data in H^1, and recently this result was extended to the whole of $\mathbb{R}^2$ by Lei and Zhou in [4]; see also [5,6]. By contrast, for $\nu_1 = \nu_2 > 0$ and $\mu_1 = \mu_2 = 0$, Jiu and Niu [7] established the local existence of classical solutions for initial data in $H^s(\mathbb{R}^2)(s \geq 3)$. For the existence and uniqueness of weak solutions, we refer to [8–10] and references therein. For the case that $\nu_1 = \mu_2 = 0$, $\nu_2 > 0$, and $\mu_1 > 0$, Cao and Wu in [5] obtained the global existence and uniqueness of solutions with $(\boldsymbol{u}_0, \boldsymbol{B}_0)$ in $H^2(\mathbb{R}^2)\times H^2(\mathbb{R}^2)$. If all four parameters are zero, the existence of classic solutions was proved by Schmidt [11] and Secchi [12]. Very recently, Zhai [13] considered the 3D case with fractional dissipation and established global-in-time existence of solutions for small initial data.

In the present paper, we focus on the existence and uniqueness of weak solutions of system (1.1), especially in the case of H^1 initial data. As we have mentioned, for the case with magnetic diffusion but not dissipation, system (1.1) has a global weak solution when the initial data belongs to $H^1(\mathbb{R}^2)$, whereas the uniqueness remains an open question. For the case with only dissipation, it is noteworthy that the problem of existence and uniqueness with H^1 initial data is completely unresolved. Consequently, a natural question arises: does an H^1 solution exists for the case with mixed partial dissipation and magnetic diffusion? The answer is affirmative: By means of a precise analysis of the structure of Eq (1.1), some new estimates of the convection terms are established, based on which the existence and uniqueness of solutions are derived. In fact, we have the following result:

Theorem 1.1. *Consider the Cauchy problem* (1.1) *with* $\nu_1 > 0$, $\mu_1 > 0$, *and* $\nu_2 = \mu_2 = 0$. *Suppose that* $\boldsymbol{u}_0, \boldsymbol{B}_0 \in H^1(\mathbb{R}^2)$ *are divergence-free. Then, there exists a unique solution* $\boldsymbol{u}, \boldsymbol{B} \in L^\infty(0, T_*; H^1(\mathbb{R}^2))$ *for some* $T_* > 0$.

As a direct consequence, we can obtain similar results for the system (1.1) with only vertical dissipation and vertical magnetic diffusion.

Corollary 1.1. *Consider the Cauchy problem* (1.1) *with* $\nu_2 > 0$, $\mu_2 > 0$, *and* $\nu_1 = \mu_1 = 0$. *Suppose that* $\boldsymbol{u}_0, \boldsymbol{B}_0 \in H^1(\mathbb{R}^2)$ *are divergence-free. Then, there exists a unique solution* $\boldsymbol{u}, \boldsymbol{B} \in L^\infty(0, T_*; H^1(\mathbb{R}^2))$ *for some* $T_* > 0$.

2. Proof of Theorem 1.1

For the sake of convenience, we rewrite system (1.1) with horizontal dissipation and horizontal magnetic diffusion as:

$$\boldsymbol{u}_t + \boldsymbol{u} \cdot \nabla \boldsymbol{u} - \nu \partial_{xx} \boldsymbol{u} + \nabla p = \boldsymbol{B} \cdot \nabla \boldsymbol{B}, \tag{2.1a}$$

$$\boldsymbol{B}_t + \boldsymbol{u} \cdot \nabla \boldsymbol{B} - \mu \partial_{xx} \boldsymbol{B} = \boldsymbol{B} \cdot \nabla \boldsymbol{u}, \tag{2.1b}$$

$$\nabla \cdot \boldsymbol{u} = \nabla \cdot \boldsymbol{B} = 0, \tag{2.1c}$$

$$(\boldsymbol{u}, \boldsymbol{B})|_{t=0} = (\boldsymbol{u}_0, \boldsymbol{B}_0). \tag{2.1d}$$

Here, ν and μ are positive constants.

We begin by introducing some well-known results. The following result can be found in [5].

Lemma 2.1. *Suppose that $f, g, h \in L^2(\mathbb{R}^2)$, and $\partial_x g, \partial_y h \in L^2(\mathbb{R}^2)$. Then,*

$$\int_{\mathbb{R}^2} |fgh| \leq C\|f\|_{L^2}\|g\|_{L^2}^{\frac{1}{2}}\|\partial_x g\|_{L^2}^{\frac{1}{2}}\|h\|_{L^2}^{\frac{1}{2}}\|\partial_y h\|_{L^2}^{\frac{1}{2}}. \tag{2.2}$$

In the next subsection, we will construct an approximate sequence $(\boldsymbol{u}^R, \boldsymbol{B}^R)$ for $(\boldsymbol{u}, \boldsymbol{B})$. For convenience, we also denote $\boldsymbol{u}^R$ and $\boldsymbol{B}^R$ as the velocity field and the magnetic field, respectively, and define $w^R := \operatorname{curl} \boldsymbol{u}^R$(denoting the vorticity) and $j^R := \operatorname{curl} \boldsymbol{B}^R$(denoting the current density). It follows that $(\boldsymbol{u}^R, \boldsymbol{B}^R)$ satisfies the following system.

Lemma 2.2. *Suppose that vectors $\boldsymbol{u}^R, \boldsymbol{B}^R$ are divergence-free. Then, for $\alpha \in \mathbb{N}^2$,*

$$\begin{aligned} \|D^\alpha \nabla \boldsymbol{u}^R\|_{L^2} &\leq C\|D^\alpha w^R\|_{L^2}, \\ \|D^\alpha \nabla \boldsymbol{B}^R\|_{L^2} &\leq C\|D^\alpha j^R\|_{L^2}, \end{aligned} \tag{2.3}$$

and as a result, we have

$$\begin{aligned} \|\partial_{xx}\boldsymbol{u}^R\|_{L^2} + \|\partial_{yy}u_2^R\|_{L^2} &\leq C\|\partial_x w^R\|_{L^2}, \\ \|\partial_{xx}\boldsymbol{B}^R\|_{L^2} + \|\partial_{yy}B_2^R\|_{L^2} &\leq C\|\partial_x j^R\|_{L^2}. \end{aligned} \tag{2.4}$$

Proof. The inequality (2.3) follows immediately from the Biot-Savart law and the Calderón-Zygmund theorem. To see (2.4), we first apply (2.3) to deduce that

$$\begin{aligned} \|\partial_{xx}\boldsymbol{u}^R\|_{L^2} &\leq C\|\partial_x \nabla \boldsymbol{u}^R\|_{L^2} \leq C\|\partial_x w^R\|_{L^2}, \\ \|\partial_{xx}\boldsymbol{B}^R\|_{L^2} &\leq C\|\partial_x \nabla \boldsymbol{B}^R\|_{L^2} \leq C\|\partial_x j^R\|_{L^2}. \end{aligned} \tag{2.5}$$

On the other hand, we observe that $\boldsymbol{u}^R, \boldsymbol{B}^R$ are divergence-free; therefore,

$$\partial_{yy}u_2^R = \partial_{xy}u_1^R \text{ and } \partial_{yy}B_2^R = \partial_{xy}B_1^R. \tag{2.6}$$

We then use again the inequality (2.3) to get

$$\begin{aligned} \|\partial_{yy}u_2^R\|_{L^2} &= \|\partial_{xy}u_1^R\|_{L^2} \leq C\|\partial_x w^R\|_{L^2}, \\ \|\partial_{yy}B_2^R\|_{L^2} &= \|\partial_{xy}B_1^R\|_{L^2} \leq C\|\partial_x j^R\|_{L^2}. \end{aligned} \tag{2.7}$$

Collecting the above results gives (2.4), and this completes the proof of Lemma 2.2. □

In the following section, we prove the existence of weak solutions, and then show that the solution is unique.

2.1. Proof of existence

The general strategy of the proof is similar to that for proving existence of solutions to the Navier-Stokes and Euler equations, which can be found in Chapter 3 of [14]. First, we construct a family of approximate equations of (2.1) with smooth solutions $(\boldsymbol{u}^R, \boldsymbol{B}^R)$. We then show that these solutions are uniformly bounded, and the limit vector field $(\boldsymbol{u}, \boldsymbol{B})$ solves the original equations.

To proceed the proof, we begin by defining a Fourier truncation $\mathcal{S}_R$:

$$\widehat{\mathcal{S}_R f}(\xi) := \mathbb{1}_{B_R}(\xi)\widehat{f}(\xi), f \in L^2(\mathbb{R}^2), \tag{2.8}$$

where B_R denotes the ball of radius R centered at the origin. With the Fourier truncation $\mathcal{S}_R$, we may construct the approximate MHD equations as follows:

$$\boldsymbol{u}_t^R + \mathcal{S}_R[\boldsymbol{u}^R \cdot \nabla \boldsymbol{u}^R] - \nu\partial_{xx}\boldsymbol{u}^R + \nabla p^R = \mathcal{S}_R[\boldsymbol{B}^R \cdot \nabla \boldsymbol{B}^R], \tag{2.9a}$$
$$\boldsymbol{B}_t^R + \mathcal{S}_R[\boldsymbol{u}^R \cdot \nabla \boldsymbol{B}^R] - \mu\partial_{xx}\boldsymbol{B}^R = \mathcal{S}_R[\boldsymbol{B}^R \cdot \nabla \boldsymbol{u}^R], \tag{2.9b}$$
$$\nabla \cdot \boldsymbol{u}^R = \nabla \cdot \boldsymbol{B}^R = 0, \tag{2.9c}$$
$$(\boldsymbol{u}^R, \boldsymbol{B}^R)|_{t=0} = (\mathcal{S}_R\boldsymbol{u}_0, \mathcal{S}_R\boldsymbol{B}_0). \tag{2.9d}$$

It can be checked that $\boldsymbol{u}^R$ and $\boldsymbol{B}^R$ lie in the space

$$V_R := \{f \in L^2(\mathbb{R}^2), \widehat{f} \text{ is supported in } B_R\}, \tag{2.10}$$

whence the initial data $\mathcal{S}_R\boldsymbol{u}_0, \mathcal{S}_R\boldsymbol{B}_0$ lies in V_R. Observing that the operator $\mathcal{S}_R$ acts as a mollifier that smooths the equations, we may apply Picard's theorem to deduce that (2.9) has a global smooth solution $(\boldsymbol{u}^R, \boldsymbol{B}^R)$.

We are now in the position to show that $\boldsymbol{u}^R, \boldsymbol{B}^R$ are uniformly bounded. We begin with a simple energy estimate of $(\boldsymbol{u}^R, \boldsymbol{B}^R)$.

Proposition 2.1. *Assume initial data $\boldsymbol{u}_0, \boldsymbol{B}_0 \in L^2(\mathbb{R}^2)$. Then,*

$$\|\boldsymbol{u}^R(t)\|_{L^2}^2 + \|\boldsymbol{B}^R(t)\|_{L^2}^2 + 2\nu\int_0^t \|\partial_x\boldsymbol{u}^R(s)\|_{L^2}^2\,\mathrm{d}s + 2\mu\int_0^t \|\partial_x\boldsymbol{B}^R(s)\|_{L^2}^2\,\mathrm{d}s = \|\mathcal{S}_R\boldsymbol{u}_0\|_{L^2}^2 + \|\mathcal{S}_R\boldsymbol{B}_0\|_{L^2}^2. \tag{2.11}$$

Proof. Taking the inner product of (2.9a) with $\boldsymbol{u}^R$ and integrating over $\mathbb{R}^2 \times [0, t]$ for $t > 0$, we find that

$$\|\boldsymbol{u}^R(t)\|_{L^2}^2 + 2\nu\int_0^t (\|\partial_x\boldsymbol{u}^R(s)\|_{L^2}^2\,\mathrm{d}s = \|\mathcal{S}_R\boldsymbol{u}_0\|_{L^2}^2 + 2\int_0^t (\boldsymbol{B}^R \cdot \nabla\boldsymbol{B}^R, \boldsymbol{u}^R), \tag{2.12}$$

where we have used integration by parts and the property that $\boldsymbol{u}^R$ is divergence-free. Then, we inner product (2.9b) with $\boldsymbol{B}^R$ and integrate over [0, t] to obtain

$$\|\boldsymbol{B}^R(t)\|_{L^2}^2 + 2\mu\int_0^t (\|\partial_x\boldsymbol{B}^R(s)\|_{L^2}^2\,\mathrm{d}s = \|\mathcal{S}_R\boldsymbol{B}_0\|_{L^2}^2 + 2\int_0^t (\boldsymbol{B}^R \cdot \nabla\boldsymbol{u}^R, \boldsymbol{B}^R). \tag{2.13}$$

Summing up the above two equations and using integration by parts, we obtain (2.11) immediately. □

To obtain bounds of high-order derivatives, we turn to the evolution equations of the vorticity w^R and the current density j^R.

Proposition 2.2. *Let $(\boldsymbol{u}^R, \boldsymbol{B}^R)$ be the solution to* (2.9)*. There is a time $T_* = T_*(\nu, \mu, \boldsymbol{u}_0, \boldsymbol{B}_0) > 0$ such that, for $t \in [0, T_*]$,*

$$\|w^R(t)\|_{L^2}^2 + \|j^R(t)\|_{L^2}^2 + \nu\int_0^t \|\partial_x w^R\|_{L^2}^2 + \mu\int_0^t \|\partial_x j^R\|_{L^2}^2 \le C(\nu, \mu, \|\boldsymbol{u}_0\|_{H^1}^2 + \|\boldsymbol{B}_0\|_{H^1}^2), \tag{2.14}$$

where $C(\nu, \mu, \|\boldsymbol{u}_0\|_{H^1}^2 + \|\boldsymbol{B}_0\|_{H^1}^2)$ is a constant that depends on ν, μ, and $\|\boldsymbol{u}_0\|_{H^1}^2 + \|\boldsymbol{B}_0\|_{H^1}^2$.

Proof. It is easy to see that the vorticity w^R satisfies

$$w_t^R + \boldsymbol{u}^R \cdot \nabla w^R - \boldsymbol{B}^R \cdot \nabla j^R = \nu \partial_{xx} w^R, \tag{2.15}$$

and the current density j^R obeys

$$j_t^R + \boldsymbol{u} \cdot \nabla j^R - \boldsymbol{B}^R \cdot \nabla w^R = \mu \partial_{xx} j^R + 2\partial_x B_1^R(\partial_x u_2^R + \partial_y u_1^R) - 2\partial_x u_1^R(\partial_x B_2^R + \partial_y B_1^R). \tag{2.16}$$

We multiply (2.15) by w^R and (2.16) by j^R, then sum up the two resulting equations and integrate them in $\mathbb{R}^2$ to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|w^R(t)\|_{L^2}^2 + \|j^R(t)\|_{L^2}^2) + \nu\|\partial_x w^R\|_{L^2}^2 + \mu\|\partial_x j^R\|_{L^2}^2 =: \sum_{k=1}^{4} \mathcal{I}_k, \tag{2.17}$$

where

$$\begin{aligned}
\mathcal{I}_1 &:= 2\int_{\mathbb{R}^2} \partial_x B_1^R \partial_x u_2^R j^R, \\
\mathcal{I}_2 &:= 2\int_{\mathbb{R}^2} \partial_x B_1^R \partial_y u_1^R j^R, \\
\mathcal{I}_3 &:= 2\int_{\mathbb{R}^2} \partial_x u_1^R \partial_x B_2^R j^R, \\
\mathcal{I}_4 &:= 2\int_{\mathbb{R}^2} \partial_x u_1^R \partial_y B_1^R j^R.
\end{aligned} \tag{2.18}$$

We begin with estimating $\mathcal{I}_1$. Indeed, by applying the inequality (2.2) to $\mathcal{I}_1$, we arrive at

$$\left|2\int_{\mathbb{R}^2} \partial_x B_1^R \partial_x u_2^R j^R\right| \le C\|\partial_x B_1^R\|_{L^2}^{\frac{1}{2}}\|\partial_{xx} B_1^R\|_{L^2}^{\frac{1}{2}}\|\partial_x u_2^R\|_{L^2}^{\frac{1}{2}}\|\partial_{xy} u_2^R\|_{L^2}^{\frac{1}{2}}\|j^R\|_{L^2}. \tag{2.19}$$

Observing that $\partial_{xy} u_2^R = \partial_{xx} u_1^R$, it follows from Lemma 2.2 that

$$\begin{aligned}
\left|2\int_{\mathbb{R}^2} \partial_x B_1^R \partial_x u_2^R j^R\right| &\le C\|\partial_x B_1^R\|_{L^2}^{\frac{1}{2}}\|\partial_{xx} B_1^R\|_{L^2}^{\frac{1}{2}}\|\partial_x u_2^R\|_{L^2}^{\frac{1}{2}}\|\partial_{xx} u_1^R\|_{L^2}^{\frac{1}{2}}\|j^R\|_{L^2} \\
&\le C\|\partial_x \boldsymbol{B}^R\|_{L^2}^{\frac{1}{2}}\|\partial_x j^R\|_{L^2}^{\frac{1}{2}}\|\partial_x \boldsymbol{u}^R\|_{L^2}^{\frac{1}{2}}\|\partial_x w^R\|_{L^2}^{\frac{1}{2}}\|j^R\|_{L^2} \\
&\le C\|\partial_x j^R\|_{L^2}^{\frac{1}{2}}\|\partial_x w^R\|_{L^2}^{\frac{1}{2}}\|w^R\|_{L^2}^{\frac{1}{2}}\|j^R\|_{L^2}^{\frac{3}{2}}.
\end{aligned} \tag{2.20}$$

Therefore, by Young's inequality,

$$|\mathcal{I}_1| \le \frac{\nu}{8}\|\partial_x w^R\|_{L^2}^2 + \frac{\mu}{8}\|\partial_x j^R\|_{L^2}^2 + C(\nu,\mu)(\|w^R\|_{L^2}^2 + \|j^R\|_{L^2}^2)\|j^R\|_{L^2}^2. \tag{2.21}$$

The second term $\mathcal{I}_2$ can be handled similarly. Indeed, using the inequality (2.2), we have that

$$\left|2\int_{\mathbb{R}^2} \partial_x B_1^R \partial_y u_1^R j^R\right| \le C\|\partial_x B_1^R\|_{L^2}^{\frac{1}{2}}\|\partial_{xy} B_1^R\|_{L^2}^{\frac{1}{2}}\|\partial_y u_1^R\|_{L^2}^{\frac{1}{2}}\|\partial_{xy} u_1^R\|_{L^2}^{\frac{1}{2}}\|j^R\|_{L^2}. \tag{2.22}$$

As $\boldsymbol{u}$ and $\boldsymbol{B}$ are divergence-free, we have $\partial_{xy} u_1^R = \partial_{yy} u_2^R$, and we get that

$$\left|2\int_{\mathbb{R}^2} \partial_x B_1^R \partial_y u_1^R j^R\right| \le C\|\partial_x B_1^R\|_{L^2}^{\frac{1}{2}}\|\partial_{yy} B_2^R\|_{L^2}^{\frac{1}{2}}\|\partial_y u_1^R\|_{L^2}^{\frac{1}{2}}\|\partial_{yy} u_2^R\|_{L^2}^{\frac{1}{2}}\|j^R\|_{L^2}. \tag{2.23}$$

From Lemma 2.2, we find that

$$\left|2\int_{\mathbb{R}^2}\partial_x B_1^R\partial_y u_1^R j^R\right| \le C\|\partial_x j^R\|_{L^2}^{\frac12}\|w^R\|_{L^2}^{\frac12}\|\partial_x w^R\|_{L^2}^{\frac12}\|j^R\|_{L^2}^{\frac32}. \tag{2.24}$$

Therefore,

$$|\mathcal{I}_2| \le C\|\partial_x j^R\|_{L^2}^{\frac12}\|\partial_x w^R\|_{L^2}^{\frac12}\|w^R\|_{L^2}^{\frac12}\|j^R\|_{L^2}^{\frac32} \le \frac{\nu}{8}\|\partial_x w^R\|_{L^2}^2 + \frac{\mu}{8}\|\partial_x j\|_{L^2}^2 + C(\nu,\mu)(\|w^R\|_{L^2}^2 + \|j^R\|_{L^2}^2)\|j^R\|_{L^2}^2.$$

$\mathcal{I}_3$ and $\mathcal{I}_4$ can be handled similarly; it can be checked that

$$|\mathcal{I}_3| + |\mathcal{I}_4| \le \frac{\nu}{4}\|\partial_x w^R\|_{L^2}^2 + \frac{\mu}{4}\|\partial_x j\|_{L^2}^2 + C(\nu,\mu)(\|w^R\|_{L^2}^2 + \|j^R\|_{L^2}^2)\|j^R\|_{L^2}^2. \tag{2.25}$$

Substituting the estimates about $\mathcal{I}_k$(k=1,...,4) into (2.17) and integrating over [0, t), we find that

$$\|w^R(t)\|_{L^2}^2+\|j^R(t)\|_{L^2}^2+\nu\int_0^t\|\partial_x w^R\|_{L^2}^2+\mu\int_0^t\|\partial_x j^R\|_{L^2}^2 \le \|w_0^R\|_{L^2}^2+\|j_0^R\|_{L^2}^2+C(\nu,\mu)\int_0^t(\|w^R\|_{L^2}^2+\|j^R\|_{L^2}^2)^2, \tag{2.26}$$

where $C(\nu,\mu)$ is a constant independent of R. We set $M^R := \|w_0^R\|_{L^2}^2 + \|j_0^R\|_{L^2}^2$ and then define $T(R)$ by

$$T(R) := \sup\{T \ge 0 : \sup_{t\in[0,T]}\|w^R\|_{L^2}^2 + \|j^R\|_{L^2}^2 \le 2M^R\}. \tag{2.27}$$

As $(\boldsymbol{u}^R, \boldsymbol{B}^R)$ are smooth solutions to (2.9), we therefore have $T(R) > 0$ for all $R > 0$. Moreover, for all $t < T(R)$, it follows from (2.26) that

$$\|w^R(t)\|_{L^2}^2 + \|j^R(t)\|_{L^2}^2 \le M^R(1 + C(\nu,\mu)M^R t). \tag{2.28}$$

On the other hand, from the definition of M^R, we find that M^R is uniformly bounded with respect to R. Indeed,

$$\begin{aligned}\|w_0^R\|_{L^2} &= \|\widehat{\mathcal{S}_R w_0}\|_{L^2} = \|\mathbb{1}_R w_0\|_{L^2} \le \|\boldsymbol{u}_0\|_{H^1},\\ \|j_0^R\|_{L^2} &= \|\widehat{\mathcal{S}_R j_0}\|_{L^2} = \|\mathbb{1}_R j_0\|_{L^2} \le \|\boldsymbol{B}_0\|_{H^1},\end{aligned} \tag{2.29}$$

which implies that there exists a constant M independent of R such that $M^R \le M$. Let us set $T_* := \frac{1}{C(\nu,\mu)M}$, and then we may assert that $T(R) \ge T_*$ for all $R > 0$. To see this, we suppose that $T(R) < T_*$; it then follows from (2.28) that

$$\sup_{t\in[0,T(R)]}\|w^R(t)\|_{L^2}^2 + \|j^R(t)\|_{L^2}^2 < M^R(1 + C(\nu,\mu)M^R T_*) \le 2M^R, \tag{2.30}$$

which contradicts with the definition of $T(R)$. Therefore, the inequality (2.14) holds, and this completes the proof of Proposition 2.2. □

With uniform bounds of $\boldsymbol{u}^R, \boldsymbol{w}^R$, we next establish uniform bounds on the time derivatives $\partial_t\boldsymbol{u}^R$ and $\partial_t\boldsymbol{B}^R$. By applying the Leray projector $\mathbb{P}$ to (2.9a), we obtain the following equations:

$$\boldsymbol{u}_t^R = -\mathbb{P}\mathcal{S}_R[\boldsymbol{u}^R\cdot\nabla\boldsymbol{u}^R] + \nu\partial_{xx}\boldsymbol{u}^R + \mathbb{P}\mathcal{S}_R[\boldsymbol{B}^R\cdot\nabla\boldsymbol{B}^R], \tag{2.31a}$$

$$\boldsymbol{B}_t^R = -\mathcal{S}_R[\boldsymbol{u}^R\cdot\nabla\boldsymbol{B}^R] + \mu\partial_{xx}\boldsymbol{B}^R + \mathcal{S}_R[\boldsymbol{B}^R\cdot\nabla\boldsymbol{u}^R]. \tag{2.31b}$$

We assert that $\boldsymbol{u}_t^R$ and $\boldsymbol{B}_t^R$ are uniformly bounded in $L^\infty(0,T_*;H^{-1}(\mathbb{R}^2))$. Indeed, for arbitrary function $\varphi\in H^1(\mathbb{R}^2)$, we have

$$\begin{aligned}\left\langle \boldsymbol{u}_t^R,\varphi\right\rangle &= (-\mathbb{P}\mathcal{S}_R[\boldsymbol{u}^R\cdot\nabla\boldsymbol{u}^R]+\nu\partial_{xx}\boldsymbol{u}^R+\mathbb{P}\mathcal{S}_R[\boldsymbol{B}^R\cdot\nabla\boldsymbol{B}^R],\varphi)\\ &= (\boldsymbol{u}^R\cdot\nabla[\mathbb{P}\mathcal{S}_R\varphi],\boldsymbol{u}^R)-\nu(\partial_x\boldsymbol{u}^R,\partial_x\varphi)-(\boldsymbol{B}^R\cdot\nabla[\mathbb{P}\mathcal{S}_R\varphi],\boldsymbol{B}^R),\end{aligned}\tag{2.32}$$

where we used the fact that the projector $\mathbb{P}$ and the Fourier multiplier $\mathcal{S}_R$ commute, because the approximate solution $(\boldsymbol{u}^R,\boldsymbol{B}^R)$ lies in Schwartz space. We use a 2D Ladyzhenskaya inequality to deduce that

$$|\left\langle \boldsymbol{u}_t^R,\varphi\right\rangle|\le C(\|\boldsymbol{u}^R\|_{H^1}+\|\boldsymbol{B}^R\|_{H^1})(\|\boldsymbol{u}^R\|_{H^1}+\|\boldsymbol{B}^R\|_{H^1}+1)\|\varphi\|_{H^1}.\tag{2.33}$$

Observing that $\boldsymbol{u}^R$ and $\boldsymbol{B}^R$ are uniformly bounded in $L^\infty(0,T_*;H^1(\mathbb{R}^2))$, we obtain that

$$\boldsymbol{u}_t^R \text{ is uniformly bounded in } L^\infty(0,T_*;H^{-1}(\mathbb{R}^2)).\tag{2.34}$$

Similarly, we have that

$$\boldsymbol{B}_t^R \text{ is uniformly bounded in } L^\infty(0,T_*;H^{-1}(\mathbb{R}^2)).\tag{2.35}$$

We then apply the Banach-Alaoglu theorem to extract a weakly-* convergent subsequence such that

$$\begin{aligned}&\boldsymbol{u}_t^{R_m}\overset{*}{\rightharpoonup}\boldsymbol{u}\text{ in }L^\infty(0,T_*;H^{-1}(\mathbb{R}^2)),\\ &\boldsymbol{B}_t^{R_m}\overset{*}{\rightharpoonup}\boldsymbol{B}\text{ in }L^\infty(0,T_*;H^{-1}(\mathbb{R}^2)),\\ &\boldsymbol{u}^{R_m}\overset{*}{\rightharpoonup}\boldsymbol{u}\text{ in }L^\infty(0,T_*;H^{1}(\mathbb{R}^2)),\\ &\boldsymbol{B}^{R_m}\overset{*}{\rightharpoonup}\boldsymbol{B}\text{ in }L^\infty(0,T_*;H^{1}(\mathbb{R}^2)).\end{aligned}\tag{2.36}$$

It is time to verify that $(\boldsymbol{u},\boldsymbol{B})$ is a weak solution of the equations. Indeed, by the Aubin-Lions compactness lemma, we find that there exists a subsequence of $(\boldsymbol{u}^{R_m},\boldsymbol{B}^{R_m})$ such that

$$\begin{aligned}&\boldsymbol{u}^{R_m}\to\boldsymbol{u}\text{ in }L^2(0,T_*;L^2(K)),\\ &\boldsymbol{B}^{R_m}\to\boldsymbol{B}\text{ in }L^2(0,T_*;L^2(K))\end{aligned}\tag{2.37}$$

for any compact subset $K\subset\mathbb{R}^2$. This local convergence allows us to pass the limit in nonlinear terms. Thus, $(\boldsymbol{u},\boldsymbol{B})$ is a weak solution of (1.1).

2.2. *Proof of uniqueness*

We first state and prove a new estimate for the nonlinear terms, which is important for proving uniqueness of solutions.

Lemma 2.3. *Suppose that $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in H^1(\mathbb{R}^2)$ are divergence-free vector fields, and $\partial_{yy}v_2\in L^2(\mathbb{R}^2)$. Then,*

$$|(\boldsymbol{u}\cdot\nabla\boldsymbol{v},\boldsymbol{w})|\le C\|\boldsymbol{u}\|_{L^2}\|\nabla\boldsymbol{v}\|_{L^2}^{\frac12}\|\partial_{yy}v_2\|_{L^2}^{\frac12}\|\boldsymbol{w}\|_{L^2}^{\frac12}\|\partial_x\boldsymbol{w}\|_{L^2}^{\frac12}+C\|\boldsymbol{u}\|_{L^2}^{\frac12}\|\partial_x\boldsymbol{u}\|_{L^2}^{\frac12}\|\nabla\boldsymbol{v}\|_{L^2}\|\boldsymbol{w}\|_{L^2}^{\frac12}\|\partial_x\boldsymbol{w}\|_{L^2}^{\frac12}.\tag{2.38}$$

Proof. We rewrite $(\boldsymbol{u}\cdot\nabla\boldsymbol{v},\boldsymbol{w})$ as

$$(\boldsymbol{u}\cdot\nabla\boldsymbol{v},\boldsymbol{w})=\int_{\mathbb{R}^2}u_1\partial_x v_1 w_1+\int_{\mathbb{R}^2}u_2\partial_y v_1 w_1+\int_{\mathbb{R}^2}(u_1\partial_x v_2+u_2\partial_y v_2)w_2. \tag{2.39}$$

We next deal with each term on the right-hand side of the above equation. Let us begin with the first term. Indeed, from Lemma 2.1, it follows that

$$\begin{aligned}\left|\int_{\mathbb{R}^2}u_1\partial_x v_1 w_1\right|&=\left|\int_{\mathbb{R}^2}u_1\partial_y v_2 w_1\right|\\&\le C\|u_1\|_{L^2}\|\partial_y v_2\|_{L^2}^{\frac12}\|\partial_{yy}v_2\|_{L^2}^{\frac12}\|w_1\|_{L^2}^{\frac12}\|\partial_x w_1\|_{L^2}^{\frac12}\\&\le C\|\boldsymbol{u}\|_{L^2}\|\nabla\boldsymbol{v}\|_{L^2}^{\frac12}\|\partial_{yy}v_2\|_{L^2}^{\frac12}\|\boldsymbol{w}\|_{L^2}^{\frac12}\|\partial_x\boldsymbol{w}\|_{L^2}^{\frac12}.\end{aligned}$$

We use again Lemma 2.1 to estimate the second term,

$$\left|\int_{\mathbb{R}^2}u_2\partial_y v_1 w_1\right|\le C\|u_2\|_{L^2}^{\frac12}\|\partial_y u_2\|_{L^2}^{\frac12}\|\partial_y v_1\|_{L^2}\|w_1\|_{L^2}^{\frac12}\|\partial_x w_1\|_{L^2}^{\frac12}. \tag{2.40}$$

Observing that $\boldsymbol{u}^R$ is divergence-free, it follows that

$$\begin{aligned}\left|\int_{\mathbb{R}^2}u_2\partial_y v_1 w_1\right|&\le C\|u_2\|_{L^2}^{\frac12}\|\partial_x u_1\|_{L^2}^{\frac12}\|\partial_y v_1\|_{L^2}\|w_1\|_{L^2}^{\frac12}\|\partial_x w_1\|_{L^2}^{\frac12}\\&\le C\|\boldsymbol{u}\|_{L^2}^{\frac12}\|\partial_x\boldsymbol{u}\|_{L^2}^{\frac12}\|\nabla\boldsymbol{v}\|_{L^2}\|\boldsymbol{w}\|_{L^2}^{\frac12}\|\partial_x\boldsymbol{w}\|_{L^2}^{\frac12}.\end{aligned} \tag{2.41}$$

The third term can be handled in the same way, and we have

$$\begin{aligned}\left|\int_{\mathbb{R}^2}(u_1\partial_x v_2+u_2\partial_y v_2)w_2\right|&\le C\|\boldsymbol{u}\|_{L^2}^{\frac12}\|\partial_x\boldsymbol{u}\|_{L^2}^{\frac12}\|\nabla\boldsymbol{v}\|_{L^2}\|w_2\|_{L^2}^{\frac12}\|\partial_y w_2\|_{L^2}^{\frac12}\\&\le C\|\boldsymbol{u}\|_{L^2}^{\frac12}\|\partial_x\boldsymbol{u}\|_{L^2}^{\frac12}\|\nabla\boldsymbol{v}\|_{L^2}\|\boldsymbol{w}\|_{L^2}^{\frac12}\|\partial_x\boldsymbol{w}\|_{L^2}^{\frac12},\end{aligned} \tag{2.42}$$

where we use the property that $\partial_y w_2=-\partial_x w_1$. Collecting (2.39)–(2.42) yields (2.38) immediately, which completes the proof of Lemma 2.3. □

We now proceed with the proof of uniqueness. Let $(\boldsymbol{u}^i,\boldsymbol{B}^i)$, i=1,2, be two solutions of (1.1) with the same initial data $(\boldsymbol{u}_0,\boldsymbol{B}_0)\in H^1(\mathbb{R}^2)\times H^1(\mathbb{R}^2)$. We recall that, for $t\in[0,T_*]$, $(\boldsymbol{u}^i,\boldsymbol{B}^i)$ satisfies

$$\begin{aligned}&\|\boldsymbol{u}^i\|_{L^2}^2+\|\boldsymbol{B}^i\|_{L^2}^2+\nu\int_0^t\|\partial_x\boldsymbol{u}^i\|_{L^2}^2+\mu\int_0^t\|\partial_x\boldsymbol{B}^i\|_{L^2}^2\le C,\\&\|w^i(t)\|_{L^2}^2+\|j^i(t)\|_{L^2}^2+\nu\int_0^t\|\partial_x w^i\|_{L^2}^2+\mu\int_0^t\|\partial_x j^i\|_{L^2}^2\le C.\end{aligned} \tag{2.43}$$

Furthermore, it follows Lemma 2.2 that

$$\nu\int_0^{T_*}\left[\|\partial_{xx}\boldsymbol{u}^i\|_{L^2}^2+\|\partial_{yy}u_2^i\|_{L^2}^2\right]+\mu\int_0^{T_*}\left[\|\partial_{xx}\boldsymbol{B}^i\|_{L^2}^2+\|\partial_{yy}B_2^i\|_{L^2}^2\right]\le C. \tag{2.44}$$

Let $\widetilde{\boldsymbol{u}} := \boldsymbol{u}^1 - \boldsymbol{u}^2$ and $\widetilde{\boldsymbol{B}} := \boldsymbol{B}^1 - \boldsymbol{B}^2$. As $(\boldsymbol{u}^i, \boldsymbol{B}^i)$(i=1,2) are solutions of Eq (1.1), by using elementary energy methods, we may obtain the following inequality:

$$\begin{aligned} &\|\widetilde{\boldsymbol{u}}\|_{L^2}^2 + \|\widetilde{\boldsymbol{B}}\|_{L^2}^2 + \nu \int_0^t \|\partial_x \widetilde{\boldsymbol{u}}\|_{L^2}^2 + \mu \int_0^t \|\partial_x \widetilde{\boldsymbol{B}}\|_{L^2}^2 \\ \leq &\left| \int_0^t (\widetilde{\boldsymbol{u}} \cdot \nabla \boldsymbol{u}^1, \widetilde{\boldsymbol{u}})_{L^2} \right| + \left| \int_0^t (\widetilde{\boldsymbol{u}} \cdot \nabla \boldsymbol{B}^1, \widetilde{\boldsymbol{B}})_{L^2} \right| + \left| \int_0^t (\widetilde{\boldsymbol{B}} \cdot \nabla \boldsymbol{B}^1, \widetilde{\boldsymbol{u}})_{L^2} \right| + \left| \int_0^t (\widetilde{\boldsymbol{B}} \cdot \nabla \boldsymbol{u}^1, \widetilde{\boldsymbol{B}})_{L^2} \right|. \end{aligned} \tag{2.45}$$

We then deal with each term on the right-hand side of the above inequality. From Lemma 2.3, we know that the first term satisfies

$$\left| \int_0^t (\widetilde{\boldsymbol{u}} \cdot \nabla \boldsymbol{u}^1, \tilde{\boldsymbol{u}})_{L^2} \right| \leq \int_0^t \|\nabla \boldsymbol{u}^1\|_{L^2}^{\frac{1}{2}} \|\partial_{yy} u_2^1\|_{L^2}^{\frac{1}{2}} \|\tilde{\boldsymbol{u}}\|_{L^2}^{\frac{3}{2}} \|\partial_x \tilde{\boldsymbol{u}}\|_{L^2}^{\frac{1}{2}} + \int_0^t \|\nabla \boldsymbol{u}^1\|_{L^2} \|\tilde{\boldsymbol{u}}\|_{L^2} \|\partial_x \tilde{\boldsymbol{u}}\|_{L^2}. \tag{2.46}$$

Similarly, we obtain the estimate of the second term,

$$\left| \int_0^t (\widetilde{\boldsymbol{u}} \cdot \nabla \boldsymbol{B}^1, \tilde{\boldsymbol{B}}) \right| \leq \int_0^t \|\widetilde{\boldsymbol{u}}\|_{L^2} \|\nabla \boldsymbol{B}^1\|_{L^2}^{\frac{1}{2}} \|\partial_{yy} B_2^1\|_{L^2}^{\frac{1}{2}} \|\tilde{\boldsymbol{B}}\|_{L^2}^{\frac{1}{2}} \|\partial_x \tilde{\boldsymbol{B}}\|_{L^2}^{\frac{1}{2}} + \int_0^t \|\widetilde{\boldsymbol{u}}\|_{L^2}^{\frac{1}{2}} \|\partial_x \widetilde{\boldsymbol{u}}\|_{L^2}^{\frac{1}{2}} \|\nabla \boldsymbol{B}^1\|_{L^2} \|\tilde{\boldsymbol{B}}\|_{L^2}^{\frac{1}{2}} \|\partial_x \tilde{\boldsymbol{B}}\|_{L^2}^{\frac{1}{2}}. \tag{2.47}$$

Using (2.43), it follows that (2.46) and (2.47) that

$$\begin{aligned} &\left| \int_0^t (\widetilde{\boldsymbol{u}} \cdot \nabla \boldsymbol{u}^1, \tilde{\boldsymbol{u}})_{L^2} \right| + \left| \int_0^t (\widetilde{\boldsymbol{u}} \cdot \nabla \boldsymbol{B}^1, \tilde{\boldsymbol{B}}) \right| \\ \leq &\frac{\nu}{2} \int_0^t \|\partial_x \widetilde{\boldsymbol{u}}\|_{L^2}^2 + \frac{\mu}{2} \int_0^t \|\partial_x \widetilde{\boldsymbol{B}}\|_{L^2}^2 + \int_0^t (\|\widetilde{\boldsymbol{u}}\|_{L^2}^2 + \|\widetilde{\boldsymbol{B}}\|_{L^2}^2)(1 + \|\partial_{yy} u_2^1\|_{L^2} + \|\partial_{yy} B_2^1\|_{L^2}). \end{aligned} \tag{2.48}$$

The third term and the fourth term on the right hand side of (2.45) can be handled in the same way, and we obtain

$$\begin{aligned} &\left| \int_0^t (\widetilde{\boldsymbol{B}} \cdot \nabla \boldsymbol{B}^1, \tilde{\boldsymbol{u}})_{L^2} \right| + \left| \int_0^t (\widetilde{\boldsymbol{B}} \cdot \nabla \boldsymbol{u}^1, \tilde{\boldsymbol{B}}) \right| \\ \leq &\frac{\nu}{2} \int_0^t \|\partial_x \widetilde{\boldsymbol{u}}\|_{L^2}^2 + \frac{\mu}{2} \int_0^t \|\partial_x \widetilde{\boldsymbol{B}}\|_{L^2}^2 + \int_0^t (\|\widetilde{\boldsymbol{u}}\|_{L^2}^2 + \|\widetilde{\boldsymbol{B}}\|_{L^2}^2)(1 + \|\partial_{yy} u_2^1\|_{L^2} + \|\partial_{yy} B_2^1\|_{L^2}). \end{aligned} \tag{2.49}$$

We substitute (2.48) and (2.49) into (2.45) to deduce that

$$\|\widetilde{\boldsymbol{u}}\|_{L^2}^2 + \|\widetilde{\boldsymbol{B}}\|_{L^2}^2 \leq C \int_0^t (\|\widetilde{\boldsymbol{u}}\|_{L^2}^2 + \|\widetilde{\boldsymbol{B}}\|_{L^2}^2)(1 + \|\partial_{yy} u_2^1\|_{L^2} + \|\partial_{yy} B_2^1\|_{L^2}), \tag{2.50}$$

so ineq (2.44) and Grönwall's inequality tell that $\widetilde{\boldsymbol{u}} \equiv \widetilde{\boldsymbol{B}} \equiv 0$. Thus, the solution is unique, and this completes the proof of Theorem 1.1.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares that he has no conflict of interest.

References

1. G. Duvaut, J. L. Lions, Inéquations en thermoélasticité et magnétohydrodynamique, *Arch. Ration. Mech. Anal.*, **46** (1972), 241–279. https://doi.org/10.1007/BF00250512
2. M. Sermange, R. Temam, Some mathematical questions related to the MHD equations, *Commun. Pur. Appl. Math.*, **36** (1983), 635–664. https://doi.org/10.1002/cpa.3160360506
3. H. Kozono, Weak and classical solutions of the two-dimensional magnetohydrodynamic equations, *Tohoku Math. J.*, **41** (1989), 471–488. https://doi.org/10.2748/tmj/1178227774
4. Z. Lei, Y. Zhou, BKM's criterion and global weak solutions for magnetohydrodynamics with zero viscosity, *Discrete Cont. Dyn.*, **25** (2009), 575–583. https://doi.org/10.3934/dcds.2009.25.575
5. C. Cao, J. Wu, Global regularity for the 2D MHD equations with mixed partial dissipation and magnetic diffusion, *Adv. Math.*, **226** (2011), 1803–1822. https://doi.org/10.1016/j.aim.2010.08.017
6. X. Zhai, Stability for the 2D incompressible MHD equations with only magnetic diffusion, *J. Differ. Equations*, **374** (2023), 267–278. https://doi.org/10.1016/j.jde.2023.07.033
7. Q. Jiu, D. Niu, Mathematical results related to a two-dimensional magnetohydrodynamic equations, *Acta Math. Sci.*, **26** (2006), 744–756. https://doi.org/10.1016/S0252-9602(06)60101-X
8. C. L. Fefferman, D. S. McCormick, J. C. Robinson, J. L. Rodrigo, Higher order commutator estimates and local existence for the non-resistive MHD equations and related models, *J. Funct. Anal.*, **267** (2014), 1035–1056. https://doi.org/10.1016/j.jfa.2014.03.021
9. J. Y. Chemin, D. S. McCormick, J. C. Robinson, J. L. Rodrigo, Local existence for the non-resistive MHD equations in Besov spaces, *Adv. Math.*, **286** (2016), 1–31. https://doi.org/10.1016/j.aim.2015.09.004
10. C. L. Fefferman, D. S. McCormick, J. C. Robinson, J. L. Rodrigo, Local existence for the non-resistive MHD equations in nearly optimal Sobolev spaces, *Arch. Ration. Mech. Anal.*, **223** (2017), 677–691. https://doi.org/10.1007/s00205-016-1042-7
11. P. G. Schmidt, On a magnetohydrodynamic problem of Euler type, *J. Differ. Equations*, **74** (1988), 318–335. https://doi.org/10.1016/0022-0396(88)90008-3
12. P. Secchi, On the equations of ideal incompressible magneto-hydrodynamics, *Rend. Semin. Mat. Univ. Pad.*, **90** (1993), 103–119.
13. X. Zhai, Global solutions to 3D MHD equations with fractional dissipation, *Appl. Math. Lett.*, **176** (2026), 109873. https://doi.org/10.1016/j.aml.2026.109873
14. A. J. Majda, A. L. Bertozzi, *Vorticity and incompressible flow*, Cambridge: Cambridge University Press, 2010. https://doi.org/10.1017/CBO9780511613203

AIMS Press

© 2026 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (https://creativecommons.org/licenses/by/4.0)