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*Research article*

# On generalizations of strongly $ss$-discrete modules

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**Abstract:** In this paper, we introduced the notions of $\mathcal{LA}$-lifting modules and $\mathcal{LA}$-discrete modules, which are strong concepts of lifting modules and discrete modules, respectively. We provided some conditions of these modules and generalizations of $ss$-lifting modules and $ss$-discrete modules, respectively. Finally, we characterized $\mathcal{LA}$-lifting modules via left perfect rings and strongly $\mathcal{LA}$-discrete rings by using the notion of semiperfect rings.

**Keywords:** locally Artinian modules; locally Artinian supplemented modules; lifting modules; $\mathcal{LA}$-lifting modules; discrete modules; $\mathcal{LA}$-discrete modules; perfect rings; semiperfect rings
**Mathematics Subject Classification:** 16D10, 16D40, 16D70

## 1. Introduction

Throughout this work, the notation $\mathcal{R}$ will always stand for an associative ring with unity, and all modules under consideration are assumed to be unital left $\mathcal{R}$-modules. The Jacobson radical of $\mathcal{R}$ is denoted by $J(\mathcal{R})$. For a submodule $\mathcal{U}$ of a module $\mathcal{M}$ (denoted by $\mathcal{U} \leq \mathcal{M}$), we write $\mathcal{U} \ll \mathcal{M}$ to indicate that $\mathcal{U}$ is small in $\mathcal{M}$. The symbols $Rad(\mathcal{M})$ and $Soc(\mathcal{M})$ represent, respectively, the radical of $\mathcal{M}$ (that is, the sum of all the small submodules of $\mathcal{M}$) and the socle of $\mathcal{M}$ (i.e., the sum of all the semisimple submodules of $\mathcal{M}$). When all of $\mathcal{M}$'s proper submodules are small, the module is said to be hollow; when all of $\mathcal{M}$'s proper submodules sum is a proper submodule, then it is said to be local. The module $\mathcal{M}$ is said to be *locally Artinian* if each finitely generated submodule is Artinian [13].

A submodule $\mathcal{V}$ of a module $\mathcal{M}$ is said to serve as a *supplement* to another submodule $\mathcal{U}$ whenever the condition $\mathcal{M} = \mathcal{U} + \mathcal{V}$ is satisfied and, in addition, the intersection $\mathcal{U} \cap \mathcal{V}$ is a small submodule of $\mathcal{V}$ (that is, $\mathcal{U} \cap \mathcal{V} \ll \mathcal{V}$). A module in which every submodule possesses at least one such supplement is referred to as a *supplemented module*. Moreover, a submodule $\mathcal{U}$ is described as having *ample* *supplements* in $\mathcal{M}$ provided that for each submodule $\mathcal{V}$ with $\mathcal{M} = \mathcal{U} + \mathcal{V}$, there exists a submodule $\mathcal{V}' \leq \mathcal{V}$ which is a supplement of $\mathcal{U}$. In this text, module $\mathcal{M}$ is termed *amply supplemented* whenever every submodule of $\mathcal{M}$ has the property of admitting ample supplements. This refined concept extends the notion of supplemented modules and plays a significant role in module theory, especially in connection with decomposition properties and structures of modules [13].

The notions of small submodules, supplements, and lifting modules play a central role in the structural study of modules and their decomposition properties. In many situations, however, classical supplemented or lifting conditions prove to be too restrictive, particularly when one considers modules whose radicals are not necessarily semisimple or when contexts of infinitely generated modules arise. For this reason, several weakened versions of these concepts have been introduced in the literature to extend decomposition theory to broader classes of modules.

In particular, the concept of *ss-supplemented modules* was developed as a refinement of the classical supplemented condition by controlling intersections via the generalized socle $\mathrm{Soc}_s(M)$ in [4]. This concept allows one to capture situations in which the intersection of supplements is not necessarily small, yet still exhibits semisimple behavior. Combining these perspectives provides a natural *conceptual setting* for investigating decomposition properties of modules whose radicals satisfy certain Artinian restrictions.

On the other hand, locally Artinian conditions arise naturally when studying radicals and chain conditions in modules whose finitely generated submodules retain the notion of being Artinian. Combining these two perspectives provides a useful context for investigating decomposition properties of modules. In [14], the concept of the socle was generalized by defining

$$Soc_s(\mathcal{M}) = \sum\{\,\mathcal{U} \ll \mathcal{M} \mid \mathcal{U} \text{ is simple}\,\}.$$

It follows that $Soc_s(\mathcal{M}) \subseteq Rad(\mathcal{M})$ and $Soc_s(\mathcal{M}) \subseteq Soc(\mathcal{M})$. According to [5], if $Rad(\mathcal{M})$ is semisimple and module $\mathcal{M}$ is local, then $\mathcal{M}$ is *strongly local*. Let $\mathcal{U}$ be a submodule of a module $\mathcal{M}$. A submodule $\mathcal{V}$ is said to be an *ss-supplement* of $\mathcal{U}$ in $\mathcal{M}$ whenever $\mathcal{M} = \mathcal{U} + \mathcal{V}$ and the intersection $\mathcal{U} \cap \mathcal{V}$ is contained in $Soc_s(\mathcal{V})$.

The concept of $ss$-lifting modules arises as a natural generalization of classical lifting modules obtained by weakening the smallness condition through the socle of the module. Recall that a module $\mathcal{M}$ is called lifting if for every submodule $\mathcal{A} \leq \mathcal{M}$, there exists a direct summand $\mathcal{D}$ of $\mathcal{M}$ such that $\mathcal{D} \subseteq \mathcal{A}$ and $\frac{\mathcal{A}}{\mathcal{D}}$ is small in $\frac{\mathcal{M}}{\mathcal{D}}$. By replacing the classical notion of smallness with $ss$-smallness, which controls intersections via the socle, one obtains the notion of $ss$-lifting modules. More precisely, a module $\mathcal{M}$ is called $ss$-lifting if for every submodule $\mathcal{A} \leq \mathcal{M}$, there exists a direct summand $\mathcal{D}$ of $\mathcal{M}$ such that $\mathcal{D} \subseteq \mathcal{A}$ and $\frac{\mathcal{A}}{\mathcal{D}}$ is $ss$-small in $\frac{\mathcal{M}}{\mathcal{D}}$ [3]. This modification allows the lifting property to be extended to a wider class of modules in which strict smallness may fail, but the obstruction to decomposition is governed by semisimple components.

Closely related to this notion is the concept of $ss$-discrete modules, which can be viewed as a socle-controlled analog of discrete modules. A module $\mathcal{M}$ is called $ss$-discrete if it satisfies the following two conditions:

(1) $\mathcal{M}$ is $ss$-lifting, and

(2) every submodule $\mathcal{A} \leq \mathcal{M}$ with $\mathcal{A} \cap \mathcal{D} = 0$ for some direct summand $\mathcal{D}$ of $\mathcal{M}$ is contained in a direct summand of $\mathcal{M}$ up to an $ss$-small factor [3].

Thus, $ss$-discrete modules refine the $ss$-lifting property by ensuring that submodules interact with direct summands in a controlled manner through $ss$-small components.

From another perspective, lifting, discrete, and quasi-discrete modules constitute fundamental classes in module decomposition theory, as they provide strong structural control over the interaction between submodules and direct summands. These classes have played a central role in understanding how modules decompose and how their internal structure can be systematically analyzed. However, many naturally occurring modules do not satisfy these classical conditions in their original form, particularly in situations where the structure of the radical becomes more intricate.

This limitation motivates the development of suitable generalizations that preserve the essential decomposition behavior while allowing greater flexibility in the structure of the radical. In this direction, the notions of $\mathcal{LA}$-lifting and $\mathcal{LA}$-discrete modules arise naturally by incorporating locally Artinian conditions into the supplement and lifting theory.

In essence, these concepts extend the classical theory by requiring that the radical components appearing in supplements be accompanied by a locally Artinian condition. Such a refinement makes it possible to investigate modules whose radicals are not necessarily semisimple but still exhibit a controlled Artinian structure. As a result, it establishes a meaningful connection between the decomposition properties of the modules and the locally Artinian behavior of their radicals, thus enriching the structural analysis of modules within modern algebra.

The main objective of this research is to systematically investigate $\mathcal{LA}$-lifting and $\mathcal{LA}$-discrete modules, to determine their structural properties, and to clarify their relationships with $ss$-supplemented modules and other generalized decomposition conditions. The originality of this study lies in introducing new results with generalized lifting and supplement concepts, thereby extending classical decomposition theory to broader classes of modules by the notion of being locally Artinian.

In [9], the concepts of strongly local modules and of (amply) $ss$-supplemented modules have been generalized and adapted to broader classes of modules, namely to the classes of *RLA*-local modules (R:=Radical is L:=Locally A:=Artinian), and of (amply) locally Artinian supplemented modules, respectively. More precisely, a local module $\mathcal{M}$ is called *RLA-local* whenever its Jacobson radical $Rad(\mathcal{M})$ forms a submodule that is locally Artinian, that is, every finitely generated submodule of $Rad(\mathcal{M})$ is Artinian. Extending the notion of supplemented modules, recall from [9] that a module $\mathcal{M}$ is *locally Artinian supplemented* in the case where for every submodule $\mathcal{U} \leq \mathcal{M}$, there exists another submodule $\mathcal{V} \leq \mathcal{M}$ such that $\mathcal{M} = \mathcal{U} + \mathcal{V}$, the intersection $\mathcal{U} \cap \mathcal{V}$ is small in $\mathcal{V}$, and in addition, $\mathcal{U} \cap \mathcal{V}$ is a locally Artinian submodule. Furthermore, this notion can be strengthened in the following way: if each submodule $\mathcal{U}$ of $\mathcal{M}$ has the property that, for every submodule $\mathcal{V}$ with $\mathcal{M} = \mathcal{U} + \mathcal{V}$, one can always find a submodule $\mathcal{V}' \leq \mathcal{V}$ that is a supplement of $\mathcal{U}$ and whose intersection with $\mathcal{U}$ is locally Artinian, then the module $\mathcal{M}$ is said to be *amply locally Artinian supplemented*. This last concept provides a natural refinement of the classical supplemented and amply supplemented modules and plays a central role in the structural study of modules satisfying radical and locally Artinian submodules.

We begin by recalling several definitions that will be used throughout this study.

Let $\alpha$ denote a cardinal number. A module $\mathcal{M}$ is said to have an *$\alpha$-internal exchange property* if, for every direct sum decomposition

$$\mathcal{M} = \bigoplus_{\omega \in \Omega} \mathcal{M}_{\omega}$$

with card$(\Omega) \leq \alpha$, the direct summands $\mathcal{M}_\omega$ can be rearranged or replaced so that the resulting decomposition remains valid. If this requirement holds for all (respectively, finite) cardinals $\alpha$, then the module $\mathcal{M}$ is said to satisfy the *(finite) internal exchange property* [2, 11.34].

A module $\mathcal{M}$ is said to satisfy the property $(D_3)$ if, whenever $\mathcal{U}$ and $\mathcal{V}$ are direct summands of $\mathcal{M}$ with $\mathcal{M} = \mathcal{U} + \mathcal{V}$, their intersection $\mathcal{U} \cap \mathcal{V}$ is also a direct summand of $\mathcal{M}$ (see [7]). In [2, 4.29], $\cap$-*direct projective* modules are introduced as an equivalent formulation of $(D_3)$. The property $(D_2)$ is defined as follows: for every submodule $\mathcal{U} \leq \mathcal{M}$ such that $\mathcal{M}/\mathcal{U}$ is isomorphic to a direct summand of $\mathcal{M}$, it follows that $\mathcal{U}$ is itself a direct summand of $\mathcal{M}$. Clearly, $(D_2)$ implies $(D_3)$.

A module $\mathcal{M}$ is said to be *direct projective* if, for every direct summand $\mathcal{U}$ of $\mathcal{M}$, any epimorphism from $\mathcal{M}$ onto $\mathcal{U}$ admits a splitting. According to [2, 4.21], this is equivalent to $\mathcal{M}$ satisfying condition $(D_2)$. A *lifting module* is defined as a module $\mathcal{M}$ such that, for any submodule $\mathcal{U} \leq \mathcal{M}$, there exists a decomposition $\mathcal{M} = \mathcal{U}' \oplus \mathcal{V}$ where $\mathcal{U}' \leq \mathcal{U}$ and $\mathcal{U} \cap \mathcal{V} \ll \mathcal{V}$ in [7]. A module that is both supplemented and self-projective is referred to as *strongly discrete*. A *quasi-discrete* module is one that is simultaneously $\pi$-projective and supplemented. A *discrete* module is a supplemented module that is both $\pi$-projective and direct projective. Finally, following [2], a module is termed *semi-discrete* if it is a lifting module possessing the finite internal exchange property.

In [8], for a module $\mathcal{M}$ and for submodules $\mathcal{K}$ and $\mathcal{N}$:

(1) $\mathcal{M}$ is called *semi-ss-discrete* if it is $ss$-lifting with the finite exchange property.

(2) $\mathcal{M}$ is called *quasi-ss-disrete* when $\mathcal{M}$ is both $\pi$-projective and $ss$-supplemented.

(3) $\mathcal{K}$ is said to be $\mathcal{N}$-$ss$-lifting when each homomorphism $\mathcal{M} \to \frac{\mathcal{M}}{\mathcal{N} \cap \mathcal{K}}$ lifts to an endomorphism of $\mathcal{M}$, where $\mathcal{N} \cap \mathcal{K}$ is semisimple.

For a comprehensive overview of recent developments in $ss$-supplemented modules and locally Artinian submodules, the reader is referred to the seminal works and subsequent studies found in [6, 10, 12].

In the Figure 1 below, the diagram presents a structured overview of the interrelationships among certain classes of modules considered in this work. These classes are defined through the interplay of specific properties of modules, such as being *supplemented*, *$\pi$-projective*, *direct projective*, *lifting*, and *self-projective*. Each rectangular node in the diagram corresponds to a distinct class of modules, characterized by a particular combination of these properties.

The directed edges (arrows) in the diagram indicate *logical implication* or *definitional inclusion* between the connected concepts. More precisely, an arrow from node $\mathcal{A}$ to node $\mathcal{B}$ means that every module belonging to class $\mathcal{A}$ necessarily satisfies the defining conditions of class $\mathcal{B}$; in some cases, this reflects an equivalence, whereas in others it is a proper inclusion. Thus, the diagram serves both as a taxonomy of module classes and as a visual map of their definitional dependencies.

The horizontal positioning of the boxes reflects conceptual hierarchy and independence between certain properties. In particular, "quasi-discrete" and "discrete" modules are placed on the far right to emphasize that they are characterized by the simultaneous satisfaction of multiple structural properties, while not being necessarily derivable from all other classes depicted on the left.

In Section 2, it is established that a module $\mathcal{M}$ is *$\mathcal{LA}$-lifting* if and only if $\mathcal{M}$ is amply locally Artinian supplemented and every locally Artinian supplement in $\mathcal{M}$ is a direct summand. It is further shown that $\mathcal{R}$ is a left perfect ring with $J(\mathcal{R})$ locally Artinian if and only if every free $\mathcal{R}$-module is $\mathcal{LA}$-

lifting. Additionally, $\mathcal{LA}$-lifting modules are characterized over left and right Artinian serial rings with $J(\mathcal{R})^2 = 0$.

In Section 3, we investigate the connection between quasi-$\mathcal{LA}$-discrete modules and $\mathcal{LA}$-lifting modules, using the concept of relatively projective direct summands in decompositions. It is proved that a projective module $\mathcal{M}$ is strongly $\mathcal{LA}$-discrete precisely when $\mathcal{M}$ is strongly discrete and $Rad(\mathcal{M})$ is locally Artinian. Furthermore, for $\mathcal{M} = \bigoplus_{i\in I} \mathcal{M}_i$, if each $Rad(\mathcal{M}_i)$ is locally Artinian, then $\mathcal{M}$ is strongly $\mathcal{LA}$-discrete if and only if each $\mathcal{M}_i$ is strongly discrete. Finally, the class of strongly $\mathcal{LA}$-discrete rings is characterized via the concept of semiperfect rings.

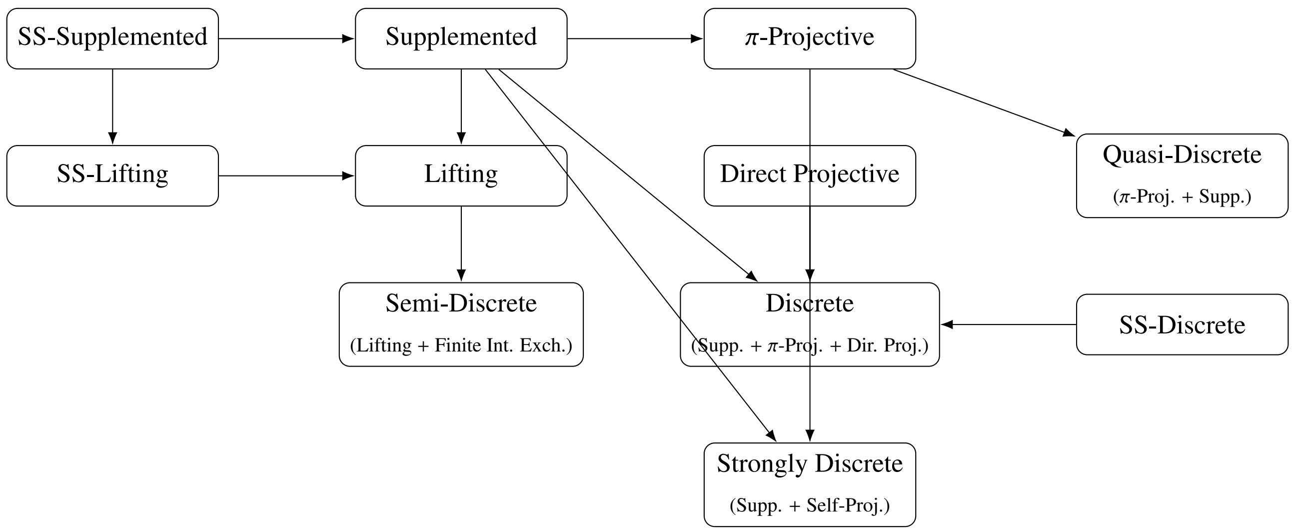


**Figure 1.** An overview of the interrelations among certain classes of modules.

## 2. $\mathcal{LA}$-lifting modules

In this section, we introduce and investigate the notion of $\mathcal{LA}$-*lifting modules*, which is a proper generalization of $ss$-lifting modules by incorporating locally Artinian submodules as a generalization of lifting modules. We begin by recalling the definition of an $\mathcal{LA}$-lifting module, which requires that for each submodule of a given module, an appropriate decomposition exists where the small intersection is locally Artinian. Subsequently, we establish several characterizations of this property, demonstrating its equivalence with the existence of certain decompositions of submodules and with the condition that all locally Artinian supplements are direct summands. The results further explore the behavior of $\mathcal{LA}$-lifting modules under various module structures: they are preserved when taking direct summands, quotient modules by fully invariant submodules, and finite direct sums. Connections with other classes of modules are also highlighted, such as $\pi$-projective modules and weakly distributive modules, where the $\mathcal{LA}$-lifting property arises naturally under suitable conditions. Moreover, ring characterizations are provided, showing that the $\mathcal{LA}$-lifting property for free modules is closely related to the structure of the underlying ring, in particular, to the perfect ring and Jacobson radical of the locally Artinian modules. Finally, applications to Artinian serial rings with nilpotent radicals are discussed, illustrating the depth and breadth of the $\mathcal{LA}$-lifting concept in module theory.

**Definition 1.** *Let $\mathcal{M}$ be a module. $\mathcal{M}$ is called $\mathcal{LA}$-*lifting *if, for every submodule $\mathcal{N}$ of $\mathcal{M}$, there exists* *a decomposition $\mathcal{M} = \mathcal{N}' \oplus \mathcal{K}$ such that $\mathcal{N}' \leq \mathcal{N}$, $\mathcal{N} \cap \mathcal{K} \ll \mathcal{K}$, and $\mathcal{N} \cap \mathcal{K}$ is a locally Artinian submodule of $\mathcal{M}$.*

$\mathcal{LA}$-lifting modules introduce another type of weakening by incorporating locally Artinian conditions into the lifting concept. In this setting, the radical components involved in supplements are required by the notion of being locally Artinian, which guaranties that finitely generated submodules of the radical possess an Artinian structure. Consequently, $\mathcal{LA}$-lifting modules extend the lifting property to modules whose radicals are not necessarily semisimple but still satisfy controlled chain conditions.

These classes are closely connected. Every lifting module is also $ss$-lifting, $ss$-discrete, and $\mathcal{LA}$-lifting, since the classical smallness condition implies $ss$-smallness and the stronger structural requirements of the generalized concepts. However, the converses do not hold in general. In particular, $ss$-lifting and $ss$-discrete modules enlarge the classical theory by weakening smallness via the socle, whereas $\mathcal{LA}$-lifting modules extend it by imposing locally Artinian control on radical components. Therefore, these concepts provide broader research study in which decomposition properties analogous to those of lifting and discrete modules can still be investigated.

The systematic study of the relationships between $ss$-lifting, $ss$-discrete, and $\mathcal{LA}$-lifting modules offers new insight into how decomposition properties interact with socle structure and locally Artinian behavior of radicals, thereby contributing to the ongoing development of generalized decomposition theory in module theory.

**Lemma 1.** *Let $\mathcal{M}$ be an $\mathcal{LA}$-lifting module. Then $\mathcal{M}$ is amply locally Artinian supplemented.*

*Proof.* Let $\mathcal{N}$ be any submodule of $\mathcal{M}$. Since $\mathcal{M}$ is $\mathcal{LA}$-lifting, there exists a decomposition $\mathcal{M} = \mathcal{N}' \oplus \mathcal{K}$ such that $\mathcal{N}' \leq \mathcal{N}$, $\mathcal{N} \cap \mathcal{K} \ll \mathcal{K}$, and $\mathcal{N} \cap \mathcal{K}$ is locally Artinian. It follows from $\mathcal{M} = \mathcal{N} + \mathcal{K}$, $\mathcal{N} \cap \mathcal{K} \ll \mathcal{K}$, and the fact that $\mathcal{N} \cap \mathcal{K}$ is locally Artinian that $\mathcal{K}$ is a locally Artinian supplement of $\mathcal{N}$ in $\mathcal{M}$. Thus, $\mathcal{M}$ is locally Artinian supplemented by [9, Proposition 2.16]. Assume that $\mathcal{N}'$ is a locally Artinian supplemented module. We observe that

$$\mathcal{N} = \mathcal{N} \cap (\mathcal{N}' \oplus (\mathcal{N} \cap \mathcal{K})) = \mathcal{N}' \oplus (\mathcal{N} \cap \mathcal{K}).$$

Since $\mathcal{N} \cap \mathcal{K}$ is locally Artinian, it follows from [9, Corollary 2.15] that $\mathcal{N}$ itself is locally Artinian supplemented. Consequently, by [9, Proposition 2.17], we deduce that $\mathcal{M}$ is amply locally Artinian supplemented. □

**Proposition 1.** *Let $\mathcal{M}$ be a hollow module. Then the following statements are equivalent for $\mathcal{M}$:*

*(1) $\mathcal{M}$ is an RLA-local module;*

*(2) $\mathcal{M}$ is an $\mathcal{LA}$-lifting module;*

*(3) $\mathcal{M}$ is an amply locally Artinian supplemented module.*

*Proof.* (1) ⇒ (2) Let $\mathcal{N}$ be a proper submodule of $\mathcal{M}$. Since $\mathcal{M}$ is hollow, we have $\mathcal{N} \leq Rad(\mathcal{M})$, that is, $\mathcal{N} \ll \mathcal{M}$. Given that $Rad(\mathcal{M})$ is locally Artinian, it follows from [13, 31.2 (1) (i)] that $\mathcal{N}$ is also locally Artinian. Hence, we may consider the decomposition $\mathcal{M} = 0 \oplus \mathcal{M}$, where $0 \leq \mathcal{N}$, $\mathcal{N} \ll \mathcal{M}$, and $\mathcal{N}$ is locally Artinian. Therefore, $\mathcal{M}$ satisfies the definition of an $\mathcal{LA}$-lifting module.

(2) ⇒ (3) This implication follows directly from Lemma 1.

(3) ⇒ (1) Under the given assumptions, $\mathcal{M}$ is locally Artinian supplemented and $Rad(\mathcal{M}) \ll \mathcal{M}$. By [9, Theorem 2.9], this implies that $\mathcal{M}$ is supplemented and that $Rad(\mathcal{M})$ is locally Artinian. Moreover, since $\mathcal{M}$ is local, it follows that $\mathcal{M}$ is hollow. Consequently, $\mathcal{M}$ is an *RLA*-local module. □

It is clear that every $\mathcal{LA}$-lifting module is lifting. The converse is not always correct, as can be seen from the example below.

**Example 1.** *(1) Consider the hollow $\mathbb{Z}$-module $M = \mathbb{Z}_8$. By [9, Example 2.2(1)], M is an RLA-local module. Hence, by Proposition 1, it follows that M is an $\mathcal{LA}$-lifting module.*

*Notice that $M = \mathbb{Z}_8$ is* not *a lifting module. To see this, recall that M is a uniserial $\mathbb{Z}$-module with the submodule lattice*

$$0 \subsetneq \langle \bar{4} \rangle \cong \mathbb{Z}_2 \subsetneq \langle \bar{2} \rangle \cong \mathbb{Z}_4 \subsetneq \mathbb{Z}_8 = M.$$

*Since $\mathbb{Z}_8$ is indecomposable, the only direct summands of M are* 0 *and M itself. For M to be lifting, every submodule N of M must lie over a direct summand; that is, there must exist a decomposition $M = A \oplus B$ with $A \leq N$ and $N \cap B \ll M$. Consider $N = \langle \bar{2} \rangle \cong \mathbb{Z}_4$. Since M is indecomposable, we must have $A = 0$, which forces $N \cap B = N$ to be small in M. But $N = \langle \bar{2} \rangle = Rad(M)$ is the unique maximal submodule of M, and in particular, N is small in M (as M is hollow). Thus, the lifting condition is satisfied for N vacuously through $A = 0$.*

*In fact, since M is* hollow, *every proper submodule of M is small, and therefore M is trivially lifting: for any proper submodule N, taking $A = 0$ yields $N = N \cap M \ll M$. Hence $M = \mathbb{Z}_8$ is both lifting and $\mathcal{LA}$-lifting. This example illustrates that the $\mathcal{LA}$-lifting property is accessible even in modules with very simple structure, and that the locally Artinian behavior of the radical suffices to guarantee the $\mathcal{LA}$-lifting condition in elementary settings.*

*(2) Let R be a local Dedekind domain (i.e., a discrete valuation ring) with quotient field $K \neq R$. Then the R-module $_RK$ is hollow. By [9, Example 2.7], the module $_RK$ is amply supplemented but not amply locally Artinian supplemented. Consequently, by Proposition 1, $_RK$ is a lifting module but not an $\mathcal{LA}$-lifting module.*

*Let us briefly explain why $_RK$ is a lifting module. Since R is a discrete valuation ring with maximal ideal $\mathfrak{m}$, the R-module K is hollow: indeed, K is a uniserial R-module (as every R-submodule of K is of the form $\mathfrak{m}^n$ for some $n \in \mathbb{Z}$, forming a single chain under inclusion), and any proper submodule of K is small. A hollow module is lifting if and only if it is local, i.e., it has a unique maximal submodule. Here, $Rad(M)(_RK) = \mathfrak{m} \cdot K = K \neq K$ does not hold directly; rather, the proper submodules of $_RK$ form the chain*

$$\cdots \subsetneq \mathfrak{m}^{-1} \subsetneq R \subsetneq \mathfrak{m}^{-1} \subsetneq \cdots \subsetneq K,$$

*and K has no maximal submodule, so $Rad(K) = K$. A hollow module with $Rad(M) = M$ is* not *local and hence,* not *lifting in the classical sense.*

*However, the result in [9, Example 2.7] establishes that $_RK$ is amply supplemented and so, it is lifting by the characterization of lifting modules via ample supplements for hollow modules. More precisely, $_RK$ is lifting because it is a hollow supplemented module: for a hollow module, the lifting property reduces to the condition that for every submodule N of M, either $N \ll M$ or $N = M$, which holds since every proper submodule of K is small in K.*

*On the other hand, ${}_RK$ is* not *$\mathcal{LA}$-lifting because it fails to be amply locally Artinian supplemented: the supplements that arise in ${}_RK$ do not satisfy the locally Artinian condition on the corresponding radical layers. Specifically, for certain submodules of K, the supplements have radical quotients that are not locally Artinian, which obstructs the $\mathcal{LA}$-lifting condition. This demonstrates that even in the presence of the lifting property and ample supplementation, the additional locally Artinian constraint may fail.*

These two examples together demonstrate a key structural context: the class of $\mathcal{LA}$-lifting modules is *not* comparable to the class of lifting modules by simple inclusion in general, but rather the $\mathcal{LA}$-lifting condition imposes an orthogonal constraint. However, when restricted to appropriate classes of modules, we obtain the proper inclusion:

$$\mathcal{LA}\text{-lifting modules} \subsetneq \text{lifting modules}.$$

The theoretical significance of this distinction lies in the fact that $\mathcal{LA}$-lifting modules provide a more refined structural setting in which decomposition properties are studied under additional locally Artinian constraints on radical components, thereby allowing a deeper analysis of the interaction between supplements, radicals, and direct summands in module theory.

**Lemma 2.** *The following conditions are equivalent for a submodule $\mathcal{N}$ of a module $\mathcal{M}$:*

*(1) There exists a direct summand $\mathcal{N}'$ of $\mathcal{M}$ satisfying $\mathcal{N}' \leq \mathcal{N}$, the factor module $\frac{\mathcal{N}}{\mathcal{N}'}$ is small in $\frac{\mathcal{M}}{\mathcal{N}'}$ (i.e., $\frac{\mathcal{N}}{\mathcal{N}'} \ll \frac{\mathcal{M}}{\mathcal{N}'}$), and moreover, $\frac{\mathcal{N}}{\mathcal{N}'}$ is a locally Artinian submodule of $\frac{\mathcal{M}}{\mathcal{N}'}$.*

*(2) There exist a direct summand $\mathcal{N}'$ of $\mathcal{M}$ and a submodule $\mathcal{K} \leq \mathcal{M}$ such that $\mathcal{N}' \leq \mathcal{N}$, $\mathcal{N} = \mathcal{N}' + \mathcal{K}$, $\mathcal{K} \ll \mathcal{M}$, and $\mathcal{K}$ is locally Artinian in $\mathcal{M}$.*

*(3) The module $\mathcal{M}$ admits a decomposition $\mathcal{M} = \mathcal{N}' \oplus \mathcal{K}$, where $\mathcal{N}' \leq \mathcal{N}$, $\mathcal{N} \cap \mathcal{K} \ll \mathcal{M}$, and $\mathcal{N} \cap \mathcal{K}$ is locally Artinian in $\mathcal{M}$.*

*(4) The submodule $\mathcal{N}$ possesses a locally Artinian supplement $\mathcal{K}$ in $\mathcal{M}$ such that $\mathcal{N} \cap \mathcal{K}$ forms a direct summand of $\mathcal{N}$.*

*(5) There exists an idempotent endomorphism $\alpha\colon \mathcal{M} \to \mathcal{M}$ (i.e., $\alpha^2 = \alpha$) such that $\alpha(\mathcal{M}) \leq \mathcal{N}$, $(1-\alpha)(\mathcal{N}) \ll (1-\alpha)(\mathcal{M})$, and $(1-\alpha)(\mathcal{N})$ is locally Artinian as a submodule of $(1-\alpha)(\mathcal{M})$.*

*Proof.* (1) $\Rightarrow$ (2) Assume that $\mathcal{N}'$ is a direct summand of $\mathcal{M}$ with $\mathcal{N}' \leq \mathcal{N}$. Then there exists a submodule $\mathcal{K}'$ such that $\mathcal{M} = \mathcal{N}' \oplus \mathcal{K}'$. From this, we obtain $\mathcal{M} = \mathcal{N} + \mathcal{K}'$ and

$$\mathcal{N} = \mathcal{N} \cap (\mathcal{N}' \oplus \mathcal{K}') = \mathcal{N}' \oplus (\mathcal{N} \cap \mathcal{K}').$$

Consequently, $\frac{\mathcal{N}}{\mathcal{N}'} \cong \mathcal{N} \cap \mathcal{K}'$. By the given assumptions, the intersection $\mathcal{N} \cap \mathcal{K}'$ is small in $\mathcal{K}'$ (hence also in $\mathcal{M}$), and moreover, it is locally Artinian. This establishes the implication.

(2) $\Rightarrow$ (3) Suppose that $\mathcal{M} = \mathcal{N}' \oplus \mathcal{K}'$ with $\mathcal{K}' \leq \mathcal{M}$ as provided by the hypothesis. Then $\mathcal{K}'$ is a locally Artinian supplement of $\mathcal{N}'$ in $\mathcal{M}$. Let us now consider a submodule of the form $\mathcal{N} = \mathcal{N}' + \mathcal{K}$, where $\mathcal{N}' \leq \mathcal{N}$, the submodule $\mathcal{K}$ is small in $\mathcal{M}$, and $\mathcal{K}$ is locally Artinian. By [9, Lemma 2.13], the submodule $\mathcal{K}$ admits a locally Artinian supplement in $\mathcal{M}$, denoted again by $\mathcal{N}$. Therefore, $\mathcal{N} \cap \mathcal{K}'$ is locally Artinian, as required.

(3) ⇒ (4) From the assumption, we may choose a decomposition $\mathcal{M} = \mathcal{N}' \oplus \mathcal{K}$ with $\mathcal{N}' \leq \mathcal{N}$, $\mathcal{N} \cap \mathcal{K} \ll \mathcal{M}$, and where the intersection $\mathcal{N} \cap \mathcal{K}$ is locally Artinian. This yields the representation

$$\mathcal{N} = \mathcal{N}' \oplus (\mathcal{N} \cap \mathcal{K}).$$

Since $\mathcal{K}$ is a direct summand of $\mathcal{M}$, the smallness condition $\mathcal{N} \cap \mathcal{K} \ll \mathcal{M}$ implies that $\mathcal{N} \cap \mathcal{K} \ll \mathcal{K}$. Consequently, $\mathcal{M} = \mathcal{N} + \mathcal{K}$.

(4) ⇒ (5) Under our assumption, $\mathcal{N}$ can be written as $\mathcal{N} = (\mathcal{N} \cap \mathcal{K}) \oplus \mathcal{T}$, where $\mathcal{N} \cap \mathcal{K}$ is small in $\mathcal{K}$ and locally Artinian, and $\mathcal{T} \leq \mathcal{N}$ with $\mathcal{M} = \mathcal{N} + \mathcal{K}$. In this situation, $\mathcal{M} = \mathcal{K} + \mathcal{T}$ and $(\mathcal{N} \cap \mathcal{K}) \cap \mathcal{T} = 0$. Consider now the endomorphism $\alpha : \mathcal{M} \to \mathcal{M}$ defined by $\alpha(m) = n$ whenever $m = n + k$ with $n \in \mathcal{N}$ and $k \in \mathcal{K}$. Clearly $\alpha^2 = \alpha$ and $\alpha(\mathcal{M}) \subseteq \mathcal{N}$. Since $(1-\alpha)(\mathcal{M}) = \mathcal{K}$, we observe that $(1-\alpha)(\mathcal{N}) = \mathcal{N} \cap \mathcal{K}$ is both small in $(1 - \alpha)(\mathcal{M})$ and locally Artinian, in accordance with [13, 31.2 (1) (i)].

(5) ⇒ (1) Finally, take $\mathcal{N}' = \alpha(\mathcal{M})$, where $\alpha$ is an idempotent endomorphism of $\mathcal{M}$ satisfying $\alpha^2 = \alpha$, $\alpha(\mathcal{M}) \leq \mathcal{N}$, and such that $(1 - \alpha)(\mathcal{N})$ is small in $(1 - \alpha)(\mathcal{M})$ and locally Artinian. Then

$$\mathcal{M} = \alpha(\mathcal{M}) \oplus (1 - \alpha)(\mathcal{M}) = \mathcal{N}' \oplus (1 - \alpha)(\mathcal{M}).$$

Moreover,

$$\mathcal{N} = \mathcal{N}' \oplus \mathcal{N} \cap (1 - \alpha)(\mathcal{M}) = \mathcal{N}' \oplus (1 - \alpha)(\mathcal{N}),$$

which shows that $\frac{\mathcal{N}}{\mathcal{N}'} \cong (1 - \alpha)(\mathcal{N})$. Since $(1 - \alpha)(\mathcal{N})$ is small in $(1 - \alpha)(\mathcal{M})$ and locally Artinian, it follows that $\frac{\mathcal{N}}{\mathcal{N}'}$ is small in $\frac{\mathcal{M}}{\mathcal{N}'}$ and locally Artinian as well. This completes the cycle of equivalences. □

Recall from [9, Proposition 2.16] that every factor module (direct summand) of a locally Artinian supplemented module is locally Artinian supplemented.

**Theorem 1.** *The following statements are equivalent for a module $\mathcal{M}$.*

*(1) $\mathcal{M}$ is $\mathcal{LA}$-lifting;*

*(2) For every submodule $\mathcal{N}$ of $\mathcal{M}$, there exists a decomposition $\mathcal{N} = \mathcal{K} \oplus \mathcal{T}$, where $\mathcal{K}$ is a direct summand of $\mathcal{M}$, $\mathcal{T} \ll \mathcal{M}$, and $\mathcal{T}$ is locally Artinian.*

*(3) $\mathcal{M}$ is an amply locally Artinian supplemented module and every locally Artinian supplement in $\mathcal{M}$ is a direct summand.*

*Proof.* (1) ⇔ (2) This equivalence follows directly from Lemma 2.

(1) ⇒ (3) Assume that $\mathcal{M}$ is an $\mathcal{LA}$-lifting module. By Lemma 1, it follows that $\mathcal{M}$ is also an amply locally Artinian supplemented module. Hence, every locally Artinian supplemented submodule of $\mathcal{M}$ must be a direct summand of $\mathcal{M}$.

(3) ⇒ (1) Let $\mathcal{N}$ be an arbitrary submodule of $\mathcal{M}$. Then $\mathcal{N}$ possesses a locally Artinian supplement $\mathcal{K}$. Moreover, $\mathcal{K}$ itself admits a locally Artinian supplement $\mathcal{T}$ in $\mathcal{M}$ with $\mathcal{T} \leq \mathcal{N}$ and $\mathcal{T}$ being a direct summand of $\mathcal{M}$. Thus, we can write $\mathcal{M} = \mathcal{T} \oplus \mathcal{V}$ for some submodule $\mathcal{V}$. Consequently,

$$\mathcal{N} = \mathcal{T} \oplus (\mathcal{N} \cap \mathcal{V}).$$

Since $\mathcal{V} \leq \mathcal{K}$, it follows that $\mathcal{N} = \mathcal{T} + (\mathcal{N} \cap \mathcal{K})$. Consider now the canonical projection $\gamma : \mathcal{M} \to \mathcal{V}$. Then

$$\gamma(\mathcal{N}) = \gamma(\mathcal{T} + (\mathcal{N} \cap \mathcal{K})) = \mathcal{N} \cap \mathcal{V}.$$

Because $\mathcal{M} = \mathcal{T} \oplus \mathcal{V}$, with $\mathcal{T} \leq \mathcal{N}$, and since $\mathcal{N} \cap \mathcal{K} \ll \mathcal{K}$ and $\mathcal{N} \cap \mathcal{K}$ is locally Artinian, it follows from [13, 31.2 (1) (i)] that $\mathcal{N} \cap \mathcal{V}$ is also locally Artinian. □

Recall from [13] that a left $\mathcal{R}$-module $\mathcal{M}$ is called *π-projective* if for any submodules $\mathcal{U}, \mathcal{V} \subseteq \mathcal{M}$ satisfying $\mathcal{M} = \mathcal{U} + \mathcal{V}$, there exists an endomorphism $\phi \in \mathrm{End}_{\mathcal{R}}(\mathcal{M})$ such that $\mathrm{im}(\phi) \subseteq \mathcal{U}$ and $\mathrm{im}(1 - \phi) \subseteq \mathcal{V}$.

**Theorem 2.** *Let $\mathcal{M}$ be a π-projective module. If $\mathcal{M}$ is locally Artinian supplemented, then it is $\mathcal{LA}$-lifting.*

*Proof.* Assume that $\mathcal{M}$ is π-projective and locally Artinian supplemented. By [9, Proposition 2.20], this implies that $\mathcal{M}$ is in fact *amply* locally Artinian supplemented. Hence, for any submodule $\mathcal{N} \leq \mathcal{M}$, there exists a submodule $\mathcal{K} \leq \mathcal{M}$ such that

$$\mathcal{M} = \mathcal{N} + \mathcal{K}, \quad \mathcal{N} \cap \mathcal{K} \ll \mathcal{K}, \quad \text{and} \quad \mathcal{N} \cap \mathcal{K} \text{ is locally Artinian.}$$

Moreover, the ample condition ensures the existence of a submodule $\mathcal{N}' \leq \mathcal{N}$ with

$$\mathcal{M} = \mathcal{N}' + \mathcal{K}, \quad \mathcal{N}' \cap \mathcal{K} \ll \mathcal{K}, \quad \text{and} \quad \mathcal{N}' \cap \mathcal{K} \text{ locally Artinian.}$$

In this situation, $\mathcal{N}'$ and $\mathcal{K}$ form a pair of mutual locally Artinian supplements in $\mathcal{M}$, as established in [9, Lemma 2.22]. By [13, 41.14(2)], such a pair must satisfy $\mathcal{N}' \cap \mathcal{K} = 0$, which yields the direct sum decomposition

$$\mathcal{M} = \mathcal{N}' \oplus \mathcal{K}.$$

Consequently, $\mathcal{M}$ fulfills the definition of an $\mathcal{LA}$-lifting module. □

**Corollary 1.** *Every projective locally Artinian supplemented module is $\mathcal{LA}$-lifting.*

**Theorem 3.** *Let $\mathcal{M}$ be an $\mathcal{LA}$-lifting module. Then every direct summand of $\mathcal{M}$ is $\mathcal{LA}$-lifting.*

*Proof.* Suppose that $\mathcal{K}$ is a direct summand of $\mathcal{M}$. Hence, there exists a submodule $\mathcal{T}$ of $\mathcal{M}$ such that

$$\mathcal{M} = \mathcal{K} \oplus \mathcal{T}.$$

Let $\mathcal{N}$ be any submodule of $\mathcal{K}$. Since $\mathcal{M}$ is assumed to be $\mathcal{LA}$-lifting, we can find submodules $\mathcal{V} \leq \mathcal{M}$ and $\mathcal{N}' \leq \mathcal{N}$ satisfying

$$\mathcal{M} = \mathcal{N}' \oplus \mathcal{V}, \qquad \mathcal{N} \cap \mathcal{V} \ll \mathcal{V},$$

and moreover, $\mathcal{N} \cap \mathcal{V}$ is locally Artinian.

From the decomposition $\mathcal{M} = \mathcal{N} + \mathcal{V}$, it follows that

$$\mathcal{K} = \mathcal{N}' \oplus (\mathcal{K} \cap \mathcal{V}), \qquad \mathcal{K} = \mathcal{N} + (\mathcal{K} \cap \mathcal{V}).$$

In particular,

$$\mathcal{N} \cap (\mathcal{K} \cap \mathcal{V}) = \mathcal{N} \cap \mathcal{V}.$$

Since $\mathcal{N} \cap \mathcal{V} \ll \mathcal{V}$ and is locally Artinian, the same property holds inside $\mathcal{K}$. Therefore, $\mathcal{K}$ is also an $\mathcal{LA}$-lifting module. □

**Theorem 4.** *Let $\mathcal{M}$ be a module with $Rad(\mathcal{M}) \ll \mathcal{M}$. Then $\mathcal{M}$ is $\mathcal{LA}$-lifting if and only if $\mathcal{M}$ is lifting and $Rad(\mathcal{M})$ is locally Artinian.*

*Proof.* ($\Rightarrow$) Assume that $\mathcal{M}$ is an $\mathcal{LA}$-lifting module. By definition, every $\mathcal{LA}$-lifting module is in particular a lifting module, so $\mathcal{M}$ is lifting. Moreover, since $Rad(\mathcal{M}) \ll \mathcal{M}$, it follows that $Rad(\mathcal{M})$ is a locally Artinian supplement of $\mathcal{M}$. Hence, we conclude that $Rad(\mathcal{M})$ is locally Artinian.

($\Leftarrow$) Conversely, suppose that $Rad(\mathcal{M})$ is locally Artinian and $\mathcal{M}$ is lifting. Let $\mathcal{N}$ be a submodule of $\mathcal{M}$. By the lifting property, there exists a decomposition

$$\mathcal{M} = \mathcal{N}' \oplus \mathcal{T},$$

where $\mathcal{N}' \leq \mathcal{N}$ for some submodule $\mathcal{N}'$ of $\mathcal{M}$, $\mathcal{T}$ is a submodule of $\mathcal{M}$, and $\mathcal{N} \cap \mathcal{T} \ll \mathcal{T}$. Since $\mathcal{N} \cap \mathcal{T} \subseteq Rad(\mathcal{M})$ and the radical is locally Artinian by assumption, it follows from [13, 31.2 (1) (i)] that $\mathcal{N} \cap \mathcal{T}$ is also locally Artinian. Therefore, $\mathcal{M}$ satisfies the defining condition of an $\mathcal{LA}$-lifting module. □

The following corollary is obtained directly as a result of every projective supplemented module having a small radical.

**Corollary 2.** *Let $\mathcal{M}$ be a projective module. The following statements are equivalent for $\mathcal{M}$.*

*(1) $\mathcal{M}$ is $\mathcal{LA}$-lifting;*

*(2) $\mathcal{M}$ is lifting and $Rad(\mathcal{M})$ is locally Artinian.*

Recall from [13, 42.6] that a ring $\mathcal{R}$ is *semiperfect* if every finitely generated $\mathcal{R}$-module is a(amply) supplemented. Now we give a characterization of semiperfect rings via $\mathcal{LA}$-lifting modules.

**Lemma 3.** *Let $\mathcal{R}$ be a ring. Then ${}_{\mathcal{R}}\mathcal{R}$ is $\mathcal{LA}$-lifting if and only if $\mathcal{R}$ is a semiperfect ring and $J(\mathcal{R})$ is locally Artinian.*

*Proof.* This is the result of Corollary 2 and [9, Corollary 2.10]. □

Recall that a ring is called *left perfect* if every left $\mathcal{R}$-module has a projective cover, or equivalently, if $\mathcal{R}$ satisfies the descending chain condition on principal right ideals. The *Jacobson radical* of $\mathcal{R}$, denoted by $J(\mathcal{R})$, is said to be *locally Artinian* when, for every finitely generated left ideal $\mathcal{I} \subseteq \mathcal{R}$, the quotient module $\mathcal{I}/J(\mathcal{R})\mathcal{I}$ is Artinian.

With these definitions in mind, we have the following.

**Theorem 5.** *For a ring $\mathcal{R}$, the following conditions are mutually equivalent:*

*(1) The ring $\mathcal{R}$ is left perfect and $J(\mathcal{R})$ is locally Artinian;*

*(2) Every free $\mathcal{R}$-module is $\mathcal{LA}$-lifting;*

*(3) Every free $\mathcal{R}$-module is locally Artinian supplemented;*

*(4) Every $\mathcal{R}$-module is locally Artinian supplemented.*

*Proof.* (1) ⇒ (2) Let $\mathcal{F} = \mathcal{R}^{(I)}$ for some index set $I$. By [13, 43.9], $\mathcal{F}$ is supplemented. Then $\mathcal{F}$ is lifting as a result of $\mathcal{F}$ being a projective module. Since $J(\mathcal{R})$ is locally Artinian, $Rad(\mathcal{F}) = Rad(\mathcal{R}^{\mathbb{N}}) = J(\mathcal{R})^{(\mathbb{N})}$ is locally Artinian by [13, 31.2(2)]. Since every projective supplemented module has a small radical, $Rad(\mathcal{F}) \ll \mathcal{F}$. So $\mathcal{F}$ is $\mathcal{LA}$-lifting by Theorem 4.

(2) ⇒ (3) This is clear by Lemma 1.

(3) ⇒ (4) and (4) ⇒ (1) These are clear by [9, Theorem 2.23]. □

**Lemma 4.** *Let $\mathcal{M}$ be a lifting module and $Rad(\mathcal{M})$ is locally Artinian. Then $\mathcal{M}$ is $\mathcal{LA}$-lifting.*

**Theorem 6.** *Let $\mathcal{R}$ be a left and right Artinian serial ring with $J(\mathcal{R})^2 = 0$. Then $J(\mathcal{R})$ is locally Artinian if and only if every $\mathcal{R}$-module is $\mathcal{LA}$-lifting.*

*Proof.* (⇒) Suppose that $\mathcal{M}$ is an $\mathcal{R}$-module. From [13, 21.12(4)] together with [2, 29.10], it follows that $\mathcal{M}$ is a lifting module. Since $\mathcal{R}$ is assumed to be Artinian, then $Rad(\mathcal{M}) = J(\mathcal{R})\mathcal{M}$. By hypothesis, $J(\mathcal{R})$ is locally Artinian, and hence, $Rad(\mathcal{M})$ inherits the locally Artinian property. Applying Lemma 4, we deduce that $\mathcal{M}$ is indeed $\mathcal{LA}$-lifting.

(⇐) Conversely, assume that every $\mathcal{R}$-module is $\mathcal{LA}$-lifting. In particular, this forces $J(\mathcal{R})$ to be locally Artinian. Furthermore, since each $\mathcal{R}$-module is lifting, it follows by [2, 29.10] that $\mathcal{R}$ must be a left and right Artinian serial ring satisfying $J(\mathcal{R})^2 = 0$. This completes the argument. □

Consider the left and right Artinian serial ring $\mathcal{R} = \mathbb{Z}_4$ with $J(R)^2 = 0$. By using Theorem 6, we have that every $\mathcal{R}$-module is $\mathcal{LA}$-lifting.

**Proposition 2.** *Let $\mathcal{M}$ be a $\mathcal{LA}$-lifting module. If $\frac{\mathcal{U}'+\mathcal{K}}{\mathcal{K}}$ is a direct summand of $\frac{\mathcal{M}}{\mathcal{K}}$ for a direct summand $\mathcal{U}'$ of $\mathcal{M}$, then $\frac{\mathcal{M}}{\mathcal{K}}$ is $\mathcal{LA}$-lifting.*

*Proof.* Let $\frac{\mathcal{U}}{\mathcal{K}}$ be a submodule of $\frac{\mathcal{M}}{\mathcal{K}}$. Then $\mathcal{U}$ is a submodule of $\mathcal{M}$. Since $\mathcal{M}$ is $\mathcal{LA}$-lifting, there exists a direct summand $\mathcal{U}'$ of $\mathcal{M}$ such that $\mathcal{U}' \leq \mathcal{U}$, $\frac{\mathcal{U}}{\mathcal{U}'} \ll \frac{\mathcal{M}}{\mathcal{U}'}$, and $\frac{\mathcal{U}}{\mathcal{U}'}$ is a locally Artinian submodule of $\frac{\mathcal{M}}{\mathcal{U}'}$ according to Lemma 2. $\frac{\frac{\mathcal{U}}{\mathcal{U}'}}{\frac{\mathcal{U}'+\mathcal{K}}{\mathcal{U}'}} \cong \frac{\mathcal{U}}{\mathcal{U}'+\mathcal{K}} \cong \frac{\frac{\mathcal{U}}{\mathcal{K}}}{\frac{\mathcal{U}'+\mathcal{K}}{\mathcal{K}}}$ is locally Artinian by [13, 31.2(1) (i)] and $\frac{\mathcal{U}}{\mathcal{U}'+\mathcal{K}} \ll \frac{\mathcal{M}}{\mathcal{U}'+\mathcal{K}}$. Again by Lemma 2, $\frac{\mathcal{M}}{\mathcal{K}}$ is $\mathcal{LA}$-lifting. □

Recall from [13] that a submodule $\mathcal{K}$ of $\mathcal{M}$ is called *fully invariant* if $\alpha(\mathcal{K}) \leq \mathcal{K}$ for every endomorphism $\alpha$ of $\mathcal{M}$. In [11], a module $\mathcal{M}$ is called *duo* if every submodule of $\mathcal{M}$ is fully invariant.

**Theorem 7.** *Let $\mathcal{M}$ be an $\mathcal{LA}$-lifting module and $\mathcal{K}$ be a fully invariant submodule of $\mathcal{M}$. Then the factor module $\frac{\mathcal{M}}{\mathcal{K}}$ is $\mathcal{LA}$-lifting.*

*Proof.* Assume that $\mathcal{U}$ is a direct summand of $\mathcal{M}$. Hence, there exists a submodule $\mathcal{V} \leq \mathcal{M}$ with $\mathcal{M} = \mathcal{U} \oplus \mathcal{V}$. Consider the canonical endomorphism $\alpha\colon \mathcal{M} \to \mathcal{M}$ determined by $\alpha(\mathcal{M}) = \mathcal{U}$ and $(1-\alpha)(\mathcal{M}) = \mathcal{V}$. Since $\mathcal{K}$ is fully invariant in $\mathcal{M}$, we obtain $\alpha(\mathcal{K}) = \mathcal{K} \cap \mathcal{U}$ and $(1-\alpha)(\mathcal{K}) = \mathcal{K} \cap \mathcal{V}$. Consequently,

$$\mathcal{K} = \alpha(\mathcal{K}) \oplus (1-\alpha)(\mathcal{K}) = (\mathcal{K} \cap \mathcal{U}) \oplus (\mathcal{K} \cap \mathcal{V}).$$

Now, observe that

$$\frac{\mathcal{U}+\mathcal{K}}{\mathcal{K}} = \frac{\mathcal{U} + [(\mathcal{K} \cap \mathcal{U}) \oplus (\mathcal{K} \cap \mathcal{V})]}{\mathcal{K}} = \frac{\mathcal{U} \oplus (\mathcal{K} \cap \mathcal{V})}{\mathcal{K}},$$

and similarly,

$$\frac{\mathcal{V}+\mathcal{K}}{\mathcal{K}}=\frac{\mathcal{V}+[(\mathcal{K}\cap\mathcal{U})\oplus(\mathcal{K}\cap\mathcal{V})]}{\mathcal{K}}=\frac{\mathcal{V}\oplus(\mathcal{K}\cap\mathcal{U})}{\mathcal{K}}.$$

Therefore,

$$\mathcal{M}=\mathcal{U}+\mathcal{V}+\mathcal{K}=[\mathcal{U}\oplus(\mathcal{K}\cap\mathcal{V})]+\mathcal{V}+\mathcal{K}.$$

It follows that:

$$[\mathcal{U}\oplus(\mathcal{K}\cap\mathcal{V})]\cap(\mathcal{V}+\mathcal{K})=[\mathcal{U}\oplus(\mathcal{K}\cap\mathcal{V})]\cap[\mathcal{V}+(\mathcal{K}\cap\mathcal{U})]=(\mathcal{K}\cap\mathcal{U})\oplus(\mathcal{K}\cap\mathcal{V})=\mathcal{K}.$$

Hence, we obtain the decomposition

$$\frac{\mathcal{M}}{\mathcal{K}}=\Big(\frac{\mathcal{U}\oplus(\mathcal{K}\cap\mathcal{V})}{\mathcal{K}}\Big)\oplus\Big(\frac{\mathcal{V}+\mathcal{K}}{\mathcal{K}}\Big).$$

By Proposition 2, it follows that $\mathcal{M}/\mathcal{K}$ is an $\mathcal{LA}$-lifting module, as desired. □

Recall from [1] that a submodule $\mathcal{K}$ of $\mathcal{M}$ is called *weakly distributive* of $\mathcal{M}$ if $\mathcal{K}=(\mathcal{K}\cap\mathcal{U})+(\mathcal{K}\cap\mathcal{V})$ for every submodule $\mathcal{U},\mathcal{V}\leq\mathcal{M}$ with $\mathcal{M}=\mathcal{U}+\mathcal{V}$. A module $\mathcal{M}$ is said to be *weakly distributive* if every submodule of $\mathcal{M}$ is a weak distributive submodule of $\mathcal{M}$.

**Theorem 8.** *Let $\mathcal{M}$ be a weakly distributive module and $K$ be a submodule of $\mathcal{M}$. Then $\frac{\mathcal{M}}{\mathcal{K}}$ is $\mathcal{LA}$-lifting.*

*Proof.* Let $\mathcal{M}=\mathcal{U}\oplus\mathcal{V}$. Then $\mathcal{M}=\mathcal{U}+\mathcal{V}$ and $\frac{\mathcal{M}}{\mathcal{K}}=\frac{\mathcal{U}+\mathcal{K}}{\mathcal{K}}+\frac{\mathcal{V}+\mathcal{K}}{\mathcal{K}}$. Since $\mathcal{K}$ is a weak distributive submodule of $\mathcal{M}$, $\mathcal{K}=(\mathcal{K}\cap\mathcal{U})+(\mathcal{K}\cap\mathcal{V})$. Since $\frac{\mathcal{U}+\mathcal{K}}{\mathcal{K}}\cap\frac{\mathcal{V}+\mathcal{K}}{\mathcal{K}}=\frac{(\mathcal{U}+\mathcal{K})\cap(\mathcal{V}+\mathcal{K})}{\mathcal{K}}=\frac{[\mathcal{U}+(\mathcal{K}\cap\mathcal{V})]\cap[\mathcal{V}+(\mathcal{K}\cap\mathcal{U})]}{\mathcal{K}}=\frac{(\mathcal{K}\cap\mathcal{U})+[\mathcal{U}+(\mathcal{K}\cap\mathcal{V})]\cap\mathcal{V}}{\mathcal{K}}=\frac{(\mathcal{K}\cap\mathcal{U})+(\mathcal{K}\cap\mathcal{V})+(\mathcal{U}\cap\mathcal{V})}{\mathcal{K}}=\frac{\mathcal{K}+(\mathcal{U}\cap\mathcal{V})}{\mathcal{K}}=\frac{\mathcal{K}}{\mathcal{K}}$, $\frac{\mathcal{M}}{\mathcal{K}}=\frac{\mathcal{U}+\mathcal{K}}{\mathcal{K}}\oplus\frac{\mathcal{V}+\mathcal{K}}{\mathcal{K}}$ by using Proposition 2, and we obtain that $\frac{\mathcal{M}}{\mathcal{K}}$ is $\mathcal{LA}$-lifting. □

**Theorem 9.** *Let $\mathcal{M}=\mathcal{M}_1\oplus\mathcal{M}_2$ be a duo module. If $\mathcal{M}_1$ and $\mathcal{M}_2$ are $\mathcal{LA}$-lifting, then $\mathcal{M}$ is $\mathcal{LA}$-lifting.*

*Proof.* Let $\mathcal{N}$ be a submodule of $\mathcal{M}$. It follows from [11, Lemma 2.1] that $\mathcal{N}=(\mathcal{N}\cap\mathcal{M}_1)\oplus(\mathcal{N}\cap\mathcal{M}_2)$. Then, by the hypothesis, there exist direct summands $\mathcal{K}_1$ of $\mathcal{M}_1$ and $\mathcal{K}_2$ of $\mathcal{M}_2$ such that $\mathcal{K}_1\leq\mathcal{N}\cap\mathcal{M}_1$, $\mathcal{K}_2\leq\mathcal{N}\cap\mathcal{M}_2$, $\mathcal{N}\cap\mathcal{K}_1'\ll\mathcal{K}_1'$, $\mathcal{N}\cap\mathcal{K}_2'\ll\mathcal{K}_2'$, and $\mathcal{N}\cap\mathcal{K}_1'$ and $\mathcal{N}\cap\mathcal{K}_2'$ are locally Artinian, where $\mathcal{M}_1=\mathcal{K}_1\oplus\mathcal{K}_1'$, $\mathcal{M}_2=\mathcal{K}_2\oplus\mathcal{K}_2'$. Note that $\mathcal{M}=\mathcal{M}_1\oplus\mathcal{M}_2=(\mathcal{K}_1\oplus\mathcal{K}_1')\oplus(\mathcal{K}_2\oplus\mathcal{K}_2')$. It is clear that $\mathcal{K}_1\oplus\mathcal{K}_2\leq\mathcal{N}$. Since $\mathcal{N}\cap\mathcal{K}_1'$ and $\mathcal{N}\cap\mathcal{K}_2'$ are locally Artinian, $\mathcal{N}\cap(\mathcal{K}_1'\oplus\mathcal{K}_2')$ is locally Artinian by [13, 31.2(2)]. As $\mathcal{N}\cap\mathcal{K}_1'\ll\mathcal{K}_1'$ and $\mathcal{N}\cap\mathcal{K}_2'\ll\mathcal{K}_2'$, then $\mathcal{N}\cap(\mathcal{K}_1'\oplus\mathcal{K}_2')\ll\mathcal{K}_1'\oplus\mathcal{K}_2'$. Hence, $\mathcal{M}$ is $\mathcal{LA}$-lifting. □

### *2.1. Semi-$\mathcal{LA}$-discrete modules and quasi-$\mathcal{LA}$-discrete modules*

In this section, we extend the study of $\mathcal{LA}$-lifting modules by introducing two new and closely related classes, namely the *semi-$\mathcal{LA}$-discrete* and *quasi-$\mathcal{LA}$-discrete* modules. A semi-$\mathcal{LA}$-discrete module is defined as an $\mathcal{LA}$-lifting module that also satisfies the finite internal exchange property, thereby connecting lifting conditions with exchange-theoretic behavior. On the other hand, a quasi-$\mathcal{LA}$-discrete module is characterized by the simultaneous satisfaction of $\pi$-projectivity and the property of being locally Artinian supplemented. Various equivalences are established to clarify these notions: for instance, semi-$\mathcal{LA}$-discreteness can be described through the interplay between direct summands, locally Artinian supplements, and relative projectivity. Furthermore, it is shown that quasi-$\mathcal{LA}$-discrete modules necessarily inherit the $\mathcal{LA}$-lifting property and yield relatively projective decompositions. Additional results highlight the stability of these classes under direct sum decompositions, the interaction with $\pi$-projectivity, and the relationship with amply locally Artinian supplemented modules. Altogether, these findings provide a deeper structural understanding of how lifting, exchange, and projectivity conditions coalesce in the context of $\mathcal{LA}$-lifting.

Recall from [8] that an *ss*-supplemented module $\mathcal{M}$ is defined as an *ss-discrete module* if it is both $\pi$-projective and direct projective. In that paper, $\mathcal{M}$ is called a *strongly ss-discrete module* if $\mathcal{M}$ is self-projective.

**Definition 2.** *A module $\mathcal{M}$ is said to be* semi-$\mathcal{LA}$-discrete *provided that it is $\mathcal{LA}$-lifting and satisfies the finite internal exchange property. Likewise, $\mathcal{M}$ is termed a* quasi-$\mathcal{LA}$-discrete module *if it is both $\pi$-projective and locally Artinian supplemented.*

As recalled from [2], a module $\mathcal{K}$ is called *generalized $\mathcal{M}$-projective* if, for every epimorphism $g : \mathcal{M} \to \mathcal{N}$ and homomorphism $f : \mathcal{K} \to \mathcal{N}$, there exist decompositions $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$ and $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$, together with a homomorphism $h_1 : \mathcal{K}_1 \to \mathcal{M}_1$ and an epimorphism $h_2 : \mathcal{M}_2 \to \mathcal{K}_2$, such that

$$g \circ h_1 = f_{|\mathcal{K}_1} \quad \text{and} \quad f \circ h_2 = g_{|\mathcal{M}_2}.$$

**Proposition 3.** *For a module $\mathcal{M}$, the following statements are equivalent:*

*(1) $\mathcal{M}$ is semi-$\mathcal{LA}$-discrete.*

*(2) $\mathcal{M}$ is locally Artinian supplemented, where each locally Artinian supplement in $\mathcal{M}$ is a direct summand, and for every decomposition $\mathcal{M} = \mathcal{U} \oplus \mathcal{V}$, the summands $\mathcal{U}$ and $\mathcal{V}$ are relatively generalized projective.*

*(3) $\mathcal{M}$ is $\mathcal{LA}$-lifting, and in any decomposition $\mathcal{M} = \mathcal{U} \oplus \mathcal{V}$, both $\mathcal{U}$ and $\mathcal{V}$ are relatively generalized projective.*

*Proof.* (1) $\Rightarrow$ (2) Assume that $\mathcal{M}$ is a semi-$\mathcal{LA}$-discrete module. By definition, this implies that $\mathcal{M}$ is $\mathcal{LA}$-lifting. According to Theorem 1, $\mathcal{M}$ is (amply) locally Artinian supplemented and, moreover, each locally Artinian supplement in $\mathcal{M}$ is a direct summand.

Now let $\mathcal{M} = \mathcal{X} + \mathcal{U}$. Since $\mathcal{X}$ contains a locally Artinian supplement $\mathcal{X}'$ of $\mathcal{U}$, which is itself a direct summand of $\mathcal{M}$, and since $\mathcal{M}$ satisfies the finite internal exchange property, there exists a decomposition

$$\mathcal{M} = \mathcal{X}' \oplus \mathcal{V}' \oplus \mathcal{U}',$$

with $\mathcal{V}' \leq \mathcal{V}$ and $\mathcal{U}' \leq \mathcal{U}$. Hence, $\mathcal{V}$ is generalized $\mathcal{U}$-projective by [2, 4.42]. By symmetry, the same reasoning shows that $\mathcal{U}$ is generalized $\mathcal{V}$-projective.

(2) $\Rightarrow$ (3) It remains to show that $\mathcal{M}$ is $\mathcal{LA}$-lifting. Take an arbitrary submodule $\mathcal{X} \leq \mathcal{M}$. By the assumption, $\mathcal{X}$ has a locally Artinian supplement $\mathcal{Y}$ that is also a direct summand of $\mathcal{M}$. Thus, we may write $\mathcal{M} = \mathcal{W} \oplus \mathcal{Y}$ with $\mathcal{W}$ generalized $\mathcal{Y}$-projective. From [2, 4.42], one obtains a decomposition

$$\mathcal{M} = \mathcal{X}'' \oplus \mathcal{W}' \oplus \mathcal{Y}' = \mathcal{X}'' + \mathcal{Y},$$

where $\mathcal{X}'' \subseteq \mathcal{X}$, $\mathcal{Y}' \subseteq \mathcal{Y}$, and $\mathcal{W}' \subseteq \mathcal{W}$. Consequently, $\mathcal{X} = \mathcal{X}'' + (\mathcal{X} \cap \mathcal{Y})$.

Since $\mathcal{X} \cap \mathcal{Y}$ is locally Artinian, it follows from [13, 31.2 (1) (i)] that $\mathcal{X}'' \cap \mathcal{Y}$ is also locally Artinian. Furthermore, $\mathcal{X} \cap \mathcal{Y} \ll \mathcal{Y}$. Hence, $\mathcal{M}$ is amply locally Artinian supplemented and every such supplement is a direct summand. By Theorem 1, it follows that $\mathcal{M}$ is $\mathcal{LA}$-lifting.

(3) $\Rightarrow$ (1) Suppose $\mathcal{M} = \mathcal{U} \oplus \mathcal{V}$. Since $\mathcal{M}$ is $\mathcal{LA}$-lifting, Theorem 3 implies that both $\mathcal{U}$ and $\mathcal{V}$ are $\mathcal{LA}$-lifting. By the hypothesis, $\mathcal{U}$ and $\mathcal{V}$ are also relatively generalized projective. Therefore, by [2, 23.10], $\mathcal{M}$ satisfies the 2-internal exchange property. This establishes the desired conclusion. □

**Proposition 4.** *Consider a duo module of the form $\mathcal{M} = \bigoplus_{i=1}^{n} \mathcal{M}_i$, where each summand $\mathcal{M}_i$ is assumed to be semi-$\mathcal{LA}$-discrete. Under this assumption, the following assertions are equivalent:*

*(1) The module $\mathcal{M}$ itself is semi-$\mathcal{LA}$-discrete.*

*(2) $\mathcal{M}$ possesses the $\mathcal{LA}$-lifting property, and the decomposition $\mathcal{M} = \bigoplus_{i=1}^{n} \mathcal{M}_i$ constitutes an exchange decomposition.*

*(3) For any pair of non-empty, mutually disjoint index sets $\mathcal{I}, \mathcal{J} \subseteq \{1, 2, \ldots, n\}$, if $\mathcal{U}$ is a direct summand of $\bigoplus_{i \in \mathcal{I}} \mathcal{M}_i$ and $\mathcal{V}$ is a direct summand of $\bigoplus_{j \in \mathcal{J}} \mathcal{M}_j$, then $\mathcal{U}$ and $\mathcal{V}$ are relatively generalized projective.*

*(4) For each index $i$ with $1 \leq i \leq n$, whenever $\mathcal{M}'_i$ is a direct summand of $\mathcal{M}_i$ and $\mathcal{W}$ is a direct summand of $\bigoplus_{j \neq i} \mathcal{M}_j$, the modules $\mathcal{M}'_i$ and $\mathcal{W}$ are relatively generalized projective.*

*Proof.* This can be found by Theorem 9 and [2, 23.14]. □

As a direct consequence of Proposition 4, we obtain the following corollary.

**Corollary 3.** *Let $\mathcal{M} = \bigoplus_{i=1}^{n} \mathcal{M}_i$ be a duo module in which each $\mathcal{M}_i$ is semi-$\mathcal{LA}$-discrete. If, for every pair of distinct indices $i \neq j$, the modules $\mathcal{M}_i$ and $\mathcal{M}_j$ are relatively generalized projective, then $\mathcal{M}$ is semi-$\mathcal{LA}$-discrete.*

**Lemma 5.** *If a module $\mathcal{M}$ is quasi-$\mathcal{LA}$-discrete, then $\mathcal{M}$ is $\mathcal{LA}$-lifting.*

*Proof.* By the hypothesis, $\mathcal{M}$ is $\pi$-projective and locally Artinian supplemented. It follows from Theorem 2 that $\mathcal{M}$ is $\mathcal{LA}$-lifting. □

With the help of [13, 41.15], it can be easily proven that if the intersection of any pair of mutually locally Artinian supplements is zero in a module of locally Artinian supplements, then the submodules of locally Artinian supplements of $\mathcal{M}$ are direct summands.

**Lemma 6.** *Let $\mathcal{M}$ be a $\pi$-projective module. If $\mathcal{M}$ is $\mathcal{LA}$-lifting, then $\mathcal{M}$ is amply locally Artinian supplemented and the intersection of any pair of mutual locally Artinian supplements in $\mathcal{M}$ is zero.*

*Proof.* This is clear by Lemma 1 and [2, 20.9]. □

**Corollary 4.** *Let $\mathcal{M}$ be a module. If $\mathcal{M}$ is quasi-$\mathcal{LA}$-discrete, then $\mathcal{M}$ is amply locally Artinian supplemented and the intersection of any pair of mutual locally Artinian supplements in $\mathcal{M}$ is zero.*

*Proof.* This follows from Lemmas 5 and 6. □

**Corollary 5.** *Let $\mathcal{M}$ be an $\mathcal{LA}$-lifting module. If $\mathcal{M}$ has the property $(D_3)$, then $\mathcal{M}$ is a quasi-$\mathcal{LA}$-discrete module.*

*Proof.* It can be seen clearly by Lemma 6. □

**Theorem 10.** *Let $\mathcal{M}$ be a module. If $\mathcal{M}$ is quasi-$\mathcal{LA}$-discrete, then it is $\mathcal{LA}$-lifting, and for every decomposition $\mathcal{M} = \mathcal{U} \oplus \mathcal{V}$, the modules $\mathcal{U}$ and $\mathcal{V}$ are relatively projective.*

*Proof.* From Lemmas 5 and 6, we know that $\mathcal{M}$ is amply locally Artinian supplemented and that the intersection of any two mutual locally Artinian supplements in $\mathcal{M}$ is zero. Since $\mathcal{M}$ is locally Artinian supplemented, every such supplement is necessarily a direct summand; hence, by Theorem 1, $\mathcal{M}$ is $\mathcal{LA}$-lifting.

Now assume that $\mathcal{M} = \mathcal{X} + \mathcal{Y}$, where both $\mathcal{X}$ and $\mathcal{Y}$ are direct summands of $\mathcal{M}$. Let $\mathcal{W}$ be a locally Artinian supplement of $\mathcal{Y}$ contained in $\mathcal{X}$. This yields the decomposition $\mathcal{M} = \mathcal{W} \oplus \mathcal{Y}$. Furthermore, since $\mathcal{X} = \mathcal{W} \oplus (\mathcal{X} \cap \mathcal{Y})$, it follows that $\mathcal{X} \cap \mathcal{Y}$ is a direct summand of $\mathcal{M}$. By [2, 4.14(2)], this implies that $\mathcal{M}$ is $\cap$-direct projective, as required. □

The following example shows that every $\mathcal{LA}$-lifting module is not always quasi-$\mathcal{LA}$-discrete.

**Example 2.** *Consider the Artinian $\mathbb{Z}$-modules*

$$\mathcal{X} = \frac{\mathbb{Z}}{p\mathbb{Z}} \quad \text{and} \quad \mathcal{Y} = \frac{\mathbb{Z}}{p^2\mathbb{Z}},$$

*where $p$ is a prime number. Both $\mathcal{X}$ and $\mathcal{Y}$ are relatively generalized projective modules, meaning that each is projective relative to certain submodules in the classical sense. However, $\mathcal{X}$ is* not *$\mathcal{Y}$-projective, i.e., there exists a surjective homomorphism $f : \mathcal{Y} \to \mathcal{L}$ for some submodule $\mathcal{L}$ of $\mathcal{Y}$ and a map $g : \mathcal{X} \to \mathcal{L}$ that does not lift to a homomorphism from $\mathcal{X}$ to $\mathcal{Y}$.*

*Now, consider the direct sum*

$$\mathcal{M} = \mathcal{X} \oplus \mathcal{Y}.$$

*Since $\mathcal{X}$ and $\mathcal{Y}$ are both Artinian and satisfy the relatively generalized projective conditions, $\mathcal{M}$ inherits the $\mathcal{LA}$-lifting property by Proposition 1. That is, for every submodule $\mathcal{A} \leq \mathcal{M}$, there exists a direct summand $\mathcal{D}$ of $\mathcal{M}$ such that $\mathcal{D} \subseteq \mathcal{A}$ and $\frac{\mathcal{A}}{\mathcal{D}}$ are $\mathcal{LA}$-lifting modules by the concept of being locally Artinian.*

*However, $\mathcal{M}$ is* not *quasi-$\mathcal{LA}$-discrete. This follows from the fact that quasi-$\mathcal{LA}$-discrete modules require additional restrictions on the interaction of summands and submodules relative to the socle, which fail in this example because $\mathcal{X}$ is not $\mathcal{Y}$-projective. In particular, there exists a submodule of $\mathcal{M}$ that cannot be complemented up to an ss-small submodule while preserving the locally Artinian structure, demonstrating that quasi-$\mathcal{LA}$-discreteness is a strictly stronger condition than $\mathcal{LA}$-lifting.*

This example illustrates a key role in generalized decomposition theory: *$\mathcal{LA}$-lifting modules* can accommodate direct sums of Artinian components with controlled radicals, yet may fail to satisfy stronger quasi-discrete conditions, highlighting the subtle differences between these related classes.

**Proposition 5.** *Let $\mathcal{M}$ be an amply locally Artinian supplemented module. Then $\mathcal{M}$ is quasi-$\mathcal{LA}$-discrete if and only if $\mathcal{M}$ is $\pi$-projective.*

*Proof.* This follows from [9, Proposition 2.16] and [13, 41.15]. □

Recall from [2, 4.13] that any factor module $\frac{\mathcal{M}}{\mathcal{N}}$ of a $\pi$-projective module $\mathcal{M}$ with a fully invariant submodule $\mathcal{N}$ of $\mathcal{M}$ is $\pi$-projective.

The following corollary is obtained directly with regard to Proposition 5.

**Corollary 6.** *Let $\mathcal{M}$ be a quasi-$\mathcal{LA}$-discrete module and $\mathcal{N}$ be a fully invariant submodule of $\mathcal{M}$. Then $\frac{\mathcal{M}}{\mathcal{N}}$ is quasi-$\mathcal{LA}$-discrete.*

### *2.2. $\mathcal{LA}$-discrete modules and strongly $\mathcal{LA}$-discrete modules*

In this part of the study, we introduce and analyze the notions of *$\mathcal{LA}$-discrete* and *strongly $\mathcal{LA}$-discrete* modules, which extend the text of discrete modules and strongly discrete modules by incorporating the condition of being locally Artinian supplemented. A module is $\mathcal{LA}$-discrete precisely when it is locally Artinian supplemented and enjoys both $\pi$-projectivity and direct projectivity, while the strongly $\mathcal{LA}$-discrete condition additionally requires self-projectivity. Several structural properties of these classes are established: for instance, both $\mathcal{LA}$-discrete modules and strongly $\mathcal{LA}$-discrete modules are closed when taking direct summands, and they arise naturally from classical discrete modules once the Jacobson radical is locally Artinian. Moreover, for projective modules, the strongly $\mathcal{LA}$-discrete property can be characterized through the interplay between strong discreteness and the concept of being Artinian of the radical. A decomposition theorem is also provided, showing that finite direct sums of modules with locally Artinian radicals are strongly $\mathcal{LA}$-discrete if and only if each summand is strongly discrete and pairwise projective relative to the others. Finally, ring-theoretic consequences are explored: in particular, the equivalence between the strongly $\mathcal{LA}$-discrete property of the regular module ${}_{\mathcal{R}}\mathcal{R}$, semiperfectness of the ring, and the locally Artinian nature of its Jacobson radical is established. These results highlight the central role of $\mathcal{LA}$-discrete structures in bridging module discreteness with Artinian radical conditions in ring theory.

**Definition 3.** *Let $\mathcal{M}$ be a module. We say that $\mathcal{M}$ is* $\mathcal{LA}$-discrete *if it is a locally Artinian supplemented module that is both $\pi$-projective and direct projective. Similarly, $\mathcal{M}$ is called* strongly $\mathcal{LA}$-discrete *if it is locally Artinian supplemented and self-projective.*

It is clear that every $\mathcal{LA}$-lifting module with the property $(D_2)$ is $\mathcal{LA}$-discrete.

**Proposition 6.** *Let $\mathcal{M}$ be a module in which $Rad(\mathcal{M})$ is locally Artinian. If $\mathcal{M}$ is a (quasi-)discrete module, then $\mathcal{M}$ is a (quasi-)$\mathcal{LA}$-discrete module.*

*Proof.* This follows from [9, Theorem 2.9]. □

**Proposition 7.** *Let $\mathcal{M}$ be an $\mathcal{LA}$-discrete module. Then each direct summand of $\mathcal{M}$ is also $\mathcal{LA}$-discrete.*

*Proof.* Let $\mathcal{K}$ be a direct summand of $\mathcal{M}$. By [2, 4.22], $\mathcal{K}$ is direct projective and therefore satisfies the property $(D_2)$. Since $\mathcal{M}$ is assumed to be locally Artinian supplemented and $\pi$-projective, Theorem 2 implies that $\mathcal{M}$ is $\mathcal{LA}$-lifting. Applying Theorem 3 then yields the result that $\mathcal{K}$ is $\mathcal{LA}$-lifting. Therefore, $\mathcal{K}$ is indeed $\mathcal{LA}$-discrete. □

**Example 3.** *The theory of $\mathcal{LA}$-lifting and $\mathcal{LA}$-discrete modules provides a refined perspective on classical decomposition properties in module theory, particularly when radicals are not necessarily semisimple. To illustrate these notions, consider first the Artinian $\mathbb{Z}$-modules*

$$\mathcal{X} = \frac{\mathbb{Z}}{p\mathbb{Z}} \quad \textit{and} \quad \mathcal{Y} = \frac{\mathbb{Z}}{p^2\mathbb{Z}},$$

*where p is a prime number. Both $\mathcal{X}$ and $\mathcal{Y}$ are relatively generalized projective; however, $\mathcal{X}$ is not $\mathcal{Y}$-projective. Considering the direct sum $\mathcal{M} = \mathcal{X} \oplus \mathcal{Y}$, we observe that $\mathcal{M}$ is an $\mathcal{LA}$-lifting module, since each submodule admits a direct summand whose quotient satisfies as a locally Artinian submodule. Nevertheless, $\mathcal{M}$ fails to be quasi-$\mathcal{LA}$-discrete, as the interaction between $\mathcal{X}$ and $\mathcal{Y}$ prevents certain submodules from being supplemented up to an ss-small factor while maintaining the notion of being locally Artinian. This example demonstrates the subtle distinction between $\mathcal{LA}$-lifting and stronger quasi-discrete conditions.*

*Another illustrative case is the self-projective $\mathbb{Z}$-module*

$$\mathcal{M} = \frac{\mathbb{Z}}{2\mathbb{Z}}.$$

*Here, $\mathcal{M}$ is locally Artinian supplemented, meaning that every submodule has a supplement whose intersection satisfies the notion of being a locally Artinian submodule. Due to its simple structure and self-projectivity, $\mathcal{M}$ automatically satisfies these requirements and thus serves as a prototypical example of a strongly $\mathcal{LA}$-discrete module, in which both supplement and decomposition properties are tightly governed by the locally Artinian submodule.*

These examples collectively highlight that the classes of $\mathcal{LA}$-lifting, quasi-$\mathcal{LA}$-discrete, and strongly $\mathcal{LA}$-discrete modules are strictly related yet distinct. In general, we have the proper inclusions

$$\text{strongly } \mathcal{LA}\text{-discrete} \subsetneq \text{quasi-}\mathcal{LA}\text{-discrete} \subsetneq \mathcal{LA}\text{-lifting} \subsetneq \text{lifting modules.}$$

The theoretical significance of these distinctions lies in their capacity to provide a hierarchy of decomposition properties, allowing researchers to study modules under increasingly refined structural constraints. By notions of being locally Artinian and having socle structure such as $ss$-smallness, these concepts enable a deeper analysis of the interaction between submodules, supplements, and radicals, thereby extending classical decomposition theory to a broader and more flexible context.

**Proposition 8.** *Let $\mathcal{M}$ be a projective module. Then $\mathcal{M}$ is strongly $\mathcal{LA}$-discrete if and only if it is strongly discrete and $Rad(\mathcal{M})$ is locally Artinian.*

*Proof.* This follows directly from [9, Theorem 2.9], noting that every projective module is self-projective. □

**Proposition 9.** *If $\mathcal{M}$ is strongly $\mathcal{LA}$-discrete, then each direct summand of $\mathcal{M}$ is also strongly $\mathcal{LA}$-discrete.*

*Proof.* Since every direct summand of a self-projective module is itself self-projective, the claim follows from [9, Proposition 2.16]. □

**Theorem 11.** *Let $\{\mathcal{M}_i\}_{i\in I}$ be a finite collection of $\mathcal{R}$-modules and set $\mathcal{M} = \bigoplus_{i\in I} \mathcal{M}_i$. Assume that for each $i \in I$, $Rad(\mathcal{M}_i)$ is locally Artinian. Then the following are equivalent:*

*(1) $\mathcal{M}$ is strongly $\mathcal{LA}$-discrete;*

*(2) (a) each $\mathcal{M}_i$ is strongly discrete;*

*(b) for every $i \in I$ and $j \neq i$, $\mathcal{M}_i$ is $\mathcal{M}_j$-projective.*

*Proof.* This equivalence is a direct consequence of [9, Theorem 2.9], [2, 27.16], and [13, 31.2(2)]. □

We can now describe *strongly$\mathcal{LA}$-discrete* rings via the notion of semiperfectness.

**Corollary 7.** *For a ring $\mathcal{R}$, the following statements are equivalent:*

*(1) ${}_{\mathcal{R}}\mathcal{R}$ is locally Artinian supplemented;*

*(2) ${}_{\mathcal{R}}\mathcal{R}$ is semiperfect and $J(\mathcal{R})$ is locally Artinian;*

*(3) Every $\mathcal{R}$-module $\mathcal{M} = \bigoplus_{i\in I} \mathcal{M}_i$, where $I$ is finite and each $\mathcal{M}_i$ is an RLA-local $\mathcal{M}$-projective module, satisfies the condition in (1);*

*(4) ${}_{\mathcal{R}}\mathcal{R}$ is strongly $\mathcal{LA}$-discrete.*

*Proof.* The equivalence follows from [9, Corollary 2.10]. □

## 3. Conclusions

Here, the following chain illustrates the hierarchy among the classes of modules considered in this paper:

strongly $\mathcal{LA}$-discrete ⇒ $\mathcal{LA} - discrete$ ⇒ quasi-$\mathcal{LA}$-discrete ⇒ semi-$\mathcal{LA}$-discrete ⇒ $\mathcal{LA}$-lifting.

## Author contributions

Yıldız Aydın and Burcu Nişancı Türkmen: Conceptualization, Methodology, Validation, Writing-original draft, Writing-review & editing. All authors contributed equally in this work.

## Use of Generative-AI tools declaration

AI tools were used for presenting the diagram and its explanations given in the Introduction section, and also for the linguistic improvement of the text.

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## Conflict of interest

All authors declare no conflicts of interest in this paper.

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