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Research article

The well-posedness of incompressible magneto-viscoelastic hydrodynamics

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Abstract: In this paper, we focused on the evolutionary model for magneto-viscoelasticity. First, we proved the local-in-time existence of classical solutions with finite initial energy for the system by the energy method. Furthermore, the global well-posedness of the magneto-viscoelastic system with small initial data was established in the absence of an external magnetic field.

Keywords: magneto-viscoelasticity; Landau-Lifshitz-Gilbert equation; complex fluids; global well-posedness
Mathematics Subject Classification: 35A01, 35B45, 35B65, 35Q35, 76D03

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1. Introduction

Magneto-viscoelastic materials, which exhibit coupled mechanical, viscous and magnetic responses under external stimuli, have attracted considerable attention in both applied mathematics and engineering due to their rich physical behaviors and broad applications in sensors, actuators, and smart devices. The dynamics of such materials are governed by a system of partial differential equations coupling the Navier–Stokes equations for incompressible fluids, the transport equation for the deformation gradient, and the Landau–Lifshitz–Gilbert (LLG) equation for magnetization evolution.

The evolution of magnetoelastic materials dates back to the 1960s [5], when Brown rectified the limitations of the classical magnetoelastic theory in his monograph and established a self-consistent foundational theory for magnetoelastic interactions in ferromagnetic materials, thereby furnishing a theoretical framework for empirical investigations in the magnetoelastic domain. Earlier, Tiersten [23] formulated the first set of coupled magnetomechanical equations for magnetically saturated insulating materials and further developed a variational principle for this theoretical system in [24]. These two seminal works collectively laid the theoretical groundwork for saturated magnetoelastic insulators. In the decades that followed, the field has yielded a rich corpus of research findings, with in-depth explorations of magnetoelasticity advancing considerably in recent years. For rate-independent evolution models, studies [7, 8, 13] investigated the static cases relying on energy minimization, while [16, 22] derived rate-independent evolution models by using the energy method. As for the magnetic component, the evolution of magnetization is usually described by the LLG equations [11, 12, 17], and relevant research is relatively comprehensive [2, 6, 21]. In recent years, certain results have been obtained regarding the well-posedness of the coupled system consisting of the LLG equation and the viscoelastic equation [1, 3, 25]. Among them, De Anna-Kortum-Schlömerkemper proved the existence and uniqueness of Struwe-like solutions (smooth except for discrete time singularities) for 2D magneto-viscoelastic fluids via harmonic analysis and energy estimates [3]. Ai-Shen established finite-fractal-dimensional pullback attractors for 2D non-autonomous magneto-viscoelastic flows using the l-trajectories method under the assumption of a spatiotemporally bounded external force [1]. Wang-Yang-Yuan investigated the well-posedness of an undamped incompressible magneto-viscoelastic model, covering a blow-up criterion, global solutions under small perturbations or strains, and weak-strong uniqueness [25]. For the case with boundary conditions, Kalousek-Schlömerkemper proposed the dissipative solution of the magneto-viscoelastic fluid system under general initial values with boundary conditions in [15]. For the numerical analysis of the evolutionary magneto-thermo-viscoelastic model, see [9, 10].

In this paper, we focus on investigating the well-posedness of the evolutionary behavior of the following magneto-viscoelastic system:

$$\begin{cases} \partial_t u + (u \cdot \nabla)u + \nabla p + \nabla \cdot (\nabla d \odot \nabla d - FF^\top) - \nu \Delta u = 0\,, \\ \nabla \cdot u = 0\,, \\ \partial_t F + (u \cdot \nabla)F - \nabla u F = \kappa \Delta F\,, \\ \partial_t d + (u \cdot \nabla)d = \Delta d + |\nabla d|^2 d - d \times \Delta d\,, \\ |d| = 1, \end{cases} \tag{1.1}$$

in $\mathbb{R}^+ \times \mathbb{R}^n$, $n = 2, 3$. The system is proposed to describe magneto-viscoelastic fluids, a type of smart composite material composed of magnetizable particles suspended in a viscoelastic fluid matrix. It captures the coupled behaviors of flow, viscoelastic deformation, and magnetization rotation in such materials, applicable to magnetorheological fluids and magnetoelastic gels. The first equation of (1.1) is the balance of momentum in the Eulerian coordinates, where $u(t, x) \in \mathbb{R}^n$ denotes the velocity field and $p(t, x) \in \mathbb{R}$ is the hydrodynamic pressure. The parameter $\nu \geq 0$ represents the fluid viscosity, which characterizes the viscous resistance of the magneto-viscoelastic material. In practical actuators such as magnetorheological valves and magnetoelastic actuators, the kinematic viscosity of magneto-viscoelastic fluids typically ranges from 10^{-6} to 10^{-3} m^2/s, depending on the base fluid, particle volume fraction, and temperature. The second equation of (1.1) represents the incompressibility property. The third equation of (1.1) is an evolution for $F(t, x) \in \mathbb{R}^{n\times n}$, which represents the deformation gradient with respect to the velocity $u(t, x)$. The parameter $\kappa \geq 0$ here introduces a regularization in the deformation transport. Physically, it corresponds to the microscopic relaxation processes in the material, such as the viscous motion of magnetic domain walls and the elastic relaxation of polymer chains. In real magneto-viscoelastic materials, κ is taken as a small positive value, usually on the order of 10^{-5} to 10^{-3} m^2/s, to ensure the well-posedness of the evolution equation for the deformation gradient. In fact, when $\kappa = 0$ is set, the third equation of (1.1) can be rewritten into Eq (5) in [20]. In this case, although the third equation of (1.1) acts as the evolution equation for the deformation gradient, the setting of $\kappa = 0$ complicates the proof of the existence of

$F(t,x)$, and additional assumptions have to be imposed on $F(t,x)$ to complete the proof. For this reason, we introduce a regularization term [18], where the parameter κ is intended to take a small value. The fourth equation of (1.1) is the LLG equation with $d(t,x) \in \mathbb{S}^{n-1} = \{u(t,x) \in \mathbb{R}^n; |u(t,x)| = 1\}$ standing for the magnetization and satisfying the geometric constraint $|d(t,x)| = 1$. Moreover, since constants are irrelevant for mathematical analysis, here we have set $A = \frac{1}{2}, \mu_0 = \gamma = \lambda = 1$ in [4]. The initial data of (1.1) is imposed on

$$u(0,x) = u_0(x) \in \mathbb{R}^n\,,\ F(0,x) = F_0(x) \in \mathbb{R}^{n\times n}\,,\ d(0,x) = d_0(x) \in \mathbb{R}^n, \tag{1.2}$$

and compatibilities $\nabla \cdot u_0(x) = 0$, $\det F_0(x) = 1$, and $|d_0(x)| = 1$.

The mathematical analysis of this coupled system poses significant challenges due to its strong nonlinearity, geometric constraint $|d(t,x)| = 1$, and the lack of a general maximum principle. Existing well-posedness theories have heavily relied on the presence of dissipation mechanisms: viscosity ($\nu \geq 0$), regularization ($\kappa \geq 0$) and magnetic damping. Viscosity, deformation relaxation, and magnetic damping in the LLG equation together ensure the stability and regularity of the coupled magnetoelastic system. The fluid viscosity ν dissipates kinetic energy, suppresses velocity instabilities, and regularizes the convective nonlinearity. The parameter κ regularizes the deformation gradient transport equation, smoothing steep gradients and stabilizing the microstructural evolution. Magnetic dissipation dampens fast precession of the magnetization, preserves the constraint $|d(t,x)| = 1$, and prevents uncontrolled growth of magnetic energy. Without these dissipative or regularizing effects, the strong nonlinearity and geometric constraint would typically lead to a loss of regularity or even finite-time blowup, making the system mathematically ill-posed. Thus, ν, κ, and magnetic damping are not only physically realistic but also indispensable for the global stability and well-posedness of the model.

For instance, the work of Benešová-Forster-Liu-Schlömerkemper [4] and Kalousek-Kortum-Schlömerkemper [14] established the global existence of weak solutions for the regularized system with $\kappa > 0$. In [4], global-in-time weak solutions were constructed by using the Galerkin method and a fixed point argument. The proof in [4] combined the ideas of Lin-Zhang [19] on the liquid crystal flow and Carbou-Fabrie [6] on the Landau-Lifshitz equation. See also the recent progress in this direction in [14] and [26]. We emphasize that the so-called weak solutions in [4] and [14] are not in the usual sense: the spatial regularity requirement on d is not H^1, but higher (H^3 in [4] and H^2 in [14]). Furthermore, the size of the initial deviation of d from the identity matrix I is required to be small. All of these unsatisfactory assumptions are caused by the essential difficulties of the lack of regularity of the transport equation and the geometric constraint of $|d(t,x)| = 1$, which is extremely hard to approximate in a weaker norm. In fact, it is still highly nontrivial even in the higher regular norm considered in this paper. We remark that in [26], a local well-posedness and blow-up criteria of classical solutions were established for the modified (1.1), i.e., replacing the constraint $|d(t,x)| = 1$ by the usual Ginzburg-Landau approximation. In this paper, instead of Ginzburg-Landau approximation, we consider the incompressible version of model (1.1) with the geometric constraint $|d(t,x)| = 1$. Furthermore, we work in the context of classical solutions, rather than the weak-strong solutions studied in [4] and [14]. First, we prove the local-in-time existence of classical solutions with finite initial energy to the system by the energy method. Furthermore, the global well-posedness of the magneto-viscoelastic system with small initial data is established in the absence of an external magnetic field.

For convenience, we give some notational conventions in the following section. In this paper, the standard L^p-space in $\mathbb{R}^n$ for $1 \le p \le +\infty$ is endowed with the norm

$$\|f\|_{L^p} := \begin{cases} \Big(\int_{\mathbb{R}^n} |f(x)|^p \mathrm{d}x\Big)^{\frac{1}{p}} & \text{for } 1 \le p < \infty\,, \\ \operatorname{ess\,sup}_{x\in\mathbb{R}^n} |f(x)| & \text{for } p = \infty\,. \end{cases}$$

For $p = 2$, we use the following notation to denote the inner product of the Hilbert space L^2:

$$\langle f, g\rangle := \int_{\mathbb{R}^n} fg \mathrm{d}x\,.$$

For any multi-indexes $m = (m_1, m_2, ..., m_n) \in \mathbb{N}^n$, we use

$$\partial^m f = \frac{\partial^{|m|} f}{\partial_{x_1}^{m_1}\partial_{x_2}^{m_2}...\partial_{x_n}^{m_n}}$$

to denote the m-th order derivative operator "∂^m" with $|m| = m_1 + m_2 + ... + m_n$. The notation $m \le \tilde{m}$ means each component of $m \in \mathbb{N}^n$ is not greater than those of $\tilde{m}$. The symbol $m < \tilde{m}$ means $m \le \tilde{m}$ and $|m| < |\tilde{m}|$. What is more, we denote $A \lesssim B$ to represent $A \le CB$ for some constant C. Then we define the Sobolev space $W^{s,p} = W^{s,p}(\mathbb{R}^n)$ by the norm

$$\|f\|_{W^{s,p}} = \Big(\sum_{|m|\le s}\int_{\mathbb{R}^n} |\partial^m f(x)|^p \mathrm{d}x\Big)^{\frac{1}{p}}\,.$$

If $p = 2$, we denote the Hilbert space $H^s = H^s(\mathbb{R}^n) := W^{s,2}$. For brevity, we define the following initial energy before presenting the main results. The energy functional $\mathcal{E}_N(t)$ and energy dissipative rate functional $\mathcal{D}_N(t)$ are denoted by

$$\begin{aligned} \mathcal{E}_N(t) &:= \|u\|_{H^N}^2 + \|F\|_{H^N}^2 + \|\nabla d\|_{H^N}^2\,, \\ \mathcal{D}_N(t) &:= \nu\|\nabla u\|_{H^N}^2 + \kappa\|\nabla F\|_{H^N}^2 + \|\Delta d\|_{H^N}^2\,. \end{aligned}$$

Correspondingly, we also define the local initial energy function $\mathcal{E}_N(0)$ as

$$\mathcal{E}_N(0) := \|u_0\|_{H^N}^2 + \|F_0\|_{H^N}^2 + \|\nabla d_0\|_{H^N}^2\,.$$

2. Energy method

In this part, we will derive the *a priori* estimate of system (1.1). Now we state our main lemma as follows:

Lemma 2.1. *Let $N \ge 3$ be any fixed integer. Assume that $(u, F, d)(t, x)$ is a sufficiently smooth solution to (1.1) on the interval $[0, T]$. Then there is a positive constant C, depending only on ν, N, n, κ, such that*

$$\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_N(t) + \mathcal{D}_N(t) \le C(\mathcal{E}_N(t) + \mathcal{E}_N^{\frac{3}{2}}(t))\mathcal{D}_N^{\frac{1}{2}}(t) \tag{2.1}$$

holds for all $t \in [0, T]$.

Proof of Lemma 2.1. We divide the proof into two steps.
Step 1. L^2-estimates. Taking the inner product for the first equation of (1.1) with u, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|u\|_{L^2}^2\right) + \nu\|\nabla u\|_{L^2}^2 + \langle \nabla d \cdot \Delta d),\ u\rangle + \langle FF^T,\ \nabla u\rangle = 0, \tag{2.2}$$

where we have used the fact that $\nabla \cdot (\nabla d \odot \nabla d) = \frac{1}{2}\nabla|\nabla d|^2 + \nabla d \cdot \Delta d$ and $\nabla \cdot u = 0$. Then, taking the inner product for the third equation of (1.1) with F, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|F\|_{L^2}^2\right) + \langle u \cdot \nabla F, F\rangle - \langle \nabla u F, F\rangle + \kappa\|\nabla F\|_{L^2}^2 = 0. \tag{2.3}$$

Similarly, taking the inner product for the fourth equation of (1.1) with Δd, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|\nabla d\|_{L^2}^2\right) + \|\Delta d\|_{L^2}^2 - \langle (u \cdot \nabla)d, \Delta d\rangle + \langle |\nabla d|^2 d - d \times \Delta d, \Delta d\rangle = 0. \tag{2.4}$$

The cancellations

$$\big\langle \nabla \cdot (\nabla M \odot \nabla M), v\big\rangle - \big\langle (v \cdot \nabla)M, \Delta M\big\rangle = 0$$

and

$$\big\langle \nabla \cdot (FF^{\top}), v\big\rangle + \big\langle \nabla v F, F\big\rangle = 0$$

play an essential role in the derivation of the a *priori* estimates of system (1.1).

Then adding the resultant equalities (2.2)–(2.4) together gives us

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u\|_{L^2}^2 + \|F\|_{L^2}^2 + \|\nabla d\|_{L^2}^2\right) + \nu\|\nabla u\|_{L^2}^2 + \kappa\|\nabla F\|_{L^2}^2 + \|\Delta d\|_{L^2}^2 \lesssim \|\nabla d\|_{L^2}^2\|\Delta d\|_{L^2}. \tag{2.5}$$

Step 2. H^N-estimates. Applying the derivative operator $\partial^m(1 \le |m| \le N)$ to the first equation of (1.1), taking the L^2-inner product for the resultant equation by dot with $\partial^m u$, and integrating by parts over $x \in \mathbb{R}^n$, one has

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial^m u\|_{L^2}^2 + \langle \partial^m(u \cdot \nabla u), \partial^m u\rangle + \langle \nabla \cdot \partial^m(\nabla d \odot \nabla d - FF^T), \partial^m u\rangle + \nu\|\nabla\partial^m u\|_{L^2}^2 = 0.$$

Then direct calculation yields

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial^m u\|_{L^2}^2 + \nu\|\nabla\partial^m u\|_{L^2}^2 + \langle \Delta d \cdot \Delta\partial^m d, \partial^m u\rangle + \langle (\partial^m F)F^T, \nabla\partial^m u\rangle \\
&= -\sum_{0 \neq m' \le m} C_m^{m'}\left[\langle \partial^{m'} u \cdot \nabla\partial^{m-m'} u, \partial^m u\rangle + \langle \nabla\partial^{m'} d \Delta\partial^{m-m'} d, \partial^m u\rangle\right] \\
&\quad - \sum_{0 \neq m' \le m} C_m^{m'}\langle \partial^{m-m'} F \partial^{m'} F^T, \nabla\partial^m u\rangle .
\end{aligned} \tag{2.6}$$

Next, applying the derivative operator $\partial^m(1 \le |m| \le N)$ to the third equation of (1.1), taking the L^2-inner product for the resultant equation by dot with $\partial^m F$, and integrating by parts over $x \in \mathbb{R}^n$, one has

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial^m F\|_{L^2}^2 + \langle \partial^m(u \cdot \nabla F),\ \partial^m F\rangle - \langle \partial^m(\nabla u F),\ \partial^m F\rangle + \kappa\|\nabla\partial^m F\|_{L^2}^2 = 0.$$

Direct calculation yields

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial^m F\|_{L^2}^2+\kappa\|\nabla\partial^m F\|_{L^2}^2-\langle\nabla\partial^m uF,\,\partial^m F\rangle\\&=-\sum_{0\neq m'\leq m}C_m^{m'}\langle\partial^{m'}u\cdot\nabla\partial^{m-m'}F,\,\partial^m F\rangle+\sum_{0\neq m'\leq m}C_m^{m'}\langle\nabla\partial^{m-m'}u\partial^{m'}F,\,\partial^m F\rangle\,.\end{aligned}$$

Finally, applying the derivative operator $\partial^m(1\leq|m|\leq N)$ to the fourth equation of (1.1), taking the L^2-inner product for the resultant equation by dot with $\Delta\partial^m d$, and integrating by parts over $x\in\mathbb{R}^n$, one has

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\partial^m\nabla d\|_{L^2}^2\right)+\|\Delta\partial^m d\|_{L^2}^2-\langle\partial^m[(u\cdot\nabla)d],\Delta\partial^m d\rangle+\langle\partial^m(|\nabla d|^2 d),\Delta\partial^m d\rangle-\langle\partial^m(d\times\Delta d),\Delta\partial^m d\rangle=0.$$

Then

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla\partial^m d\|_{L^2}^2\right)+\|\Delta\partial^m d\|_{L^2}^2-\langle\partial^m u\cdot\nabla d,\Delta\partial^m d\rangle\\&=-\langle\partial^m(|\nabla d|^2)d,\Delta\partial^m d\rangle+\sum_{0\neq m'\leq m}C_m^{m'}\langle\partial^{m-m'}u\cdot\nabla\partial^{m'}d,\Delta\partial^m d\rangle\\&\quad-\sum_{0\neq m'\leq m}C_m^{m'}\partial^{m-m'}(|\nabla d|^2)\partial^{m'}d,\Delta\partial^m d\rangle-\sum_{0\neq m'\leq m}C_m^{m'}\langle\partial^{m'}d\times\Delta\partial^{m-m'}d,\Delta\partial^m d\rangle\,.\end{aligned}\tag{2.7}$$

Then adding the resultant equalities (2.6) and (2.7) together gives us

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\partial^m u\|_{L^2}^2+\|\partial^m F\|_{L^2}^2+\|\nabla\partial^m d\|_{L^2}^2\right)+\nu\|\nabla\partial^m u\|_{L^2}^2+\kappa\|\nabla\partial^m F\|_{L^2}^2+\|\Delta\partial^m d\|_{L^2}^2\\&=\underbrace{-\sum_{0\neq m'\leq m}C_m^{m'}\langle\partial^{m'}u\cdot\nabla\partial^{m-m'}u,\partial^m u\rangle}_{I_1}\underbrace{-\sum_{0\neq m'\leq m}C_m^{m'}\langle\nabla\partial^{m'}d\Delta\partial^{m-m'}d,\partial^m u\rangle}_{I_2}\\&\quad\underbrace{-\sum_{0\neq m'\leq m}C_m^{m'}\langle\partial^{m-m'}F\partial^{m'}F^T,\nabla\partial^m u\rangle}_{I_3}\underbrace{-\sum_{0\neq m'\leq m}C_m^{m'}\langle\partial^{m'}u\cdot\nabla\partial^{m-m'}F,\,\partial^m F\rangle}_{I_4}\\&\quad\underbrace{+\sum_{0\neq m'\leq m}C_m^{m'}\langle\nabla\partial^{m-m'}u\partial^{m'}F,\,\partial^m F\rangle}_{I_5}+\underbrace{\sum_{0\neq m'\leq m}C_m^{m'}\langle\partial^{m-m'}u\cdot\nabla\partial^{m'}d,\Delta\partial^m d\rangle}_{I_6}\\&\quad\underbrace{-\langle\partial^m(|\nabla d|^2)d,\Delta\partial^m d\rangle}_{I_7}\underbrace{-\sum_{0\neq m'\leq m}C_m^{m'}\partial^{m-m'}(|\nabla d|^2)\partial^{m'}d,\Delta\partial^m d\rangle}_{I_8}\\&\quad\underbrace{-\sum_{0\neq m'\leq m}C_m^{m'}\langle\partial^{m'}d\times\Delta\partial^{m-m'}d,\Delta\partial^m d\rangle}_{I_9}\,.\end{aligned}\tag{2.8}$$

Now we estimate the above terms one by one. For $1 \leq |m| \leq N(N \geq 3)$, I_1 can be estimated as

$$\begin{aligned}
I_1 &= -\sum_{0\neq m'\leq m} C_m^{m'}\langle \partial^{m'}u\cdot\nabla\partial^{m-m'}u,\ \partial^m u\rangle \\
&\leq C\sum_{0\neq m'\leq m} \|\partial^{m'}u\|_{L^4}\|\nabla\partial^{m-m'}u\|_{L^4}\|\partial^m u\|_{L^2} \\
&\leq C\sum_{0\neq m'\leq m} \|\partial^{m'}u\|_{H^1}\|\nabla\partial^{m-m'}u\|_{H^1}\|\partial^m u\|_{L^2} \\
&\leq C\|u\|_{H^N}^2\|\nabla u\|_{H^N} \\
&\leq C\mathcal{E}_N(t)\mathcal{D}_N^{\frac{1}{2}}(t)\,,
\end{aligned}\tag{2.9}$$

where $C = C(\nu, N, n, \kappa)$. Here, we have used the Hölder inequality and the Sobolev embedding $H^1(\mathbb{R}^n) \hookrightarrow L^4(\mathbb{R}^n)$. Similarly, I_2 and I_3 can be estimated as

$$\begin{aligned}
I_2 &= -\sum_{0\neq m'\leq m} C_m^{m'}\langle \nabla\partial^{m'}d\cdot\Delta\partial^{m-m'}d, \partial^m u\rangle \\
&\leq C\sum_{|m'|=1} \|\nabla\partial^{m'}d\|_{L^\infty}\|\Delta\partial^{m-m'}d\|_{L^2}\|\partial^m u\|_{L^2} \\
&\quad + C\sum_{|m'|=m} \|\nabla\partial^{m}d\|_{L^2}\|\Delta d\|_{L^\infty}\|\partial^m u\|_{L^2} \\
&\quad + C\sum_{2\leq m'<|m|} \|\nabla\partial^{m'}d\|_{L^4}\|\Delta\partial^{m-m'}d\|_{L^4}\|\partial^m u\|_{L^2} \\
&\leq C\|\nabla d\|_{H^N}\|\Delta d\|_{H^N}\|u\|_{H^N} \\
&\leq C\mathcal{E}_N(t)\mathcal{D}_N^{\frac{1}{2}}(t)\,,
\end{aligned}\tag{2.10}$$

and

$$\begin{aligned}
I_3 &= -\sum_{0\neq m'\leq m} C_m^{m'}\langle \partial^{m-m'}F\partial^{m'}F^T, \nabla\partial^m u\rangle \\
&\leq C\|F\|_{L^\infty}\|\partial^m F^\top\|_{L^2}\|\nabla\partial^m u\|_{L^2} + C\sum_{0\neq m'<m}\|\partial^{m-m'}F\|_{L^4}\|\partial^m F^\top\|_{L^4}\|\nabla\partial^m u\|_{L^2} \\
&\leq C\|F\|_{H^N}^2\|\nabla u\|_{H^N} \\
&\leq C\mathcal{E}_N(t)\mathcal{D}_N^{\frac{1}{2}}(t)\,.
\end{aligned}\tag{2.11}$$

For the terms I_4, I_5, from the Hölder inequality, the Sobolev embedding $H^1(\mathbb{R}^n) \hookrightarrow L^4(\mathbb{R}^n)$, and $H^2(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$ $(n = 2, 3)$, we deduce that

$$I_4 = -\sum_{0\neq m'\leq m} C_m^{m'}\langle \partial^{m'}u\cdot\nabla\partial^{m-m'}F,\ \partial^m F\rangle \tag{2.12}$$

$$\begin{aligned}
&= C\sum_{|m'|=1} \|\partial^{m'}u\|_{L^\infty}\|\nabla\partial^{m-m'}F\|_{L^2}\|\partial^m F\|_{L^2} + C\sum_{2\leq|m'|\leq|m|} \|\partial^{m'}u\|_{L^4}\|\nabla\partial^{m-m'}F\|_{L^4}\|\partial^m F\|_{L^2} \\
&\leq C\|\nabla u\|_{H^N}\|F\|_{H^N}^2 \\
&\leq C\mathcal{E}_N(t)\mathcal{D}_N^{\frac{1}{2}}(t)\,,
\end{aligned}$$

and

$$
\begin{aligned}
I_5 &= \sum_{0\neq m'\leq m} C_m^{m'}\langle \nabla\partial^{m-m'}u\partial^{m'}F, \partial^m F\rangle\\
&\leq C\sum_{0\neq m'\leq m}\|\nabla\partial^{m-m'}u\|_{L^4}\|\partial^{m'}F\|_{L^4}\|\partial^m F\|_{L^2} + C\|\nabla u\|_{L^\infty}\|\partial^m F\|_{L^2}^2 \qquad (2.13)\\
&\leq C\|\nabla u\|_{H^N}\|F\|_{H^N}^2 \leq C\mathcal{E}_N(t)\mathcal{D}_N^{\frac{1}{2}}(t)\,.
\end{aligned}
$$

Similarly, by the Hölder inequality and the Sobolev embedding theorem, we have

$$
\begin{aligned}
I_6 &= \sum_{0\neq m'\leq m} C_m^{m'}\langle \partial^{m-m'}u\cdot\nabla\partial^{m'}d, \Delta\partial^m d\rangle\\
&\leq C\|u\|_{L^\infty}\|\nabla\partial^m d\|_{L^2}\|\Delta\partial^m d\|_{L^2} + C\sum_{0\neq m'<m}\|\partial^{m-m'}u\|_{L^4}\|\nabla\partial^{m'}d\|_{L^4}\|\Delta\partial^m d\|_{L^2} \qquad (2.14)\\
&\leq C\|u\|_{H^N}\|\nabla d\|_{H^N}\|\Delta d\|_{H^N}\\
&\leq C\mathcal{E}_N(t)\mathcal{D}_N^{\frac{1}{2}}(t)\,,
\end{aligned}
$$

$$
I_7 = -\langle\partial^m(|\nabla d|^2)d, \Delta\partial^m d\rangle \leq C\|\nabla d\|_{H^N}^2\|\Delta d\|_{H^N} \leq C\mathcal{E}_N(t)\mathcal{D}_N^{\frac{1}{2}}(t)\,, \qquad (2.15)
$$

$$
\begin{aligned}
I_8 &= -\sum_{0\neq m'\leq m} C_m^{m'}\partial^{m-m'}(|\nabla d|^2)\partial^{m'}d, \Delta\partial^m d\rangle \qquad (2.16)\\
&\leq C\|\nabla d\|_{H^N}^3\|\Delta d\|_{H^N} \leq C\mathcal{E}_N^{\frac{3}{2}}(t)\mathcal{D}_N^{\frac{1}{2}}(t)\,,
\end{aligned}
$$

and

$$
\begin{aligned}
I_9 &= -\sum_{0\neq m'\leq m} C_m^{m'}\langle\partial^{m'}d\times\Delta\partial^{m-m'}d, \Delta\partial^m d\rangle\\
&\leq C\sum_{|m'|=1}\|\partial^{m'}d\|_{L^\infty}\|\Delta\partial^{m-m'}d\|_{L^2}\|\Delta\partial^m d\|_{L^2} + C\sum_{\substack{m'\leq m,\\ |m'|\geq 2}}\|\partial^{m'}d\|_{L^4}\|\Delta\partial^{m-m'}d\|_{L^4}\|\Delta\partial^m d\|_{L^2} \qquad (2.17)\\
&\leq C\|\nabla d\|_{H^N}^2\|\Delta d\|_{H^N} \leq C\mathcal{E}_N(t)\mathcal{D}_N^{\frac{1}{2}}(t)\,.
\end{aligned}
$$

Substituting (2.9)–(2.17) into (2.8), we have

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_N(t) + \mathcal{D}_N(t) \leq C(\mathcal{E}_N(t) + \mathcal{E}_N^{\frac{3}{2}}(t))\mathcal{D}_N^{\frac{1}{2}}(t)\,. \qquad (2.18)
$$

Then the proof of Lemma 2.1 is finished. □

3. Results

In this article, first, we prove the local-in-time existence of a classical solution for the system with finite initial energy via the energy method. Furthermore, global well-posedness is established for the magneto-viscoelastic system with small initial data in the absence of an external magnetic field. We have the following main theorems.

Theorem 3.1 (*Local well-posedness*)**.** *Let the integer $N \geq 3$ and $n = 2, 3$. Assume that the initial data $(u_0, F_0, \nabla d_0)(x)$ in (1.2) satisfies $\mathcal{E}_N(0) < \infty$. Then there is a constant $T > 0$, depending only on $\mathcal{E}_N(0)$, such that system (1.1) admits a unique solution $(u, F, d)(t, x)$ with*

$$\begin{aligned} &u(t,x), F(t,x) \in L^\infty(0,T; H^N(\mathbb{R}^n)), \nabla u, \nabla F \in L^2(0,T; H^N(\mathbb{R}^{n\times n}))\,,\\ &\nabla d(t,x) \in L^\infty(0,T; H^N(\mathbb{R}^{n\times n})), \Delta d(t,x) \in L^2(0,T; H^N(\mathbb{R}^n))\,. \end{aligned}$$

Moreover, the solution $(u, F, d)(t, x)$ enjoys the energy bound

$$\sup_{t\in[0,T]} \mathcal{E}_N(t) + \int_0^T \mathcal{D}_N(t)\mathrm{d}t \leq C\,,$$

for some constant C depending only on N, T, and $\mathcal{E}_N(0)$.

Theorem 3.2 (Global well-posedness)**.** *Let the integer $N \geq 3$ and $n = 2, 3$. There is a small constant $\epsilon_0 > 0$ such that if the initial data $(u_0, F_0, \nabla d_0)(x)$ satisfies that $\mathcal{E}_N(0) \leq \epsilon_0$, then system (1.1) admits a unique global solution $(u, F, d)(t, x)$ with*

$$\begin{aligned} &u(t,x) \in L^\infty(0,+\infty; H^N(\mathbb{R}^n)) \cap L^2(0,+\infty; H^{N+1}(\mathbb{R}^{n\times n}))\,,\\ &F(t,x) \in L^\infty(0,+\infty; H^N(\mathbb{R}^n)) \cap L^2(0,+\infty; H^{N+1}(\mathbb{R}^{n\times n}))\,,\\ &\nabla d(t,x) \in L^\infty(0,+\infty; H^N(\mathbb{R}^{n\times n})), \Delta d(t,x) \in L^2(0,+\infty; H^N(\mathbb{R}^n))\,. \end{aligned}$$

Moreover, there is a constant $C > 0$, depending on the initial constants ν, κ such that

$$\sup_{t\geq 0} \mathcal{E}_N(t) + \int_0^{+\infty} \mathcal{D}_N(t)\mathrm{d}t \leq C\epsilon_0\,.$$

4. Well-posedness of the convective Landau-Lifshitz-Gilbert equation

In this section, we will justify the well-posedness of the following system with a given velocity field $u(t, x) \in \mathbb{R}^n$:

$$\begin{cases} \partial_t d + (u\cdot\nabla)d = \Delta d + |\nabla d|^2 d - d\times\Delta d\,,\\ d(t,x)|_{t=0} = d_0(x) \in \mathbb{S}^{n-1}\,, \end{cases} \tag{4.1}$$

with the geometric constraint $|d(t, x)| = 1$.

And we can prove that $|d(t, x)| = 1$ holds when $|d_0(x)| = 1$. That is the following lemma.

Lemma 4.1. *Assume that $(u, d)(t, x)$ is a classical solution to the LLG system (4.1) satisfying $u(t, x) \in L^\infty(0, T; H^N(\mathbb{R}^n)) \cap L^2(0, T; H^{N+1}(\mathbb{R}^n))$ and $\nabla d(t, x) \in L^\infty(0, T; H^N(\mathbb{R}^{n\times n}))$, $|d|_{L^\infty([0,T]\times\mathbb{R}^n)} < \infty$, for some $T \in (0, \infty)$, where $N \geq 3$. What is more, if $d(t, x)$ satisfies the initial condition $|d_0(x)| = 1$, then $|d(t, x)| = 1$ holds for all $t \geq 0$ and $x \in \mathbb{R}^n$.*

Proof of Lemma 4.1. Multiplying both sides of Eq (4.1) by dot with d yields the result

$$\partial_t\varphi + u\cdot\nabla\varphi - \Delta\varphi = 2|\nabla d|^2\varphi,$$

where $\varphi = |d|^2 - 1$. What is more, φ solves the Cauchy problem

$$\begin{cases} \partial_t\varphi + (u\cdot\nabla)\varphi - \Delta\varphi = 2|\nabla d|^2\varphi\,,\\ \varphi(0,x) = |d_0|^2 - 1 = 0. \end{cases} \tag{4.2}$$

Taking the L^2-inner product of (4.2) with φ and integrating over $\mathbb{R}^n$ yields that

$$\frac{1}{2}\frac{d}{dt}\|\varphi\|_{L^2}^2 + \|\nabla\varphi\|_{L^2}^2 \le 2\|\nabla d\|_{L^\infty}\|\varphi\|_{L^2}^2.$$

Thereby one has

$$\frac{d}{dt}\|\varphi\|_{L^2}^2 \le C\|\varphi\|_{L^2}^2.$$

It immediately follows from Gronwall's inequality that

$$0 \le \|\varphi(t)\|_{L^2}^2 \le \|\varphi(0)\|_{L^2}^2 e^{Ct} = 0,$$

for all $t \in [0, T]$. Thus we have $\varphi(t, x) = 0$ for all $t \in [0, T]$. Then the proof of Lemma 4.1 is finished. □

By employing the mollifier approximate method, we can easily achieve the following lemma.

Lemma 4.2. *For $N \ge 3$ and $T_0 > 0$, let vector fields $u(t, x) \in \mathbb{R}^n$ and $d_0(x) \in \mathbb{S}^{n-1} \subset \mathbb{R}^n$ satisfy $\nabla d_0(x) \in H^N(\mathbb{R}^{n\times n})$ and $u(t, x) \in C(0, T_0; H^N) \cap L^2(0, T_0; H^{N+1})$. Then there is a $T \in (0, T_0]$, depending only on $d_0(x), u(t, x)$, such that the Cauchy problem* (4.1) *has a unique classical solution $d(t, x) \in \mathbb{S}^{n-1}$ satisfying $\nabla d(t, x) \in C(0, T; H^N(\mathbb{R}^{n\times n}))$ and $\Delta d(t, x) \in L^2(0, T; H^N(\mathbb{R}^n))$. Moreover, there exists a positive $\tilde{C}$, depending only on $d_0(x), u(t, x)$ and T_0, such that the solution $d(t, x)$ satisfies the following bound:*

$$\|\nabla d\|_{L^\infty(0,T;H^N)}^2 + \|\Delta d\|_{L^2(0,T;H^N)}^2 \le \tilde{C}\,. \tag{4.3}$$

Proof of Lemma 4.2. First, we construct the approximate system by the mollifier approximate method. We define a mollifier $\mathcal{J}_\epsilon f := \mathcal{F}^{-1}(\mathbf{1}_{|\xi|\le\frac{1}{\epsilon}}\mathcal{F}(f))$, where $\mathcal{F}$ is the Fourier transform operator. The mollifier $\mathcal{J}_\epsilon$ satisfies $\mathcal{J}_\epsilon^2 = \mathcal{J}_\epsilon$. Thus the approximate system of (4.1) is

$$\begin{cases} \partial_t d^\varepsilon = -\mathcal{J}_\epsilon((u\cdot\nabla)\mathcal{J}_\epsilon d^\varepsilon) + \Delta\mathcal{J}_\epsilon d^\varepsilon + \mathcal{J}_\epsilon[|\nabla\mathcal{J}_\epsilon d^\varepsilon|^2 d^\varepsilon] - \mathcal{J}_\epsilon(\mathcal{J}_\epsilon d^\varepsilon \times \Delta\mathcal{J}_\epsilon d^\varepsilon)\,, \\ d^\varepsilon(t, x)|_{t=0} = \mathcal{J}_\epsilon d_0(x)\,. \end{cases} \tag{4.4}$$

From standard ODE theoretical results, we deduce the following: for any given $\epsilon > 0$, there exists a maximal time horizon $T_\varepsilon \in (0, T_0]$ where the approximate system (4.4) possesses a unique solution $d^\varepsilon(t, x)$ with the regularity $d^\varepsilon(t, x) \in L^\infty([0, T_\epsilon) \times \mathbb{R}^n)$ and $\nabla d^\varepsilon(t, x) \in C(0, T_\varepsilon; H^N(\mathbb{R}^{n\times n}))$. As $\mathcal{J}_\epsilon$ is idempotent ($\mathcal{J}_\epsilon^2 = \mathcal{J}_\epsilon$), we further note that d^ε is likewise a solution to

$$\begin{cases} \partial_t d^\varepsilon = -\mathcal{J}_\epsilon((u\cdot\nabla)d^\varepsilon) + \Delta d^\varepsilon + \mathcal{J}_\epsilon(|\nabla d^\varepsilon|^2 d^\varepsilon) - \mathcal{J}_\epsilon(d^\varepsilon \times \Delta d^\varepsilon)\,, \\ d^\varepsilon(t, x)|_{t=0} = \mathcal{J}_\epsilon d_0(x)\,. \end{cases} \tag{4.5}$$

Next we derive the uniform energy bound of the approximate system on a uniform time interval $[0, T]$. We calculate the L^2-estimate of the approximate system (4.5). Multiplying by Δd^ε in the first equation of (4.5) and integrating by parts over $x \in \mathbb{R}^n$, we have

$$\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\|\nabla d^\varepsilon\|_{L^2}^2 + \|\Delta d^\varepsilon\|_{L^2}^2 = \langle (u\cdot\nabla)d^\varepsilon, \Delta d^\varepsilon\rangle - \langle |\nabla d^\varepsilon|^2 d^\varepsilon, \Delta d^\varepsilon\rangle\,. \tag{4.6}$$

By the Hölder inequality and Sobolev embedding theory, we estimate that

$$\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\|\nabla d^\varepsilon\|_{L^2}^2 + \|\Delta d^\varepsilon\|_{L^2}^2 \le C\Big(\|u\|_{H^2}\|\nabla d^\varepsilon\|_{L^2} + \|d^\varepsilon\|_{L^\infty}\|\nabla d^\varepsilon\|_{H^1}^2\Big)\|\Delta d^\varepsilon\|_{L^2}\,. \tag{4.7}$$

Next, we establish bounds on the higher-order energy of the approximate system (4.5). For every multi-index $m \in \mathbb{N}^n$ with $1 \leq |m| \leq N$, we first apply the m-th order partial derivative operator ∂^m to the first equation of system (4.5), then compute the L^2-inner product of the resulting equation with $\Delta\partial^m d^\varepsilon$, and subsequently carry out integration by parts over the spatial domain $x \in \mathbb{R}^n$. This procedure gives

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\partial^m d^\varepsilon\|_{L^2}^2 + \|\Delta\partial^m d^\varepsilon\|_{L^2}^2 = \langle\partial^m((u\cdot\nabla)d^\varepsilon), \Delta\partial^m d^\varepsilon\rangle - \langle\partial^m(|\nabla d^\varepsilon|^2 d^\varepsilon), \Delta\partial^m d^\varepsilon\rangle \\ + \langle\partial^m(d^\varepsilon \times \Delta d^\varepsilon), \Delta\partial^m d^\varepsilon\rangle\,.\end{aligned} \tag{4.8}$$

From the Hölder inequality, the Sobolev embedding theory, and Young's inequality, one can estimate that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla d^\varepsilon\|_{H^N}^2 + \|\Delta d^\varepsilon\|_{H^N}^2 \leq C(\|u\|_{H^N}^2 + \|\nabla d^\varepsilon\|_{H^N}^4 + \|d^\varepsilon\|_{L^\infty}^4 + \|\nabla d^\varepsilon\|_{H^N}^2)\|\nabla d^\varepsilon\|_{H^N}^2\,. \tag{4.9}$$

We find that the norm $\|d^\varepsilon\|_{L^\infty}$ in (4.9) is not controlled yet. To deal with it, we divide $\|d^\varepsilon\|_{L^\infty}$ into two parts, $\|d^\varepsilon - \mathcal{J}_\epsilon d_0\|_{L^\infty}$ and $\|\mathcal{J}_\epsilon d_0\|_{L^\infty}$. We estimate as follows:

$$\begin{aligned}\|d^\varepsilon\|_{L^\infty} &\leq \|d^\varepsilon - \mathcal{J}_\epsilon d_0\|_{L^\infty} + \|\mathcal{J}_\epsilon d_0\|_{L^\infty} \\ &\leq C\|d^\varepsilon - \mathcal{J}_\epsilon d_0\|_{H^2} + 1 \\ &= C\|d^\varepsilon - \mathcal{J}_\epsilon d_0\|_{L^2} + C\|\nabla d^\varepsilon - \mathcal{J}_\epsilon d_0\|_{H^1} + 1 \\ &= C\|d^\varepsilon - \mathcal{J}_\epsilon d_0\|_{L^2} + C\|\nabla d^\varepsilon\|_{H^1} + C\|\mathcal{J}_\epsilon d_0\|_{H^1} + 1 \\ &\leq C\|d^\varepsilon - \mathcal{J}_\epsilon d_0\|_{L^2} + C\|\nabla d^\varepsilon\|_{H^s} + C\|d_0\|_{H^s} + 1\,,\end{aligned} \tag{4.10}$$

where the relation $\|\mathcal{J}_\epsilon d_0\|_{L^\infty} = |d_0| = 1$ and the Sobolev embedding $H^2(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$ are utilized. It remains to control the norm $\|d^\varepsilon - \mathcal{J}_\epsilon d_0\|_{L^2}$, which vanishes at $t = 0$. To be more precise, from the LLG Eq (4.1), we deduce that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|d^\varepsilon - \mathcal{J}_\epsilon d_0\|_{L^2}^2 = \langle\partial_t d^\varepsilon, d^\varepsilon - \mathcal{J}_\epsilon d_0\rangle \\ &\quad= \langle-\mathcal{J}_\epsilon(u\cdot\nabla d^\varepsilon) + \Delta d^\varepsilon + H_{ext} + \mathcal{J}_\epsilon[|\nabla d^\varepsilon|^2 d^\varepsilon] - \mathcal{J}_\epsilon(d^\varepsilon \times \Delta d^\varepsilon), d^\varepsilon - \mathcal{J}_\epsilon d_0\rangle \\ &\quad\leq C(1 + \|u\|_{H^s} + \|\nabla d^\varepsilon\|_{H^s} + \|d^\varepsilon - \mathcal{J}_\epsilon d_0\|_{L^2})\|\nabla d^\varepsilon\|_{H^s}\|d^\varepsilon - \mathcal{J}_\epsilon d_0\|_{L^2} \\ &\qquad+ C\|d^\varepsilon - \mathcal{J}_\epsilon d_0\|_{L^2}^2(1 + \|d^\varepsilon - \mathcal{J}_\epsilon d_0\|_{L^2})\,.\end{aligned} \tag{4.11}$$

By inequality (4.10), the higher-order estimate (4.9) can be written as

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla d^\varepsilon\|_{H^s}^2 + \|\Delta d^\varepsilon\|_{H^s}^2 \leq +C(1 + \|u\|_{H^s}^2 + \|\nabla d^\varepsilon\|_{H^s}^4 + \|d^\varepsilon - \mathcal{J}_\epsilon d_0\|_{L^2}^4 + \|\nabla d^\varepsilon\|_{H^s}^2)\|\nabla d^\varepsilon\|_{H^s}^2\,. \tag{4.12}$$

Consequently, combining (4.11) and (4.12), we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla d^\varepsilon\|_{H^N}^2 + \|d^\varepsilon - \mathcal{J}_\epsilon d_0\|_{L^2}) + \|\Delta d^\varepsilon\|_{H^N}^2 \\ &\leq C(1 + \|u\|_{H^N}^2 + \|\nabla d^\varepsilon\|_{H^N}^4 + \|d^\varepsilon - \mathcal{J}_\epsilon d_0\|_{L^2}^4 + \|\nabla d^\varepsilon\|_{H^N}^2)\|\nabla d^\varepsilon\|_{H^N}^2 \\ &\quad+ C(1 + \|u\|_{H^N} + \|\nabla d^\varepsilon\|_{H^N} + \|d^\varepsilon - \mathcal{J}_\epsilon d_0\|_{L^2})\|\nabla d^\varepsilon\|_{H^N}\|d^\varepsilon - \mathcal{J}_\epsilon d_0\|_{L^2} \\ &\quad+ C\|d^\varepsilon - \mathcal{J}_\epsilon d_0\|_{L^2}^2(1 + \|d^\varepsilon - \mathcal{J}_\epsilon d_0\|_{L^2})\,.\end{aligned} \tag{4.13}$$

Then we define the approximate energy functional

$$E_\varepsilon(t) = \|\nabla d^\varepsilon\|_{H^N}^2 + \|d^\varepsilon - \mathcal{J}_\epsilon d_0\|_{L^2}^2,$$

and the approximate energy dissipative rate functional

$$D_\varepsilon(t) = \|\Delta d^\varepsilon\|_{H^N}^2 .$$

Then inequality (4.13) can be rewritten as

$$\tfrac{\mathrm{d}}{\mathrm{d}t}E_\varepsilon(t) + D_\varepsilon(t) \le C(1 + \|u\|_{H^N}^2)(1 + E_\varepsilon(t))^3 \tag{4.14}$$

for all $t \in [0, T_\varepsilon)$.

Notice that

$$E_\varepsilon(0) = \|\nabla d^\varepsilon\|_{H^N}^2 = \|\nabla \mathcal{J}_\epsilon d_0\|_{H^N}^2 \le \|\nabla d_0\|_{H^N}^2 := E_0 . \tag{4.15}$$

We now define

$$T_\varepsilon^1 = \sup\Big\{\tau \in [0, T_\varepsilon);\ \sup_{t\in[0,\tau]} E_\varepsilon(t) \le 2E_0\Big\} \ge 0 . \tag{4.16}$$

By the continuity of $E_\varepsilon(t)$ on $[0, T_\varepsilon)$, we know that $T_\varepsilon^1 > 0$. Then inequality (4.14) implies that for all $t \in [0, T_\varepsilon^1)$,

$$\tfrac{\mathrm{d}}{\mathrm{d}t}E_\varepsilon(t) + D_\varepsilon(t) \le \mathcal{H}(t)(1 + E_\varepsilon(t)) , \tag{4.17}$$

where $\mathcal{H}(t) = C(1 + 2E_0)^2(1 + \|u(t)\|_{H^N}^2) \in L^\infty(0, T_0)$ by the conditions of Lemma 4.2. Then, by the Grönwall inequality, we have

$$E_\varepsilon(t) \le (E_0 + \|\mathcal{H}\|_{L^\infty(0,T_0)} t) \exp(\|\mathcal{H}\|_{L^\infty(0,T_0)} t) := \mathcal{G}(t) . \tag{4.18}$$

Since the function $\mathcal{G}(t)$ is independent of ε, increasing on $[0, T_\varepsilon^1]$, and satisfies $\mathcal{G}(0) = E_0$, we deduce the existence of a universal time $T > 0$ (also independent of ε) such that $\mathcal{G}(t) \le 2E_0$ for all $t \in [0, T]$. By the definition of T_ε^1, we immediately have $T_\varepsilon^1 \ge T > 0$. Consequently, for all $\varepsilon > 0$ and $t \in [0, T]$, we obtain the uniform energy bound $E_\varepsilon(t) \le 2E_0$ and

$$E_\varepsilon(t) + \int_0^t D_\varepsilon(\tau)\mathrm{d}\tau \le c_0 T \tag{4.19}$$

holds for all $t \in [0, T]$. In the final step, we take the limit from the approximate system to obtain the solutions to the problem (4.1). Since

$$\begin{aligned} \|d^\varepsilon\|_{L^\infty([0,T]\times\mathbb{R}^n)} &\le C\|d^\varepsilon - \mathcal{J}_\epsilon d_0\|_{L^2} + C\|\nabla d^\varepsilon\|_{H^N} + C\|\nabla d_0\|_{H^N} + 1 \\ &\le C E_\varepsilon^{\frac{1}{2}}(t) + C\|\nabla d_0\|_{H^N} + 1 \le C\sqrt{c_0 T} + C\|\nabla d_0\|_{H^N} + 1 , \end{aligned} \tag{4.20}$$

consequently, the uniform bounds (4.19) and (4.20) imply the existence of a limiting vector field $d(t, x) \in L^\infty([0, T] \times \mathbb{R}^n)$ with the regularity

$$\nabla d(t, x) \in C(0, T; H^N(\mathbb{R}^{n\times n})), \quad \Delta d(t, x) \in L^2(0, T; H^N(\mathbb{R}^n)),$$

such that the following weak (and weak-$\star$) convergences hold as $\epsilon \to 0$:

$$\begin{aligned} &d^\epsilon \rightharpoonup^\star d \ \text{ in } \ L^\infty([0, T] \times \mathbb{R}^n), \\ &\nabla d^\epsilon \rightharpoonup^\star \nabla d \ \text{ in } \ L^\infty(0, T; H^N(\mathbb{R}^{n\times n})), \\ &\Delta d^\epsilon \rightharpoonup \Delta d \ \text{ in } \ L^2(0, T; H^N(\mathbb{R}^n)). \end{aligned}$$

Passing to the limit $\varepsilon \to 0$ in the approximate system (4.5) then shows that the limit function d satisfies the limiting equation

$$\partial_t d + (u\cdot\nabla)d = \Delta d + |\nabla d|^2 d - d\times\Delta d,$$

which is precisely the first equation of (4.1). Moreover, Lemma 4.1 guarantees that the constraint $|d(t,x)| = 1$ is preserved in the limit, i.e., $d(t,x)\in\mathbb{S}^{n-1}$ for all $(t,x)\in[0,T]\times\mathbb{R}^n$.

Finally, the uniform boundedness and equicontinuity of the energy functional $E_\varepsilon(t)$ on $[0,T]$, provided by (4.14) and (4.19), allow us to apply the Arzelà–Ascoli theorem. We conclude that there is a subsequence ϵ_j, such that d^{ϵ_j} converges smoothly to a limit $\nabla d\in H^N$. By taking $\epsilon_j\to 0$ in (4.5), we can verify that d is a smooth solution to system (4.5). This yields the stronger regularity $\nabla d(t,x) \in C(0,T;H^N(\mathbb{R}^{n\times n}))$ and $\Delta d(t,x) \in L^2(0,T;H^N(\mathbb{R}^n))$, thereby completing the proof of Lemma 4.2. □

5. Local existence: Proof of Theorem 3.1

In this section, we prove the local well-posedness of the evolutionary model for the magneto-viscoelastic system (1.1) with large initial data. The key point is to justify the positive lower bound of T^*_{s+1} and the uniform energy bounds of the iterative approximate system (5.1), which will be shown in Lemma 5.1. Finally, by the compactness arguments and the geometric constraint $|d(t,x)| = 1$, we can reach our goal.

The iterative approximate system is constructed as follows:

$$\begin{cases} \partial_t u^{s+1} + (u^s\cdot\nabla)u^{s+1} + \nabla p^{s+1} + \nabla\cdot(\nabla d^s\odot\nabla d^s - F^s(F^s)^\top) - \nu\Delta u^{s+1} = 0\,,\\ \nabla\cdot u^{s+1} = 0\,,\\ \partial_t F^{s+1} + (u^s\cdot\nabla)F^{s+1} - \nabla u^s F^s = \kappa\Delta F^{s+1}\,,\\ \partial_t d^{s+1} + (u^s\cdot\nabla)d^{s+1} = \Delta d^{s+1} + |\nabla d^{s+1}|^2 d^{s+1} - d^{s+1}\times\Delta d^{s+1}\,,\\ |d^{s+1}| = 1\,. \end{cases} \tag{5.1}$$

The iterative starts from $n = 0$ with

$$(u^0(t,x), F^0(t,x), d^0(t,x)) = (u_0(x), F_0(x), d_0(x))\,,$$

which holds for all $s\geq 0$.

Now we start the existence result of the iterative approximate system (5.1) as follows.

Lemma 5.1. *Suppose that $N\geq 3$, and the initial data $(u_0,F_0,d_0)(x)\in\mathbb{R}^n\times\mathbb{R}^{n\times n}\times\mathbb{S}^{n-1}$ satisfies $u_0(x)\in H^N(\mathbb{R}^n), F_0(x)\in H^N(\mathbb{R}^{n\times n}), \nabla d_0(x)\in H^N(\mathbb{R}^{n\times n})$. Then there is a maximal number $T^*_{s+1} > 0$ such that system* (5.1) *admits a unique solution $(u^{s+1},F^{s+1},d^{s+1})(t,x)$ satisfying $u^{s+1}(t,x)\in C(0,T^*_{s+1};H^N(\mathbb{R}^n))\cap L^2(0,T^*_{s+1};H^{N+1}(\mathbb{R}^n))$ and $F^{s+1}(t,x),\nabla d^{s+1}(t,x)\in C(0,T^*_{s+1};H^N(\mathbb{R}^{n\times n}))$.*

Proof of Lemma 5.1. For the case $s+1$, the vectors $u^s, F^s, \nabla d^s$ are the known functions. That is, the velocity equation u^{s+1} is a linear Stokes type system:

$$\begin{cases} \partial_t u^{s+1} + (u^s\cdot\nabla)u^{s+1} + \nabla p^{s+1} + \nabla\cdot(\nabla d^s\odot\nabla d^s - F^s(F^s)^\top) - \nu\Delta u^{s+1} = 0\,,\\ \nabla\cdot u^{s+1} = 0\,,\\ u^{s+1}(t,x)|_{t=0} = u_0(x)\in\mathbb{R}^n\,, \end{cases} \tag{5.2}$$

which admits a unique solution $u^{s+1}(t,x) \in C(0,\hat{T}_{s+1};H^N) \cap L^2(0,\hat{T}_{s+1};H^{N+1})$ on the maximal time interval $[0,\hat{T}_{s+1})$. Moreover, the matrix-valued function F^{s+1} obeys a linear ODE system

$$\begin{cases} \partial_t F^{s+1} + u^s \cdot \nabla F^s = \nabla u^s F^s\,, \\ F^{s+1}|_{t=0} = F_0 \in \mathbb{R}^{n\times n}\,, \end{cases} \tag{5.3}$$

in which the spatial variables can be regarded as the parameters. Thus the evolution of F^{s+1} admits a unique solution on the maximal interval $[0,\overline{T}_{s+1})$. By the regularities of u^s, F^s, and F_0, one easily derives $F^{s+1} \in C(0,\overline{T}_{s+1};H^N)$. The orientation equation of $d^{s+1}(t,x)$ is given bulk velocity $u^s(t,x)$:

$$\begin{cases} \partial_t d^{s+1} + (u^s \cdot \nabla) d^{s+1} = \Delta d^{s+1} + |\nabla d^{s+1}|^2 d^{s+1} - d^{s+1} \times \Delta d^{s+1}\,, \\ (u^{s+1}, d^{s+1})(t,x)|_{t=0} = (u_0(x), d_0(x)) \in \mathbb{R}^n \times \mathbb{S}^{n-1}\,. \end{cases} \tag{5.4}$$

By Lemma 4.2, (5.4) has a unique solution $d^{s+1}(t,x)$ satisfying $\nabla d^{s+1} \in C(0,\tilde{T}_{s+1};H^N)$ on the maximal time interval $[0,\tilde{T}_{s+1})$. We denote $T^*_{s+1} = \min\{\hat{T}_{s+1}, \overline{T}_{s+1}, \tilde{T}_{s+1}\} > 0$, and then the proof of Lemma 5.1 is finished. We remark that $T^*_{s+1} \le T^*_s$. □

Now we define the iterative approximate energy functional and dissipative rate functional:

$$\begin{aligned} E_{s+1}(t) &:= \|u^{s+1}\|^2_{H^N} + \|F^{s+1}\|^2_{H^N} + \|\nabla d^{s+1}\|^2_{H^N}\,, \\ D_{s+1}(t) &:= \nu\|\nabla u^{s+1}\|^2_{H^N} + \kappa\|\nabla F^{s+1}\|_{H^N} + \|\Delta d^{s+1}\|^2_{H^N}\,, \end{aligned}$$

and our key lemma is stated as follows.

Lemma 5.2. *Assume that* $(u^{s+1}, F^{s+1}, d^{s+1})(t,x)$ *is the solution to the iterative approximate system* (5.1) *and we define*

$$T_{s+1} = \sup\Big\{t^* \in [0,T^*_{s+1});\ \sup_{t\in[0,t^*]} E_{s+1}(t) + \int_0^{t^*} D_{s+1}(t)\mathrm{d}t \le M\Big\}, \tag{5.5}$$

where $T^*_{s+1} > 0$ *is the existence time of the iterative approximate system* (5.1). *Then for any fixed* $M > \mathcal{E}_N(0)$*, there is a constant* $T > 0$*, depending only on* $\mathcal{E}_N(0), \nu, \kappa$ *such that* $T_{s+1} > T > 0$ *holds for all integers* $s \ge 0$.

Proof of Lemma 5.2. If the sequence $\{T_s; s = 1,2,3,...\}$ is increasing, the conclusion obviously holds. So we consider that the sequence T_s is not increasing. Now we choose a strictly increasing sequence $\{S_q\}_{q=1}^{\wedge}$ as follows:

$$S_1 = 1, S_{q+1} = \min\{s; s > S_q; T_s < T_{S_q}\}.$$

If $\wedge < \infty$, the conclusion holds, and consequently, we consider the case $\wedge = \infty$. By the definition of n_q, the sequence $\{T_{S_q}\}_{q=1}^{\infty}$ is strictly decreasing, so our goal is to prove $\lim_{q\to\infty} T_{S_q} > 0$. Next, from employing virtually the same arguments and processes as deriving the a *priori* estimates in Lemma 2.1, we first estimate the energy of the evolution model for the iterative approximate system (5.1).

Step 1. u^{s+1}*-estimates.* For all $|m| \le N$, we take "∂^m" in the first equation, and multiply by $\partial^m u^{s+1}$, and integrate on $\mathbb{R}^n$. Then by a similar estimate as that in Lemma (2.1), we get

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial^m u^{s+1}\|^2_{L^2} + \nu\|\nabla\partial^m u^{s+1}\|^2_{L^2} \\ &= -\langle \partial^m(u^s \cdot \nabla u^s), \partial^m u^{s+1}\rangle - \langle \partial^m(\nabla\frac{|\nabla d^s|^2}{2} + \nabla d^s \cdot \Delta d^s), \partial^m u^{s+1}\rangle \\ &\quad - \langle \partial^m F^s (F^s)^\top, \nabla\partial^m u^{s+1}\rangle\,. \end{aligned} \tag{5.6}$$

Since $\nabla \cdot u^{s+1} = 0$, from the Hölder inequality and the Sobolev embedding theory, we deduce that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u^{s+1}\|_{H^N}^2 + \nu\|\nabla u^{s+1}\|_{H^N}^2 \leq C(\|u^s\|_{H^N}\|\nabla u^s\|_{H^N} + \|\nabla d^s\|_{H^N}^2)\|u^{s+1}\|_{H^N} + C\|F^s\|_{H^N}^2\|\nabla u^{s+1}\|_{H^N}. \tag{5.7}$$

Step 2. F^{s+1}-estimates. We take the m-order derivative on the third F-equation of (5.1) for all $|m| \leq N$, multiply the L^2-inner product by dot with $\partial^m F^{s+1}$, and integrate by parts over $x \in \mathbb{R}^n$. We thereby have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\partial^m F^{s+1}\right\|_{L^2}^2 + k\left\|\nabla\partial^m F^{s+1}\right\|_{L^2}^2 = \langle\partial^m(u^s \cdot \nabla F^s), \partial^m F^{s+1}\rangle + \langle\partial^m(\nabla u^s F^s), \partial^m F^{s+1}\rangle. \tag{5.8}$$

Then the right-hand side of (5.8) can be estimated as

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|F^{s+1}\right\|_{H^N}^2 + k\left\|\nabla F^{s+1}\right\|_{H^N}^2 \leq C(\|u^s\|_{H^N}\|\nabla F^s\|_{H^N} + \|\nabla u^s\|_{H^N}\|F^s\|_{H^N}) \times \|F^{s+1}\|_{H^N}. \tag{5.9}$$

Step 3. d^{s+1}-estimates. We take the m-order derivative on the fourth d-equation of (5.1) for all $|m| \leq N$, multiply the L^2-inner product by dot with $\Delta\partial^m d^{s+1}$, and integrate by parts over $x \in \mathbb{R}^n$. Then we can have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla\partial^m d^{s+1}\right\|_{L^2}^2 + \|\Delta\partial^m d^{s+1}\|_{L^2}^2 &= \langle\partial^m((u^s \cdot \nabla)d^{s+1}), \Delta\partial^m d^{s+1}\rangle \\ &\quad - \langle\partial^m(|\nabla d^{s+1}|^2 d^{s+1}), \Delta\partial^m d^{s+1}\rangle + \langle\partial^m(d^{s+1} \times \Delta d^{s+1}), \Delta\partial^m d^{s+1}\rangle\,.\end{aligned}$$

From the Hölder inequality, the Sobolev embedding theory, and Young's inequality, we deduce that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla d^{s+1}\right\|_{H^N}^2 + \|\Delta d^{s+1}\|_{H^N}^2 \leq C(\|u^s\|_{H^N}^2 + \|\nabla d^{s+1}\|_{H^N}^2 + \|\nabla d^{s+1}\|_{H^N}^4)\|\nabla d^{s+1}\|_{H^N}^2\,. \tag{5.10}$$

Together with (5.7), (5.9), and (5.10), one has

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|u^{s+1}\right\|_{H^N}^2 + \left\|F^{s+1}\right\|_{H^N}^2 + \left\|\nabla d^{s+1}\right\|_{H^N}^2\right) + \nu\left\|\nabla u^{s+1}\right\|_{H^N}^2 + \kappa\left\|\nabla F^{s+1}\right\|_{H^N}^2 + \|\Delta d^{s+1}\|_{H^N}^2 \\ &\lesssim \left(\|u^s\|_{H^N}\|\nabla u^s\|_{H^N} + \|\nabla d^s\|_{H^N}^2\right)\|u^{s+1}\|_{H^N} + \|F^s\|_{H^N}^2\|\nabla u^{s+1}\|_{H^N} \\ &\quad + (\|u^s\|_{H^N}\|\nabla F^s\|_{H^N} + \|\nabla u^s\|_{H^N}\|F^s\|_{H^N})\,\|F^{s+1}\|_{H^N} \\ &\quad + \left(\|u^s\|_{H^N}^2 + \|\nabla d^{s+1}\|_{H^N}^2 + \|\nabla d^{s+1}\|_{H^N}^4\right)\|\nabla d^{s+1}\|_{H^N}^2 \\ &\lesssim E_s^{\frac{1}{2}}(t)D_s^{\frac{1}{2}}(t)E_{s+1}^{\frac{1}{2}}(t) + E_s(t)E_{s+1}^{\frac{1}{2}}(t) + E_s(t)D_{s+1}^{\frac{1}{2}}(t) \\ &\quad + E_s^{\frac{1}{2}}(t)D_s^{\frac{1}{2}}(t)E_{s+1}^{\frac{1}{2}}(t) + D_s^{\frac{1}{2}}(t)E_s^{\frac{1}{2}}(t)E_{s+1}^{\frac{1}{2}}(t) \\ &\quad + \left(E_s(t)E_{s+1}(t) + E_{s+1}(t)E_{s+1}(t) + E_{s+1}^3(t)\right) \\ &\leq C\left(1 + E_s^2(t) + E_s^{\frac{1}{2}}(t)D_s^{\frac{1}{2}}(t)\right)\left(1 + E_{s+1}^3(t)\right) + \frac{1}{2}D_{s+1}(t)\,,\end{aligned}$$

and further simplification can be obtained as

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E_{s+1}(t) + \frac{1}{2}D_{s+1}(t) \leq C(1 + E_s^2(t) + E_s^{\frac{1}{2}}(t)D_s^{\frac{1}{2}}(t))(1 + E_{s+1}(t))^3\,, \tag{5.11}$$

for all $t \in [0, T_{s+1}^*]$. Here $E_s(t)$ and $D_s(t)$ are well-defined, since $T_{s+1}^* \leq T_s^*$. This indicates that the existence interval of the solution at the next layer depends on that at the previous layer, thus we need to prove T_{s+1}^* is bounded.

Since $E_{s+1}(t)$ is continuous and $E_{s+1}(0) \le \mathcal{E}_N(0) < M$, then $T_{s+1} > 0$.
Case I. If $\{T_s\}_{s=1}^{\infty}$ is increasing, then $T_s \ge T_1 > 0$.
Case II. If $\{T_s\}_{s=1}^{\infty}$ is not increasing, we can choose a sequence $\{s_q\}_{q=1}^{\Lambda}$ that is strictly increasing, where

$$s_1 = 1, s_{q+1} = \min\{s : s > s_q; T_s < T_{s_q}\}. \tag{5.12}$$

Case i. If $\Lambda < \infty$, then $T_{s_q} \ge T_s > 0$.
Case ii. If $\Lambda = \infty$, $\{T_{s_q}\}_{q=1}^{\infty}$ is strictly decreasing. We need to prove $\lim_{q\to\infty} T_{s_q} > 0$. From (5.11), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}E_{s+1}(t) + D_{s+1}(t) \le 2C\Lambda_s(t)(1 + E_{s+1}(t))^3\,, \tag{5.13}$$

where $\Lambda_s(t) = 1 + E_s^2(t) + E_s^{\frac{1}{2}}(t)D_s^{\frac{1}{2}}(t)$, for all $t \in [0, T_{s+1}^*]$. By the definition of $\{s_q\}_{q=1}^{\Lambda}$ in (5.12), we have that $T_p > T_{s_q}$ when $p < s_q$. When we take $s = s_q - 1$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}E_{s_q}(t) + D_{s_q}(t) \le 2C\Lambda_{s_q-1}(t)(1 + E_{s_q-1}(t))^3\,, \tag{5.14}$$

where

$$\Lambda_{s_q-1}(t) = 1 + E_{s_q-1}^2(t) + E_{s_q-1}^{\frac{1}{2}}(t)D_{s_q-1}^{\frac{1}{2}}(t)\,, \tag{5.15}$$

with initial data

$$E_{s_q}(0) = \mathcal{E}_N(0), \tag{5.16}$$

for all $t \in [0, T_{s_q}]$. Next, we solve the above ordinary differential equations (5.14)–(5.16). For all $t \in [0, T_{s_q}] \subset [0, T_{s_q-1}]$, by the definition of T_{s+1} in (5.5), we have

$$T_{s_q-1} = \sup\Big\{t^* \in [0, T_{s_q-1}^*);\ \sup_{t\in[0,t^*]} E_{s_q-1}(t) + \int_0^{t^*} D_{s_q-1}(t)\mathrm{d}t \le M\Big\}.$$

Then

$$\sup_{t\in[0,T_{s_q}]} E_{s_q-1}(t) \le M;\ \int_0^{T_{s_q}} D_{s_q-1}(t)\mathrm{d}t \le M.$$

So

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E_{s_q}(t) + D_{s_q}(t) &\le 2C\Lambda_{s_q-1}(t)(1 + E_{s_q}(t))^3\\ &\le 2C(1+M)^2\Lambda_{s_q-1}(t)(1 + E_{s_q}(t))\,,\end{aligned} \tag{5.17}$$

for all $t \in [0, T_{s_q}]$. Then one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\ln(1 + E_{s_q}(t)) \le 2C(1+M)^2\Lambda_{s_q-1}(t)\,.$$

Then

$$\ln(1 + E_{s_q}(t)) - \ln(1 + \mathcal{E}_N(0)) \le 2C(1+M)^2\int_0^t \Lambda_{s_q-1}(\tau)\mathrm{d}\tau\,. \tag{5.18}$$

By (5.15),

$$\begin{aligned}\int_0^t \Lambda_{s_q-1}(\tau)\mathrm{d}\tau &= \int_0^t 1 + E^2_{s_q-1}(\tau) + E^{\frac{1}{2}}_{s_q-1}(\tau)D^{\frac{1}{2}}_{s_q-1}(\tau)\mathrm{d}\tau \\ &= t + \int_0^t E^2_{s_q-1}(\tau)\mathrm{d}\tau + \int_0^t E^{\frac{1}{2}}_{s_q-1}(\tau)D^{\frac{1}{2}}_{s_q-1}(\tau)\mathrm{d}\tau \\ &\le t + M^2 t + M^{\frac{1}{2}}\int_0^t D^{\frac{1}{2}}_{s_q-1}(\tau)\mathrm{d}\tau \\ &\le (1+M^2)t + Mt^{\frac{1}{2}}\,. \end{aligned} \tag{5.19}$$

Plugging (5.19) into (5.18), we have

$$\begin{aligned}\ln(1+E_{s_q}(t)) - \ln(1+\mathcal{E}_N(0)) &\le 2C(1+M)^2[(1+M^2)t + Mt^{\frac{1}{2}}] \\ &\le 2C(1+M)^2[(1+M^2)t + (1+M^2)t^{\frac{1}{2}}] \\ &\le 2C(1+M)^4(t+t^{\frac{1}{2}})\,,\end{aligned}$$

which immediately reduces to

$$\ln(1+E_{s_q}(t)) \le \ln(1+\mathcal{E}_N(0)) + 2C(1+M)^4(t+t^{\frac{1}{2}})\,.$$

Then

$$1+E_{s_q}(t) \le (1+\mathcal{E}_N(0))e^{2C(1+M)^4(t+t^{\frac{1}{2}})}\,.$$

So one has

$$E_{s_q}(t) \le (1+\mathcal{E}_N(0))e^{2C(1+M)^4(t+t^{\frac{1}{2}})} - 1\,,$$

for all $t \in [0, T_{s_q}]$. Let $G(t) = (1+\mathcal{E}_N(0))e^{2C(1+M)^4(t+t^{\frac{1}{2}})} - 1$, where $G(0) = \mathcal{E}_N(0)$, and $G(t)$ is increasing in the time interval $[0, T_{s_q}]$. Then (5.17) can be estimated as

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E_{s_q}(t) + D_{s_q}(t) &\le 2C\Lambda_{s_q-1}(t)(1+E_{s_q}(t))^3 \\ &\le 2C(1+G(t))^3\Lambda_{s_q-1}(t)\,.\end{aligned}$$

Integrating both sides from 0 to t yields

$$\begin{aligned}E_{s_q}(t) - \mathcal{E}_N(0) + \int_0^t D_{s_q}(\tau)\mathrm{d}\tau &\le 2C\int_0^t (1+G(\tau))^3\Lambda_{s_q-1}(\tau)\mathrm{d}\tau \\ &\le 2C(1+G(t))^3\int_0^t \Lambda_{s_q-1}(\tau)\mathrm{d}\tau \\ &\le 2C(1+M)^2(1+G(t))^3(t+t^{\frac{1}{2}})\,.\end{aligned}$$

So

$$E_{s_q}(t) + \int_0^t D_{s_q}(\tau)\mathrm{d}\tau \le \underbrace{\mathcal{E}_N(0) + 2C(1+M)^2(1+G(t))^3(t+t^{\frac{1}{2}})}_{Q(t)}\,,$$

for all $t \in [0, T_{s_q}]$, where $Q(0) = \mathcal{E}_N(0)$ and $Q(t)$ is increasing in the time interval $[0, T_{s_q}]$. For all $\mathcal{E}_N(0) < M$, there exists a $\tau = \tau(M) > 0$, such that for all $t \in [0, \tau]$, $Q(t) \le M$. Then $E_{s_q}(t) + \int_0^t D_{s_q}(\tau)\mathrm{d}\tau \le M$, for all $t \in [0, \tau]$. By the definition of T_{s_q}, we have $T_{s_q} \ge \tau > 0$, and thus

$$\lim_{q\to\infty} T_{s_q} \ge \tau > 0.$$

With this, the proof of Lemma 5.2 is finished. □

Proof of Theorem 3.1: Local well-posedness. By Lemma 5.2, we know that for any fixed $\mathcal{E}_N(0) < M$, there is a $T > 0$ such that for all integer $s \ge 0$ and $t \in [0, T]$,

$$\begin{aligned}&\sup_{t\in[0,T]} (\|u^{s+1}\|_{H^N}^2 + \|F^{s+1}\|_{H^N}^2 + \|\nabla d^{s+1}\|_{H^N}^2)\\ &+ \int_0^T (\nu\|\nabla u^{s+1}\|_{H^N}^2 + \kappa\|\nabla F^{s+1}\|_{H^N}^2 + \|\Delta d^{s+1}\|_{H^N}^2)(\tau)\mathrm{d}\tau \le M\,.\end{aligned}$$

By the above uniform bound and the Arzelà–Ascoli theorem, there exists a subsequence such that $u^s \rightharpoonup^\star u$ in $L^\infty(0, T; H^N)$, $u^s \rightharpoonup u$ in $L^2(0, T; H^{N+1})$, and similarly for d^s, F^s. Passing to the limit in the approximate system yields that (u, d, F) is a solution of the original problem. Combined with Lemma 5.1, we have vectors $(u, d) \in \mathbb{R}^n\times\mathbb{S}^{n-1}$, $F \in \mathbb{R}^{n\times n}$, satisfying $u \in L^\infty(0, +\infty; H^N(\mathbb{R}^n))\cap L^2(0, +\infty; H^{N+1}(\mathbb{R}^{n\times n}))$, $F \in L^\infty(0, +\infty; H^N(\mathbb{R}^n)) \cap L^2(0, +\infty; H^{N+1}(\mathbb{R}^{n\times n}))$ and $\nabla d \in L^\infty(0, +\infty; H^N(\mathbb{R}^{n\times n}))$, $\Delta d \in L^2(0, +\infty; H^N(\mathbb{R}^n))$, which solve the evolutionary model for magneto-viscoelastic system (1.1). What is more, $(u, F, \nabla d)$ satisfies the bound

$$\sup_{t\in[0,T]} (\|u\|_{H^N}^2 + \|F\|_{H^N}^2 + \|\nabla d\|_{H^N}^2) + \int_0^T (\nu\|\nabla u\|_{H^N}^2 + \kappa\|\nabla F\|_{H^N}^2 + \|\Delta d\|_{H^N}^2)(\tau)\mathrm{d}\tau \le M\,.$$

Thus the proof of Theorem 3.1 is complete. □

6. Global existence: Proof of Theorem 3.2

In this section, we prove the global well-posedness of system (1.1) with small initial data. We first introduce the following energy functional:

$$\mathcal{E}_N(t) := \|u\|_{H^N}^2 + \|F\|_{H^N}^2 + \|\nabla d\|_{H^N}^2\,,$$

and energy dissipative rate functional:

$$\mathcal{D}_N(t) := \nu\|\nabla u\|_{H^N}^2 + \kappa\|\nabla F\|_{H^N}^2 + \|\Delta d\|_{H^N}^2\,.$$

We have the following lemma.

Lemma 6.1. *For $N \ge 3$, assume that $(u, F, d)(t, x)$ is a sufficiently smooth solution to* (1.1). *Then there is a positive constant $C' = C'(\nu, n, N, \kappa) > 0$, such that*

$$\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_N(t) + \mathcal{D}_N(t) \le C'(\mathcal{E}_N(t) + \mathcal{E}_N^{\frac{1}{2}}(t))\mathcal{D}_N(t) \tag{6.1}$$

holds for all $t \ge 0$.

Proof of Lemma 6.1. In this part, we re-estimate all the items contained in (2.8). For $1 \leq |m| \leq N (N \geq 3)$, I_1 can be estimated as

$$\begin{aligned}
I_1 &= -\sum_{0 \neq m' \leq m} C_m^{m'} \langle \partial^{m'} u \cdot \nabla \partial^{m-m'} u,\ \partial^m u \rangle \\
&\leq C' \sum_{0 \neq m' \leq m} \|\partial^{m'} u\|_{L^4} \|\nabla \partial^{m-m'} u\|_{L^4} \|\partial^m u\|_{L^2} \\
&\leq C' \sum_{0 \neq m' \leq m} \|\partial^{m'} u\|_{H^1} \|\nabla \partial^{m-m'} u\|_{H^1} \|\partial^m u\|_{L^2} \\
&\leq C' \|u\|_{H^N} \|\nabla u\|_{H^N}^2 \\
&\leq C' \mathcal{E}_N^{\frac{1}{2}}(t) \mathcal{D}_N(t),
\end{aligned} \tag{6.2}$$

where $C' = C'(\nu, N, n, \kappa)$. Here we have used the Hölder inequality and the Sobolev embedding $H^1(\mathbb{R}^n) \hookrightarrow L^4(\mathbb{R}^n)$. I_2 can be estimated as

$$\begin{aligned}
I_2 &\leq C' \|\nabla \partial^m d\|_{L^2} \|\Delta d\|_{L^\infty} \|\partial^m u\|_{L^2} + C' \sum_{0 \neq m' < m} \|\nabla \partial^{m'} d\|_{L^4} \|\nabla \partial^{m-m'} d\|_{L^4} \|\partial^m u\|_{L^2} \\
&\leq C' \|\nabla d\|_{H^N} \|\Delta d\|_{H^N} \|\nabla u\|_{H^N} \\
&\leq C' \mathcal{E}^{\frac{1}{2}}(t) \mathcal{D}(t),
\end{aligned} \tag{6.3}$$

where we utilize the Sobolev embedding $H^2(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$ for $n = 2, 3$. Similarly, I_3 can be estimated as

$$\begin{aligned}
I_3 &= -\sum_{0 \neq m' \leq m} C_m^{m'} \langle \partial^{m-m'} F \partial^{m'} F^T, \nabla \partial^m u \rangle \\
&\leq C' \|F\|_{L^\infty} \|\partial^m F^\top\|_{L^2} \|\nabla \partial^m u\|_{L^2} + C' \sum_{0 \neq m' < m} \|\partial^{m-m'} F\|_{L^4} \|\partial^m F^\top\|_{L^4} \|\nabla \partial^m u\|_{L^2} \\
&\leq C' \|F\|_{H^N} \|\nabla F\|_{H^N} \|\nabla u\|_{H^N} \\
&\leq C' \mathcal{E}_N^{\frac{1}{2}}(t) \mathcal{D}_N(t).
\end{aligned} \tag{6.4}$$

Furthermore, from the Hölder inequality and the Sobolev embeddings $H^1(\mathbb{R}^n) \hookrightarrow L^4(\mathbb{R}^n)$ and $H^2(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$ ($n = 2, 3$), we deduce that

$$\begin{aligned}
I_4 &= -\sum_{0 \neq m' \leq m} C_m^{m'} \langle \partial^{m'} u \cdot \nabla \partial^{m-m'} F,\ \partial^m F \rangle \\
&= C' \sum_{|m'|=1} \|\partial^{m'} u\|_{L^\infty} \|\nabla \partial^{m-m'} F\|_{L^2} \|\partial^m F\|_{L^2} + C' \sum_{2 \leq |m'| \leq |m|} \|\partial^{m'} u\|_{L^4} \|\nabla \partial^{m-m'} F\|_{L^4} \|\partial^m F\|_{L^2} \\
&\leq C' \|\nabla u\|_{H^N} \|\nabla F\|_{H^N} \|F\|_{H^N} \\
&\leq C' \mathcal{E}_N^{\frac{1}{2}}(t) \mathcal{D}_N(t).
\end{aligned} \tag{6.5}$$

Similarly, by the Hölder inequality and the Sobolev embedding theorem, we have

$$\begin{aligned}
I_5 &= \sum_{0 \neq m' \leq m} C_m^{m'} \langle \nabla \partial^{m-m'} u \partial^{m'} F, \partial^m F \rangle \\
&\leq C' \|\nabla u\|_{H^N} \|\nabla F\|_{H^N} \|F\|_{H^N} \\
&\leq C' \mathcal{E}_N^{\frac{1}{2}}(t) \mathcal{D}_N(t),
\end{aligned} \tag{6.6}$$

and

$$
\begin{aligned}
I_6 &= \sum_{0\neq m'\leq m} C_m^{m'}\langle \partial^{m-m'}u\cdot\nabla\partial^{m'}d,\Delta\partial^m d\rangle\\
&\leq C'\|u\|_{L^\infty}\|\nabla\partial^m d\|_{L^2}\|\Delta\partial^m d\|_{L^2}+C'\sum_{0\neq m'<m}\|\partial^{m-m'}u\|_{L^4}\|\nabla\partial^{m'}d\|_{L^4}\|\Delta\partial^m d\|_{L^2}\\
&\leq C'\|u\|_{H^N}\|\Delta d\|_{H^N}^2\\
&\leq C'\mathcal{E}_N^{\frac{1}{2}}(t)\mathcal{D}_N(t)\,.
\end{aligned}
\tag{6.7}
$$

From the Hölder inequality, Sobolev embedding $H^1(\mathbb{R}^n)\hookrightarrow L^6(\mathbb{R}^n)$, and Gagliardo-Nirenberg interpolation inequality, one has

$$
\begin{aligned}
I_7 &= -\langle\partial^m(|\nabla d|^2)d,\Delta\partial^m d\rangle\\
&= -\sum_{0\neq m'<m} C_m^{m'}\langle(\nabla\partial^{m'}d\cdot\nabla\partial^{m-m'}d)d,\Delta\partial^m d\rangle-2\langle(\nabla\partial^m d\cdot\nabla d)d,\Delta\partial^m d\rangle\\
&\leq C'\sum_{0\neq m'\leq m}\|\nabla\partial^{m'}d\|_{L^4}\|\nabla\partial^{m-m'}d\|_{L^4}\|\Delta\partial^m d\|_{L^2}\\
&\quad+2\|\nabla\partial^m d\|_{L^6}\|\nabla d\|_{L^3}\|\Delta\partial^m d\|_{L^2}\\
&\leq C'\|\nabla d\|_{H^N}\|\Delta d\|_{H^N}^2\\
&\leq C'\mathcal{E}_N^{\frac{1}{2}}(t)\mathcal{D}_N(t)\,.
\end{aligned}
\tag{6.8}
$$

Finally, I_8 and I_9 can be estimated as

$$
\begin{aligned}
I_8 &= -\sum_{0\neq m'\leq m} C_m^{m'}\partial^{m-m'}(|\nabla d|^2)\partial^{m'}d,\Delta\partial^m d\rangle\\
&\leq C'\|\nabla d\|_{H^N}^2\|\Delta d\|_{H^N}^2\leq C'\mathcal{E}_N(t)\mathcal{D}_N(t)\,,
\end{aligned}
\tag{6.9}
$$

and

$$
\begin{aligned}
I_9 &= -\sum_{0\neq m'\leq m} C_m^{m'}\langle\partial^{m'}d\times\Delta\partial^{m-m'}d,\Delta\partial^m d\rangle\\
&\leq C'\sum_{0\neq m'\leq m}\|\partial^{m'}d\|_{L^4}\|\Delta\partial^{m-m'}d\|_{L^4}\|\Delta\partial^m d\|_{L^2}\\
&\leq C'\|\nabla d\|_{H^N}\|\Delta d\|_{H^N}^2\\
&\leq C'\mathcal{E}_N^{\frac{1}{2}}(t)\mathcal{D}_N(t)\,.
\end{aligned}
\tag{6.10}
$$

Substituting (6.2)–(6.10) into (2.8), we have

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_N(t)+\mathcal{D}_N(t)\leq C'(\mathcal{E}_N(t)+\mathcal{E}_N^{\frac{1}{2}}(t))\mathcal{D}_N(t)\,.
\tag{6.11}
$$

Then the proof of Lemma 6.1 is finished. □

Proof of Theorem 3.2: Global well-posedness with small initial data. One observes that

$$
\mathcal{E}_N(0)=\|u_0\|_{H^N}^2+\|F_0\|_{H^N}^2+\|\nabla d_0\|_{H^N}^2.
$$

Denote $P(\mathcal{E}_N(t)) = C'(\mathcal{E}_N(t) + \mathcal{E}_N^{\frac{1}{2}}(t))$, so we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_N(t) + \mathcal{D}_N(t) \leq \mathcal{D}_N(t)P(\mathcal{E}_N(t)) .$$

Here, $P(0) = 0$ and $P(\mathcal{E}_N(t))$ is strictly increasing. Then there is a sufficiently small $\epsilon_0 = \epsilon_0(\nu, \kappa) > 0$, such that if $\mathcal{E}_N(0) \leq \epsilon_0$, then $P(\mathcal{E}_N(0)) = P(\mathcal{E}_N(0)) \leq \frac{1}{4}$. Now we define

$$\tilde{T} = sup\{t^* > 0; sup_{t\in[0,t^*]}P(\mathcal{E}_N(t)) \leq \frac{1}{2}\} \geq 0.$$

By the continuity of $\mathcal{E}_N(t)$, we have $\tilde{T} > 0$. We claim that $\tilde{T} = +\infty$. Indeed, if $\tilde{T} < +\infty$, then for all $t \in [0, \tilde{T}]$, $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_N(t) + \mathcal{D}_N(t) \leq 0$, which immediately means

$$\mathcal{E}_N(t) + \int_0^{\tilde{T}} \mathcal{D}_N(t)\mathrm{d}t \leq \mathcal{E}_N(0) .$$

Then the above bound reduces to

$$\sup_{t\in[0,\tilde{T}]} P(\mathcal{E}_N(t)) \leq P(\mathcal{E}_N(0)) \leq \frac{1}{4} .$$

By the continuity of $\mathcal{E}_N(t)$, there is a $\hat{t} > 0$, such that for all $t \in [0, \tilde{T} + \hat{t}]$, $P(\mathcal{E}_N(t)) \leq \frac{1}{2}$, which is opposite to the definition of $\tilde{T}$. Thus the claim holds. Finally, by the definition of the energy functional $\mathcal{E}_N(t)$ and energy dissipative rate functional $\mathcal{D}_N(t)$, we have the following bound:

$$\sup_{t\geq 0}(\|u\|_{H^N}^2 + \|F\|_{H^N}^2 + \|\nabla d\|_{H^N}^2) + \int_0^{+\infty} (\nu\|\nabla u\|_{H^N}^2 + \kappa\|\nabla F\|_{H^N}^2 + \|\Delta d\|_{H^N}^2)\mathrm{d}t \leq C\epsilon_0 ,$$

where C depends on ν, κ. This is uniformly bounded in time. Then we can extend the solution constructed in Theorem 3.1 to the time interval $[0, \infty)$ and finish the proof of Theorem 3.2. □

7. Conclusions

In this paper, we focus on an evolutionary model for magneto-viscoelasticity, which describes the coupling between magnetic, viscous and elastic properties in magnetic materials. First, we introduce the evolutionary magneto-viscoelastic system (1.1) in Section 1. In Section 2, we derive the a *priori* estimate of system (1.1). In Section 3, we precisely state our main theorems. In Section 4, we establish the well-posedness of the LLG evolution with a given field. Furthermore, we construct an iterative approximation system for the initial model and obtain the local-in-time solution by using the standard compactness argument in Section 5. Finally, the global-in-time solution for large initial data is obtained in Section 6.

The evolutionary model of magneto-viscoelasticity broadens our understanding of fluid behavior and holds significant value for both theoretical research and practical applications. Future extensions of this model can be pursued in several directions. The compressible case can be explored to analyze how compressibility effects influence the dynamic behavior of the model. Magnetic field problems with boundary conditions can also be investigated, incorporating non-uniform external magnetic fields and complex boundaries, such as mixed or moving boundaries to better align with engineering applications and enhance practicality. In addition, multi-field coupling can be examined. The existing magneto-mechanical-fluid coupling model can be extended to a magneto-thermo-mechanical-fluid framework to systematically explore how temperature and other key factors affect the dynamic response of magneto-viscoelastic materials. Magnetoelectric field coupling can be further considered to study dynamic characteristics under the synergistic effects of electric and magnetic fields, enriching the multi-field coupling dimension of the model and supporting its theoretical development and practical application.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares that she has no conflict of interest.

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