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Research article

Some results on pointwise pseudo-slant submanifolds in a cosymplectic manifold

Noura M. Alhouiti[*]

Department of Mathematics, Faculty of Science, University of Tabuk, 71491 Tabuk, Saudi Arabia

* **Correspondence:** Email: nalhouiti@ut.edu.sa.

Abstract: In this article, we study submanifolds of a cosymplectic manifold and establish some fundamental results. We then introduce and investigate pointwise pseudo-slant submanifolds in this setting. Conditions for the integrability of the associated distributions are obtained. Furthermore, we examine geometric properties such as totally geodesic, totally umbilical, and minimal pointwise pseudo-slant submanifolds are studied.

Keywords: cosymplectic manifold; pointwise-slant submanifold; pointwise pseudo-slant submanifold
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1. Introduction

The notion of pointwise (quasi)-slant submanifold in the setting of an almost Hermitian manifold was presented by F. Etayo [10] and further studied by Chen and O. J. Garay [7] as a natural generalization of a slant submanifold. Afterwards, these types of submanifolds have been investigated in various Riemannian manifold structures (see [2, 11, 16]).

In contrast, pseudo-slant (or hemi-slant) submanifolds have also attracted considerable attention as they generalize invariant, anti-invariant, semi-invariant and slant submanifolds. Several authors have studied their geometric properties in different ambient spaces (see [4, 8, 9]). The authors of [1] defined the totally umbilical pseudo-slant submanifolds and provided descriptions of such submanifolds in a Kaehler manifold.

Motivated by these developments, the aim of this research is to study pointwise pseudo-slant submanifolds in the context of cosymplectic manifolds. This class of submanifolds has been previously considered in other geometric settings, such as almost contact metric 3-structures [13] and in Kaehler manifolds [3, 15].

The paper is organized as follows: In Section 2, we recall basic definitions and fundamental formulas related to cosymplectic manifolds. In Section 3, we present some general results on submanifolds. Section 4 is devoted to the study of pointwise pseudo-slant submanifolds, including integrability conditions. Finally, in Section 5, we investigate totally geodesic, totally umbilical, and minimal cases.

2. Preliminaries

Let $\tilde{\mathcal{M}}$ be a $(2p+1)$-dimensional smooth manifold endowed with the almost contact metric structure $(\varphi, \xi, \eta, \langle\,,\,\rangle)$, where φ is a $(1,1)$ tensor field, ξ is a vector field, η is a 1-form, and $\langle\,,\,\rangle$ is a Riemannian metric on $\tilde{\mathcal{M}}$. The structure satisfies [6]

$$\varphi^2 U = -U + \eta(U)\xi, \;\; \varphi\xi = 0, \;\; \eta(\varphi U) = 0, \;\; \eta(\xi) = 1, \tag{2.1}$$

and

$$\langle \varphi U, \varphi V\rangle = \langle U, V\rangle - \eta(U)\eta(V), \quad \eta(U) = \langle U, \xi\rangle, \tag{2.2}$$

for every $U, V \in \Gamma(\mathcal{T}\tilde{\mathcal{M}})$, where $\Gamma(\mathcal{T}\tilde{\mathcal{M}})$ is the set of all smooth vector fields on $\tilde{\mathcal{M}}$. Then, $\tilde{\mathcal{M}}$ is said to be an *almost contact metric manifold*. Furthermore, from (2.2), φ is skew symmetric with respect to the Riemannian metric $\langle\,,\,\rangle$, that is

$$\langle \varphi U, V\rangle = -\langle U, \varphi V\rangle. \tag{2.3}$$

If the relations mentioned above are added to

$$(\tilde{\nabla}_U \varphi)V = 0, \quad \tilde{\nabla}_U \xi = 0, \tag{2.4}$$

then $\tilde{\mathcal{M}}$ is said to be *cosymplectic manifold*, where $\tilde{\nabla}$ is the Levi-Civita connection of $\tilde{\mathcal{M}}$. The covariant derivative of the tensor field φ is provided by

$$(\tilde{\nabla}_U \varphi)V = \tilde{\nabla}_U \varphi V - \varphi \tilde{\nabla}_U V.$$

Therefore, in a cosymplectic manifold, we have

$$\tilde{\nabla}_U \varphi V = \varphi \tilde{\nabla}_U V, \tag{2.5}$$

for every $U, V \in \Gamma(\mathcal{T}\tilde{\mathcal{M}})$. Equivalently, an almost contact metric manifold $\tilde{\mathcal{M}}$ is said to be *cosymplectic* if it is normal and both Φ and η are closed, i.e., $d\eta = 0$ and $d\Phi = 0$, where $\Phi(U, V) = \langle U, \varphi V\rangle$ is the fundamental 2-form.

Now, let $\mathcal{M}$ be a q-dimensional submanifold of an almost contact metric manifold $\tilde{\mathcal{M}}$ with the generated metric $\langle\,,\,\rangle$ and the vector field ξ be tangent to $\mathcal{M}$. Denote by ∇ and $\nabla^{\perp}$ the generated connections on the tangent bundle $\mathcal{T}\mathcal{M}$ and the normal bundle $\mathcal{T}^{\perp}\mathcal{M}$ of $\mathcal{M}$, respectively. The Gauss and Weingarten formulas of $\mathcal{M}$ into $\tilde{\mathcal{M}}$ are given, respectively, by

$$\tilde{\nabla}_U V = \nabla_U V + \alpha(U, V), \tag{2.6}$$

and

$$\tilde{\nabla}_U Y = \nabla^{\perp}_U Y - S_Y U, \tag{2.7}$$

for any $U, V \in \Gamma(\mathcal{T}\mathcal{M})$ and $Y \in \Gamma(\mathcal{T}^{\perp}\mathcal{M})$, where α is the second fundamental form of $\mathcal{M}$ and S_Y is the shape operator in the direction of Y. It is well known that α and S_Y satisfy the following relation:

$$\langle S_Y U, V\rangle = \langle \alpha(U, V), Y\rangle. \tag{2.8}$$

If $\alpha(U, V) = 0$, for each $U, V \in \Gamma(\mathcal{T}\mathcal{M})$, then the submanifold $\mathcal{M}$ of $\tilde{\mathcal{M}}$ is said to be *totally geodesic*. The *mean curvature* H of $\mathcal{M}$ is provided by

$$H = \frac{1}{q}\sum_{i=1}^{q} \alpha(e_i, e_i),$$

where $\{e_1, \cdots, e_q\}$ is a local orthonormal frame of $\Gamma(\mathcal{T}\mathcal{M})$. The submanifold $\mathcal{M}$ is of $\tilde{\mathcal{M}}$ is said to be *totally umbilical* if

$$\alpha(U, V) = \langle U, V\rangle H,$$

and $\mathcal{M}$ is said to be *minimal* if $H = 0$.

Since $\mathcal{M}$ is tangent to ξ, using (2.4) and (2.6), we gain

$$\nabla_U \xi = 0, \qquad \alpha(U, \xi) = 0, \tag{2.9}$$

for any $U \in \Gamma(\mathcal{T}\mathcal{M})$.

3. Results on submanifolds of a cosymplectic manifold

Let $\mathcal{M}$ be a submanifold of a cosymplectic manifold $\tilde{\mathcal{M}}$. Then, for any $U \in \Gamma(\mathcal{T}\mathcal{M})$, we have

$$\varphi U = TU + NU, \tag{3.1}$$

where TU is the tangential projection of φU on $\mathcal{T}\mathcal{M}$ and NU is the normal projection of φU on $\mathcal{T}^{\perp}\mathcal{M}$. Moreover, for any $Y \in \Gamma(\mathcal{T}^{\perp}\mathcal{M})$, we have

$$\varphi Y = tY + nY, \tag{3.2}$$

where tY and nY are the tangential and normal projections of φY, respectively.

A submanifold $\mathcal{M}$ is called an *invariant* if N is identically zero, i.e., $\varphi U \in \Gamma(\mathcal{T}\mathcal{M})$, for every $U \in \Gamma(\mathcal{T}\mathcal{M})$, while $\mathcal{M}$ is called an *anti-invariant* if T is identically zero, i.e., $\varphi U \in \Gamma(\mathcal{T}^{\perp}\mathcal{M})$, for every $U \in \Gamma(\mathcal{T}\mathcal{M})$ [17].

From, (2.3), (3.1), and (3.2), we can describe the behavior of the tensor fields T, N, t and n with respect to the metric $\langle\, ,\, \rangle$ as follows:

$$\langle U, TV\rangle = -\langle TU, V\rangle, \quad \langle Y, nZ\rangle = -\langle nY, Z\rangle, \quad \langle NU, Y\rangle = -\langle U, tY\rangle, \tag{3.3}$$

for every $U, V \in \Gamma(\mathcal{TM})$ and $Y, Z \in \Gamma(\mathcal{T}^{\perp}\mathcal{M})$.

Moreover, the covariant derivatives of these tensor fields are provided by

$$(\nabla_U T)V = \nabla_U TV - T\nabla_U V, \tag{3.4}$$

$$(\nabla_U N)V = \nabla^{\perp}_U NV - N\nabla_U V, \tag{3.5}$$

$$(\nabla_U t)Y = \nabla_U tY - t\nabla^{\perp}_U Y, \tag{3.6}$$

and

$$(\nabla_U n)Y = \nabla^{\perp}_U nY - n\nabla^{\perp}_U Y, \tag{3.7}$$

for every $U, V \in \Gamma(\mathcal{TM})$ and $Y \in \Gamma(\mathcal{T}^{\perp}\mathcal{M})$.

Therefore, it follows from the Eqs (2.6)–(3.7) that

$$(\nabla_U T)V = S_{NV}U + t\alpha(U, V), \tag{3.8}$$

$$(\nabla_U N)V = n\alpha(U, V) - \alpha(U, TV), \tag{3.9}$$

$$(\nabla_U t)Y = S_{nY}U - TS_Y U, \tag{3.10}$$

and

$$(\nabla_U n)Y = -\Big(\alpha(tY, U) + NS_Y U\Big), \tag{3.11}$$

for each $U, V \in \Gamma(\mathcal{TM})$ and $Y \in \Gamma(\mathcal{T}^{\perp}\mathcal{M})$.

Now, we will introduce the conditions for the parallelism of these tensor fields as follows.

Proposition 3.1. *Let $\mathcal{M}$ be a submanifold of a cosymplectic manifold $\tilde{\mathcal{M}}$. Then, the tensor field T is parallel if and only if the shape operator S satisfies*

$$S_{NV}W = S_{NW}V,$$

for every $V, W \in \Gamma(\mathcal{TM})$.

Proof. Let $U, V, W \in \Gamma(\mathcal{TM})$, then from (2.8), (3.3), and (3.8), we obtain

$$\langle(\nabla_U T)V, W\rangle = \langle S_{NV}U, W\rangle + \langle t\alpha(U, V), W\rangle = \langle\alpha(U, W), NV\rangle - \langle\alpha(U, V), NW\rangle = \langle S_{NV}W, U\rangle - \langle S_{NW}V, U\rangle,$$

which proves our assertion. □

Proposition 3.2. *Let $\mathcal{M}$ be a submanifold of a cosymplectic manifold $\tilde{\mathcal{M}}$. Then, the tensor field N is parallel if and only if*

$$S_{nY}V = -S_Y TV,$$

for every $V \in \Gamma(\mathcal{TM})$, and $Y \in \Gamma(\mathcal{T}^{\perp}\mathcal{M})$.

Proof. Consider $U, V \in \Gamma(\mathcal{TM})$ and $Y \in \Gamma(\mathcal{T}^{\perp}\mathcal{M})$; by (2.8), (3.3), and (3.9), we derive

$$\langle(\nabla_U N)V, Y\rangle = \langle n\alpha(U, V), Y\rangle - \langle\alpha(U, TV), Y\rangle = -\langle\alpha(U, V), nY\rangle - \langle S_Y TV, U\rangle = -\langle S_{nY}V, U\rangle - \langle S_Y TV, U\rangle,$$

which is the required result. □

Proposition 3.3. *Let $\mathcal{M}$ be a submanifold of a cosymplectic manifold $\tilde{\mathcal{M}}$. Then, the tensor field n is parallel if and only if*

$$S_Y tZ = S_Z tY, \tag{3.12}$$

for every $Y, Z \in \Gamma(\mathcal{T}^{\perp}\mathcal{M})$.

Proof. Given any $U \in \Gamma(\mathcal{TM})$, and $Y, Z \in \Gamma(\mathcal{T}^{\perp}\mathcal{M})$, from (2.8), (3.3), and (3.11), we deduce that

$$\langle(\nabla_U n)Y, Z\rangle = -\langle\alpha(tY, U), Z\rangle - \langle NS_Y U, Z\rangle = -\langle S_Z tY, U\rangle + \langle S_Y U, tZ\rangle = \langle S_Y tZ - S_Z tY, U\rangle.$$

This completes the proof. □

The following propositions provide the properties of the covariant derivation of such tensor fields.

Proposition 3.4. *Let $\mathcal{M}$ be a submanifold of a cosymplectic manifold $\tilde{\mathcal{M}}$. Then, the covariant derivation of the tensor field T satisfies*

$$\langle(\nabla_U T)V, W\rangle = -\langle(\nabla_U T)W, V\rangle,$$

for each $U, V, W \in \Gamma(\mathcal{TM})$, i.e., the tensor field T is skew-symmetric.

Proof. Let $U, V, W \in \Gamma(\mathcal{TM})$; according to (2.8), (3.3), and (3.8), we derive

$$\begin{aligned}\langle(\nabla_U T)V, W\rangle &= \langle S_{NV}U, W\rangle + \langle t\sigma(U, V), W\rangle = \langle\alpha(U, W), NV\rangle - \langle\alpha(U, V), NW\rangle \\ &= -\langle t\alpha(U, W), V\rangle - \langle S_{NW}U, V\rangle = -\langle S_{NW}U + t\alpha(U, W), V\rangle = -\langle(\nabla_U T)W, V\rangle.\end{aligned}$$

Hence, the desired result is achieved. □

Proposition 3.5. *Let $\mathcal{M}$ be a submanifold of a cosymplectic manifold $\tilde{\mathcal{M}}$. Then, the covariant derivation of the tensor field n satisfies*

$$\langle(\nabla_U n)Y, Z\rangle = -\langle(\nabla_U n)Z, Y\rangle,$$

for every $U \in \Gamma(\mathcal{TM})$, and $Y, Z \in \Gamma(\mathcal{T}^{\perp}\mathcal{M})$, i.e., the tensor field n is skew-symmetric.

Proof. Consider $U \in \Gamma(\mathcal{TM})$, and $Y, Z \in \Gamma(\mathcal{T}^{\perp}\mathcal{M})$; from (2.8), (3.3), and (3.11), we find

$$\begin{aligned}\langle(\nabla_U n)Y, Z\rangle &= -\langle\alpha(tY, U), Z\rangle - \langle NS_Y U, Z\rangle = -\langle S_Z U, tY\rangle + \langle S_Y U, tZ\rangle \\ &= \langle NS_Z U, Y\rangle + \langle\alpha(tZ, U), Y\rangle = -\langle(\nabla_U n)Z, Y\rangle,\end{aligned}$$

which is the intended result. □

Proposition 3.6. *Let $\mathcal{M}$ be a submanifold of a cosymplectic manifold $\tilde{\mathcal{M}}$. Then, the relation between the covariant derivation of the tensor fields N, and t is given by*

$$\langle(\nabla_U t)Y, V\rangle = -\langle(\nabla_U N)V, Y\rangle, \tag{3.13}$$

for every $U, V \in \Gamma(\mathcal{T}\mathcal{M})$, and $Y \in \Gamma(\mathcal{T}^\perp\mathcal{M})$.

Proof. Consider $U, V \in \Gamma(\mathcal{T}\mathcal{M})$, and $Y \in \Gamma(\mathcal{T}^\perp\mathcal{M})$. By (2.8), (3.3), (3.9), and (3.10), it follows that

$$\begin{aligned}\langle(\nabla_U t)Y, V\rangle &= \langle S_{nY}U, V\rangle - \langle TS_YU, V\rangle = \langle\alpha(U, V), nY\rangle + \langle S_YU, TV\rangle \\ &= -\langle n\alpha(U, V), Y\rangle + \langle\alpha(U, TV), Y\rangle = -\langle(\nabla_U N)V, Y\rangle.\end{aligned}$$

Consequently, the proof is completed □

4. Pointwise pseudo-slant submanifolds of a cosymplectic manifold

In the present section, we obtain a basic introduction to pointwise pseudo-slant submanifolds in a cosymplectic manifold and we provide certain results.

First, we give the following definitions of slant and pointwise slant submanifolds in contact geometry.

Definition 4.1. *[5, 14] A submanifold $\mathcal{M}$ of an almost contact metric manifold $\tilde{\mathcal{M}}$ is said to be a slant submanifold if the Wirtinger angle $\theta(U)$ between φU and $\mathcal{T}_x\mathcal{M}$ is constant for any point $x \in \mathcal{M}$ and for any nonzero $U \in \mathcal{T}_x\mathcal{M}\backslash\{\xi\}$.*

Definition 4.2. *[12] Let $\mathcal{M}$ be a submanifold of an almost contact metric manifold $\tilde{\mathcal{M}}$. Then, $\mathcal{M}$ is called a pointwise slant submanifold if at any point $x \in \mathcal{M}$ the Wirtinger angle $\theta(U)$ between φU and $\mathcal{T}_x\mathcal{M}$ is constant for each nonzero $U \in \mathcal{T}_x\mathcal{M}\backslash\{\xi\}$. That is, $\theta(U)$ only depends on the selection of x and does not depend on the selection of U.*

On the pointwise slant submanifold, $\theta(U)$ can be considered a function called a slant function. If for any point $x \in \mathcal{M}$, $\theta = 0$ (resp. $\theta = \frac{\pi}{2}$), then $\mathcal{M}$ is called an invariant submanifold (resp. anti-invariant submanifold). Otherwise, $\mathcal{M}$ is called proper pointwise slant submanifold.

We remember the following theorem for a pointwise slant submanifold in an almost contact metric manifold $\tilde{M}$ [12] .

Theorem 4.1. *A submanifold $\mathcal{M}$ tangent to the structure vector field ξ is a pointwise slant submanifold of an almost contact metric manifold $\tilde{\mathcal{M}}$ if and only if there exists a real-valued function θ on $\mathcal{M}$ such that*

$$T^2U = -\cos^2\theta\{U - \eta(U)\xi\}, \tag{4.1}$$

for any $U \in \Gamma(\mathcal{T}\mathcal{M})$.

This theorem leads to the following lemma.

Lemma 4.1. *Let $\mathcal{M}$ be a pointwise slant submanifold of an almost contact metric manifold $\tilde{\mathcal{M}}$ with slant function θ. Then, we have*

$$\langle TU, TV\rangle = \cos^2\theta\,\{\langle U, V\rangle - \eta(U)\eta(V)\}, \tag{4.2}$$

$$\langle NU, NV\rangle = \sin^2\theta\{\langle U, V\rangle - \eta(U)\eta(V)\}, \tag{4.3}$$

for every $U, V \in \Gamma(\mathcal{TM})$.

Also, there is a crucial relation for a pointwise slant submanifold of $\tilde{M}$ comes from (2.1), and (4.1) as follows [16]:

$$(a)\quad tNU = -\sin^2\theta\{U - \eta(U)\xi\}, \quad (b)\quad nNU = -NTU, \tag{4.4}$$

for any $U \in \Gamma(\mathcal{TM})$.

We now introduce the main concept studied in this paper as follows.

Definition 4.3. *[13] Let $\mathcal{M}$ be a submanifold of an almost contact metric manifold $\tilde{\mathcal{M}}$. Then, $\mathcal{M}$ is called a pointwise pseudo-slant submanifold if there exist three orthogonal complementary distributions $\mathcal{D}^\theta$, $\mathcal{D}^\perp$ and $\langle\xi\rangle$ such that*

(a) $\mathcal{TM} = \mathcal{D}^\theta \oplus \mathcal{D}^\perp \oplus \langle\xi\rangle$.
(b) $\mathcal{D}^\theta$ is a pointwise slant distribution with slant function θ.
(c) $\mathcal{D}^\perp$ is an anti-invariant distribution, that is, $\varphi\mathcal{D}^\perp \subset \mathcal{T}^\perp\mathcal{M}$.

It ought to be observed that if $dim(\mathcal{D}^\perp) = 0$, then $\mathcal{M}$ is a pointwise slant submanifold and if $dim(\mathcal{D}^\theta) = 0$, then $\mathcal{M}$ is an anti-invariant submanifold. Thus, we claim that the dimensions of all three distributions are non-zero and such submanifold is called a *proper pointwise pseudo-slant submanifold.*

Now, we construct the following example of proper pointwise pseudo-slant submanifold in the Euclidean space.

Example 4.1. Given the Euclidean 11-space $\mathbb{R}^{11}$ with the standard cartesian coordinates $(x_1, \cdots x_5, y_1, \cdots y_5, z)$ and the almost contact metric structure $(\varphi, \xi, \eta, \langle\,,\,\rangle)$ given by

$$\varphi\Big(\frac{\partial}{\partial x_i}\Big) = -\frac{\partial}{\partial y_i}, \ \ \varphi\Big(\frac{\partial}{\partial y_j}\Big) = \frac{\partial}{\partial x_j}, \ \ \varphi\Big(\frac{\partial}{\partial z}\Big) = 0, \ \ 1 \le i, j \le 5,$$

with $\xi = \dfrac{\partial}{\partial z}$, $\eta = dz$ and the standard Euclidean metric $\langle\,,\,\rangle$ on $\mathbb{R}^{11}$.

Let us consider a submanifold $\mathcal{M}$ of $\mathbb{R}^{11}$ defined by the immersion $f : \mathbb{R}^5 \to \mathbb{R}^{11}$ as follows:

$$f(r,\ s,\ u,\ v,\ z) = (e^u,\ v,\ r,\ r^2,\ \frac{2s^3}{3},\ u^2 + 3u,\ -v,\ s,\ \frac{2r^3}{3},\ s^2,\ z),$$

for non vanishing real valued functions $r,\ s$ on $\mathcal{M}$. Thus, the tangent bundle of $\mathcal{M}$ is generated by the following orthogonal vector fields:

$$e_1 = \frac{\partial}{\partial x_3} + 2r\frac{\partial}{\partial x_4} + 2r^2\frac{\partial}{\partial y_4}, \ \ e_2 = 2s^2\frac{\partial}{\partial x_5} + \frac{\partial}{\partial y_3} + 2s\frac{\partial}{\partial y_5},$$
$$e_3 = e^u\frac{\partial}{\partial x_1} + (2u+3)\frac{\partial}{\partial y_1}, \ \ e_4 = \frac{\partial}{\partial x_2} - \frac{\partial}{\partial y_2}, \ \ e_5 = \frac{\partial}{\partial z}.$$

Then, we have

$$\varphi e_1 = -\frac{\partial}{\partial y_3} - 2r\frac{\partial}{\partial y_4} + 2r^2\frac{\partial}{\partial x_4}, \ \ \varphi e_2 = -2s^2\frac{\partial}{\partial y_5} + \frac{\partial}{\partial x_3} + 2s\frac{\partial}{\partial x_5},$$

$$\varphi e_3 = -e^u \frac{\partial}{\partial y_1} + (2u+3)\frac{\partial}{\partial x_1}, \ \ \varphi e_4 = -\frac{\partial}{\partial y_2} - \frac{\partial}{\partial x_2}, \ \ \varphi e_5 = 0.$$

It is clear that, $\mathcal{D}^\theta = \mathrm{Span}\{e_1, e_2\}$ is a pointwise slant distribution with slant function $\theta = \cos^{-1}\left(\frac{1}{(1+2r^2)(1+2s^2)}\right)$, as $r, s\,(r \neq s)$ are non vanishing real valued functions on $\mathcal{M}$ while $\mathcal{D}^\perp = \mathrm{Span}\{e_3, e_4\}$ is an anti-invariant distribution since φe_3 and φe_4 are orthogonal to $\mathcal{TM}$. Thus, $\mathcal{M}$ is a 5- dimensional proper pointwise pseudo-slant submanifold of $\mathbb{R}^{11}$.

Let $\mathcal{M}$ be a proper pointwise pseudo-slant submanifold of a cosymplectic manifold $\tilde{\mathcal{M}}$. By definition, the distributions $\mathcal{D}^\theta$ and $\mathcal{D}^\perp$ are orthogonal on $\mathcal{M}$. Thus, $\langle U, W\rangle = 0$, for any $U \in \Gamma(\mathcal{D}^\theta)$ and $W \in \Gamma(\mathcal{D}^\perp)$. From (2.2) and (3.1), we observe that

$$\langle NU, NW\rangle = \langle \varphi U, \varphi W\rangle = \langle U, W\rangle = 0,$$

for any $U \in \Gamma(\mathcal{D}^\theta)$ and $W \in \Gamma(\mathcal{D}^\perp)$, which shows that the distributions $N(\mathcal{D}^\theta)$ and $N(\mathcal{D}^\perp)$ are also orthogonal. Therefore, the normal bundle $\mathcal{T}^\perp\mathcal{M}$ of $\mathcal{M}$ can be divided as the following orthogonal decomposition:

$$\mathcal{T}^\perp\mathcal{M} = N(\mathcal{D}^\theta) \oplus N(\mathcal{D}^\perp) \oplus \beta,$$

where β is the orthogonal complementary distribution of $\varphi(\mathcal{TM})$ in $\mathcal{T}^\perp\mathcal{M}$. The following lemma can be immediately obtained.

Lemma 4.2. *Let $\mathcal{M}$ be a proper pointwise pseudo-slant submanifold in a cosymplectic manifold $\tilde{\mathcal{M}}$. Then, we have*

$$T(\mathcal{D}^\theta) = \mathcal{D}^\theta \quad \textit{and} \quad T(\mathcal{D}^\perp) = \{0\}. \tag{4.5}$$

Lemma 4.3. *Let $\mathcal{M}$ be a proper pointwise pseudo-slant submanifold in a cosymplectic manifold $\tilde{\mathcal{M}}$. Then, the shape operator satisfies*

$$S_{NW}X = S_{NX}W, \tag{4.6}$$

for every $W, X \in \Gamma(\mathcal{D}^\perp)$.

Proof. Let $U \in \Gamma(\mathcal{TM})$ and $W, X \in \Gamma(\mathcal{D}^\perp)$; by (2.3), (2.5)–(2.8) and (4.5) with the symmetry property of the shape operator, we obtain

$$\begin{aligned}\langle S_{NW}X, U\rangle &= \langle \alpha(X, U), NW\rangle = \langle \tilde{\nabla}_U X, \varphi W\rangle - \langle \nabla_U X, \varphi W\rangle \\ &= -\langle \varphi\tilde{\nabla}_U X, W\rangle = -\langle \tilde{\nabla}_U \varphi X, W\rangle = -\langle \tilde{\nabla}_U NX, W\rangle \\ &= \langle S_{NX}U, W\rangle - \langle \nabla^\perp_U NX, W\rangle = \langle S_{NX}W, U\rangle.\end{aligned}$$

Thus, we get the relation (4.6). □

Theorem 4.2. *Let $\mathcal{M}$ be a proper pointwise pseudo-slant submanifold in a cosymplectic manifold $\tilde{\mathcal{M}}$. Then, we receive that*

$$\nabla^\perp_W NX - \nabla^\perp_X NW \in N(\mathcal{D}^\perp),$$

for every $W, X \in \Gamma(\mathcal{D}^\perp)$.

Proof. Consider $W, X \in \Gamma(\mathcal{D}^\perp)$ and $\zeta \in \Gamma(\beta)$. According to (2.3), (2.5)–(2.7), (3.1), (4.5) and (4.6), we find

$$\begin{aligned}\langle \nabla^\perp_W NX - \nabla^\perp_X NW, \zeta\rangle &= \langle \tilde{\nabla}_W NX - S_{NX}W - \tilde{\nabla}_X NW + S_{NW}X, \zeta\rangle = \langle \tilde{\nabla}_W \varphi X, \zeta\rangle - \langle \tilde{\nabla}_X \varphi W, \zeta\rangle \\ &= \langle \varphi\tilde{\nabla}_W X, \zeta\rangle - \langle \varphi\tilde{\nabla}_X W, \zeta\rangle = \langle \tilde{\nabla}_X W, \varphi\zeta\rangle - \langle \tilde{\nabla}_W X, \varphi\zeta\rangle \\ &= \langle \nabla_X W, \varphi\zeta\rangle + \langle \alpha(X, W), \varphi\zeta\rangle - \langle \nabla_W X, \varphi\zeta\rangle - \langle \alpha(W, X), \varphi\zeta\rangle = 0,\end{aligned}$$

which is the intended result. □

In the following theories, we present the integrability condition of the distributions for the pointwise pseudo-slant submanifold $\mathcal{M}$ in a cosymplectic manifold $\tilde{\mathcal{M}}$.

Theorem 4.3. *Let $\mathcal{M}$ be a proper pointwise pseudo-slant submanifold of a cosymplectic manifold $\tilde{\mathcal{M}}$. Then, the pointwise slant distribution $\mathcal{D}^\theta$ is integrable if and only if*

$$\alpha(U, TV) - \alpha(V, TU) + \nabla^\perp_U NV - \nabla^\perp_V NU \in N(D^\theta) \oplus \beta, \tag{4.7}$$

for all $U, V \in \Gamma(\mathcal{D}^\theta)$.

Proof. Given any $U, V \in \Gamma(\mathcal{D}^\theta)$ and $W \in \Gamma(\mathcal{D}^\perp)$, by the use of (2.2), (2.5)–(2.7), (3.1) and (4.5), we derive

$$\begin{aligned}\langle [U, V], W\rangle &= \langle \tilde{\nabla}_U V, W\rangle - \langle \tilde{\nabla}_V U, W\rangle \\ &= \langle \varphi\tilde{\nabla}_U V, \varphi W\rangle + \eta(\tilde{\nabla}_U V)\eta(W) - \langle \varphi\tilde{\nabla}_V U, \varphi W\rangle - \eta(\tilde{\nabla}_V U)\eta(W) \\ &= \langle \tilde{\nabla}_U \varphi V, NW\rangle - \langle \tilde{\nabla}_V \varphi U, NW\rangle \\ &= \langle \tilde{\nabla}_U TV + \tilde{\nabla}_U NV, NW\rangle - \langle \tilde{\nabla}_V TU - \tilde{\nabla}_V NU, NW\rangle \\ &= \langle \alpha(U, TV) + \nabla^\perp_U NV - \alpha(V, TU) - \nabla^\perp_V NU, NW\rangle.\end{aligned}$$

As, $NW \in N(\mathcal{D}^\perp) \subseteq \mathcal{T}^\perp\mathcal{M}$. So, by the above relation, we conclude that the distribution $\mathcal{D}^\theta$ is integrable if and only if the relation (4.7) holds. □

Theorem 4.4. *Let $\mathcal{M}$ be a proper pointwise pseudo-slant submanifold in a cosymplectic manifold $\tilde{\mathcal{M}}$. Then, the anti-invariant distribution $\mathcal{D}^\perp$ is always integrable.*

Proof. Consider $U \in \Gamma(\mathcal{D}^\theta)$ and $W, X \in \Gamma(\mathcal{D}^\perp)$. From (2.2), (2.3), (2.5)–(2.7), (3.1) and (4.5), we obtain

$$\begin{aligned}\langle [W, X], U\rangle &= \langle \tilde{\nabla}_W X - \tilde{\nabla}_X W, U\rangle = \langle \tilde{\nabla}_X U, W\rangle - \langle \tilde{\nabla}_W U, X\rangle \\ &= \langle \varphi\tilde{\nabla}_X U, \varphi W\rangle + \eta(\tilde{\nabla}_X U)\eta(W) - \langle \varphi\tilde{\nabla}_W U, \varphi X\rangle - \eta(\tilde{\nabla}_W U)\eta(X) \\ &= \langle \tilde{\nabla}_X \varphi U, \varphi W\rangle - \langle \tilde{\nabla}_W \varphi U, \varphi X\rangle \\ &= \langle \tilde{\nabla}_X TU + \tilde{\nabla}_X NU, NW\rangle - \langle \tilde{\nabla}_W TU + \tilde{\nabla}_W NU, NX\rangle \\ &= \langle \alpha(X, TU), NW\rangle + \langle \nabla^\perp_X NU, NW\rangle - \langle \alpha(W, TU), NX\rangle - \langle \nabla^\perp_W NU, NX\rangle.\end{aligned}$$

In view of (2.8), (3.5), (3.9), and (4.3)–(4.6), we arrive at

$$\langle [W, X], U\rangle = \langle S_{NW}X, TU\rangle + \langle (\nabla_X N)U, NW\rangle + \langle N\nabla_X U, NW\rangle - \langle S_{NX}W, TU\rangle$$

$$
\begin{aligned}
&-\langle(\nabla_W N)U, NX\rangle - \langle N\nabla_W U, NX\rangle\\
=&\langle n\alpha(X,U), NW\rangle - \langle\alpha(X,TU), NW\rangle + \sin^2\theta\langle\nabla_X U, W\rangle\\
&-\langle n\alpha(W,U), NX\rangle + \langle\alpha(W,TU), NX\rangle - \sin^2\theta\langle\nabla_W U, X\rangle\\
=&-\langle\alpha(X,U), nNW\rangle - \langle S_{NW}X, TU\rangle - \sin^2\theta\langle\nabla_X W, U\rangle\\
&+\langle\alpha(W,U), nNX\rangle + \langle S_{NX}W, TU\rangle + \sin^2\theta\langle\nabla_W X, U\rangle\\
=&\langle\alpha(X,U), NTW\rangle - \sin^2\theta\langle\nabla_X W, U\rangle - \langle\alpha(W,U), NTX\rangle + \sin^2\theta\langle\nabla_W X, U\rangle\\
=&\sin^2\theta\langle[W,X], U\rangle.
\end{aligned}
$$

Therefore, we achieve

$$\cos^2\theta\langle[W,X], U\rangle = 0.$$

By the above relation, we conclude that $[W,X] \in \Gamma(\mathcal{D}^\perp)$ for every $W, X \in \Gamma(\mathcal{D}^\perp)$. That is, the distribution $\mathcal{D}^\perp$ is always integrable. Thus, the theorem is proved. □

Corollary 4.1. *Let $\mathcal{M}$ be a proper pointwise pseudo-slant submanifold in a cosymplectic manifold $\tilde{\mathcal{M}}$. Then, we acquire that*

$$(\nabla_W T)X = (\nabla_X T)W,$$

for each $W, X \in \Gamma(D^\perp)$.

Proof. Since the distribution $\mathcal{D}^\perp$ is always integrable, then $T[W,X] = 0$, for any $W, X \in \Gamma(\mathcal{D}^\perp)$. That is

$$T\nabla_W X = T\nabla_X W.$$

Thus, from (3.4) and (4.5), we get the required result. □

Theorem 4.5. *Let $\mathcal{M}$ be a proper pointwise pseudo-slant submanifold of a cosymplectic manifold $\tilde{\mathcal{M}}$. Then, the distribution $\mathcal{D}^\theta \oplus \mathcal{D}^\perp$ is always integrable.*

Proof. Let $U, W \in \Gamma(\mathcal{D}^\theta \oplus \mathcal{D}^\perp)$; then, we have

$$\langle[U,W], \xi\rangle = \langle\nabla_U W, \xi\rangle - \langle\nabla_W U, \xi\rangle = -\langle\nabla_U \xi, W\rangle + \langle\nabla_W \xi, U\rangle.$$

From (2.9), we obtain

$$\langle[U,W], \xi\rangle = 0.$$

Thus, we conclude that $[U,W] \in \Gamma(\mathcal{D}^\theta \oplus \mathcal{D}^\perp)$, for each $U, W \in \Gamma(\mathcal{D}^\theta \oplus \mathcal{D}^\perp)$. Therefore, the distribution $\mathcal{D}^\theta \oplus \mathcal{D}^\perp$ is always integrable. This completes the proof. □

Remark 4.1. *Since the distribution $\mathcal{TM} - \langle\xi\rangle$ is a proper subset of $Ker\,\eta = \{U : \langle U, \xi\rangle = 0\}$ and from the closure of η it follows that the distribution $Ker\,\eta$ is integrable, i.e., for any $U, W \in \mathcal{TM} - \langle\xi\rangle$ we have $\langle[U,W], \xi\rangle = 0$ automatically.*

5. Totally geodesic and totally umbilical pointwise pseudo-slant submanifolds in a cosymplectic manifold

In the present section, we give some conditions for the pointwise pseudo-slant submanifold to be $\mathcal{D}^\theta$ or $\mathcal{D}^\perp$-geodesic and mixed geodesic submanifold of a cosymplectic manifold $\tilde{\mathcal{M}}$. Then, we study the geometry of the leaves of pointwise pseudo-slant submanifold. At last, we examine the minimality of the pointwise slant distribution $\mathcal{D}^\theta$ on the proper pointwise pseudo-slant submanifold $\mathcal{M}$ of a cosymplectic manifold $\tilde{\mathcal{M}}$.

Definition 5.1. *The pointwise pseudo-slant submanifold $\mathcal{M}$ of a cosymplectic manifold $\tilde{\mathcal{M}}$ is called $\mathcal{D}^\theta$-geodesic (resp. $\mathcal{D}^\perp$-geodesic) if $\alpha(U,V) = 0$ for every $U,V \in \Gamma(\mathcal{D}^\theta)$ (resp. $\alpha(W,X) = 0$ for every $W,X \in \Gamma(\mathcal{D}^\perp)$) and $\mathcal{M}$ is called mixed geodesic submanifold if $\alpha(U,W) = 0$ for every $U \in \Gamma(\mathcal{D}^\theta)$ and $W \in \Gamma(\mathcal{D}^\perp)$.*

Theorem 5.1. *Let $\mathcal{M}$ be a proper pointwise pseudo-slant submanifold in a cosymplectic manifold $\tilde{\mathcal{M}}$. If the tensor n is parallel, then $\mathcal{M}$ is $\mathcal{D}^\theta$-geodesic submanifold of $\tilde{\mathcal{M}}$.*

Proof. Assume that n is parallel, by the consequence of (3.1), (3.11) and (4.5), we obtain

$$\alpha(tY,U) + \varphi S_Y U = 0$$

for any $U \in \Gamma(\mathcal{D}^\theta)$ and $Y \in \Gamma(\mathcal{T}^\perp\mathcal{M})$. Operating the almost contact structure φ to the above relation by using (2.1) and (3.2), we get

$$t\alpha(tY,U) + n\alpha(tY,U) - S_Y U + \eta(S_Y U) = 0.$$

By taking the inner product with $V \in \Gamma(\mathcal{D}^\theta)$ into the above expression, then applying (2.8), (3.3), (3.12) and (4.4), we arrive at

$$\begin{aligned}\langle\alpha(U,V),Y\rangle &= \langle t\alpha(tY,U),V\rangle = -\langle\alpha(tY,U),NV\rangle = -\langle S_{NV}tY,U\rangle\\ &= -\langle S_Y tNV,U\rangle = \sin^2\theta\,\langle S_Y V,U\rangle = \sin^2\theta\,\langle\alpha(U,V),Y\rangle.\end{aligned}$$

Thus,

$$\cos^2\theta\,\langle\alpha(U,V),Y\rangle = 0.$$

Using the fact that $\mathcal{M}$ is a proper pointwise pseudo-slant submanifold, we conclude that $\alpha(U,V) = 0$ for every $U,V \in \Gamma(\mathcal{D}^\theta)$. Hence, $\mathcal{M}$ is $\mathcal{D}^\theta$-geodesic submanifold of $\tilde{\mathcal{M}}$. □

Theorem 5.2. *Let $\mathcal{M}$ be a proper pointwise pseudo-slant submanifold of a cosymplectic manifold $\tilde{\mathcal{M}}$. If the tensor t is parallel, then, $\mathcal{M}$ is $\mathcal{D}^\perp$-geodesic submanifold of $\tilde{\mathcal{M}}$.*

Proof. Assume that t is parallel, then using (3.10) and taking the inner product with $W \in \Gamma(\mathcal{D}^\perp)$, we derive

$$\langle S_{nY}X,W\rangle - \langle TS_Y X,W\rangle = 0$$

for any $W,X \in \Gamma(\mathcal{D}^\perp)$ and $Y \in \Gamma(\mathcal{T}^\perp\mathcal{M})$. In view of (2.8) and (3.3), we obtain that

$$\langle\alpha(X,W),nY\rangle + \langle S_Y X,TW\rangle = 0.$$

But, from (4.5), we get

$$\langle \alpha(X, W), nY \rangle = 0.$$

Thus, $\alpha(X, W) = 0$. That is, $\mathcal{M}$ is a $\mathcal{D}^{\perp}$-geodesic submanifold of $\tilde{\mathcal{M}}$. ☐

Theorem 5.3. *Let $\mathcal{M}$ be a proper pointwise pseudo-slant submanifold of a cosymplectic manifold $\tilde{\mathcal{M}}$. If the tensor t is parallel, then $\mathcal{M}$ is a mixed-geodesic submanifold of $\tilde{\mathcal{M}}$.*

Proof. If t is parallel, then from (3.9), (3.13) and (4.5), we obtain

$$n\alpha(U, W) = 0$$

for every $U \in \Gamma(D^{\theta})$ and $W \in \Gamma(D^{\perp})$. In other words, we can get

$$n\alpha(W, U) - \alpha(W, TU) = 0.$$

Replacing U with TU in the above relation and using (4.1), we derive

$$\cos^2\theta\, \alpha(W, U) - \cos^2\theta\, \eta(U)\, \alpha(\xi, W) = 0.$$

Using the fact that $\mathcal{M}$ is a proper pointwise pseudo-slant submanifold and the relation (2.9), we find $\alpha(U, W) = 0$. This shows that $\mathcal{M}$ is mixed-geodesic submanifold of $\tilde{\mathcal{M}}$. ☐

Theorem 5.4. *Let $\mathcal{M}$ be a proper pointwise pseudo-slant submanifold of a cosymplectic manifold $\tilde{\mathcal{M}}$. If the leaves of $\mathcal{D}^{\theta}$ are totally geodesic in $\mathcal{M}$, then*

$$\langle \alpha(U, V), NW \rangle = 0$$

for any $U, V \in \Gamma(\mathcal{D}^{\theta})$ and $W \in \Gamma(\mathcal{D}^{\perp})$.

Proof. The leaves of $\mathcal{D}^{\theta}$ are totally geodesic in $\mathcal{M}$ if for any $U, V \in \Gamma(\mathcal{D}^{\theta})$, $\nabla_U V \in \Gamma(\mathcal{D}^{\theta})$. Using (2.5)–(2.7), we obtain

$$\nabla_U \varphi V + \alpha(U, \varphi V) = \varphi \nabla_U V + \varphi \alpha(U, V).$$

Taking the inner product with $W \in \Gamma(\mathcal{D}^{\perp})$, we find

$$\langle \nabla_U \varphi V, W \rangle = \langle \varphi \nabla_U V, W \rangle - \langle \alpha(U, V), \varphi W \rangle.$$

Thus,

$$\langle \varphi \nabla_U V, W \rangle - \langle \nabla_U \varphi V, W \rangle = \langle \alpha(U, V), \varphi W \rangle.$$

Hence, the leaves of $\mathcal{D}^{\theta}$ are totally geodesic in $\mathcal{M}$ if the above relation is satisfied. ☐

Theorem 5.5. *Let $\mathcal{M}$ be a proper pointwise pseudo-slant submanifold of a cosymplectic manifold $\tilde{\mathcal{M}}$. If the leaves of $\mathcal{D}^{\perp}$ are totally geodesic in $\mathcal{M}$, then*

$$\langle \alpha(U, W), NX \rangle = 0,$$

for any $U \in \Gamma(\mathcal{D}^{\theta})$ and $W, X \in \Gamma(\mathcal{D}^{\perp})$.

Proof. By the consequence of (2.5)–(2.7) and (3.1), we obtain

$$\nabla_W TX + \alpha(W, TX) - S_{NX}W + \nabla^{\perp}_W NX = \varphi\nabla_W X + \varphi\alpha(W, X),$$

for any $W, X \in \Gamma(\mathcal{D}^{\perp})$. Taking the inner product with $U \in \Gamma(\mathcal{D}^{\theta})$, we find

$$\langle\nabla_W TX, U\rangle - \langle S_{NX}W, U\rangle = \langle\varphi\nabla_W X, U\rangle,$$

which gives that

$$\langle\nabla_W TX, U\rangle + \langle\nabla_W X, \varphi U\rangle = \langle\alpha(U, W), NX\rangle.$$

Thus, the leaves of $\mathcal{D}^{\perp}$ are totally geodesic in $\mathcal{M}$ if the above relation holds. □

Theorem 5.6. *Let $\mathcal{M}$ be a totally umbilical proper pointwise pseudo-slant submanifold of a cosymplectic manifold $\tilde{\mathcal{M}}$. In that case, at least one of the following assertions is true:*

- *(i) $dim(\mathcal{D}^{\perp}) = 1$,*
- *(ii) $H \in \Gamma(\beta)$,*
- *(iii) $\mathcal{M}$ is proper pointwise pseudo-slant submanifold.*

Proof. As $\tilde{\mathcal{M}}$ is a cosymplectic manifold, then for any $W \in \Gamma(\mathcal{D}^{\perp})$, we have

$$\tilde{\nabla}_W \varphi W - \varphi\tilde{\nabla}_W W = 0.$$

In view of (2.6), (3.1) and (4.5), we can write

$$\tilde{\nabla}_W NW - \varphi\nabla_W W - \varphi\alpha(W, W) = 0.$$

Using (2.7), (3.1), (3.2) and (4.5), we find that

$$-S_{NW}W + \nabla^{\perp}_W NW - N\nabla_W W - t\alpha(W, W) - n\alpha(W, W) = 0.$$

Comparing the tangential parts in the above expression, we obtain

$$S_{NW}W + t\alpha(W, W) = 0.$$

By taking the inner product with $X \in \Gamma(\mathcal{D}^{\perp})$ and using (2.8), the above relation becomes

$$\langle\alpha(W, X), NW\rangle + \langle t\alpha(W, W), X\rangle = 0.$$

Using the fact that $\mathcal{M}$ is totally umbilical submanifold, we get

$$\langle W, X\rangle\langle NW, H\rangle + \langle W, W\rangle\langle tH, X\rangle = 0,$$

or

$$\langle W, W\rangle\langle tH, X\rangle - \langle W, X\rangle\langle tH, W\rangle = 0.$$

The solution of the previous equation holds if either $dim(\mathcal{D}^{\perp}) = 1$, otherwise $H \in \Gamma(\beta)$ or $\mathcal{M}$ is proper pointwise pseudo-slant submanifold. □

Finally, we prove the minimality of the pointwise slant distribution $\mathcal{D}^\theta$ on the proper pointwise pseudo-slant submanifold $\mathcal{M}$ of a cosymplectic manifold $\tilde{\mathcal{M}}$.

Theorem 5.7. *Let $\mathcal{M}$ be a pointwise pseudo-slant submanifold of a cosymplectic manifold $\tilde{\mathcal{M}}$. Then the pointwise slant distribution $\mathcal{D}^\theta$ is minimal if and only if the normal connection is parallel and the following relation is verified:*

$$\langle (1 + \sec^2\theta + \sec^2\theta \tan\theta\, U(\theta))\, S_{NW}U, TU\rangle = 0$$

for every $U \in \Gamma(\mathcal{D}^\theta)$ and $W \in \Gamma(\mathcal{D}^\perp)$.

Proof. Given $U \in \Gamma(\mathcal{D}^\theta)$ and $W \in \Gamma(\mathcal{D}^\perp)$, we have

$$\langle \nabla_U U + \nabla_{\sec\theta TU} \sec\theta TU, W\rangle = \langle \tilde{\nabla}_U U, W\rangle + \langle \tilde{\nabla}_{\sec\theta TU} \sec\theta TU, W\rangle.$$

Using (2.2), (2.7), (2.8), (3.1) and (4.5), we derive

$$\begin{aligned}
&\langle \nabla_U U + \nabla_{\sec\theta TU} \sec\theta TU, W\rangle \\
=&\langle \tilde{\nabla}_U \varphi U, \varphi W\rangle + \eta(\tilde{\nabla}_U U)\eta(W) + \sec^2\theta\, [\langle \tilde{\nabla}_U \varphi U, \varphi W\rangle + \eta(\tilde{\nabla}_U U)\eta(W)] \\
&+ \sec^2\theta \tan\theta\, U(\theta)\, [\langle \tilde{\nabla}_U \varphi U, \varphi W\rangle + \eta(\tilde{\nabla}_U U)\eta(W)] \\
=&\langle \tilde{\nabla}_U TU, NW\rangle + \langle \tilde{\nabla}_U NU, NW\rangle + \sec^2\theta\, \langle \tilde{\nabla}_U TU, NW\rangle + \sec^2\theta\, \langle \tilde{\nabla}_U NU, NW\rangle \\
&+ \sec^2\theta \tan\theta\, U(\theta)\, [\langle \tilde{\nabla}_U TU, NW\rangle + \langle \tilde{\nabla}_U NU, NW\rangle] \\
=&-\langle \tilde{\nabla}_U NW, TU\rangle + \langle \nabla^\perp_U NU, NW\rangle - \sec^2\theta\, \langle \tilde{\nabla}_U NW, TU\rangle + \sec^2\theta\, \langle \nabla^\perp_U NU, NW\rangle \\
&- \sec^2\theta \tan\theta\, U(\theta)\, [\langle \tilde{\nabla}_U NW, TU\rangle - \langle \tilde{\nabla}_U NU, NW\rangle] \\
=&\langle S_{NW}U + \sec^2\theta\, S_{NW}U + \sec^2\theta \tan\theta\, U(\theta)\, S_{NW}U, TU\rangle \\
&+ \langle \nabla^\perp_U NU + \sec^2\theta\, \nabla^\perp_U NU + \sec^2\theta \tan\theta\, U(\theta) \nabla^\perp_U NU, NW\rangle.
\end{aligned}$$

Hence, we conclude that the distribution $\mathcal{D}^\theta$ is minimal if and only if the normal connection is parallel and

$$\langle (1 + \sec^2\theta + \sec^2\theta \tan\theta\, U(\theta))\, S_{NW}U, TU\rangle = 0.$$

□

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares that he has no conflicts of interest.

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