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Research article

New applications of Carleson measures on $\mathcal{N}_K(p,q)$-type spaces in the unit ball of $\mathbb{C}^n$

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Abstract: This paper aims to examine a weighted class of function spaces, referred to as $\mathcal{N}_K(p,q)$-type spaces, defined on the unit ball of $\mathbb{C}^n$. The study explores the characterization of Carleson measures associated with these spaces and establishes embedding theorems connecting $\mathcal{N}_K(p,q)$-type spaces with weighted Hardy and Bergman spaces. In addition, applications involving Hadamard products and random power series within $\mathcal{N}_K(p,q)$-type spaces are discussed.

Keywords: $\mathcal{N}_K(p,q)$-type classes; Carleson measures; Hadamard products; random power series
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1. Introduction

In this work, the unit ball in $\mathbb{C}^n$ will be denoted by $\mathbb{B}$ ($\mathbb{B}$ is a complete Reinhardt domain in $\mathbb{C}^n$). $\mathbb{S}$ denotes the boundary of $\mathbb{B}$, and $\mathbb{U}$ represents the unit polydisc in $\mathbb{C}^n$. We denote by $\mathcal{HO}(\mathbb{B})$ and $\mathcal{HO}(\mathbb{U})$ the sets of holomorphic functions on $\mathbb{B}$ and $\mathbb{U}$, respectively.

As is customary, the Banach space $H^\infty(\mathbb{B})$ consists of all bounded functions $g \in \mathcal{HO}(\mathbb{B})$, furnished with the norm $\|g\|_\infty = \sup_{\xi\in\mathbb{B}} |g(\xi)|$. Similarly, $H^\infty(\mathbb{U})$ the Banach space in the unit polydisc $\mathbb{U}$.

The Beurling-type space $B^{-\gamma}(\mathbb{B})$ with $\gamma > 0$ (see for example, [19]) consists of all functions g belonging to $\mathcal{HO}(\mathbb{B})$, furnished with the norm

$$\|g\|_\gamma = \sup_{\xi\in\mathbb{B}} |g(\xi)|(1-|\xi|)^\gamma < \infty.$$

The little space $B_0^{-\gamma}(\mathbb{B})$ denotes the closed subspace of $B^{-\gamma}(\mathbb{B})$ satisfying

$$\lim_{|\xi|\to 1} |g(\xi)|(1-|\xi|)^\gamma = 0.$$

Denote by dV the normalized Lebesgue volume measure on $\mathbb{C}^n$ such that $V(\mathbb{B}) \equiv 1$. Likewise, let $d\sigma$ be the surface measure on $\mathbb{S}$, normalized by $\sigma(\mathbb{S}) \equiv 1$. For any $\xi \in \mathbb{B}$, the measures V and σ satisfy the following relation:

$$\int_{\mathbb{B}} g(\xi)dV(\xi) = 2n\int_0^1 s^{2n-1}ds\int_{\mathbb{S}} g(s\xi)d\sigma(\xi), \ \ 0 \le s < 1.$$

For any $w, z \in \mathbb{C}^n$ and $m \in \mathbb{Z}_+^n$, where $w = (w_1, \cdots, w_n), z = (z_1, \cdots, z_n)$ and $m = (m_1, \cdots, m_n)$, we adopt the notation

$$\begin{aligned}
&\overline{w} = (\overline{w_1}, \cdots, \overline{w_n}), \quad w^m = w_1^{m_1} \cdots w_n^{m_n}, \\
&|m| = m_1 + \cdots + m_n, \quad m! = m_1! \cdots m_n!, \\
&\partial_\alpha = \frac{\partial}{\partial_\alpha}, \ 1 \le \alpha \le n, \quad \partial^m = \partial_1^{m_1} \cdots \partial_n^{m_n}, \\
&\langle w, z\rangle = w_1\overline{z_1} + \cdots + w_n\overline{z_n}), \quad |z| = \sqrt{\langle z, z\rangle}.
\end{aligned}$$

Because $\mathbb{B}$ is a complete Reinhardt domain in $\mathbb{C}^n$, it follows that whenever $w \in \mathbb{B}$, we also have

$$w \cdot \zeta = (w_1\zeta_1, \cdots, w_n\zeta_n) \in \mathbb{B},$$

for every $\zeta \in \overline{\mathbb{U}}$, where $\overline{\mathbb{U}}$ denotes the closure of the unit polydisc in $\mathbb{C}^n$.

Then, every function $g \in \mathcal{HO}(\mathbb{B})$ possesses a unique power series representation $g = \sum_m b_m w^m$, and so $g \in \mathcal{HO}(\mathbb{B})$ can be viewed as the space of multi-index sequences $\{b_m\}$, where

$$b_m = \frac{\partial^m g(0)}{m!}, \ \ m \in \mathbb{Z}_+^n.$$

The weighted Bloch space $\mathcal{B}^\gamma(\mathbb{B}), \gamma > 0$ is the set of all functions g belonging to $\mathcal{HO}(\mathbb{B})$, furnished with the norm

$$||g||_{\mathcal{B}^\gamma} = |g(0)| + \sup_{\xi\in\mathbb{B}} |\nabla g(\xi)|(1 - |\xi|)^\gamma < \infty,$$

where $\nabla = (\partial_1, \cdots, \partial_n)$. The space $\mathcal{B}^1(\mathbb{B})$ corresponds to the classical Bloch space $\mathcal{B}(\mathbb{B})$.

For $d > 0$ and $\kappa > -1$, the weighted Bergman-type space $A_\kappa^d(\mathbb{B})$ is the set of functions $h \in \mathcal{HO}(\mathbb{B})$, such that

$$||h||_{A_\kappa^d(\mathbb{B})}^d = \int_{\mathbb{B}} |h(\xi)|^d dV_\kappa(\xi) < \infty,$$

where $dV_\kappa(\xi) = C_\kappa(1-|\xi|^2)^\kappa dV(\xi)$ is the weighted Lebesgue measure, and C_κ is a normalizing constant. When $\kappa > -1$, the positive measure dV_κ was normalized to be a probability measure. We know that the space $A_\kappa^d(\mathbb{B}), 1 \le d < \infty$ is a Banach space, and $A_\kappa^d(\mathbb{B}), 0 < d < 1$, is a complete metric space. Furthermore, $A_\kappa^d(\mathbb{B})$ with $\kappa = 0$ is the classical Bergman space. Several properties of $A_\kappa^d(\mathbb{B})$ spaces are discussed in [17, 20].

The Möbius invariant $d\lambda$ on $\mathbb{B}$ (see, e.g., [14]) is denoted by

$$d\lambda(\xi) = dV_{-n-1} = (1 - |\xi|^2)^{-n-1}dV.$$

Let ψ be an automorphism, and then the Möbius invariant $d\lambda$ is such that

$$\int_{\mathbb{B}} g(\xi)d\lambda(\xi) = \int_{\mathbb{B}} g \circ \psi(\xi)d\lambda(\xi), \quad \forall \ g \in L^1(\mathbb{B}).$$

Let $\Theta_b(\xi)$ be the automorphism of $\mathbb{B}$, for $b \in \mathbb{B}$, that is

$$\Theta_b(\xi) = \frac{b - P_b\xi - \sqrt{1 - |b|^2}(I - P_b)\xi}{1 - \langle \xi, b\rangle},$$

where P_b is the orthogonal projection into the space spanned by b (see, e.g., [14, 18]). The transformation Θ_b satisfies that $\Theta_b(0) = b, \Theta_b(b) = 0$ and $\Theta_b = \Theta_b^{-1}$. Moreover, for $\xi \in \mathbb{B}$, we obtain

$$1 - |\Theta_b(\xi)|^2 = \frac{(1 - |\xi|^2)(1 - |b|^2)}{|1 - \langle \xi, b\rangle|^2}.$$

With respect to a nonzero weighted function $K \in \mathcal{RC}^+$ (where $\mathcal{RC}^+$ denotes the set of all positive, nondecreasing and right-continuous functions $K : [0, \infty) \longrightarrow [0, \infty)$). In [1], we have provided an extensive class $\mathcal{N}_K(p, q)$-type space defined on $\mathbb{B}$.

Suppose that $K \in \mathcal{RC}^+$ and $p, q > 0$. The large family $\mathcal{N}_K(p, q)$-type class is defined as

$$\mathcal{N}_K(p, q) := \{g \in \mathcal{HO}(\mathbb{B}) : \|g\|_K < \infty\}$$

wherever

$$\|g\|_K^p = \sup_{b\in\mathbb{B}} \int_{\mathbb{B}} |g(\xi)|^p(1 - |\xi|^2)^q K(1 - |\Theta_b(\xi)|^2)d\lambda(\xi).$$

Moreover, the corresponding little space $\mathcal{N}_{K,0}(p, q)$ is the class of functions $g \in \mathcal{N}_K(p, q)$, such that

$$\lim_{|b|\to 1} \int_{\mathbb{B}} |g(\xi)|^p(1 - |\xi|^2)^q K(1 - |\Theta_b(\xi)|^2)d\lambda(\xi) = 0.$$

The functional spaces $\mathcal{N}_K(p, q)$ and $\mathcal{N}_{K,0}(p, q)$ were shown to be Banach spaces in [1]. The definition of the broad family of $\mathcal{N}_K(p, q)$-type spaces encompasses most Möbius-invariant function spaces, many of which appear as special cases, including the following examples.

- If $p = 2, q = n + 1$, then $\mathcal{N}_K(2, n + 1) = \mathcal{N}_K$, as well as $\mathcal{N}_{K,0}(2, n + 1) = \mathcal{N}_{K,0}$.
- If we replace $g \in \mathcal{HO}(\mathbb{B})$ by its gradient ∇g, then we obtain $Q_K(p, q)$-spaces. Moreover, $\mathcal{N}_K(2, n+1)$ becomes Q_K-spaces.
- If we let $K(x) = x^s$, $s \geq 0$ and replace $g \in \mathcal{HO}(\mathbb{B})$ by ∇g, we obtain $F(p, q, s)$-spaces.
- Note that $F(p, q, 0) = \mathcal{D}_q^p(\mathbb{B})$ (the Dirichlet type space), $F(2, 0, s) = Q_s(\mathbb{B})$ and $F(2, 0, 1) = BMOA(\mathbb{B})$.
- If we let $K(x) = 1$ and replace $g \in \mathcal{HO}(\mathbb{B})$ by ∇g for $q = p - n - 1, p > n$, then we obtain the Besov space $B_p(\mathbb{B})$.
- If we let $K(x) = x^{ns}$ for $s > 0$, then we obtain an $\mathcal{N}(p, q, s)$-type space. Furthermore, $\mathcal{N}_K(p, q)$ is $F(p, p + q - n - 1, ns)$, and $\mathcal{N}_{K,0}(p, q)$ becomes $F_0(p, p + q - n - 1, ns)$.

For detailed information on these spaces, readers are directed to articles [3, 6, 13]. Embedding theorems for various analytic function spaces were established in [8, 11]. Additionally, various studies investigate Hadamard products—defined as termwise multiplication of power series—in conjunction with random power series, particularly within analytic function theory, moments of random vectors, and algebraic properties (see, e.g., [5, 7, 9]). These results link to random power series through moment estimates and asymptotic analyses in probabilistic contexts on the unit disc or ball.

Our study of $\mathcal{N}_K(p,q)$-type spaces relies heavily on the supplementary function

$$\Phi_K(x) = \sup_{t\in(0,1]} \frac{K(xt)}{K(t)}.$$

For $x > 0$, the function Φ_K has the following constraints:

$$\int_0^1 \Phi_K(x)\frac{dx}{x} < \infty, \quad \text{and} \quad \int_1^\infty \Phi_K(x)\frac{dx}{x^2} < \infty. \tag{1.1}$$

If the weighted function K satisfies (1.1), then we obtain $K(2x) \approx K(x),\ \forall\, x > 0$; see, for example, [16].

Wulan and Zhou introduced $Q_K(p,q)$ spaces on the unit disk in an earlier work [15]. As far as we know, studies on $Q_K(p,q)$ and $F(p,q,s)$ spaces in the unit ball are still limited due to the intricate nature of the parameters p and q, the function K, and the difficulties introduced by the complex gradient in higher dimensions.

The principal novel contribution herein is the substitution of the complex gradient $|\nabla g(\xi)|$ by $|g(\xi)|$ for all $g \in \mathcal{HO}(\mathbb{B})$. This modification unveils new properties for the subclass of $\mathcal{N}_K(p,q)$-type class within the space $Q_K(p,q)$. Specifically, computation shows that $\mathcal{N}_K(p,q)$-spaces coincide with $Q_K(p,q)$ spaces, thereby allowing direct transfer of results from the former to the latter. Furthermore, $\mathcal{N}_K(p,q)$-type spaces are distinct from Q_K spaces, namely $Q_K(2,2-(n+1))$ spaces, underscoring the independent significance of this class. For further motivation, examples, and intuition concerning $\mathcal{N}_K(p,q)$-type spaces, see [1].

In [1], we provide a detailed discussion of some fundamental properties of the classes $\mathcal{N}_K(p,q)$ and $\mathcal{N}_{K,0}(p,q)$. For $n > 2$, we describe $\mathcal{N}_K(p,q)$-type spaces for using Greene's function. Additionally, we investigate the behavior of Hadamard gaps in $\mathcal{N}_K(p,q)$ functions.

This article is structured as follows: Section 2 begins by reviewing the primary findings from [1] and introducing the concept of Carleson measures associated with $\mathcal{N}_K(p,q)$-type spaces. In Section 3, embedding results are investigated using the Carleson measure approach to establish links between $\mathcal{N}_K(p,q)$-type spaces and both weighted Hardy spaces and weighted Bergman spaces. Section 4 presents applications of these embedding theorems, including discussions on Hadamard products and random power series within $\mathcal{N}_K(p,q)$-type spaces.

Throughout this article, for the real or complex quantities q_1 and q_2, the notation $q_1 \lesssim q_2$ (respectively, $q_1 \gtrsim q_2$) indicates that there is a constant $c_0 > 0$, which is independent of q_1 and q_2 and is such that $q_1 \le c_0 q_2$ (respectively, $q_1 \ge c_0 q_2$). Moreover, when both $q_1 \lesssim q_2$ and $q_1 \gtrsim q_2$ hold, we write $q_1 \approx q_2$.

2. $\mathcal{N}_K(p,q)$-type functions and Carleson measures

The first part of this section focuses on describing some fundamental properties of functions within $\mathcal{N}_K(p,q)$-type spaces.

For any $r \in (0,1)$ and $b \in \mathbb{B}$, we let $\mathbb{B}(b,r)$ be the set $\{\xi \in \mathbb{B} : |\Theta_b(\xi)| < r\}$, and we let $\mathbb{B}_r$ be the set $\{\xi \in \mathbb{B} : |\xi| < r\}$. Further, $\mathbf{P}(\mathbb{B})$ consist of all polynomials defined on the unit ball $\mathbb{B}$. It is worth noting that any polynomial $P \in \mathbf{P}(\mathbb{B})$ is bounded on the unit ball so that there is a constant $M > 0$ for all $w \in \overline{\mathbb{B}}$ such that $|P(w)| \leq M$. Here, we define the poler integral I_K as follows:

$$I_K = \int_0^1 \frac{s^{2n-1}}{(1-s^2)^{n+1}} K(1-s^2)ds,$$

and $I_{K,q}$ is defined by

$$I_{K,q} = \int_0^1 \frac{s^{2n-1}}{(1-s^2)^{n+1-q}} K(1-s^2)ds.$$

The following results established in [1] are essential for subsequent developments.

Theorem 2.1. *The following results are confirmed, for all $K \in \mathcal{RC}^+$, $p \geq 1$, and $q > 0$.*

(1) *The point evaluation $E_w : h \longmapsto h(w)$ is continuous on $\mathcal{N}_K(p,q)$ spaces; in addition, $\mathcal{N}_K(p,q) \subseteq B^{-q/p}(\mathbb{B})$.*

(2) *If $I_K < \infty$, then $\mathcal{N}_K(p,q) = B^{-q/p}(\mathbb{B})$.*

(3) *The set $\mathbf{P}(\mathbb{B})$ is densely contained in $\mathcal{N}_K(p,q)$ if and only if $I_{K,q} < \infty$.*

Theorem 2.2. *Let $K \in \mathcal{RC}^+, n \geq 2, p \geq 1$, and $q > 0$. If the integral $I_{K,q}$ is divergent, then $\mathcal{N}_K(p,q)$ are trivial spaces.*

The second part of this section introduces a broader concept of K-Carleson-type measures on $\mathbb{B}$. This approach is beneficial for studying $\mathcal{N}_K(p,q)$-type spaces.

Recall that, for any $s > 0$, the set $U_s(\xi)$ denotes the Carleson tube at $\xi \in \mathbb{S}$ defined by (see [18])

$$U_s(\xi) \;=\; \{w \in \mathbb{B} : |1 - \langle w, \xi \rangle| < s\}.$$

Likewise, the nonisotropic metric ball of radius s and center $\xi \in \mathbb{S}$ is given by

$$O_s(\xi) \;=\; \{w \in \mathbb{S} : |1 - \langle w, \xi \rangle| < s\}.$$

For all $s > 0$ and $\xi \in \mathbb{S}$, we know that a positive Borel measure μ is said to be a p-CM($\mathbb{B}$) (p-Carleson measure on $\mathbb{B}$) if $\mu(U_s(\xi)) \lesssim s^p$.

Moreover, if $\lim_{s\to 0} \mu(U_s(\xi))s^{-p} = 0$ uniformly for $\xi \in \mathbb{S}$, then μ is said to be a compact p-CM($\mathbb{B}$).

A positive measure μ is referred to as a K-CM($\mathbb{B}$) (K-Carleson measure on $\mathbb{B}$) when

$$\sup_{\xi\in\mathbb{S}, s\in(0,1)} \int_{U_s(\xi)} K(s^{-2}(1-|w|))d\mu(w) < \infty. \tag{2.1}$$

Furthermore, μ is referred to as a compact K-Carleson measure on $\mathbb{B}$ (CK-CM($\mathbb{B}$)) when

$$\lim_{s\to 0} \int_{U_s(\xi)} K(s^{-2}(1-|w|))d\mu(w) = 0. \tag{2.2}$$

Here, for $z \in \mathbb{B}$, we define the integral $J_{K,\mu}$ as follows:

$$J_{K,\mu} = \int_{\mathbb{B}} K(1 - |\Theta_z(w)|^2)d\mu(w).$$

Theorem 2.3. *Whenever $K \in \mathcal{RC}^+$ satisfies Condition (1.1), we let μ be a positive measure. Then, for $z \in \mathbb{B}$, we have*

(1) *μ is a K-$CM(\mathbb{B}) \Longleftrightarrow J_{K,\mu} < \infty$.*

(2) *μ is a CK-$CM(\mathbb{B}) \Longleftrightarrow \lim_{|z|\to 1} J_{K,\mu} = 0$.*

Proof. For any $\xi \in \mathbb{S}$ and $s \in (0,1)$, if we consider $w = (1-s)\xi$, then

$$1 - \langle w, z\rangle = (1-s)(1 - \langle \xi, z\rangle) + s$$

for all $z \in \mathbb{B}$, so

$$|1 - \langle w, z\rangle| \le (1-s)s + s < 2s$$

for all $z \in U_s(\xi)$, which gives

$$\frac{(1-|z|^2)}{|1-\langle z,w\rangle|^2} \ge \frac{(1-|z|)}{|1-\langle z,w\rangle|^2} \gtrsim \frac{1}{s^2}.$$

It follows that

$$\begin{aligned} J_{K,\mu} &\ge \int_{U_s(\xi)} K(1 - |\Theta_z(w)|^2)d\mu(w) \\ &\gtrsim \int_{U_s(\xi)} K(s^{-2}(1-|w|))d\mu(w). \end{aligned}$$

This shows the sufficiency conditions in both assertions **(1)** and **(2)**.

For the necessary conditions, we let some $z \in \mathbb{B}$ with $|z| > 3/4$, and we let $\xi = z/|z|$.

Now, we set $s_k = 2^{k+1}(1-|z|)$ for each $k \ge 0$, and we set

$$U_k = U_{s_k}(\xi) - U_{s_{k-1}}(\xi).$$

Then, we have

$$\begin{aligned} J_{K,\mu} &= \int_{U_s(\xi)} K(1 - |\Theta_z(w)|^2)d\mu(w) \\ &\quad + \sum_{k=1}^{\infty} \int_{U_k} K(1 - |\Theta_z(w)|^2)d\mu(w). \end{aligned}$$

For any $w \in \mathbb{B}$, we obtain

$$\begin{aligned} 1 - |\Theta_z(w)|^2 &= \frac{(1-|z|^2)(1-|w|^2)}{|1-\langle z,w\rangle|^2} \\ &\le \frac{4(1-|z|)(1-|w|)}{|1-\langle z,w\rangle|^2} \approx \frac{1-|w|}{s^2}. \end{aligned}$$

Moreover, for $w \in U_k$, we have

$$\begin{aligned} |1-\langle z,w\rangle| &= \big||z|(1-\langle \xi,w\rangle)+(1-|z|)\big| \\ &\geq |z|\,|1-\langle \xi,w\rangle|-(1-|z|) \\ &\geq 2^{k-1}(1-|w|). \end{aligned}$$

We then get $1-|\Theta_z(w)|^2 \approx (1-|w|)/(2^k s^2)$. Therefore, putting $t=(1-|w|)/(2^k s^2)$, we have

$$\begin{aligned} J_{K,\mu} &\lesssim \mu_K(U_s(\xi)) + \sum_{k=1}^{\infty}\int_{U_k} K(1-|\Theta_z(w)|^2)d\mu(w) \\ &\lesssim \mu_K(U_s(\xi)) + \sum_{k=1}^{\infty}\sup_{w\in U_{s_k}(\xi)}\frac{K(2^{-k}t)}{K(t)}\int_{U_{s_k}(\xi)} K\Big(\frac{(1-|w|)}{2^k s^2}\Big)d\mu(w), \end{aligned}$$

where

$$\mu_K(U_s(\xi)) = \int_{U_s(\xi)} K\Big(\frac{(1-|w|)}{s^2}\Big)d\mu(w).$$

By using the definition of Φ_K for all $0<t<1$, we obtain

$$\sup_{w\in U_{s_k}(\xi)}\frac{K(2^{-k}t)}{K(t)} \leq \sup_{0<t<1}\frac{K(2^{-k}t)}{K(t)} \approx \Phi_K(2^{-k}).$$

This denotes that

$$\begin{aligned} J_{K,\mu} &\lesssim \mu_K(U_s(\xi)) + \sum_{k=1}^{\infty}\Phi_K(2^{-k})\mu_K(U_{s_k}(\xi)) \\ &\lesssim \mu_K(U_s(\xi))\Big\{1+\int_0^1 \Phi_K(s)\frac{ds}{s}\Big\} < \infty. \end{aligned}$$

This proves the necessary condition in **(1)**.

Now, suppose that μ is CK-CM($\mathbb{B}$). Then, (2.1) holds for any $\varepsilon > 0$ when $\delta > 0$ such that $\mu_K(U_s(\xi)) < \varepsilon$ whenever $|s| < \delta$.

Because $\sum_{k=1}^{\infty}\Phi_K(2^{-k})$ is convergent, then there is an integer $N > 0$ such that $\sum_{k\geq N}^{\infty}\Phi_K(2^{-k}) < \varepsilon$. Thus, we obtain $J_{K,\mu} < \varepsilon$. This proves the necessary condition in **(2)**. □

The next result delineates a relationship between K-Carleson measures and the functions of $\mathcal{N}_K(p,q)$ and $\mathcal{N}_{K,0}(p,q)$ classes.

Corollary 2.1. *For $K \in \mathcal{RC}^+$, $p \geq 1$, and $q > 0$, let $g \in \mathcal{HO}(\mathbb{B})$, and assume the measure $d\mu_g^{p,q}(w) = |g(w)|^p(1-|w|^2)^q d\lambda(w)$. Then, we obtain*

(1) *$g \in \mathcal{N}_K(p,q) \Longleftrightarrow d\mu_g^{p,q}$ is a K-CM($\mathbb{B}$).*

(2) *$g \in \mathcal{N}_{K,0}(p,q) \Longleftrightarrow d\mu_g^{p,q}$ is a CK-CM($\mathbb{B}$).*

Moreover,

$$\|g\|_K^p \approx \sup_{\xi\in\mathbb{S},s\in(0,1)}\int_{U_s(\xi)} |g(w)|^p(1-|w|^2)^q K(s^{-2}(1-|w|))d\lambda(w).$$

Proof. The proof is a direct consequence of Theorem 2.3. □

Lemma 2.1. *For any $K \in \mathcal{RC}^+$, with $K(t) = t^{n\kappa}, \kappa > 0$, let μ be a positive measure. Then, μ is a K-CM($\mathbb{B}$) if and only if $d\mu_\kappa = (1-|w|^2)^\kappa d\mu(w)$ is an $(n\kappa)$-CM($\mathbb{B}$).*

Proof. By Theorem 2.3, μ is a K-CM($\mathbb{B}$) if and only if

$$\sup_{\xi\in\mathbb{S}, s\in(0,1)} \int_{U_s(\xi)} \left(\frac{1-|w|}{s^2}\right)^{n\kappa} d\mu(w) < \infty,$$

which implies that $d\mu_\kappa$ is an $(n\kappa)$-CM($\mathbb{B}$). □

3. Embedding relationship among $\mathcal{N}_K(p,q)$ with weighted spaces

For $\rho > 0$ and $h \in \mathcal{HO}(\mathbb{B})$, we define

$$[M_\rho(s,h)]^\rho = \int_{\mathbb{S}} |h(s\xi)|^\rho d\sigma(\xi), \quad \text{where } 0 \le s < 1.$$

For any $h \in \mathcal{HO}(\mathbb{B}), \beta \ge 0$ and $\rho > 0$, the norm of the weighted Hardy space $H^\rho_\beta(\mathbb{B})$ is defined as

$$\|h\|_{H^\rho_\beta} = \sup_{0<s<1} (1-s)^\beta M_\rho(s,h) < \infty.$$

Moreover, the little weighted Hardy space $H^\rho_{\beta,0}(\mathbb{B})$ is the closed subspace of $H^\rho_\beta(\mathbb{B})$ defined by the norm

$$\|h\|_{H^\rho_{\beta,0}} = \lim_{s\to1} (1-s)^\beta M_\rho(s,h) = 0.$$

The weighted Hardy space $H^\rho_\beta(\mathbb{B})$ is a Banach space when $1 \le \rho < \infty$, whereas $H^\rho_\beta(\mathbb{B})$ with $0 < \rho < 1$ is a complete metric space. The space $H^\rho_\beta(\mathbb{B}), \beta = 1$ is the classical Hardy space $H^\rho(\mathbb{B})$. Further details about the properties of a weighted Hardy spaces appear in [10, 18].

Now, we give a generalization of the constraints (1.1):

$$T_{K,\gamma} = \int_0^1 \Phi_K(t) \frac{dt}{t^\gamma} < \infty, \quad 1 \le \gamma \le 2 \tag{3.1}$$

and

$$\Upsilon_{K,\gamma} = \int_1^\infty \Phi_K(t) \frac{dt}{t^{\gamma+1}} < \infty, \quad \gamma > 0. \tag{3.2}$$

Proposition 3.1. *For any $h \in \mathcal{HO}(\mathbb{B})$ and $K \in \mathcal{RC}^+$ which satisfy (3.1), let $p \ge 1, q > 0$, and $\rho > 0$. If $\beta = q/p - n/\rho$, then*

$$H^\rho_\beta(\mathbb{B}) \subsetneq \mathcal{N}_K(p,q).$$

Proof. For fixed $\xi \in \mathbb{S}$ and $s \in (0,1)$, if we consider $w \in U_s(\xi)$, we have

$$s > |1 - \langle w, \xi\rangle| \ge 1 - |\langle w, \xi\rangle| \ge 1 - |w||\xi|.$$

Thus,

$$s > 1 - |w| \implies 1 - s < |w| < 1. \tag{3.3}$$

It is clear that for the parameter ρ, there are two different cases.

The first case: If $p = \rho \geq np/q \Longrightarrow q \geq n$, for a fixed $\xi \in \mathbb{S}$ and $s \in (0,1)$ by (3.3), we obtain

$$\begin{aligned} I^{p,q}_{K(s,\xi)} &= \int_{U_s(\xi)} |h(w)|^p (1-|w|^2)^q K\Big(\frac{(1-|w|)}{s^2}\Big) d\lambda(w) \\ &= \int_{U_s(\xi)} |h(w)|^p (1-|w|^2)^{q-n-1} K\Big(\frac{(1-|w|)}{s^2}\Big) dV(w) \\ &\leq \int_{1-s}^1 (1-r^2)^{q-n-1} K\Big(\frac{(1-r)}{s^2}\Big)\Big(\int_{O_{4s}(\xi)} |h(\theta r)|^p d\sigma(\theta)\Big) dr \\ &\leq \int_{1-s}^1 (1-r^2)^{q-n-1} K\Big(\frac{(1-r)}{s^2}\Big) M_p^p(r,h) dr \\ &\leq \|h\|^p_{H^p_{\frac{q-n}{p}}} \int_{1-s}^1 K\Big(\frac{(1-r)}{s^2}\Big)\frac{dr}{1-r} \lesssim \|h\|^p_{H^p_{\frac{q-n}{p}}}, \end{aligned}$$

which results in the intended result.

The second case: If $p < \rho$, for a fixed $\xi \in \mathbb{S}$ and $s \in (0,1)$, by Hölder's inequality and Lemma 4.6 in [18], we obtain

$$\begin{aligned} \int_{O_s(\xi)} |h(\theta r)|^p d\sigma(\theta) &\leq \sigma^{1-\frac{p}{\rho}}(O_s(\xi))\Big(\int_{O_s(\xi)} |h(\theta r)|^p d\sigma(\theta)\Big)^{\frac{p}{\rho}} \\ &\lesssim s^{n(1-\frac{p}{\rho})} M_\rho^p(r,h) \\ &\leq s^{n(1-\frac{p}{\rho})}(1-r)^{\frac{np}{\rho}-q}\|h\|^p_{H^\rho_{\frac{q}{p}-\frac{n}{\rho}}}. \end{aligned}$$

Then, by previous calculation and the fact $1-s < |w| < 1$, we obtain

$$\begin{aligned} I^{p,q}_{K(s,\xi)} &\leq \int_{1-s}^1 (1-r^2)^{q-n-1} K\Big(\frac{(1-r)}{s^2}\Big)\Big(\int_{O_{4s}(\xi)} |h(\theta r)|^p d\sigma(\theta)\Big) dr \\ &\lesssim \|h\|^p_{H^\rho_{\frac{q}{p}-\frac{n}{\rho}}} \int_{1-s}^1 s^{n(1-\frac{p}{\rho})}(1-r)^{(\frac{p}{\rho}-1)-1} K\Big(\frac{(1-r)}{s^2}\Big) dr \\ &\lesssim \|h\|^p_{H^\rho_{\frac{q}{p}-\frac{n}{\rho}}}. \end{aligned}$$

Thus,

$$\|h\|^p_K \approx \sup_{\xi\in\mathbb{S}, s\in(0,1)} I^{p,q}_{K(s,\xi)} \lesssim \|h\|^p_{H^\rho_{\frac{q}{p}-\frac{n}{\rho}}}.$$

Therefore, the desired result is obtained. □

Any such function $h \in \mathcal{HO}(\mathbb{B})$ takes the form of a homogeneous polynomial $P_{n_i}, i \geq 0$ with Hadamard gaps provided that there is a constant $C > 1$, $\forall\ \xi \in \mathbb{B}$ such that

$$h(\xi) = \sum_{i=0}^{\infty} P_{n_i}(\xi), \text{ with } \frac{n_{i+1}}{n_i} \geq C,\ n_i \in \mathbb{N}. \tag{3.4}$$

We note that all constant functions have Hadamard gaps.

Zhu, in the following result (see Theorem 1 in [18]), established a characterization of the behavior of the Hadamard gap series in $H^{\rho}_{\beta}(\mathbb{B})$ spaces; see also Proposition 3.5 in [7].

Proposition 3.2. *For $\rho > 0$ and $\beta > 0$, let $h(\xi) = \sum_{i=0}^{\infty} P_{n_i}(\xi)$ with Hadamard gaps. Then,*

(1) *$h \in H^{\rho}_{\beta}(\mathbb{B})$ if and only if $\sup\limits_{i\geq 1} L_{i,\rho}/(n_i)^{\beta} < \infty$.*

(2) *$h \in H^{\rho}_{\beta,0}(\mathbb{B})$ if and only if $\lim\limits_{i\to\infty} L_{i,\rho}/(n_i)^{\beta} = 0$,*

where

$$L_{i,\rho} = \left(\int_{\mathbb{S}} |P_{n_i}(\xi)|^{\rho} d\sigma(\xi)\right)^{\frac{1}{\rho}}.$$

Note that, for $d \geq 1$ and $\kappa > -1$, the norm of the weighted Bergman-type space $A^d_{\kappa}(\mathbb{B})$ space can be rewritten by using the expression $M_d(s, h)$ as follows:

$$\begin{aligned} \|h\|^d_{A^d_{\kappa}} &\approx \int_0^1 s^{2m-1}(1-s^2)^{\kappa} M^d_d(s,h)ds \\ &\approx \int_0^1 s^{2m-1}(1-s)^{\kappa} M^d_d(s,h)ds. \end{aligned}$$

As an application of Corollary 2.1 concerning the relationship between K-CM($\mathbb{B}$) and functions in $\mathcal{N}_K(p,q)$, we prove the embedding $\mathcal{N}_K(p,q) \hookrightarrow A^d_{\kappa}$ under suitable conditions on d and κ. Throughout this discussion, we assume that

$$I_{K,q} = \int_0^1 \frac{s^{2n-1}}{(1-s^2)^{n+1-q}} K(1-s^2)ds < \infty.$$

By Theorem 2.1, this means that $\mathbf{P}(\mathbb{B}) \subset \mathcal{N}_K(p,q)$.

Proposition 3.3. *For any $h \in \mathcal{HO}(\mathbb{B})$ and $K \in \mathcal{RC}^+$ which satisfy (3.1), let $p \geq 1$ and $q > 0$. If $0 < \kappa < q-n$ and $d = (p/q)(n+\kappa)$, then*

$$A^d_{\kappa-1} \subseteq \mathcal{N}_K(p,q).$$

Moreover, if $\kappa = q-n, n < q$, then $A^p_{q-n-1} \subseteq \mathcal{N}_K(p,q)$.

Proof. First, if $\kappa < q-n$, for fixed $\xi \in \mathbb{S}$ and $s \in (0,1)$, by Hölder's inequality and Corollary 5.24 in [18], we obtain

$$\begin{aligned} I^{p,q}_{K(s,\xi)} &= \int_{U_s(\xi)} |h(w)|^p (1-|w|^2)^q K\Big(\frac{(1-|w|)}{s^2}\Big) d\lambda(w) \\ &\approx \int_{U_s(\xi)} |h(w)|^p (1-|w|^2)^{q-n-\kappa} K\Big(\frac{(1-|w|)}{s^2}\Big) dV_{\kappa-1}(w) \\ &\leq \left(\int_{U_s(\xi)} |h(w)|^d (1-|w|^2)^{\kappa-1} dV(w)\right)^{\frac{q}{n+\kappa}} \end{aligned}$$

$$
\begin{aligned}
&\times\Big(\int_{U_s(\xi)}(1-|w|^2)^{q-n-1}K^{\frac{n+\kappa}{n+\kappa-q}}(s^{-2}(1-|w|))dV(w)\Big)^{\frac{n+\kappa-q}{n+\kappa}}\\
\approx\ & \|h\|^p_{A^d_{\kappa-1}}\times J.
\end{aligned}
$$

Then,

$$
\begin{aligned}
J &= \Big(\int_{U_s(\xi)}(1-|w|^2)^{q-n-1}K^{\frac{n+\kappa}{n+\kappa-q}}(s^{-2}(1-|w|))dV(w)\Big)^{\frac{n+\kappa-q}{n+\kappa}}\\
&= \int_{U_s(\xi)}(1-|w|^2)^{(q-n-1)\frac{n+\kappa-q}{n+\kappa}}K(s^{-2}(1-|w|))dV(w)<\infty.
\end{aligned}
$$

Thus, we have

$$
\|h\|^p_K\approx\sup_{\xi\in\mathbb{S},s\in(0,1)}I^{p,q}_{K(s,\xi)}\lesssim\|h\|^p_{A^d_{\kappa-1}}.
$$

Second, if $\kappa=q-n, n<q$, for fixed $\xi\in\mathbb{S}$ and $s\in(0,1)$, we obtain

$$
\begin{aligned}
I^{p,q}_{K(s,\xi)} &= \int_{U_s(\xi)}|h(w)|^p(1-|w|^2)^qK\Big(\frac{(1-|w|)}{s^2}\Big)d\lambda(w)\\
&\approx \int_{U_s(\xi)}|h(w)|^pK\Big(\frac{(1-|w|)}{s^2}\Big)dV_{q-n-1}(w)\\
&\leq \|h\|^p_{A^p_{q-n-1}}.
\end{aligned}
$$

Thus, we obtain

$$
\|h\|^p_K\approx\sup_{\xi\in\mathbb{S},s\in(0,1)}I^{p,q}_{K(s,\xi)}\lesssim\|h\|^p_{A^p_{q-n-1}}.
$$

□

4. Applications on $\mathcal{N}_K(p,q)$-type spaces

Initially, we leverage the Carleson measure description of $\mathcal{N}_K(p,q)$-type spaces for investigating Hadamard products within them.

For $\eta\geq 0$ and $m\in\mathbb{Z}^n_+$, we consider the multi-index sequence $\Omega_m(\eta)$, defined as

$$
\Omega_m(\eta)=\frac{m!\Gamma(\eta+n)}{\Gamma(\eta+n+|m|)}.
$$

Let $g,h\in\mathcal{HO}(\mathbb{B})$ be defined as $g=\sum_m b_mw^m$ and $h=\sum_m c_mw^m, w\in\mathbb{B}$. Then, the weighted Hadamard products, or the η-Hadamard products, with $\eta\geq 0$ of g and h are given by (see [2, 12])

$$
(g*h)_\eta(w)=(h*g)_\eta(w)=\sum_m\Omega_m(\eta)b_mc_mw^m. \tag{4.1}
$$

The following lemmas, proved by Burbea and Li in [2], are essential to the development of this subsection.

Lemma 4.1. *Let $g, h \in \mathcal{HO}(\mathbb{B})$ and $r \in [0, 1)$. Then, for all $\xi \in \mathbb{U}$, we obtain*

$$(g * h)_\eta(r\xi) = \langle g_r * \widehat{h_{\bar{\xi}}} \rangle_\eta := \int_{\mathbb{B}} g(rz)h(\xi \cdot \bar{z})dV_{\eta-1}(z), \tag{4.2}$$

where $g_r(\xi) = g(r\xi)$, $\widehat{h}(\xi) = \overline{h}(\bar{\xi})$ and $h_z(\xi) = h(z \cdot \xi)$, $\forall\, z \in \mathbb{B}$. In particular, $(g * h)_\eta \in \mathcal{HO}(\mathbb{U})$.

Lemma 4.2. *Let $\eta > 0$, and let $g, h \in \mathcal{HO}(\mathbb{B})$. Then, for $1 \leq \alpha_1, \alpha_2, \alpha_3 \leq \infty$ with*

$$\frac{1}{\alpha_1} + \frac{1}{\alpha_2} = \frac{\alpha_3 + 1}{\alpha_3},$$

we obtain $(g * h)_\eta \in \mathcal{HO}(\mathbb{U})$,

$$\|(g * h)_\eta\|_{A^{\alpha_3}_{\eta-1}} \leq \|g\|_{A^{\alpha_1}_{\eta-1}} \|h\|_{A^{\alpha_2}_{\eta-1}}.$$

Moreover,

$$\|(g * h)_\eta\|_\infty \leq \|g\|_{A^{\alpha_1}_{\eta-1}} \|h\|_{A^{\alpha_1^*}_{\eta-1}}, \;\; \forall \;\, \alpha_1^* = \frac{\alpha_1}{\alpha_1 - 1} \; (i.e., \;\alpha_3 = \infty).$$

In particular, if $g \in A^{\alpha_1}_{\eta-1}$ and $h \in A^{\alpha_1^*}_{\eta-1}$, then $(g * h)_\eta \in \mathcal{HO}(\mathbb{U})$.

Theorem 4.1. *Suppose that $K \in \mathcal{RC}^+$ satisfies (3.1), and $g, h \in \mathcal{HO}(\mathbb{B})$. Let $\rho, p \geq 1, \eta > 0$, and $q - n > 0$, so we have the following:*

(1) *If $g \in A^{\rho}_{\eta-1}$ and $h \in A^{\rho^*}_{\eta-1}$, then $(g * h)_\eta \in \mathcal{N}_K(p, q)$, where $\rho > 1$ and $\rho^* = \rho/(\rho - 1)$.*

(2) *If $g \in A^{\alpha_1}_{\eta-1}$ and $h \in A^{\alpha_2}_{\eta-1}$, then $(g * h)_\eta \in \mathcal{N}_K(p, q)$, where $\alpha_1 < q - n, \alpha_2 < q - n$ satisfy*

$$\frac{1}{\alpha_1} + \frac{1}{\alpha_2} = 1 + \frac{q}{p}(n + \eta)^{-1}.$$

(3) *If $g \in A_{\eta-1}(\mathbb{B})$ and $h \in \mathcal{B}(\mathbb{B})$, then $(g * h)_\eta \in \mathcal{N}_K(p, q)$.*

(4) *If $g \in \mathcal{N}_K(p, q)$ and $h \in A_{q-n-1}(\mathbb{B})$, then $(g * h)_{q-n} \in \mathcal{N}_K(p, q)$.*

Proof. **(1)** By Theorem 2.1 and Lemma 4.2, because $n < q$, we obtain

$$\|(g * h)_\eta\|_K \lesssim \|(g * h)_\eta\|_\infty \leq \|g\|_{A^{\rho}_{\eta-1}} \|h\|_{A^{\rho^*}_{\eta-1}}.$$

(2) By Proposition 3.3 and Lemma 4.2, we obtain

$$\|(g * h)_\eta\|_K \lesssim \|(g * h)_\eta\|_{A^{\frac{p}{q}(n+\eta)}_{\eta-1}} \leq \|g\|_{A^{\alpha_1}_{\eta-1}} \|h\|_{A^{\alpha_2}_{\eta-1}}. \tag{4.3}$$

(3) As shown by Lou and Wulan (see Theorem 1 in [12]),

$$|(g * h)_\eta(\xi)| \lesssim \|(g * h)_\eta\|_\infty \leq \|g\|_{A_{\eta-1}} \|h\|_{\mathcal{B}}. \tag{4.4}$$

Because $n < q$, the inequality (4.4) implies that

$$\|(g * h)_\eta\|_K \lesssim \|(g * h)_\eta\|_\infty \leq \|g\|_{A_{\eta-1}} \|h\|_{\mathcal{B}}.$$

(4) Let $\eta = q - n$ in Inequality (4.3) so that we obtain

$$\|(g * h)_{q-n}\|_K \lesssim \|g\|_{A^p_{q-n-1}} \|h\|_{A_{q-n-1}} \leq \|g\|_K \|h\|_{A_{q-n-1}}.$$

Therefore, all four desired results are obtained. □

The following lemma yields an estimate for the quantity $M_\rho(s, (g * h)_\eta)$ (see, Lemma 4.5 in [7]), which in turn allows us to derive a new description of the η-Hadamard products in terms of the inclusion relation between the functions in $H^\rho_\beta(\mathbb{B})$ and $\mathcal{N}_K(p, q)$.

Lemma 4.3. *Let $\eta > 0, s \in [0, 1), \rho \geq 1$ and let $g, h \in \mathcal{HO}(\mathbb{B})$. Then,*

$$M_\rho(s, (g * h)_\eta) \leq \|g\|_{A_{\eta-1}} M_\rho(\sqrt{s}, h).$$

Theorem 4.2. *Assume that $p \geq 1$, that q, ρ and η are strictly positive, and that $\beta = q/p - n/\rho$. Let $K \in \mathcal{RC}^+$ satisfy (3.1), and let $g, h \in \mathcal{HO}(\mathbb{B})$. If $g \in A_{\eta-1}$ and $h \in H^\rho_\beta(\mathbb{B})$, then $(g * h)_\eta \in \mathcal{N}_K(p, q)$.*

Proof. Suppose that $g \in A_{\eta-1}$ and $h \in H^\rho_\beta(\mathbb{B})$. By Lemma 4.3 and Proposition 3.2, we obtain

$$\begin{aligned}
\|(g * h)_\eta\|_K &\lesssim \sup_{0<s<1} (1 - s)^\beta M_\rho(s, (g * h)_\eta) \\
&\leq \|g\|_{A_{\eta-1}} \times \sup_{0<s<1} (1 - s)^\beta M_\rho(\sqrt{s}, h) \\
&\lesssim \|g\|_{A_{\eta-1}} \times \sup_{0<s<1} (1 - \sqrt{s})^\beta M_\rho(\sqrt{s}, h) \\
&\leq \|g\|_{A_{\eta-1}} \|h\|_{H^\rho_\beta}.
\end{aligned}$$

Hence, $(g * h)_\eta \in \mathcal{N}_K(p, q)$. □

The second application in this section of the previously mentioned description of spaces of type $\mathcal{N}_K(p, q)$ in terms of Carleson measures is in the analysis of random power series. Consider the Bernoulli sequence $\{\delta_m(x)\}$ of independent random variables on $(\mho, \mathcal{F}, \mathcal{P})$, where each $\delta_m(x)$ attains the values ± 1 with probability $1/2$. Here, $\mathcal{P}$ denotes the probability measure on $(\mho, \mathcal{F})$, which is countably additive and satisfies $\mathcal{P}(\mho) = 1$.

The Rademacher functions provide a canonical example of such a Bernoulli sequence, defined by

$$\{\phi_K(s)\}_{k\in\mathbb{N}} := \left\{sgn[\sin(2^k \pi s)]\right\}_{k\in\mathbb{N}}.$$

The symbol *sgn* denotes the sign function; for $\xi \neq 0$, it equals $\xi/|\xi|$.

It is straightforward to verify that the variables ϕ_k are mutually independent random variables on the interval $[0, 1]$. If $g \in \mathcal{HO}(\mathbb{B})$ with the Taylor expansion $g = \sum_m b_m w^m$, the randomization of g is defined by $g_x(w) = \sum_m \delta_m(x) b_m w^m,\ \forall\ w \in \mathbb{B}, x \in \mho$. For further information regarding Rademacher functions, readers are referred to the comprehensive exposition in [4].

For a multi-index m, consider the sequence $\{\varepsilon_{m,\alpha}\}$ defined by $\varepsilon_{m,\alpha} = \left(\int_{\mathbb{S}} |\xi^m|^\alpha \, d\sigma(\xi)\right)^{\frac{1}{\alpha}}$ (see [5]), where $m \in \mathbb{Z}_+^n, 1 \le \alpha \le n$. The following theorem provides a sufficient condition for g_x for all $x \in \mho$ to belong to the $\mathcal{N}_K(p,q)$-type spaces.

Theorem 4.3. *Suppose that $K \in \mathcal{RC}^+$ satisfy (3.1) and $g \in \mathcal{HO}(\mathbb{B})$ with the Taylor expansion $g = \sum_m b_m w^m$. Let $p \ge 1, q > 0$, and $d = (p/q)(n+\kappa)$, where $\kappa > q - n > 0$. If the sequence $\{|b_m|\varepsilon_{m,\alpha}\}_m \in \ell^{\min\{2,d\}}$ holds, then $g_x \in \mathcal{N}_K(p,q)$, where ℓ^d is the space of convergent sequences.*

Proof. In view of Proposition 3.3, because $A^d_{\kappa-1} \subseteq \mathcal{N}_K(p,q)$, it is enough to prove that $g_x \in A^d_{\kappa-1}$ for almost every $x \in \mho$; equivalently, it suffices to verify that $\mathcal{P}(\mathbb{E}) = 1$, where the event $\mathbb{E}$ is defined by

$$\mathbb{E} := \{x : g_x \in A^d_{\kappa-1}\}.$$

To prove that $\mathcal{P}(\mathbb{E}) = 1$, it suffices to show that the random integral

$$\int_0^1 (1-s)^{\kappa-1} M_d^d(s, g_x) ds$$

is finite for $\mathcal{P}$-almost every x. Consider the nonnegative function $H(x,s) = (1-s)^{\kappa-1} M_d^d(s, g_x)$ we then have

$$\mathbb{E} := \left\{x : \int_0^1 H(x,s) ds < \infty\right\}.$$

In fact, an application of Khintchine's inequality and Fubini's theorem yields

$$\begin{aligned}
\mathbb{E}\left(\int_0^1 H(x,s)ds\right) &= \int_0^1 (1-s)^{\kappa-1}\left(\int_{\mho}\int_{\mathbb{S}} |g_x(s\xi)|^d d\sigma(\xi) d\mathcal{P}\right) ds \\
&= \int_0^1 (1-s)^{\kappa-1}\left(\int_{\mathbb{S}}\int_{\mho} \left|\sum_m \delta_m(x) b_m \xi^m s^{|m|}\right|^d d\mathcal{P} d\sigma(\xi)\right) ds \\
&\lesssim \int_0^1 (1-s)^{\kappa-1}\left(\int_{\mathbb{S}} \left[\sum_m |b_m|^2 |\xi^m|^2 |s|^{2|m|}\right]^{\frac{d}{2}} d\sigma(\xi)\right) ds.
\end{aligned}$$

It is clear that for the parameter d, there are two cases.

The first case: If $d \in (0,2]$, that is, $d/2 \le 1$, then we obtain

$$\begin{aligned}
\mathbb{E}\left(\int_0^1 H(x,s)ds\right) &\lesssim \int_0^1 (1-s)^{\kappa-1}\left(\int_{\mathbb{S}} \sum_m |b_m|^d (|\xi^m||s|^{|m|})^d d\sigma(\xi)\right) ds \\
&\lesssim \sum_m |b_m|^d \varepsilon^d_{m,d} < \infty.
\end{aligned}$$

The second case: If $d/2 > 1$, by using Minkowski's inequality, we obtain

$$\begin{aligned}\mathbb{E}\left(\int_0^1 H(x,s)ds\right) &\lesssim \int_0^1 (1-s)^{\kappa-1}\left(\sum_m \left[\int_{\mathbb{S}} |b_m|^d|\xi^m|^d|s|^{d|m|}d\sigma(\xi)\right]^{\frac{2}{d}}\right)^{\frac{d}{2}} ds \\ &\lesssim \int_0^1 (1-s)^{\kappa-1}\left(\sum_m |b_m|^2|s|^{2|m|}\varepsilon_{m,d}^2\right)^{\frac{d}{2}} ds \\ &\lesssim \left(\sum_m |b_m|^2\varepsilon_{m,d}^2\right)^{\frac{d}{2}} < \infty.\end{aligned}$$

Thus, the probability $\mathcal{P}(\mathbb{E}) = 1$ holds, and then $g_x \in \mathcal{N}_K(p,q)$. □

The beta function $\mathbf{B}(a,b)$ for positive constants a and b is defined as the integral

$$\mathbf{B}(a,b) = \int_0^1 (1-s)^{b-1}s^{a-1}ds.$$

Theorem 4.4. *Suppose that $K \in \mathcal{RC}^+$ satisfy (3.1) and $g \in \mathcal{HO}(\mathbb{B})$ with the Taylor expansion $g = \sum_m b_m w^m$. For $p \geq 1, \kappa = q-n$, and $q > 0$, if there exist positive constants δ_0 and τ such that the decay condition $\mathcal{P}(\mathbb{E}) \lesssim 1/\delta^{\tau+1}$ holds, where $\mathbb{E} := \{x : \|g_x\|_K > \delta\}$ for all $\delta > \delta_0$,*

$$\left\{|b_m|\varepsilon_{m,\alpha}[\mathbf{B}(p|m|,\kappa)]^{\frac{1}{p}}\right\}_m \in \ell^\infty.$$

Proof. In light of the second part of Proposition 3.3, if $\kappa = q-n, n < q$, then $A^p_{\kappa-1} \subseteq \mathcal{N}_K(p,q)$; it is known that $H(x,s) = (1-s)^{\kappa-1}M_p^p(s,g_x)$. We then have $g_x \in A^d_{\kappa-1}$ for almost every $x \in \mho$.
Thus, for each $\delta > \delta_0$, we obtain

$$\mathbb{E} := \left\{x : \int_0^1 H(x,s)ds > \delta\right\}.$$

This immediately implies that $\mathbb{E}(\int_0^1 H(x,s)ds) < \infty$. Thus, for any δ, by Minkowski's inequality, we obtain

$$\begin{aligned}\mathbb{E}\left(\int_0^1 H(x,s)ds\right) &= \int_0^1 (1-s)^{\kappa-1}\left(\int_{\mho}\int_{\mathbb{S}} |g_x(s\xi)|^p d\sigma(\xi)d\mathcal{P}\right)ds \\ &= \int_0^1 (1-s)^{\kappa-1}\left(\int_{\mathbb{S}}\int_{\mho}\left|\sum_m \delta_m(x)b_m\xi^m s^{|m|}\right|^p d\mathcal{P}d\sigma(\xi)\right)ds \\ &\gtrsim \int_0^1 (1-s)^{\kappa-1}\left(\int_{\mathbb{S}}\left[\sum_m |b_m|^2|\xi^m|^2|s|^{2|m|}\right]^{\frac{p}{2}} d\sigma(\xi)\right)ds \\ &\gtrsim \int_0^1 (1-s)^{\kappa-1}\left(\int_{\mathbb{S}} |b_m|^p|\xi^m|^p|s|^{p|m|}d\sigma(\xi)\right)ds,\end{aligned}$$

which implies that

$$|b_m|^p\,\varepsilon_{m,d}^p\,\mathbf{B}(p|m|,\kappa)] \lesssim \mathbb{E}\left(\int_0^1 H(x,s)ds\right) < \infty.$$

Thus, $\left\{|b_m|\varepsilon_{m,\alpha}[\mathbf{B}(p|m|,\kappa)]^{\frac{1}{p}}\right\}_m \in \ell^\infty$. □

5. Conclusions

In this paper, the weighted function class $\mathcal{N}_K(p,q)$ of Möbius invariants on the unit ball of $\mathbb{C}^n$ was investigated through the characterization of their associated Carleson measures and the derivation of embedding theorems linking these spaces with weighted Hardy and Bergman spaces. The obtained results clarify the position of $\mathcal{N}_K(p,q)$-type spaces within the global function space framework and provide effective criteria for the boundedness of various operators acting on them. Furthermore, applications to Hadamard products and random power series were developed, illustrating how the structure of $\mathcal{N}_K(p,q)$-type spaces can be exploited to study coefficient multipliers and probabilistic behavior of holomorphic functions. These findings not only extend existing results for related Möbius invariant spaces but also indicate several directions for further research, including the analysis of composition and integral operators and multiplier spaces as well as more refined probabilistic models associated with $\mathcal{N}_K(p,q)$-type spaces, composition and integral operators, multiplier spaces, and more refined probabilistic models associated with $\mathcal{N}_K(p,q)$-type spaces.

Author contributions

Munirah Aljuaid and M. A. Bakhit: Conceptualization, methodology, validation, writing—original draft preparation, and writing—review and editing: all authors contributed equally to this work. All authors have read and approved the final manuscript.

Use of Generative-AI tools declaration

The authors emphasize that no artificial intelligence tools were used in the proofs or calculations for this manuscript. They employed AI-based tools to locate and review the latest, most relevant references, ensuring that the theories and applications presented in the manuscript had not been established earlier.

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Conflict of interest

There is no potential conflict of interest.

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