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Research article

Bifurcation analysis, stability analysis, modulation instability and soliton structures of the nonlinear Klein–Gordon model in particle physics

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Abstract: In this study, the nonlinear Klein-Gordon model is investigated analytically as an extension of the classical Klein-Gordon equation involving nonlinear effects. This relativistic wave equation plays an important role in describing scalar field dynamics in particle physics, condensed matter physics, and cosmology. Although several studies have reported soliton solutions for nonlinear wave equations, comprehensive analyses combining exact solutions with stability, modulation instability, and chaotic dynamics remain limited for this model. To address this gap, we employ analytical techniques to obtain different types of solutions, including solitary, periodic, and V-shaped wave structures. Graphical illustrations demonstrate the rich dynamical behavior of the system for various parameter regimes. In addition, a stability analysis is performed to determine the conditions under which the system preserves its dynamical behavior. The modulation instability and bifurcation structures are also examined through phase portraits, revealing transitions between regular and complex dynamics. Furthermore, periodic perturbations are introduced to explore chaotic behavior in the system. The results reveal previously unreported solitary and periodic solutions whose stability and chaotic phases offer insight into the underlying nonlinear mechanisms, with potential applications in wave propagation and field dynamics.

Keywords: nonlinear dynamics; periodic wave solitons; travelling wave solitons; stability analysis; bifurcation analysis; modulation instability; nonlinear Klein-Gordon equation; solitons
Mathematics Subject Classification: 35Rxx, 35Qxx, 35Q51

1. Introduction

Mathematical models must be solved repeatedly for a wide range of scientific and engineering uses [1–3]. These applications arise in areas such as design, optimisation, control, optics, engineering and uncertainty quantification [4–7]. Traditional discretisation techniques, including finite element, finite difference, and discrete volume approaches, can be used to numerically approximate the parameterised partial differential equations (PDEs) [8, 9]. However, to achieve solutions that are accurate enough, a full order model usually needs an enormous number of uncertainties. In such substantial contexts, analysing several answers frequently results in excessive demands on computational resources. The primary objective of reduced-order modelling is to alleviate this burden by constructing low-dimensional models that can be evaluated much more efficiently while retaining a comparable level of accuracy to the original full-order model [10–13].

Through variable change, several analytical techniques reliably solve classes of second-order differential equations. Using appropriate parameter adjustments, certain kinds of nonlinear differential equations have been reduced to first-order [14, 15]. The majority of the time, the resulting first-order differential equations can be transformed into well-known conventional differential equations that can be solved or integrated. Nevertheless, nonlinear partial differential equations (NLPDEs) cannot be solved using these techniques. Some NLPDE types cannot be solved with distinct factor adjustments. We illustrate a wider variety of solvable NLPDEs by utilising the advantages of standard approaches, which broadens the area of issues to which predictive procedures can be applied [16, 17]. Many mathematical approaches, such as the enhanced algebraic method [18], the first integral method [19], the exponential rational function method [20], the extended trial equation method [21], the Weierstrass elliptical function expansion method [22], the modified extended direct algebraic method method [23], the G'/G-expansion method [24], and the Adomian decomposition methods [25], are employed to analyse many complex nonlinear partial differential equations arising in various disciplines [26–28].

When nonlinear and dispersed influences in a fluid are balanced, a soliton—a self-building, single wave packet or pulse maintains its shape and rate during propagation. Solitons can clash and arise with no shift in their shape, which is in contrast to normal pulses [29]. A soliton is a solitary pulse produced by a balance between scattering and nonlinearity that maintains its shape across long distances. Elastic collisions between solitons provide a clearer example of their exceptional stability in terms of shape and momentum. The occurrence of N soliton solutions, which is a positive factor, is a common characteristic of integral systems. Elastic forces between numerous solitons can be explained by the N soliton solutions, which are typically caused by various wavelength features and are referred to as exponentially N solitons [30].

The nonlinear Klein-Gordon equation is an extension of the Klein-Gordon equation concerning nonlinear variables. The behavior of scalar fields is explained by this relativistic wave equation in particle physics, condensed matter physics, cosmology, and other areas of physics. The governing model describes optical phenomena in condensed matter physics [31], wave propagation, and related areas solitons dynamics [32],

$$\Upsilon_{tt} - \Upsilon_{xx} + j\Upsilon + \mu\Upsilon^3 = 0, \tag{1.1}$$

where the j and μ are nonzero parameters. The nonlinear Klein-Gordon equation has importance in various fields of mathematical sciences, wave propagation, optics, and soliton dynamics. In the literature, the given model is solved by different techniques such as the extended fan sub-equation method [33], the Adomian decomposition method [34], and the G'/G-expansion method [35]. However, a careful review of the literature reveals several limitations. First, most existing works concentrate on deriving exact solutions without providing a comprehensive analysis of their dynamical behavior. In particular, the interplay between stability, modulation instability, and bifurcation structures has not been systematically explored for the nonlinear Klein-Gordon model. Second, the diversity of solution structures obtained in previous studies remains limited, as many approaches yield similar functional forms under restricted parameter regimes. Third, the impact of parametric variations on the transition between regular and complex dynamics, including chaotic behavior, has received relatively little attention. Motivated by these gaps, we employ the polynomial expansion method [36] and the unified technique [37], as they have not been utilised in this model and yield modern analytical solutions with applications across different branches of the mathematical sciences and optical fiber transmission. These methods are highly significant in the literature. We apply these approaches to the present model because they have not previously been used to study the nonlinear Klein–Gordon equation. The nonlinear Klein–Gordon equation plays an important role in describing nonlinear wave propagation and scalar field dynamics in particle physics, condensed matter physics, and cosmology. In recent years, many researchers have investigated soliton solutions of nonlinear evolution equations using different analytical techniques. However, comprehensive studies combining exact solutions with stability, modulation instability, and bifurcation analysis for the nonlinear Klein–Gordon model remain limited. Motivated by these challenges, the present study aims to derive new analytical wave solutions of the model and to investigate their dynamical behavior through graphical analysis, stability conditions, and bifurcation structures.

We structure the article in the following way. Section 1, discussed the introduction of the model. In Section 2, the applied strategies are explained in detail. The solutions of the model obtained by using this approach are described in Section 3. The stability and modulation instability of the model are studied in Sections 4 and 5, respectively; the graph visuals of solutions are presented in Section 6. The dynamical features of the proposed model are discussed in Section 7. Finally, the summary of the article is described in the last section, Section 8.

2. Methodology

A partial differential equation (PDE) can take the following form

$$L(\Upsilon_t, \Upsilon_x, \Upsilon_{tt}, \Upsilon_{xx}, \Upsilon_{xt}, \cdots) = 0. \tag{2.1}$$

In the above PDE, Υ is a real function for the x, t variables, and L denotes the nonlinear term. We take the transform as

$$\Upsilon(x,t) = \mathcal{U}(\chi), \quad \chi = x - t\varrho. \tag{2.2}$$

Then, we insert Eq (2.2) into Eq (2.1), and that gives us an ordinary differential equation in form

$$M(\mathcal{U},\mathcal{U}',\mathcal{U}'',\mathcal{U}''',\cdots)=0, \tag{2.3}$$

where, $'$ symbolizes the derivative of $\mathcal{U}$.

2.1. The polynomial expansion method

The given approach is described in detail here with the following steps:
Step 1: Solution of Eq (2.3) is assumed as follow:

$$\mathcal{U}(\chi)=\sum_{i=1}^{m}h_i\delta(\chi)^{-i}+\sum_{i=1}^{m}k_i\delta(\chi)^{i}+k_0. \tag{2.4}$$

In this, h_i, k_i are the known and m denotes a real number. Further, $\delta(\chi)$ satisfies the requirement

$$\delta'(\chi)=\alpha+\nu\delta(\chi)+\delta(\chi)^2. \tag{2.5}$$

Here, α represents a real constant, and the above-mentioned equation gives different types of results under these conditions.
Case I. If $\nu=0$ then $\alpha=0$, so the solution is obtained as

$$\delta(\chi)=-\frac{1}{\chi}.$$

Case II. If $\nu\neq 0$ then $\alpha=0$, so the solution is obtained as

$$\delta(\chi)=-\frac{\nu}{R_0e^{-\nu\chi}-1}.$$

Here, R_0 denotes a constant of integration.
Case III. If $\nu=0$, and $\alpha\neq 0$ with $\alpha>0$, so the solution is obtained as

$$\delta(\chi)=\sqrt{\alpha}\tan\left(\sqrt{\alpha}\chi\right),$$
$$\delta(\chi)=-\sqrt{-\alpha}\cot\left(\sqrt{\alpha}\chi\right).$$

Case IV. If$\nu=0,\ \ \alpha\neq 0,\ \ \alpha<0$, so solution acquired as

$$\delta(\chi)=-\sqrt{-\alpha}\tanh\left(\sqrt{-\alpha}\chi\right),$$
$$\delta(\chi)=\sqrt{-\alpha}\coth\left(\sqrt{-\alpha}\chi\right).$$

Case V. If $\alpha\neq 0$ then $\nu\neq 0$, so the solution is obtained as

$$\delta(\chi)=\frac{\sigma_1-\sigma_2S_1e^{(\sigma_1-\sigma_2)\chi}}{1-S_1e^{(\sigma_1-\sigma_2)\chi}}.$$

Here, S_1 is a constant of integration, and σ_1 and σ_2 are roots of $\sigma^2+\nu\sigma+\alpha$.

$$\sigma_1=\frac{-\nu+\sqrt{\nu^2-4\alpha}}{2},\qquad \sigma_2=\frac{-\nu-\sqrt{\nu^2-4\alpha}}{2}.$$

Step 2: Substitute Eq (2.4), along with Eq (2.5), into the Eq (2.3), and set all terms with same value of $\delta(\chi)$ equal to zero.
Step 3: Further, we can find the solutions of Eq (1.1) in the last step by adding Eq (2.4) to the Eq (2.5) solution.

It is important to note that the auxiliary Eq (2.5) is a Riccati equation whose general solution is represented by Case V. The remaining cases arise as special parameter reductions of this general solution. In particular, the trigonometric and hyperbolic forms are analytically related through complex transformations and phase shifts. Therefore, the obtained solution families are not entirely independent but represent different forms of the same general solution under various parameter constraints.

2.2. The unified method

The given approach is described in detail here with the following steps:
Step 1: The solution of Eq (2.3) is assumes as follows

$$\mathcal{U}(\chi) = \xi_0 + \sum_{i=1}^{N} \left(\zeta_i \delta^{-i}(\chi) + \xi_i \delta^{i}(\chi)\right), \tag{2.6}$$

where ξ_i, ζ_i are known, N denotes a real number, and $\delta(\chi)$ satisfies the requirement

$$\delta'(\chi) = \beta + \delta(\chi)^2, \tag{2.7}$$

where β represents the constant.
Case I. If $\beta < 0$, then the solution is obtained as

$$\delta(\chi) = \frac{\pm\sqrt{\beta\left(-\left(\rho^2+\sigma^2\right)\right)} - \sqrt{-\beta}\rho\cosh\left(2\sqrt{-\beta}(\chi+\Omega)\right)}{\rho\sinh\left(2\sqrt{-\beta}(\chi+\Omega)\right)+\sigma},$$

$$\delta(\chi) = \pm\sqrt{-\beta} - \frac{2\sqrt{-\beta}\rho}{-\sinh\left(2\sqrt{-\beta}(\chi+\Omega)\right)+\cosh\left(2\sqrt{-\beta}(\chi+\Omega)\right)+\rho}.$$

Case II. If $\beta > 0$, then the solution is obtained as

$$\delta(\chi) = \frac{\pm\sqrt{\beta\left(\rho^2-\sigma^2\right)} - \sqrt{\beta}\rho\cos\left(2\sqrt{\beta}(\chi+\Omega)\right)}{\rho\sin\left(2\sqrt{\beta}(\chi+\Omega)\right)+\sigma},$$

$$\delta(\chi) = \pm i\sqrt{\beta} - \frac{\pm 2i\sqrt{\beta}\rho}{\mp i\sin\left(2\sqrt{\beta}(\chi+\Omega)\right)+\cos\left(2\sqrt{\beta}(\chi+\Omega)\right)+\rho}.$$

Case III. If $\beta = 0$, then the solution is obtained as

$$\delta(\chi) = -\frac{1}{\chi+\Omega}.$$

Step 2: By substitute Eqs (2.6) and (2.7) into Eq (2.3), then set all the terms with same value of $\delta(\chi)$ equal to zero.
Step 3: Further, we can find the solutions of Eq (1.1) in the last step by adding Eq (2.6) to the Eq (2.7) solution.

2.3. The modified extended tan hyperbolic function method

The given approach is described in detail here with steps:
Step 1: Solution of Eq (2.3) is assumed as follows:

$$\mathcal{U}(\chi) = s_0 + \sum_{i=1}^{n} s_i\delta^i(\chi) + \sum_{i=1}^{n} \frac{v_i}{\delta^i(\chi)}, \tag{2.8}$$

where s_0, s_i, and v_i are known, n denotes a real number, and $\delta(\chi)$ satisfies the requirement

$$\delta'(\chi) = \delta(\chi)^2 + \kappa, \tag{2.9}$$

where κ represents a constant.
Case I. If $\kappa < 0$, then the acquired solution is

$$\delta(\chi) = -\sqrt{-\kappa}\tanh\left(\sqrt{-\kappa}\chi\right),$$
$$\delta(\chi) = -\sqrt{-\kappa}\coth\left(\sqrt{-\kappa}\chi\right).$$

Case II. If $\kappa = 0$, then the acquired solution is

$$\delta(\chi) = -\frac{1}{\kappa}.$$

Case III. If $\kappa > 0$, then the acquired solution is

$$\delta(\chi) = \sqrt{\kappa}\tan\left(\sqrt{\kappa}\chi\right),$$
$$\delta(\chi) = -\sqrt{\kappa}\cot\left(\sqrt{\kappa}\chi\right).$$

Step 2: By submitting Eq (2.8) along with Eq (2.9) into Eq (2.3), further by selecting all the values equal to zero of the same value of $\delta(\chi)$.
Step 3: Further, we can find the solutions of Eq (1.1) in the last by adding Eq (2.8) to the Eq (2.9) solution.

3. Traveling wave solutions

For the wave solution of Eq (1.1), we utilize the modern techniques and adopt the transformation

$$\Upsilon(x,t) = \mathcal{U}(\chi), \quad \chi = x - t\varrho, \tag{3.1}$$

where ϱ are the wave numbers. Under this transformation, all partial derivatives are converted into total derivatives with respect to χ using the chain rule: $\frac{\partial \mathcal{U}}{\partial x} = \frac{d\mathcal{U}}{d\chi}$, $\frac{\partial U}{\partial t} = -\varrho\frac{dU}{d\chi}$.

Using these derivatives in Eq (1.1), we get the following ordinary differential equation (ODE):

$$(\gamma^2 - 1)\mathcal{U}'' + j\mathcal{U} + \mu\mathcal{U}^3 = 0. \tag{3.2}$$

3.1. Application to the polynomial expansion method

First of all, we find the balancing number by utilizing the latest technique called the homogeneous balance method. We attain m=1, and we express the Eq (3.2) solution in the form as written below:

$$\mathcal{U}(\chi) = k_0 + k_1\delta(\chi) + \frac{h_1}{\delta(\chi)}. \tag{3.3}$$

Here, we can get the mathematical system of equations by substituting Eq (3.3), along with Eq (2.5), into Eq (3.2), which gives

$$\begin{aligned}
&\gamma^2 h_1\nu + 6h_1k_1k_0\mu - h_1\nu + jk_0 + \alpha\gamma^2 k_1\nu - \alpha k_1\nu + k_0^3\mu = 0,\\
&2\alpha^2\gamma^2 h_1 - 2\alpha^2 h_1 + h_1^3\mu = 0,\\
&3\alpha\gamma^2 h_1\nu - 3\alpha h_1\nu + 3h_1^2 k_0\mu = 0,\\
&2\alpha\gamma^2 h_1 - 2\alpha h_1 + \gamma^2 h_1\nu^2 + h_1 j + 3h_1k_0^2\mu + 3h_1^2k_1\mu - h_1\nu^2 = 0,\\
&3h_1k_1^2\mu + jk_1 + 2\alpha\gamma^2 k_1 - 2\alpha k_1 + \gamma^2 k_1\nu^2 + 3k_0^2k_1\mu - k_1\nu^2 = 0,\\
&3\gamma^2 k_1\nu + 3k_0k_1^2\mu - 3k_1\nu = 0,\\
&2\gamma^2 k_1 + k_1^3\mu - 2k_1 = 0.
\end{aligned}$$

On solving these equations, we obtain the following results:
Family 1:

$$k_0 = -\frac{\sqrt{-2\alpha\gamma^2 + 2\alpha - j}}{\sqrt{\mu}}, \quad k_1 = 0,$$

$$h_1 = \frac{\frac{\sqrt{2}\alpha\gamma^2\sqrt{-2\alpha\gamma^2+2\alpha-j}\sqrt{2\alpha\gamma^2-2\alpha+j}}{\sqrt{\gamma^2-1}\sqrt{\mu}} - \frac{\sqrt{2}\alpha\sqrt{-2\alpha\gamma^2+2\alpha-j}\sqrt{2\alpha\gamma^2-2\alpha+j}}{\sqrt{\gamma^2-1}\sqrt{\mu}}}{2\alpha\gamma^2 - 2\alpha + j}.$$

Further, by adding the given results in Eq (3.3), we get the following outcome
Case 1: If $\nu = 0$, and $\alpha = 0$, then the required solution has the form

$$\Upsilon_1(x,t) = \frac{(t\varrho - x)\left(h_1 + \frac{\sqrt{2}\alpha\gamma^2\sqrt{2\alpha\gamma^2-2\alpha+j}\sqrt{-2\alpha\gamma^2+2\alpha-j}}{\sqrt{\gamma^2-1}\sqrt{\mu}}\right)}{2\alpha\gamma^2 - 2\alpha + j} - \frac{\sqrt{-2\alpha\gamma^2 + 2\alpha - j}}{\sqrt{\mu}}, \tag{3.4}$$

where $h_1 = -\frac{\sqrt{2}\alpha\sqrt{-2\alpha\gamma^2+2\alpha-j}\sqrt{2\alpha\gamma^2-2\alpha+j}}{\sqrt{\gamma^2-1}\sqrt{\mu}}$.
Case 2: If $\nu \neq 0$, and $\alpha = 0$ then the required solution has the form

$$\Upsilon_2(x,t) = -\frac{\left(h_2 + \frac{\sqrt{2}\alpha\gamma^2\sqrt{2\alpha\gamma^2-2\alpha+j}\sqrt{-2\alpha\gamma^2+2\alpha-j}}{\sqrt{\gamma^2-1}\sqrt{\mu}}\right)\left(R_0 e^{-\nu(x-t\varrho)} - 1\right)}{\nu\left(2\alpha\gamma^2 - 2\alpha + j\right)} - \frac{\sqrt{-2\alpha\gamma^2 + 2\alpha - j}}{\sqrt{\mu}}, \tag{3.5}$$

where $h_2 = -\frac{\sqrt{2}\alpha\sqrt{-2\alpha\gamma^2+2\alpha-j}\sqrt{2\alpha\gamma^2-2\alpha+j}}{\sqrt{\gamma^2-1}\sqrt{\mu}}$.

Case 3: If $\nu = 0$, and $\alpha \neq 0$ with $\alpha > 0$, then the required solution has the form

$$\Upsilon_3(x,t) = \frac{\left(h_3 - \frac{\sqrt{2}\alpha\sqrt{-2\alpha\gamma^2+2\alpha-j}\sqrt{2\alpha\gamma^2-2\alpha+j}}{\sqrt{\gamma^2-1}\sqrt{\mu}}\right)\cot\left(\sqrt{\alpha}(x-t\varrho)\right)}{\sqrt{\alpha}\,(2\alpha\gamma^2-2\alpha+j)} - \frac{\sqrt{-2\alpha\gamma^2+2\alpha-j}}{\sqrt{\mu}}, \tag{3.6}$$

where $h_3 = \frac{\sqrt{2}\alpha\gamma^2\sqrt{-2\alpha\gamma^2+2\alpha-j}\sqrt{2\alpha\gamma^2-2\alpha+j}}{\sqrt{\gamma^2-1}\sqrt{\mu}}$.

$$\Upsilon_4(x,t) = -\frac{\left(h_4 - \frac{\sqrt{2}\alpha\sqrt{-2\alpha\gamma^2+2\alpha-j}\sqrt{2\alpha\gamma^2-2\alpha+j}}{\sqrt{\gamma^2-1}\sqrt{\mu}}\right)\tan\left(\sqrt{\alpha}(x-t\varrho)\right)}{\sqrt{-\alpha}\,(2\alpha\gamma^2-2\alpha+j)} - \frac{\sqrt{-2\alpha\gamma^2+2\alpha-j}}{\sqrt{\mu}}, \tag{3.7}$$

where $h_4 = \frac{\sqrt{2}\alpha\gamma^2\sqrt{-2\alpha\gamma^2+2\alpha-j}\sqrt{2\alpha\gamma^2-2\alpha+j}}{\sqrt{\gamma^2-1}\sqrt{\mu}}$.

Case 4: If $\nu = 0$, and $\alpha \neq 0$, with $\alpha < 0$, then the required solution has the form

$$\Upsilon_5(x,t) = -\frac{\left(h_5 - \frac{\sqrt{2}\alpha\sqrt{-2\alpha\gamma^2+2\alpha-j}\sqrt{2\alpha\gamma^2-2\alpha+j}}{\sqrt{\gamma^2-1}\sqrt{\mu}}\right)\coth\left(\sqrt{-\alpha}(x-t\varrho)\right)}{\sqrt{-\alpha}\,(2\alpha\gamma^2-2\alpha+j)} - \frac{\sqrt{-2\alpha\gamma^2+2\alpha-j}}{\sqrt{\mu}}, \tag{3.8}$$

where $h_5 = \frac{\sqrt{2}\alpha\gamma^2\sqrt{-2\alpha\gamma^2+2\alpha-j}\sqrt{2\alpha\gamma^2-2\alpha+j}}{\sqrt{\gamma^2-1}\sqrt{\mu}}$.

$$\Upsilon_6(x,t) = \frac{\left(h_6 - \frac{\sqrt{2}\alpha\sqrt{-2\alpha\gamma^2+2\alpha-j}\sqrt{2\alpha\gamma^2-2\alpha+j}}{\sqrt{\gamma^2-1}\sqrt{\mu}}\right)\tanh\left(\sqrt{-\alpha}(x-t\varrho)\right)}{\sqrt{-\alpha}\,(2\alpha\gamma^2-2\alpha+j)} - \frac{\sqrt{-2\alpha\gamma^2+2\alpha-j}}{\sqrt{\mu}}, \tag{3.9}$$

where $h_6 = \frac{\sqrt{2}\alpha\gamma^2\sqrt{-2\alpha\gamma^2+2\alpha-j}\sqrt{2\alpha\gamma^2-2\alpha+j}}{\sqrt{\gamma^2-1}\sqrt{\mu}}$.

Case 5: If $\alpha \neq 0$, and $\nu \neq 0$, then the required solution has the form

$$\Upsilon_7(x,t) = \frac{\left(h_7 - \frac{\sqrt{2}\alpha\sqrt{-2\alpha\gamma^2+2\alpha-j}\sqrt{2\alpha\gamma^2-2\alpha+j}}{\sqrt{\gamma^2-1}\sqrt{\mu}}\right)\left(1 - S_1 e^{(\sigma_1-\sigma_2)(x-t\varrho)}\right)}{(2\alpha\gamma^2 - 2\alpha + j)\left(\sigma_1 - \sigma_2 S_1 e^{(\sigma_1-\sigma_2)(x-t\varrho)}\right)} - \frac{\sqrt{-2\alpha\gamma^2 + 2\alpha - j}}{\sqrt{\mu}}, \tag{3.10}$$

where $h_7 = \frac{\sqrt{2}\alpha\gamma^2\sqrt{-2\alpha\gamma^2+2\alpha-j}\sqrt{2\alpha\gamma^2-2\alpha+j}}{\sqrt{\gamma^2-1}\sqrt{\mu}}$.

Family 2:

$$k_0 = -\frac{\sqrt{-2\alpha\gamma^2 + 2\alpha - j}}{\sqrt{\mu}}, \quad h_1 = 0,$$

$$k_1 = \frac{\frac{\sqrt{2}\sqrt{-2\alpha\gamma^2+2\alpha-j}\sqrt{2\alpha\gamma^2-2\alpha+j}}{\sqrt{\gamma^2-1}\sqrt{\mu}} - \frac{\sqrt{2}\gamma^2\sqrt{-2\alpha\gamma^2+2\alpha-j}\sqrt{2\alpha\gamma^2-2\alpha+j}}{\sqrt{\gamma^2-1}\sqrt{\mu}}}{2\alpha\gamma^2 - 2\alpha + j}.$$

Case 1: If $\nu = 0$, and $\alpha = 0$, then the required solution has the form

$$\Upsilon_8(x,t) = -\frac{\sqrt{-2\alpha\gamma^2 + 2\alpha - j}}{\sqrt{\mu}} - \frac{j_1 - \frac{\sqrt{2}\gamma^2\sqrt{-2\alpha\gamma^2+2\alpha-j}\sqrt{2\alpha\gamma^2-2\alpha+j}}{\sqrt{\gamma^2-1}\sqrt{\mu}}}{(2\alpha\gamma^2 - 2\alpha + j)(x - t\varrho)}, \tag{3.11}$$

where $j_1 = \frac{\sqrt{2}\sqrt{-2\alpha\gamma^2+2\alpha-j}\sqrt{2\alpha\gamma^2-2\alpha+j}}{\sqrt{\gamma^2-1}\sqrt{\mu}}$.

Case 2: If $\nu \neq 0$, and $\alpha = 0$, then the required solution has the form

$$\Upsilon_9(x,t) = -\frac{\sqrt{-2\alpha\gamma^2 + 2\alpha - j}}{\sqrt{\mu}} - \frac{\nu\left(j_2 - \frac{\sqrt{2}\gamma^2\sqrt{-2\alpha\gamma^2+2\alpha-j}\sqrt{2\alpha\gamma^2-2\alpha+j}}{\sqrt{\gamma^2-1}\sqrt{\mu}}\right)}{(2\alpha\gamma^2 - 2\alpha + j)\left(R_0 e^{-\nu(x-t\varrho)} - 1\right)}, \tag{3.12}$$

where $j_2 = \frac{\sqrt{2}\sqrt{-2\alpha\gamma^2+2\alpha-j}\sqrt{2\alpha\gamma^2-2\alpha+j}}{\sqrt{\gamma^2-1}\sqrt{\mu}}$.

Case 3: If $\nu = 0$, and $\alpha \neq 0$, with $\alpha > 0$, then the required solution has the form

$$\Upsilon_{10}(x,t) = \frac{\sqrt{\alpha}\left(\frac{\sqrt{2}\sqrt{-2\alpha\gamma^2+2\alpha-j}\sqrt{2\alpha\gamma^2-2\alpha+j}}{\sqrt{\gamma^2-1}\sqrt{\mu}} - j_3\right)\tan\left(\sqrt{\alpha}(x-t\varrho)\right)}{2\alpha\gamma^2-2\alpha+j} - \frac{\sqrt{-2\alpha\gamma^2+2\alpha-j}}{\sqrt{\mu}}, \tag{3.13}$$

where $j_3 = \frac{\sqrt{2}\gamma^2\sqrt{-2\alpha\gamma^2+2\alpha-j}\sqrt{2\alpha\gamma^2-2\alpha+j}}{\sqrt{\gamma^2-1}\sqrt{\mu}}$.

$$\Upsilon_{11}(x,t) = -\frac{\sqrt{-2\alpha\gamma^2+2\alpha-j}}{\sqrt{\mu}} - \frac{\sqrt{-\alpha}\left(\frac{\sqrt{2}\sqrt{-2\alpha\gamma^2+2\alpha-j}\sqrt{2\alpha\gamma^2-2\alpha+j}}{\sqrt{\gamma^2-1}\sqrt{\mu}} - j_4\right)\cot\left(\sqrt{\alpha}(x-t\varrho)\right)}{2\alpha\gamma^2-2\alpha+j}, \tag{3.14}$$

where $j_4 = \frac{\sqrt{2}\gamma^2\sqrt{-2\alpha\gamma^2+2\alpha-j}\sqrt{2\alpha\gamma^2-2\alpha+j}}{\sqrt{\gamma^2-1}\sqrt{\mu}}$.

Case 4: If $\nu = 0$, and $\alpha \neq 0$, with $\alpha < 0$, then the required solution has the form

$$\Upsilon_{12}(x,t) = -\frac{\sqrt{-2\alpha\gamma^2+2\alpha-j}}{\sqrt{\mu}} - \frac{\sqrt{-\alpha}\left(\frac{\sqrt{2}\sqrt{-2\alpha\gamma^2+2\alpha-j}\sqrt{2\alpha\gamma^2-2\alpha+j}}{\sqrt{\gamma^2-1}\sqrt{\mu}} - j_5\right)\tanh\left(\sqrt{-\alpha}(x-t\varrho)\right)}{2\alpha\gamma^2-2\alpha+j}, \tag{3.15}$$

where $j_5 = \frac{\sqrt{2}\gamma^2\sqrt{-2\alpha\gamma^2+2\alpha-j}\sqrt{2\alpha\gamma^2-2\alpha+j}}{\sqrt{\gamma^2-1}\sqrt{\mu}}$.

$$\Upsilon_{13}(x,t) = \frac{\sqrt{-\alpha}\left(\frac{\sqrt{2}\sqrt{-2\alpha\gamma^2+2\alpha-j}\sqrt{2\alpha\gamma^2-2\alpha+j}}{\sqrt{\gamma^2-1}\sqrt{\mu}} - j\right)\coth\left(\sqrt{-\alpha}(x-t\varrho)\right)}{2\alpha\gamma^2-2\alpha+j_6} - \frac{\sqrt{-2\alpha\gamma^2+2\alpha-j}}{\sqrt{\mu}}, \tag{3.16}$$

where $j_6 = \frac{\sqrt{2}\gamma^2\sqrt{-2\alpha\gamma^2+2\alpha-j}\sqrt{2\alpha\gamma^2-2\alpha+j}}{\sqrt{\gamma^2-1}\sqrt{\mu}}$.

Case 5: If $\alpha \neq 0$, and $\nu \neq 0$, then the required solution has the form

$$\Upsilon_{14}(x,t) = \frac{\left(\frac{\sqrt{2}\sqrt{-2\alpha\gamma^2+2\alpha-j}\sqrt{2\alpha\gamma^2-2\alpha+j}}{\sqrt{\gamma^2-1}\sqrt{\mu}} - j_7\right)\left(\sigma_1-\sigma_2 S_1 e^{(\sigma_1-\sigma_2)(x-t\varrho)}\right)}{\left(2\alpha\gamma^2-2\alpha+j\right)\left(1-S_1 e^{(\sigma_1-\sigma_2)(x-t\varrho)}\right)} - \frac{\sqrt{-2\alpha\gamma^2+2\alpha-j}}{\sqrt{\mu}}. \tag{3.17}$$

where $j_7 = \frac{\sqrt{2}\gamma^2\sqrt{-2\alpha\gamma^2+2\alpha-j}\sqrt{2\alpha\gamma^2-2\alpha+j}}{\sqrt{\gamma^2-1}\sqrt{\mu}}$.

Family 3:

$$k_0 = 0,\ k_1 = -\frac{\sqrt{2}\sqrt{1-\gamma^2}}{\sqrt{\mu}},$$

$$h_1 = \frac{\frac{\sqrt{2}\sqrt{1-\gamma^2}\gamma^2 j}{(\gamma^2-1)\sqrt{\mu}} - \frac{4\sqrt{2}\sqrt{1-\gamma^2}j}{\sqrt{\mu}} - \frac{\sqrt{2}\sqrt{1-\gamma^2}j}{(\gamma^2-1)\sqrt{\mu}}}{24(\gamma^2-1)}.$$

Case 1: If $\nu = 0$, and $\alpha = 0$, then the required solution has the form

$$\Upsilon_{15}(x,t) = \frac{\left(\frac{\sqrt{2}\sqrt{1-\gamma^2}\gamma^2 j}{(\gamma^2-1)\sqrt{\mu}} - \frac{4\sqrt{2}\sqrt{1-\gamma^2}j}{\sqrt{\mu}} - \frac{\sqrt{2}\sqrt{1-\gamma^2}j}{(\gamma^2-1)\sqrt{\mu}}\right)(t\varrho - x)}{24(\gamma^2-1)} + \frac{\sqrt{2}\sqrt{1-\gamma^2}}{\sqrt{\mu}(x-t\varrho)}. \tag{3.18}$$

Case 2: If $\nu \neq 0$, and $\alpha = 0$, then the required solution has the form

$$\Upsilon_{16}(x,t) = -\frac{\left(\frac{\sqrt{2}\sqrt{1-\gamma^2}\gamma^2 j}{(\gamma^2-1)\sqrt{\mu}} - \frac{4\sqrt{2}\sqrt{1-\gamma^2}j}{\sqrt{\mu}} - \frac{\sqrt{2}\sqrt{1-\gamma^2}j}{(\gamma^2-1)\sqrt{\mu}}\right)\left(R_0 e^{-\nu(x-t\varrho)} - 1\right)}{24(\gamma^2-1)\nu} + \frac{\sqrt{2}\sqrt{1-\gamma^2}\nu}{\sqrt{\mu}\left(R_0 e^{-\nu(x-t\varrho)} - 1\right)}. \tag{3.19}$$

Case 3: If $\nu = 0$, and $\alpha \neq 0$, with $\alpha > 0$, then the required solution has the form

$$\Upsilon_{17}(x,t) = \frac{\left(\frac{\sqrt{2}\sqrt{1-\gamma^2}\gamma^2 j}{(\gamma^2-1)\sqrt{\mu}} - \frac{4\sqrt{2}\sqrt{1-\gamma^2} j}{\sqrt{\mu}} - \frac{\sqrt{2}\sqrt{1-\gamma^2} j}{(\gamma^2-1)\sqrt{\mu}}\right)\cot\left(\sqrt{\alpha}(x-t\varrho)\right)}{24\sqrt{\alpha}\,(\gamma^2-1)} - \frac{\sqrt{2}\sqrt{\alpha}\sqrt{1-\gamma^2}\tan\left(\sqrt{\alpha}(x-t\varrho)\right)}{\sqrt{\mu}}, \tag{3.20}$$

$$\Upsilon_{18}(x,t) = -\frac{\left(\frac{\sqrt{2}\sqrt{1-\gamma^2}\gamma^2 j}{(\gamma^2-1)\sqrt{\mu}} - \frac{4\sqrt{2}\sqrt{1-\gamma^2} j}{\sqrt{\mu}} - \frac{\sqrt{2}\sqrt{1-\gamma^2} j}{(\gamma^2-1)\sqrt{\mu}}\right)\tan\left(\sqrt{\alpha}(x-t\varrho)\right)}{24\sqrt{-\alpha}\,(\gamma^2-1)} + \frac{\sqrt{2}\sqrt{-\alpha}\sqrt{1-\gamma^2}\cot\left(\sqrt{\alpha}(x-t\varrho)\right)}{\sqrt{\mu}}. \tag{3.21}$$

Case 4: If $\nu = 0$, and $\alpha \neq 0$, with $\alpha < 0$, then the required solution has the form

$$\Upsilon_{19}(x,t) = -\frac{\left(\frac{\sqrt{2}\sqrt{1-\gamma^2}\gamma^2 j}{(\gamma^2-1)\sqrt{\mu}} - \frac{4\sqrt{2}\sqrt{1-\gamma^2} j}{\sqrt{\mu}} - \frac{\sqrt{2}\sqrt{1-\gamma^2} j}{(\gamma^2-1)\sqrt{\mu}}\right)\coth\left(\sqrt{-\alpha}(x-t\varrho)\right)}{24\sqrt{-\alpha}\,(\gamma^2-1)} + \frac{\sqrt{2}\sqrt{-\alpha}\sqrt{1-\gamma^2}\tanh\left(\sqrt{-\alpha}(x-t\varrho)\right)}{\sqrt{\mu}}, \tag{3.22}$$

$$\Upsilon_{20}(x,t) = \frac{\left(\frac{\sqrt{2}\sqrt{1-\gamma^2}\gamma^2 j}{(\gamma^2-1)\sqrt{\mu}} - \frac{4\sqrt{2}\sqrt{1-\gamma^2} j}{\sqrt{\mu}} - \frac{\sqrt{2}\sqrt{1-\gamma^2} j}{(\gamma^2-1)\sqrt{\mu}}\right)\tanh\left(\sqrt{-\alpha}(x-t\varrho)\right)}{24\sqrt{-\alpha}\,(\gamma^2-1)} - \frac{\sqrt{2}\sqrt{-\alpha}\sqrt{1-\gamma^2}\coth\left(\sqrt{-\alpha}(x-t\varrho)\right)}{\sqrt{\mu}}. \tag{3.23}$$

Case 5: If $\alpha \neq 0$, and $\nu \neq 0$, then the required solution has the form

$$\Upsilon_{21}(x,t) = \frac{\left(\frac{\sqrt{2}\sqrt{1-\gamma^2}\gamma^2 j}{(\gamma^2-1)\sqrt{\mu}} - \frac{4\sqrt{2}\sqrt{1-\gamma^2} j}{\sqrt{\mu}} - \frac{\sqrt{2}\sqrt{1-\gamma^2} j}{(\gamma^2-1)\sqrt{\mu}}\right)\left(1 - S_1 e^{(\sigma_1-\sigma_2)(x-t\varrho)}\right)}{24\,(\gamma^2-1)\left(\sigma_1 - \sigma_2 S_1 e^{(\sigma_1-\sigma_2)(x-t\varrho)}\right)} - \frac{\sqrt{2}\sqrt{1-\gamma^2}\left(\sigma_1 - \sigma_2 S_1 e^{(\sigma_1-\sigma_2)(x-t\varrho)}\right)}{\sqrt{\mu}\left(1 - S_1 e^{(\sigma_1-\sigma_2)(x-t\varrho)}\right)}. \tag{3.24}$$

3.2. Application to the unified method

First of all, we find the balancing number by utilizing the latest technique, called the homogeneous balance method. We obtain $N = 1$, and then we express the Eq (3.2) solution in the form written below:

$$\mathcal{U}(\chi) = \xi_0 + \xi_1\delta(\chi) + \frac{\zeta_1}{\delta(\chi)}. \tag{3.25}$$

Here we can get the mathematical system of equations through the addition of Eqs (3.25) and (2.7) to Eq (3.2), which gives

$$\begin{aligned} 6\zeta_1\mu\xi_1\xi_0 + j\xi_0 + \mu\xi_0^3 &= 0,\\ 2\beta^2\gamma^2\zeta_1 - 2\beta^2\zeta_1 + \zeta_1^3\mu &= 0,\\ 2\beta\gamma^2\zeta_1 - 2\beta\zeta_1 + 3\zeta_1\mu\xi_0^2 + 3\zeta_1^2\mu\xi_1 + \zeta_1 j &= 0,\\ 2\beta\gamma^2\xi_1 - 2\beta\xi_1 + 3\zeta_1\mu\xi_1^2 + j\xi_1 + 3\mu\xi_0^2\xi_1 &= 0,\\ 2\gamma^2\xi_1 + \mu\xi_1^3 - 2\xi_1 &= 0. \end{aligned}$$

On solving these equations, we obtain
Family 1:

$$\xi_0 = \frac{\sqrt{\beta - \beta\gamma^2}}{\sqrt{\mu}}, \;\; \xi_1 = \frac{\sqrt{2}\sqrt{1-\gamma^2}}{\sqrt{\mu}}, \;\; \zeta_1 = 0, \;\; j = \beta\left(\gamma^2 - 1\right).$$

Further, adding the given results in Eq (3.25) gives the following outcomes:
Case 1: If $\beta < 0$, the required solution is given by

$$\Upsilon_{22}(x,t) = \frac{\sqrt{\beta-\beta\gamma^2}}{\sqrt{\mu}} + \frac{\sqrt{2}\sqrt{1-\gamma^2}\left(\sqrt{\beta\left(-\rho^2-\sigma^2\right)} - \sqrt{-\beta}\rho\cosh\left(2\sqrt{-\beta}(-t\varrho+x+\Omega)\right)\right)}{\sqrt{\mu}\left(\sigma+\rho\sinh\left(2\sqrt{-\beta}(-t\varrho+x+\Omega)\right)\right)}, \tag{3.26}$$

$$\Upsilon_{23}(x,t) = \frac{\sqrt{\beta-\beta\gamma^2}}{\sqrt{\mu}} + \frac{\sqrt{2}\sqrt{1-\gamma^2}\left(\sqrt{-\beta}(-\rho)\cosh\left(2\sqrt{-\beta}(-t\varrho+x+\Omega)\right) - \sqrt{\beta\left(-\rho^2-\sigma^2\right)}\right)}{\sqrt{\mu}\left(\sigma+\rho\sinh\left(2\sqrt{-\beta}(-t\varrho+x+\Omega)\right)\right)}, \tag{3.27}$$

$$\Upsilon_{24}(x,t) = \frac{\sqrt{\beta-\beta\gamma^2}}{\sqrt{\mu}} + \frac{\sqrt{2}\sqrt{1-\gamma^2}\left(\sqrt{-\beta} - \frac{2\sqrt{-\beta}\rho}{\rho-\sinh\left(2\sqrt{-\beta}(-t\varrho+x+\Omega)\right)+\cosh\left(2\sqrt{-\beta}(-t\varrho+x+\Omega)\right)}\right)}{\sqrt{\mu}}, \tag{3.28}$$

$$\Upsilon_{25}(x,t) = \frac{\sqrt{\beta-\beta\gamma^2}}{\sqrt{\mu}} + \frac{\sqrt{2}\sqrt{1-\gamma^2}\left(\frac{2\sqrt{-\beta}\rho}{\rho+\sinh\left(2\sqrt{-\beta}(-t\varrho+x+\Omega)\right)+\cosh\left(2\sqrt{-\beta}(-t\varrho+x+\Omega)\right)} - \sqrt{-\beta}\right)}{\sqrt{\mu}}. \tag{3.29}$$

Case 2: If $\beta > 0$, the required solution is given by

$$\Upsilon_{26}(x,t) = \frac{\sqrt{\beta-\beta\gamma^2}}{\sqrt{\mu}} + \frac{\sqrt{2}\sqrt{1-\gamma^2}\left(\sqrt{\beta\left(\rho^2-\sigma^2\right)} - \sqrt{\beta}\rho\cos\left(2\sqrt{\beta}(-t\varrho+x+\Omega)\right)\right)}{\sqrt{\mu}\left(\sigma+\rho\sin\left(2\sqrt{\beta}(-t\varrho+x+\Omega)\right)\right)}, \tag{3.30}$$

$$\Upsilon_{27}(x,t) = \frac{\sqrt{\beta-\beta\gamma^2}}{\sqrt{\mu}} + \frac{\sqrt{2}\sqrt{1-\gamma^2}\left(\sqrt{\beta}(-\rho)\cos\left(2\sqrt{\beta}(-t\varrho+x+\Omega)\right) - \sqrt{\beta\left(\rho^2-\sigma^2\right)}\right)}{\sqrt{\mu}\left(\sigma+\rho\sin\left(2\sqrt{\beta}(-t\varrho+x+\Omega)\right)\right)}, \tag{3.31}$$

$$\Upsilon_{28}(x,t) = \frac{\sqrt{\beta-\beta\gamma^2}}{\sqrt{\mu}} + \frac{\sqrt{2}\sqrt{1-\gamma^2}\left(i\sqrt{\beta} - \frac{2i\sqrt{\beta}\rho}{\rho-i\sin\left(2\sqrt{\beta}(-t\varrho+x+\Omega)\right)+\cos\left(2\sqrt{\beta}(-t\varrho+x+\Omega)\right)}\right)}{\sqrt{\mu}},$$

(3.32)

$$\Upsilon_{29}(x,t) = \frac{\sqrt{\beta-\beta\gamma^2}}{\sqrt{\mu}} + \frac{\sqrt{2}\sqrt{1-\gamma^2}\left(\frac{2i\sqrt{\beta}\rho}{\rho+i\sin\left(2\sqrt{\beta}(-t\varrho+x+\Omega)\right)+\cos\left(2\sqrt{\beta}(-t\varrho+x+\Omega)\right)}-i\sqrt{\beta}\right)}{\sqrt{\mu}}. \tag{3.33}$$

Case 3: If $\beta = 0$, the required solution is given by

$$\Upsilon_{30}(x,t) = \frac{\sqrt{\beta-\beta\gamma^2}}{\sqrt{\mu}} - \frac{\sqrt{2}\sqrt{1-\gamma^2}}{\sqrt{\mu}(-t\varrho+x+\Omega)}. \tag{3.34}$$

Family 2:

$$\xi_0 = -\sqrt{-\frac{3\sqrt{1-\gamma^2}\sqrt{-\beta^2(\gamma^2-1)}}{\mu} - \frac{\beta\gamma^2}{\mu} + \frac{\beta}{\mu}},$$

$$\xi_1 = \frac{\sqrt{2}\sqrt{1-\gamma^2}}{\sqrt{\mu}},\ \zeta_1 = -\frac{\sqrt{2}\sqrt{\beta^2-\beta^2\gamma^2}}{\sqrt{\mu}}.$$

Case 1: If $\beta < 0$, the required solution is given by

$$\begin{aligned}\Upsilon_{31}(x,t) = &-\sqrt{-\frac{3\sqrt{1-\gamma^2}\sqrt{-\beta^2(\gamma^2-1)}}{\mu} - \frac{\beta\gamma^2}{\mu} + \frac{\beta}{\mu}}\\ &+\frac{\sqrt{2}\sqrt{1-\gamma^2}\left(\sqrt{\beta(-\rho^2-\sigma^2)} - \sqrt{-\beta}\rho\cosh\left(2\sqrt{-\beta}(-t\varrho+x+\Omega)\right)\right)}{\sqrt{\mu}\left(\sigma+\rho\sinh\left(2\sqrt{-\beta}(-t\varrho+x+\Omega)\right)\right)}\\ &-\frac{\sqrt{2}\sqrt{\beta^2-\beta^2\gamma^2}\left(\sigma+\rho\sinh\left(2\sqrt{-\beta}(-t\varrho+x+\Omega)\right)\right)}{\sqrt{\mu}\left(\sqrt{\beta(-\rho^2-\sigma^2)} - \sqrt{-\beta}\rho\cosh\left(2\sqrt{-\beta}(-t\varrho+x+\Omega)\right)\right)},\end{aligned} \tag{3.35}$$

$$\begin{aligned}\Upsilon_{32}(x,t) = &-\sqrt{-\frac{3\sqrt{1-\gamma^2}\sqrt{-\beta^2(\gamma^2-1)}}{\mu} - \frac{\beta\gamma^2}{\mu} + \frac{\beta}{\mu}}\\ &+\frac{\sqrt{2}\sqrt{1-\gamma^2}\left(\sqrt{-\beta}(-\rho)\cosh\left(2\sqrt{-\beta}(-t\varrho+x+\Omega)\right) - \sqrt{\beta(-\rho^2-\sigma^2)}\right)}{\sqrt{\mu}\left(\sigma+\rho\sinh\left(2\sqrt{-\beta}(-t\varrho+x+\Omega)\right)\right)}\\ &-\frac{\sqrt{2}\sqrt{\beta^2-\beta^2\gamma^2}\left(\sigma+\rho\sinh\left(2\sqrt{-\beta}(-t\varrho+x+\Omega)\right)\right)}{\sqrt{\mu}\left(\sqrt{-\beta}(-\rho)\cosh\left(2\sqrt{-\beta}(-t\varrho+x+\Omega)\right) - \sqrt{\beta(-\rho^2-\sigma^2)}\right)},\end{aligned} \tag{3.36}$$

$$\begin{aligned}\Upsilon_{33}(x,t) = &\frac{\sqrt{2}\sqrt{\beta^2-\beta^2\gamma^2}}{\sqrt{\mu}\left(\sqrt{-\beta} - \frac{2\sqrt{-\beta}\rho}{\rho-\sinh\left(2\sqrt{-\beta}(-t\varrho+x+\Omega)\right)+\cosh\left(2\sqrt{-\beta}(-t\varrho+x+\Omega)\right)}\right)}\\ &\times\frac{\sqrt{2}\sqrt{1-\gamma^2}\left(\sqrt{-\beta} - \frac{2\sqrt{-\beta}\rho}{\rho-\sinh\left(2\sqrt{-\beta}(-t\varrho+x+\Omega)\right)+\cosh\left(2\sqrt{-\beta}(-t\varrho+x+\Omega)\right)}\right)}{\sqrt{\mu}}\end{aligned}$$

$$+-\sqrt{-\frac{3\sqrt{1-\gamma^2}\sqrt{-\beta^2(\gamma^2-1)}}{\mu}-\frac{\beta\gamma^2}{\mu}+\frac{\beta}{\mu}}, \tag{3.37}$$

$$\begin{aligned}\Upsilon_{34}(x,t)=&-\sqrt{-\frac{3\sqrt{1-\gamma^2}\sqrt{-\beta^2(\gamma^2-1)}}{\mu}-\frac{\beta\gamma^2}{\mu}+\frac{\beta}{\mu}}\\&-\frac{\sqrt{2}\sqrt{\beta^2-\beta^2\gamma^2}}{\sqrt{\mu}\left(\frac{2\sqrt{-\beta}\rho}{\rho+\sinh\left(2\sqrt{-\beta}(-t\varrho+x+\Omega)\right)+\cosh\left(2\sqrt{-\beta}(-t\varrho+x+\Omega)\right)}-\sqrt{-\beta}\right)}\\&+\frac{\sqrt{2}\sqrt{1-\gamma^2}\left(\frac{2\sqrt{-\beta}\rho}{\rho+\sinh\left(2\sqrt{-\beta}(-t\varrho+x+\Omega)\right)+\cosh\left(2\sqrt{-\beta}(-t\varrho+x+\Omega)\right)}-\sqrt{-\beta}\right)}{\sqrt{\mu}}.\end{aligned} \tag{3.38}$$

Case 2: If $\beta > 0$, the required solution is given by

$$\begin{aligned}\Upsilon_{35}(x,t)=&-\sqrt{-\frac{3\sqrt{1-\gamma^2}\sqrt{-\beta^2(\gamma^2-1)}}{\mu}-\frac{\beta\gamma^2}{\mu}+\frac{\beta}{\mu}}\\&+\frac{\sqrt{2}\sqrt{1-\gamma^2}\left(\sqrt{\beta(\rho^2-\sigma^2)}-\sqrt{\beta}\rho\cos\left(2\sqrt{\beta}(-t\varrho+x+\Omega)\right)\right)}{\sqrt{\mu}\left(\sigma+\rho\sin\left(2\sqrt{\beta}(-t\varrho+x+\Omega)\right)\right)}\\&-\frac{\sqrt{2}\sqrt{\beta^2-\beta^2\gamma^2}\left(\sigma+\rho\sin\left(2\sqrt{\beta}(-t\varrho+x+\Omega)\right)\right)}{\sqrt{\mu}\left(\sqrt{\beta(\rho^2-\sigma^2)}-\sqrt{\beta}\rho\cos\left(2\sqrt{\beta}(-t\varrho+x+\Omega)\right)\right)},\end{aligned} \tag{3.39}$$

$$\begin{aligned}\Upsilon_{36}(x,t)=&-\sqrt{-\frac{3\sqrt{1-\gamma^2}\sqrt{-\beta^2(\gamma^2-1)}}{\mu}-\frac{\beta\gamma^2}{\mu}+\frac{\beta}{\mu}}\\&+\frac{\sqrt{2}\sqrt{1-\gamma^2}\left(\sqrt{\beta}(-\rho)\cos\left(2\sqrt{\beta}(-t\varrho+x+\Omega)\right)-\sqrt{\beta(\rho^2-\sigma^2)}\right)}{\sqrt{\mu}\left(\sigma+\rho\sin\left(2\sqrt{\beta}(-t\varrho+x+\Omega)\right)\right)}\\&-\frac{\sqrt{2}\sqrt{\beta^2-\beta^2\gamma^2}\left(\sigma+\rho\sin\left(2\sqrt{\beta}(-t\varrho+x+\Omega)\right)\right)}{\sqrt{\mu}\left(\sqrt{\beta}(-\rho)\cos\left(2\sqrt{\beta}(-t\varrho+x+\Omega)\right)-\sqrt{\beta(\rho^2-\sigma^2)}\right)},\end{aligned} \tag{3.40}$$

$$\begin{aligned}\Upsilon_{37}(x,t)=&-\sqrt{-\frac{3\sqrt{1-\gamma^2}\sqrt{-\beta^2(\gamma^2-1)}}{\mu}-\frac{\beta\gamma^2}{\mu}+\frac{\beta}{\mu}}\\&-\frac{\sqrt{2}\sqrt{\beta^2-\beta^2\gamma^2}}{\sqrt{\mu}\left(i\sqrt{\beta}-\frac{2i\sqrt{\beta}\rho}{\rho-i\sin\left(2\sqrt{\beta}(-t\varrho+x+\Omega)\right)+\cos\left(2\sqrt{\beta}(-t\varrho+x+\Omega)\right)}\right)}\\&+\frac{\sqrt{2}\sqrt{1-\gamma^2}\left(i\sqrt{\beta}-\frac{2i\sqrt{\beta}\rho}{\rho-i\sin\left(2\sqrt{\beta}(-t\varrho+x+\Omega)\right)+\cos\left(2\sqrt{\beta}(-t\varrho+x+\Omega)\right)}\right)}{\sqrt{\mu}},\end{aligned} \tag{3.41}$$

$$\Upsilon_{38}(x,t) = -\sqrt{-\frac{3\sqrt{1-\gamma^2}\sqrt{-\beta^2(\gamma^2-1)}}{\mu} - \frac{\beta\gamma^2}{\mu} + \frac{\beta}{\mu}}$$
$$-\frac{\sqrt{2}\sqrt{\beta^2-\beta^2\gamma^2}}{\sqrt{\mu}\left(\frac{2i\sqrt{\beta}\rho}{\rho+i\sin\left(2\sqrt{\beta}(-t\varrho+x+\Omega)\right)+\cos\left(2\sqrt{\beta}(-t\varrho+x+\Omega)\right)} - i\sqrt{\beta}\right)}$$
$$+\frac{\sqrt{2}\sqrt{1-\gamma^2}\left(\frac{2i\sqrt{\beta}\rho}{\rho+i\sin\left(2\sqrt{\beta}(-t\varrho+x+\Omega)\right)+\cos\left(2\sqrt{\beta}(-t\varrho+x+\Omega)\right)} - i\sqrt{\beta}\right)}{\sqrt{\mu}}. \tag{3.42}$$

Case 3: If $\beta = 0$, the required solution is given by

$$\Upsilon_{39}(x,t) = -\sqrt{-\frac{3\sqrt{1-\gamma^2}\sqrt{-\beta^2(\gamma^2-1)}}{\mu} - \frac{\beta\gamma^2}{\mu} + \frac{\beta}{\mu}}$$
$$-\frac{\sqrt{2}\sqrt{\beta^2-\beta^2\gamma^2}(t\varrho - x - \Omega)}{\sqrt{\mu}} - \frac{\sqrt{2}\sqrt{1-\gamma^2}}{\sqrt{\mu}(-t\varrho+x+\Omega)}. \tag{3.43}$$

Family 3:

$$\xi_0 = 0, \quad \xi_1 = \frac{\sqrt{2}\sqrt{1-\gamma^2}}{\sqrt{\mu}}, \quad \zeta_1 = \frac{\sqrt{2}\sqrt{\beta^2-\beta^2\gamma^2}}{\sqrt{\mu}},$$
$$j = 2\left(-3\sqrt{1-\gamma^2}\sqrt{-\beta^2(\gamma^2-1)} - \beta\gamma^2 + \beta\right).$$

Case 1: If $\beta < 0$, the required solution is given by

$$\Upsilon_{40}(x,t) = \frac{\sqrt{2}\sqrt{1-\gamma^2}\left(\sqrt{\beta(-\rho^2-\sigma^2)} - \sqrt{-\beta}\rho\cosh\left(2\sqrt{-\beta}(-t\varrho+x+\Omega)\right)\right)}{\sqrt{\mu}\left(\sigma+\rho\sinh\left(2\sqrt{-\beta}(-t\varrho+x+\Omega)\right)\right)}$$
$$+\frac{\sqrt{2}\sqrt{\beta^2-\beta^2\gamma^2}\left(\sigma+\rho\sinh\left(2\sqrt{-\beta}(-t\varrho+x+\Omega)\right)\right)}{\sqrt{\mu}\left(\sqrt{\beta(-\rho^2-\sigma^2)} - \sqrt{-\beta}\rho\cosh\left(2\sqrt{-\beta}(-t\varrho+x+\Omega)\right)\right)}, \tag{3.44}$$

$$\Upsilon_{41}(x,t) = \frac{\sqrt{2}\sqrt{1-\gamma^2}\left(\sqrt{-\beta}(-\rho)\cosh\left(2\sqrt{-\beta}(-t\varrho+x+\Omega)\right) - \sqrt{\beta(-\rho^2-\sigma^2)}\right)}{\sqrt{\mu}\left(\sigma+\rho\sinh\left(2\sqrt{-\beta}(-t\varrho+x+\Omega)\right)\right)}$$
$$+\frac{\sqrt{2}\sqrt{\beta^2-\beta^2\gamma^2}\left(\sigma+\rho\sinh\left(2\sqrt{-\beta}(-t\varrho+x+\Omega)\right)\right)}{\sqrt{\mu}\left(\sqrt{-\beta}(-\rho)\cosh\left(2\sqrt{-\beta}(-t\varrho+x+\Omega)\right) - \sqrt{\beta(-\rho^2-\sigma^2)}\right)}, \tag{3.45}$$

$$\Upsilon_{42}(x,t) = \frac{\sqrt{2}\sqrt{\beta^2-\beta^2\gamma^2}}{\sqrt{\mu}\left(\sqrt{-\beta} - \frac{2\sqrt{-\beta}\rho}{\rho-\sinh\left(2\sqrt{-\beta}(-t\varrho+x+\Omega)\right)+\cosh\left(2\sqrt{-\beta}(-t\varrho+x+\Omega)\right)}\right)}$$
$$+\frac{\sqrt{2}\sqrt{1-\gamma^2}\left(\sqrt{-\beta} - \frac{2\sqrt{-\beta}\rho}{\rho-\sinh\left(2\sqrt{-\beta}(-t\varrho+x+\Omega)\right)+\cosh\left(2\sqrt{-\beta}(-t\varrho+x+\Omega)\right)}\right)}{\sqrt{\mu}}, \tag{3.46}$$

$$
\Upsilon_{43}(x,t)=\frac{\sqrt{2}\sqrt{\beta^2-\beta^2\gamma^2}}{\sqrt{\mu}\left(\frac{2\sqrt{-\beta}\rho}{\rho+\sinh\left(2\sqrt{-\beta}(-t\varrho+x+\Omega)\right)+\cosh\left(2\sqrt{-\beta}(-t\varrho+x+\Omega)\right)}-\sqrt{-\beta}\right)}
+\frac{\sqrt{2}\sqrt{1-\gamma^2}\left(\frac{2\sqrt{-\beta}\rho}{\rho+\sinh\left(2\sqrt{-\beta}(-t\varrho+x+\Omega)\right)+\cosh\left(2\sqrt{-\beta}(-t\varrho+x+\Omega)\right)}-\sqrt{-\beta}\right)}{\sqrt{\mu}}. \tag{3.47}
$$

Case 2: If $\beta > 0$, the required solution is given by

$$
\Upsilon_{44}(x,t)=\frac{\sqrt{2}\sqrt{1-\gamma^2}\left(\sqrt{\beta\left(\rho^2-\sigma^2\right)}-\sqrt{\beta}\rho\cos\left(2\sqrt{\beta}(-t\varrho+x+\Omega)\right)\right)}{\sqrt{\mu}\left(\sigma+\rho\sin\left(2\sqrt{\beta}(-t\varrho+x+\Omega)\right)\right)}
+\frac{\sqrt{2}\sqrt{\beta^2-\beta^2\gamma^2}\left(\sigma+\rho\sin\left(2\sqrt{\beta}(-t\varrho+x+\Omega)\right)\right)}{\sqrt{\mu}\left(\sqrt{\beta\left(\rho^2-\sigma^2\right)}-\sqrt{\beta}\rho\cos\left(2\sqrt{\beta}(-t\varrho+x+\Omega)\right)\right)}, \tag{3.48}
$$

$$
\Upsilon_{45}(x,t)=\frac{\sqrt{2}\sqrt{1-\gamma^2}\left(\sqrt{\beta}(-\rho)\cos\left(2\sqrt{\beta}(-t\varrho+x+\Omega)\right)-\sqrt{\beta\left(\rho^2-\sigma^2\right)}\right)}{\sqrt{\mu}\left(\sigma+\rho\sin\left(2\sqrt{\beta}(-t\varrho+x+\Omega)\right)\right)}
+\frac{\sqrt{2}\sqrt{\beta^2-\beta^2\gamma^2}\left(\sigma+\rho\sin\left(2\sqrt{\beta}(-t\varrho+x+\Omega)\right)\right)}{\sqrt{\mu}\left(\sqrt{\beta}(-\rho)\cos\left(2\sqrt{\beta}(-t\varrho+x+\Omega)\right)-\sqrt{\beta\left(\rho^2-\sigma^2\right)}\right)}, \tag{3.49}
$$

$$
\Upsilon_{46}(x,t)=\frac{\sqrt{2}\sqrt{\beta^2-\beta^2\gamma^2}}{\sqrt{\mu}\left(i\sqrt{\beta}-\frac{2i\sqrt{\beta}\rho}{\rho-i\sin\left(2\sqrt{\beta}(-t\varrho+x+\Omega)\right)+\cos\left(2\sqrt{\beta}(-t\varrho+x+\Omega)\right)}\right)}
+\frac{\sqrt{2}\sqrt{1-\gamma^2}\left(i\sqrt{\beta}-\frac{2i\sqrt{\beta}\rho}{\rho-i\sin\left(2\sqrt{\beta}(-t\varrho+x+\Omega)\right)+\cos\left(2\sqrt{\beta}(-t\varrho+x+\Omega)\right)}\right)}{\sqrt{\mu}}, \tag{3.50}
$$

$$
\Upsilon_{47}(x,t)=\frac{\sqrt{2}\sqrt{\beta^2-\beta^2\gamma^2}}{\sqrt{\mu}\left(\frac{2i\sqrt{\beta}\rho}{\rho+i\sin\left(2\sqrt{\beta}(-t\varrho+x+\Omega)\right)+\cos\left(2\sqrt{\beta}(-t\varrho+x+\Omega)\right)}-i\sqrt{\beta}\right)}
+\frac{\sqrt{2}\sqrt{1-\gamma^2}\left(\frac{2i\sqrt{\beta}\rho}{\rho+i\sin\left(2\sqrt{\beta}(-t\varrho+x+\Omega)\right)+\cos\left(2\sqrt{\beta}(-t\varrho+x+\Omega)\right)}-i\sqrt{\beta}\right)}{\sqrt{\mu}}. \tag{3.51}
$$

Case 3: If $\beta = 0$, the required solution is given by

$$
\Upsilon_{48}(x,t)=\frac{\sqrt{2}\sqrt{\beta^2-\beta^2\gamma^2}(t\varrho-x-\Omega)}{\sqrt{\mu}}-\frac{\sqrt{2}\sqrt{1-\gamma^2}}{\sqrt{\mu}(-t\varrho+x+\Omega)}. \tag{3.52}
$$

3.3. *Application of the modified extended* tanh *function method*

First of all, we find the balancing number by utilizing the latest technique called the homogeneous balance method. We obtain n=1, and then we express the Eq (3.2) solution in the form as written below:

$$\mathcal{U}(\chi) = s_0 + s_1\delta(\chi) + \frac{v_1}{\delta(\chi)}. \tag{3.53}$$

Here we can get the mathematical system of equations through the addition of Eqs (3.53) and (2.9) to Eq (3.2), which gives us

$$\begin{aligned}
&js_0 + \mu s_0^3 + 6\mu s_1 s_0 v_1 = 0,\\
&2\gamma^2\kappa^2 v_1 - 2\kappa^2 v_1 + \mu v_1^3 = 0,\\
&jv_1 + 3\mu s_1 v_1^2 + 3\mu s_0^2 v_1 + 2\gamma^2\kappa v_1 - 2\kappa v_1 = 0,\\
&js_1 + 2\gamma^2\kappa s_1 - 2\kappa s_1 + 3\mu s_0^2 s_1 + 3\mu s_1^2 v_1 = 0,\\
&2\gamma^2 s_1 + \mu s_1^3 - 2s_1 = 0.
\end{aligned}$$

On solving these equations, we obtain the following:
Family 1:

$$s_0 = -\frac{\sqrt{\kappa - \gamma^2\kappa}}{\sqrt{\mu}},\ s_1 = 0,\ v_1 = \frac{\sqrt{2}\sqrt{\kappa^2 - \gamma^2\kappa^2}}{\sqrt{\mu}},\ j = \left(\gamma^2 - 1\right)\kappa.$$

Further, adding the given results in Eq (3.53) gives the following outcomes:
Case 1. If $\kappa < 0$, the required solution is given by

$$\Upsilon_{49}(x,t) = -\frac{\sqrt{\kappa - \gamma^2\kappa}}{\sqrt{\mu}} - \frac{\sqrt{2}\sqrt{\kappa^2 - \gamma^2\kappa^2}\coth\left(\sqrt{-\kappa}(x - t\varrho)\right)}{\sqrt{-\kappa}\sqrt{\mu}}, \tag{3.54}$$

$$\Upsilon_{50}(x,t) = -\frac{\sqrt{\kappa - \gamma^2\kappa}}{\sqrt{\mu}} - \frac{\sqrt{2}\sqrt{\kappa^2 - \gamma^2\kappa^2}\tanh\left(\sqrt{-\kappa}(x - t\varrho)\right)}{\sqrt{-\kappa}\sqrt{\mu}}. \tag{3.55}$$

Case 2. If $\kappa = 0$, the required solution is given by

$$\Upsilon_{51}(x,t) = -\frac{\sqrt{2}\kappa\sqrt{\kappa^2 - \gamma^2\kappa^2}}{\sqrt{\mu}} - \frac{\sqrt{\kappa - \gamma^2\kappa}}{\sqrt{\mu}}. \tag{3.56}$$

Case 3. If $\kappa > 0$, the required solution is given by

$$\Upsilon_{52}(x,t) = \frac{\sqrt{2}\sqrt{\kappa^2 - \gamma^2\kappa^2}\cot\left(\sqrt{\kappa}(x - t\varrho)\right)}{\sqrt{\kappa}\sqrt{\mu}} - \frac{\sqrt{\kappa - \gamma^2\kappa}}{\sqrt{\mu}}, \tag{3.57}$$

$$\Upsilon_{53}(x,t) = -\frac{\sqrt{\kappa - \gamma^2\kappa}}{\sqrt{\mu}} - \frac{\sqrt{2}\sqrt{\kappa^2 - \gamma^2\kappa^2}\tan\left(\sqrt{\kappa}(x - t\varrho)\right)}{\sqrt{\kappa}\sqrt{\mu}}. \tag{3.58}$$

Family 2:

$$s_0 = 0,\ s_1 = \frac{\sqrt{2}\sqrt{1-\gamma^2}}{\sqrt{\mu}},\ v_1 = \frac{\sqrt{2}\sqrt{\kappa^2-\gamma^2\kappa^2}}{\sqrt{\mu}}.$$

Further, adding the given results in Eq (3.53) gives the following outcomes:
Case 1. If $\kappa < 0$, the required solution is given by

$$\begin{aligned}\Upsilon_{54}(x,t) = &-\frac{\sqrt{2}\sqrt{\kappa^2-\gamma^2\kappa^2}\coth\left(\sqrt{-\kappa}(x-t\varrho)\right)}{\sqrt{-\kappa}\sqrt{\mu}} \\ &-\frac{\sqrt{2}\sqrt{1-\gamma^2}\sqrt{-\kappa}\tanh\left(\sqrt{-\kappa}(x-t\varrho)\right)}{\sqrt{\mu}},\end{aligned} \tag{3.59}$$

$$\begin{aligned}\Upsilon_{55}(x,t) = &-\frac{\sqrt{2}\sqrt{\kappa^2-\gamma^2\kappa^2}\tanh\left(\sqrt{-\kappa}(x-t\varrho)\right)}{\sqrt{-\kappa}\sqrt{\mu}} \\ &-\frac{\sqrt{2}\sqrt{1-\gamma^2}\sqrt{-\kappa}\coth\left(\sqrt{-\kappa}(x-t\varrho)\right)}{\sqrt{\mu}}.\end{aligned} \tag{3.60}$$

Case 2. If $\kappa = 0$, the required solution is given by

$$\Upsilon_{56}(x,t) = -\frac{\sqrt{2}\kappa\sqrt{\kappa^2-\gamma^2\kappa^2}}{\sqrt{\mu}} - \frac{\sqrt{2}\sqrt{1-\gamma^2}}{\kappa\sqrt{\mu}}. \tag{3.61}$$

Case 3. If $\kappa > 0$, the required solution is given by

$$\Upsilon_{57}(x,t) = \frac{\sqrt{2}\sqrt{\kappa^2-\gamma^2\kappa^2}\cot\left(\sqrt{\kappa}(x-t\varrho)\right)}{\sqrt{\kappa}\sqrt{\mu}} + \frac{\sqrt{2}\sqrt{1-\gamma^2}\sqrt{\kappa}\tan\left(\sqrt{\kappa}(x-t\varrho)\right)}{\sqrt{\mu}}, \tag{3.62}$$

$$\Upsilon_{58}(x,t) = -\frac{\sqrt{2}\sqrt{\kappa^2-\gamma^2\kappa^2}\tan\left(\sqrt{\kappa}(x-t\varrho)\right)}{\sqrt{\kappa}\sqrt{\mu}} - \frac{\sqrt{2}\sqrt{1-\gamma^2}\sqrt{\kappa}\cot\left(\sqrt{\kappa}(x-t\varrho)\right)}{\sqrt{\mu}}. \tag{3.63}$$

Family 3:

$$s_0 = -\frac{\sqrt{3\sqrt{1-\gamma^2}\sqrt{-\left(\gamma^2-1\right)\kappa^2}+\gamma^2(-\kappa)+\kappa}}{\sqrt{\mu}},\ s_1 = \frac{\sqrt{2}\sqrt{1-\gamma^2}}{\sqrt{\mu}},$$

$$v_1 = \frac{\sqrt{2}\sqrt{\kappa^2-\gamma^2\kappa^2}}{\sqrt{\mu}}.$$

Further, adding the given results in Eq (3.53) gives the following outcomes:
Case 1. If $\kappa < 0$, the required solution is given by

$$\Upsilon_{59}(x,t) = -\frac{\sqrt{3\sqrt{1-\gamma^2}\sqrt{(1-\gamma^2)\kappa^2+\gamma^2(-\kappa)}+\kappa}}{\sqrt{\mu}} - \frac{\sqrt{2}\sqrt{\kappa^2-\gamma^2\kappa^2}\coth\left(\sqrt{-\kappa}(x-t\varrho)\right)}{\sqrt{-\kappa}\sqrt{\mu}} - \frac{\sqrt{2}\sqrt{1-\gamma^2}\sqrt{-\kappa}\tanh\left(\sqrt{-\kappa}(x-t\varrho)\right)}{\sqrt{\mu}}, \quad (3.64)$$

$$\Upsilon_{60}(x,t) = -\frac{\sqrt{3\sqrt{1-\gamma^2}\sqrt{(1-\gamma^2)\kappa^2+\gamma^2(-\kappa)}+\kappa}}{\sqrt{\mu}} - \frac{\sqrt{2}\sqrt{\kappa^2-\gamma^2\kappa^2}\tanh\left(\sqrt{-\kappa}(x-t\varrho)\right)}{\sqrt{-\kappa}\sqrt{\mu}} - \frac{\sqrt{2}\sqrt{1-\gamma^2}\sqrt{-\kappa}\coth\left(\sqrt{-\kappa}(x-t\varrho)\right)}{\sqrt{\mu}}. \quad (3.65)$$

Case 2. If $\kappa = 0$, the required solution is given by

$$\Upsilon_{61}(x,t) = -\frac{\sqrt{2}\kappa\sqrt{\kappa^2-\gamma^2\kappa^2}}{\sqrt{\mu}} - \frac{\sqrt{3\sqrt{1-\gamma^2}\sqrt{(1-\gamma^2)\kappa^2+\gamma^2(-\kappa)}+\kappa}}{\sqrt{\mu}} - \frac{\sqrt{2}\sqrt{1-\gamma^2}}{\kappa\sqrt{\mu}}. \quad (3.66)$$

Case 3. If $\kappa > 0$, the required solution is given by

$$\Upsilon_{62}(x,t) = -\frac{\sqrt{3\sqrt{1-\gamma^2}\sqrt{(1-\gamma^2)\kappa^2+\gamma^2(-\kappa)}+\kappa}}{\sqrt{\mu}} + \frac{\sqrt{2}\sqrt{\kappa^2-\gamma^2\kappa^2}\cot\left(\sqrt{\kappa}(x-t\varrho)\right)}{\sqrt{\kappa}\sqrt{\mu}} + \frac{\sqrt{2}\sqrt{1-\gamma^2}\sqrt{\kappa}\tan\left(\sqrt{\kappa}(x-t\varrho)\right)}{\sqrt{\mu}}, \quad (3.67)$$

$$\Upsilon_{63}(x,t) = -\frac{\sqrt{3\sqrt{1-\gamma^2}\sqrt{(1-\gamma^2)\kappa^2+\gamma^2(-\kappa)}+\kappa}}{\sqrt{\mu}} - \frac{\sqrt{2}\sqrt{\kappa^2-\gamma^2\kappa^2}\tan\left(\sqrt{\kappa}(x-t\varrho)\right)}{\sqrt{\kappa}\sqrt{\mu}} - \frac{\sqrt{2}\sqrt{1-\gamma^2}\sqrt{\kappa}\cot\left(\sqrt{\kappa}(x-t\varrho)\right)}{\sqrt{\mu}}. \quad (3.68)$$

4. Stability analysis

The approach of identifying whether a system, whether biological, chemical, or mathematical, maintains or recovers to its optimum following perturbations, or if it departs, is known as stability evaluation. In order to guarantee safe, dependable functioning, it entails analysing system behaviour over time using techniques such as Lyapunov equilibrium, nonlinear stabilisation theory (small disturbances), and numerical computations. The topic of stability analysis examines how a structure reacts over time to minor changes or disruptions. It aids in predicting how the framework might veer into instability or return to optimum. Stability analysis is used to determine whether a nonlinear wave solution preserves its profile under small perturbations. In this study, we employ the well-known Vakhitov-Kolokolov (VK) stability criterion [38]. For Eq (1.1), we define the momentum functional as

$$M(\varrho) = \frac{1}{2}\int_{-\infty}^{\infty} \Upsilon^2(x,t;\varrho)\, dx, \tag{4.1}$$

where $\Upsilon(x,t;\varrho)$ represents the wave profile, which depends explicitly on the parameter ϱ (wave frequency). To obtain the soliton solution, the traveling wave transformation $\xi = x - \varrho t$ is applied, reducing Eq (1.1) to an ordinary differential equation. Consequently, the solution can be expressed as $\Upsilon(x,t;\varrho) = \Upsilon(\xi;\varrho)$, which shows that the wave profile depends parametrically on ϱ. Substituting the solution (Eq (3.18)) into Eq (4.1), we observe that the integrand depends on ϱ through $\Upsilon(\xi;\varrho)$. Therefore, the functional M is not a constant but a parameter-dependent quantity that symbolizes the momentum of the system, and Υ denotes the electricity potential. Further, the main condition of solitons stability is expressed in the form

$$\frac{\partial M}{\partial \varrho} > 0, \tag{4.2}$$

which is the standard VK criterion for soliton stability. The soliton's frequency is denoted by ϱ. By inserting Eq (3.18) results into Eq (4.1), we get

$$M = \int_{-10}^{10}\left(-\frac{\left(8\left(\gamma^2-1\right)+j(x-\varrho)^2\right)^2}{64\left(\gamma^2-1\right)\mu(x-\varrho)^2}\right)dx, \tag{4.3}$$

which gives

$$M = -\frac{5\left(192\left(\gamma^2-1\right)^2+j^2(\varrho-10)(\varrho+10)\left(3\varrho^2+100\right)+48\left(\gamma^2-1\right)j\left(\varrho^2-100\right)\right)}{48\left(\gamma^2-1\right)\mu\left(\varrho^2-100\right)}, \tag{4.4}$$

and thus we have

$$\frac{\partial M}{\partial \varrho} = -\frac{5\varrho\left(j^2\left(\varrho^2-100\right)^2-64\left(\gamma^2-1\right)^2\right)}{8\left(\gamma^2-1\right)\mu\left(\varrho^2-100\right)^2} > 0, \tag{4.5}$$

provided that $A = j^2(\varrho^2-100)^2 - 64(\gamma^2-1)^2$. Then, the derivative in Eq (4.5) is positive, $\frac{\partial M}{\partial \varrho} > 0$, under the following constraints:

$$(\gamma^2-1)\mu > 0, \quad \varrho A < 0,$$
$$(\gamma^2-1)\mu < 0, \quad \varrho A > 0,$$

$\varrho \neq \pm 10, \varrho \in \mathbb{R}$. These constraints define the parameter regime for which the momentum derivative is positive, ensuring stability according to the VK criterion. This confirms that the obtained soliton solutions of Eq (1.1) are stable.

5. Modulation instability

In optical fibres, liquids, and the bloodstream, a nonlinear phenomenon known as modulation instability occurs when tiny disturbances cause a continuous or quasi-continuous wave to split into a sequence of brief spikes. Sidebands, one outside the primary wavelength, evolve exponentially as a result of the interaction involving Kerr nonlinearity and unusual collective velocity scattering. At ideal dispersal circumstances, it usually happens when the input power surpasses a particular limit. Modulation instability is essential for the creation of supercontinuum bands in optical cables and may constitute the first step in soliton synthesis. Silence can seed the perturbations that develop into unique spectral features since the process is influenced by it. Uses for modulation destabilisation are useful in complex wave research, rapid burst production, and high-speed optical transmission. Managing signal stability in nonlinear transmission media requires a comprehension of it.

The moving waveform modulation instability can be observed in Eq (1.1) within this portion. Assuming the perturbed solution

$$\Upsilon(x,t) = \mathcal{L}\kappa(x,t) + P, \tag{5.1}$$

the terms κ and P denote the disturbance and disturbance factor and the linear flow term of Eq (1.1) can be represented by χ. Further, we linearize the results after inserting Eq (5.1) into Eq (1.1) to give

$$j\kappa + 3P^2\mu + \kappa_{\mathrm{tt}} - \kappa_{\mathrm{xx}} = 0. \tag{5.2}$$

We suppose the solution of Eq (5.2) is as fellows:

$$\kappa(x,t) = K_1 e^{i(x-t\varrho)}, \tag{5.3}$$

where the constant terms are represented by K_1 and the wave's velocity is represented by ϱ. Now, by inserting Eq (5.3) into Eq (5.2).

$$K_1\left(j + 3\mu P^2 + (1-\varrho^2)\right)e^{i(x-\varrho t)} = 0.$$

After solving the above equation for ϱ, we get

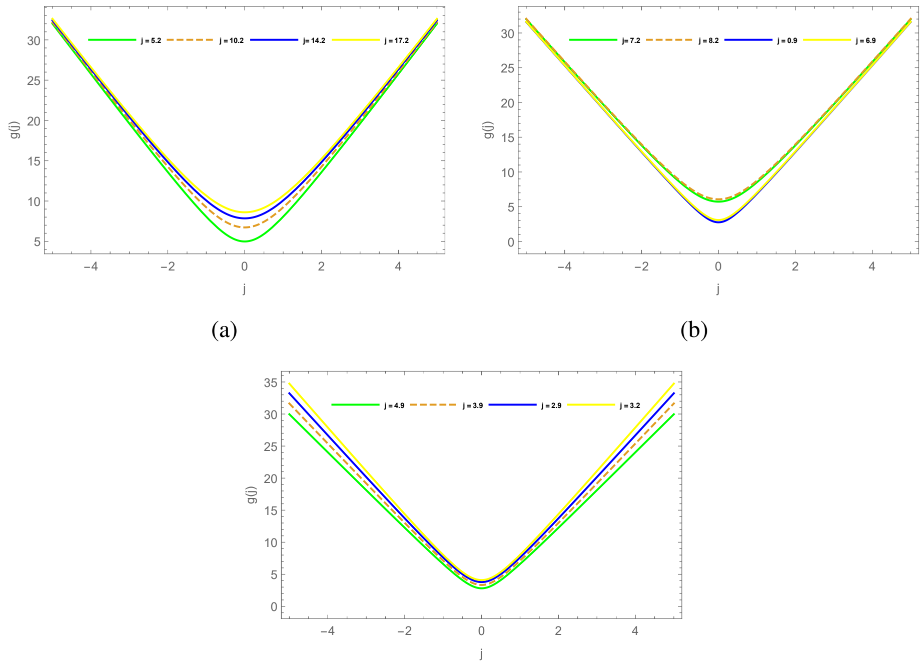
$$\varrho = \pm\sqrt{j + 3\mu P^2 + 1}. \tag{5.4}$$

Equation (5.4) expresses the stabilization phenomenon as: If ϱ is a real number, then the solution outcomes is stable, and if the ϱ is imaginary, then it shows instability behaviours and the perturbation values increase gradually. Then, we show that the modulation instability may increase in this way:

$$g(\gamma) = 2Im(\varrho),$$

$$g(\gamma) = 2Im(-\sqrt{j + 3\mu P^2 + 1}).$$

Various factors affect the modulation instability rate, such as the intensity of the incident, the combined velocity, and the detuning term, for details see Figure 1.



(a) (b) (c)

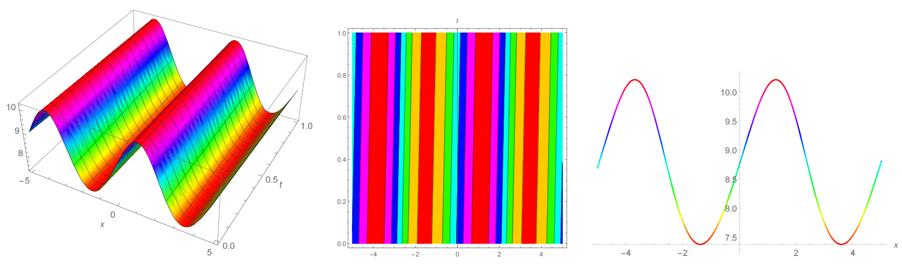
Figure 1. Representation of modulation instability behaviours with j and other system variables.

6. Discussion and results

Earlier studies primarily focused on limited classes of solutions, such as solitary waves and periodic structures derived via specific analytical techniques. For instance, methods such as the extended fan sub-equation method [33], the Adomian decomposition method [34], and the expansion method [35]. However, our study provides a wider class of solution families, including rational, trigonometric, hyperbolic, and combined functional forms. The polynomial expansion and unified methods enable the construction of solutions that incorporate richer parameter dependencies, allowing for more flexible wave structures. Moreover, several of the obtained solutions can be reduced to previously reported forms under specific parameter choices, thereby confirming the consistency of our results and demonstrating their generality. The nonlinear Klein-Gordon equation is described graphically in this section of the article using the latest computing tools, such as Mathematica. The graphical results illustrate different nonlinear wave structures supported by the model. The solitary wave solutions represent localized waves that maintain their shape during propagation due to the balance between dispersion and nonlinearity. The periodic solutions correspond to repeating nonlinear wave trains observed in many physical systems, such as plasmas and lattice structures. In contrast, singular solutions indicate strong energy localization or possible wave breaking in nonlinear media.

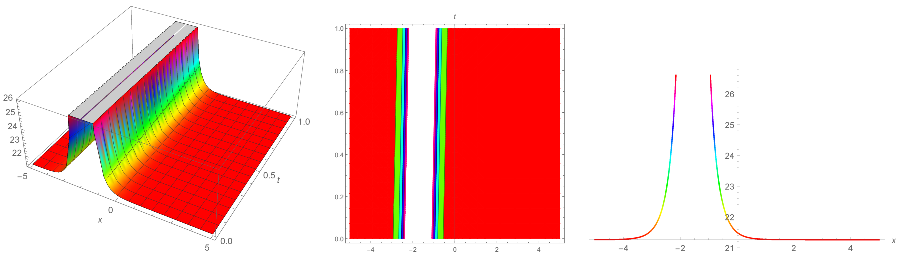
Figure 2, denotes the solution of $|\mathcal{J}_{58}(x,t)|$ by using the parameters values $\rho = 0.1,\ \Omega = 1,\ \sigma = 0.5,\ \beta = 0.4,\ \varrho = 0.2,\ \gamma = 0.61,\ \mu = 0.01$. It represents the periodic soliton wave solution. Further, the Figure 3, expresses the singular wave solution behaviour by utilizing the variable values $\rho = 0.1,\ \Omega = 1,\ \sigma = 0.5,\ \beta = -1.4,\ \varrho = 0.2,\ \gamma = -0.61$, and $\mu = 0.01$. Next, Figure 4, symbolizes the periodic wave solution by using the parameter values $\rho = 1.1,\ \Omega = 1,\ \sigma = 0.5,\ \beta = 1.4,\ \varrho = 0.2,\ \gamma = 0.61$, and $\mu = 0.01$. Figure 5, describes the singular solitons solutions behaviours by utilizing the variables $\kappa = -2.1,\ \gamma = 2.1,\ \mu = 0.3$, and $\varrho = 0.5$. Now, proceeding to Figure 6, this figure represents the periodic solitons solutions behaviours with parameters $\rho = 1.2,\ \Omega = 1,\ \sigma = 4.5,\ \beta = 0.4,\ \varrho = 0.2,\ \gamma = 0.61$, and $\mu = 0.01$. Figure 7 describes the singular solitons wave solutions utilizing the parameters values $\gamma = 0.2,\ j = 0.8,\ \alpha = 0.25,\ \gamma = 3.71,\ \mu = 1.62$, and $\varrho = 0.16$. Figure 8 which denotes periodic solitons wave solutions by using the variables values $\rho = 0.1,\ \Omega = 1,\ \sigma = 1.5,\ \beta = 0.4,\ \varrho = 0.2,\ \gamma = 1.61$, and $\mu = 2.01$. Figure 9 describes the V shaped soliton solution utilising the parameters values $\rho = 0.1,\ \Omega = 1,\ \sigma = 0.5,\ \beta = -0.7,\ \varrho = 0.2,\ \gamma = 0.61$, and $\mu = 0.01$.

In the last figure, Figure 10, the V-shaped soliton solutions utilizing the parameters $\gamma = 2.1,\ \mu = -1$, and $\varrho = 0.3$. To fully portray the extent of the solution behaviours, we include the 2D, 3D, and contour plots of the attained solutions. The obtained solutions have many applications in various fields of mathematical science, matter physics, and other related physical sciences. We get different types of solutions that give various behaviours with graphical illustrations that are described in the given section. The obtained bright, dark, and bright-dark soliton solutions not only reveal the mathematical structure of the nonlinear Klein-Gordon equation, but they also have clear physical significance. Bright solitons are localized pulses whose amplitude increases with the nonlinear coefficient and whose width grows with dispersion, modeling high-intensity light propagation in optical fibers. Dark solitons are localized dips whose depth and width similarly depend on nonlinearity and dispersion, relevant in Bose-Einstein condensates and nonlinear optical media. Bright-dark solitons represent combined peak–dip structures, illustrating energy exchange between modes in coupled waveguides or multi-component fields. Beyond analytical approaches, modern computational techniques, such as recurrent neural networks [39], time-delay estimation methods [40], and convolutional neural networks for complex pattern prediction [41], provide complementary tools for studying and controlling nonlinear wave phenomena in complex dynamical systems, as well as for handling uncertainties and disturbance rejection in such systems [42]. These approaches highlight the growing intersection between analytical soliton theory and data-driven modeling, enabling better prediction and manipulation of nonlinear wave behavior in practical applications.



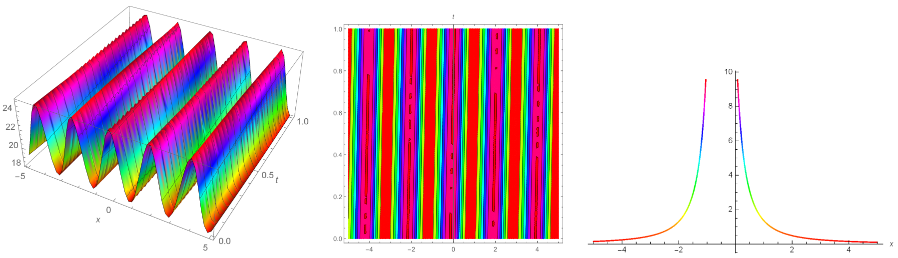
(a) (b) (c)

Figure 2. Graph visuals of solution $|\Upsilon_{22}(x,t)|$ representing the periodic soliton solution utilising $\rho = 0.1,\ \Omega = 1,\ \sigma = 0.5,\ \beta = 0.4,\ \varrho = 0.2,\ \gamma = 0.61$, and $\mu = 0.01$.



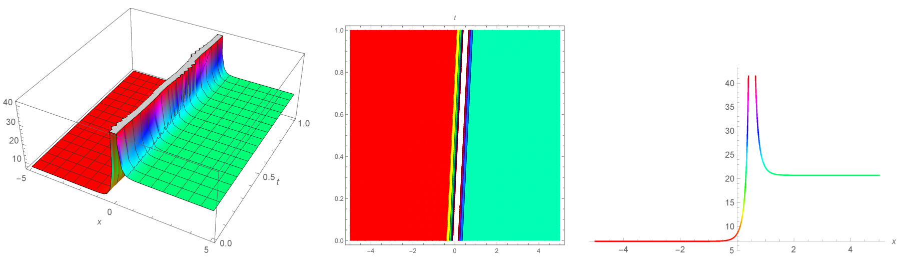
(a) (b) (c)

Figure 3. Graph visuals of solution $|\Upsilon_{23}(x,t)|$ representing the single soliton solutions utilising $\rho = 0.1$, $\Omega = 1$, $\sigma = 0.5$, $\beta = -1.4$, $\varrho = 0.2$, $\gamma = -0.61$, and $\mu = 0.01$.



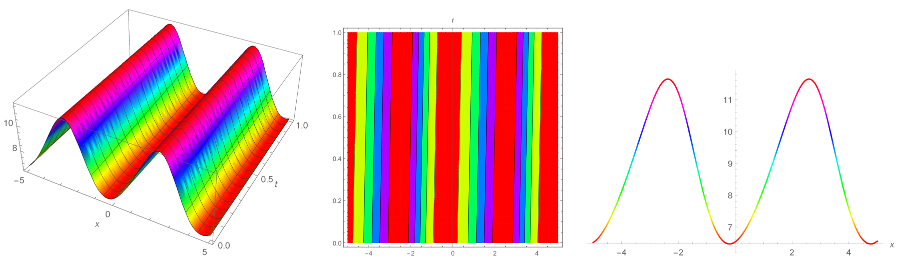
(a) (b) (c)

Figure 4. Graph visuals of solution $|\Upsilon_{27}(x,t)|$ representing the periodic soliton solution utilising $\rho = 1.1$, $\Omega = 1$, $\sigma = 0.5$, $\beta = 1.4$, $\varrho = 0.2$, $\gamma = 0.61$, and $\mu = 0.01$.



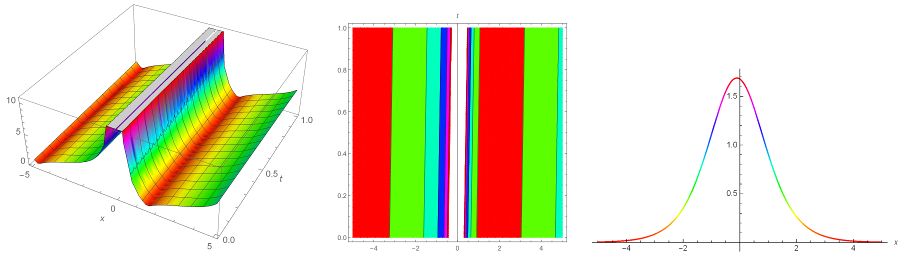
(a) (b) (c)

Figure 5. Graph visuals of solution $|\Upsilon_{62}(x,t)|$ representing the singular soliton solution utilising $\kappa = -2.1$, $\gamma = 2.1$, $\mu = 0.3$, and $\varrho = 0.5$.



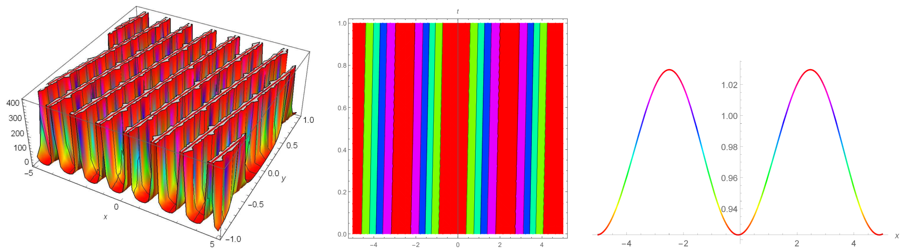
(a) (b) (c)

Figure 6. Graph visuals of solution $|\Upsilon_{30}(x,t)|$ representing the periodic soliton solutions utilising $\rho = 1.2$, $\Omega = 1$, $\sigma = 4.5$, $\beta = 0.4$, $\varrho = 0.2$, $\gamma = 0.61$, and $\mu = 0.01$.



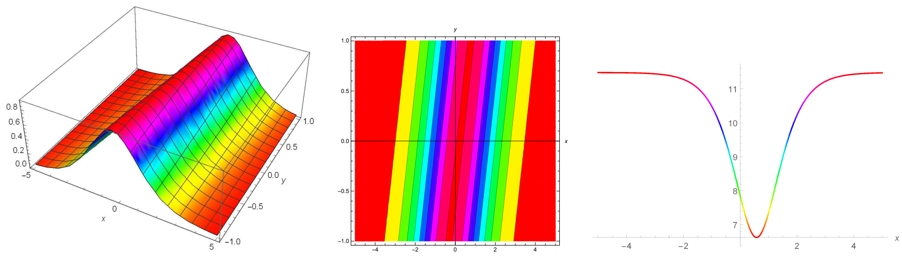
(a) (b) (c)

Figure 7. Graph visuals of solution $|\Upsilon_3(x,t)|$ representing the singular soliton solutions utilising $\gamma = 0.2,\ j = 0.8,\ \alpha = 0.25,\ \gamma = 3.71,\ \mu = 1.62$, and $\varrho = 0.16$.



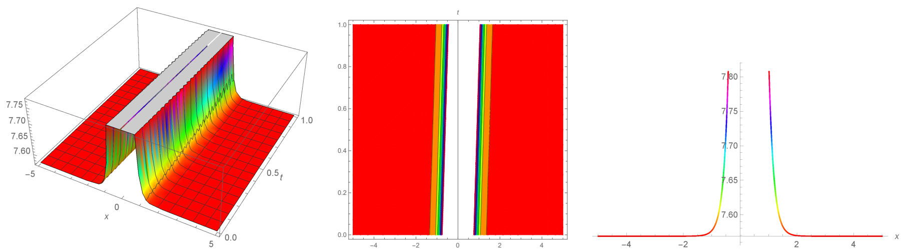
(a) (b) (c)

Figure 8. Graph visuals of solution $|\Upsilon_{36}(x,t)|$ representing the periodic soliton solution utilising $\rho = 0.1,\ \Omega = 1,\ \sigma = 1.5,\ \beta = 0.4,\ \varrho = 0.2,\ \gamma = 1.61$, and $\mu = 2.01$.



(a) (b) (c)

Figure 9. Graph visuals of solution $|\Upsilon_{26}(x,t)|$ representing the V shaped soliton solution utilising $\rho = 0.1,\ \Omega = 1,\ \sigma = 0.5,\ \beta = -0.7,\ \varrho = 0.2,\ \gamma = 0.61$, and $\mu = 0.01$.



(a) (b) (c)

Figure 10. Graph visuals of solution $|\Upsilon_{56}(x,t)|$ representing the singular solution utilising $\gamma = 2.1,\ \mu = -1$, and $\varrho = 0.3$.

7. Dynamical assessment

This section presents the dynamical analysis of the governing equation, including sensitivity analysis, bifurcation characteristics, and chaotic behavior of the resulting system. By applying a Galilean transformation, Eq (3.2) can be reduced to the following dynamical system:

$$\begin{cases} \frac{d\Upsilon(\chi)}{d\chi} = \nu, \\ \frac{d\nu(\chi)}{d\chi} = A\Upsilon(\chi) - B\Upsilon^3(\chi), \end{cases} \tag{7.1}$$

where $A = -\frac{j}{\gamma^2-1}$ and $B = \frac{\mu}{\gamma^2-1}$.

7.1. Sensitivity analysis

Sensitivity analysis is performed to examine how variations in system parameters influence the dynamical behavior of the nonlinear Klein-Gordon system. In particular, the parameters A and B play a crucial role in determining the stability of equilibrium points and the structure of the phase space. Small changes in these parameters can significantly modify the trajectories of the system and may lead to transitions between different dynamical regimes. The graphical results presented in the subsequent subsections illustrate that the system exhibits strong sensitivity to parameter variations, which is a characteristic feature of nonlinear dynamical systems. This analysis highlights the importance of parameter selection in determining the qualitative behavior of the governing model.

7.2. Bifurcation analysis

Bifurcation analysis of the proposed model is conducted to investigate how the system's qualitative behavior changes with variations in the parameters A and B. This analysis helps identify equilibrium points, examine their stability properties, and detect transitions in the system dynamics. In nonlinear wave systems, bifurcation structures provide valuable insight into the formation of localized wave patterns and the transition between different dynamical states [43].

For the dynamical system given in Eq (7.1), the Hamiltonian function is expressed as

$$H(\Upsilon, \nu) = \frac{\nu^2}{2} - A\frac{\Upsilon^2}{2} + B\frac{\Upsilon^4}{4} = h, \tag{7.2}$$

where h represents the Hamiltonian constant. The Hamiltonian is the total energy of the system and consists of kinetic and potential energy terms. This formulation provides insight into the conservation properties of the system and plays an important role in understanding the stability and evolution of the associated wave structures.

The equilibrium points of the system are obtained from

$$\begin{cases} \nu = 0, \\ A\Upsilon(\chi) - B\Upsilon^3(\chi) = 0, \end{cases} \tag{7.3}$$

which yields

$$s_1 = (0,0), \quad s_2 = \left(\sqrt{\frac{A}{B}}, 0\right), \quad s_3 = \left(-\sqrt{\frac{A}{B}}, 0\right).$$

The Jacobian matrix associated with the system and its determinant are given by

$$M(v, \Upsilon) = \begin{vmatrix} 0 & 1 \\ A - 3B\Upsilon^2 & 0 \end{vmatrix} = 3B\Upsilon^2 - A. \tag{7.4}$$

Depending on the sign of the determinant, the equilibrium points can be classified as shown in Table 1, and the corresponding phase portraits are illustrated in Figure 11.

Table 1. Classification of equilibrium points based on the determinant of the Jacobian matrix.

Condition on $M(v, \Upsilon)$	Equilibrium Points
$M(v, \Upsilon) < 0$	Saddle points
$M(v, \Upsilon) > 0$	Center points
$M(v, \Upsilon) = 0$	Cuspid points

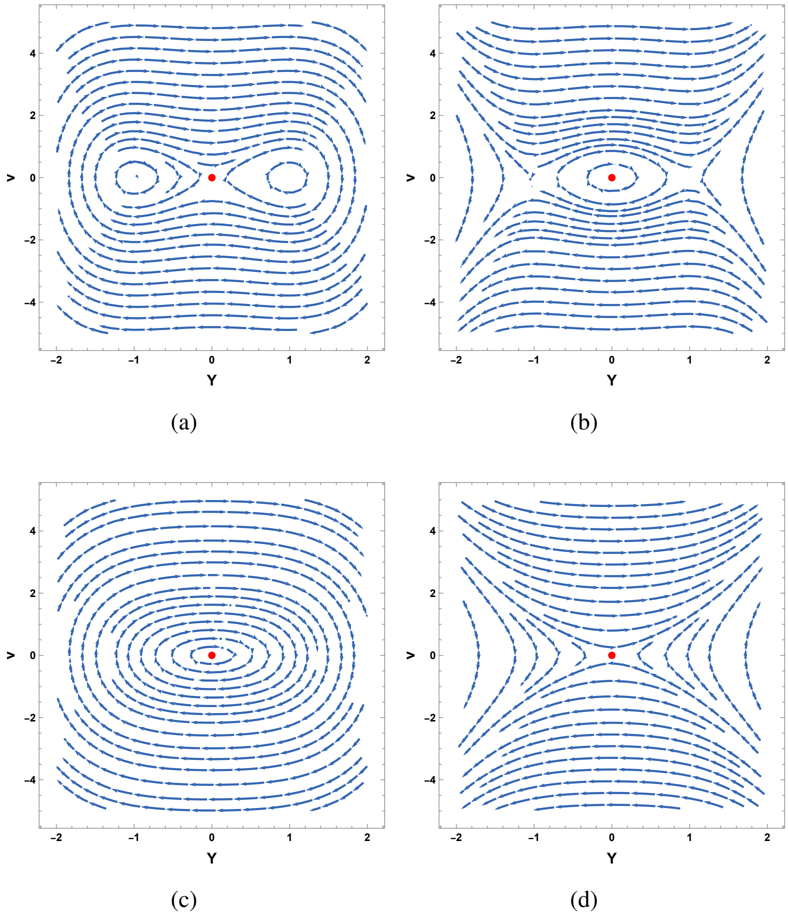


Figure 11. Phase portraits corresponding to the dynamical system in Eq (7.1) for different parameter regimes: (a) $A > 0, B > 0$; (b) $A < 0, B < 0$; (c) $A < 0, B > 0$; (d) $A > 0, B < 0$.

Case 1: $A > 0, B > 0$

For the parameter values $j = -6$, $\gamma = 2$, and $\mu = 6$, three equilibrium points are obtained: $(0, 0)$, $(1, 0)$, and $(-1, 0)$. The points $(1, 0)$ and $(-1, 0)$ correspond to center points, while $(0, 0)$ is a saddle point.

Case 2: $A < 0, B < 0$

For $j = 6$, $\gamma = 2$, and $\mu = -6$, the system again admits three equilibrium points $(0, 0)$, $(1, 0)$, and $(-1, 0)$. In this case, $(1, 0)$ and $(-1, 0)$ behave as saddle points, whereas $(0, 0)$ represents a center point.

Case 3: $A < 0, B > 0$

For parameters $j = 8$, $\gamma = 3$, and $\mu = 8$, the system possesses a single real equilibrium point $(0, 0)$, which behaves as a center point.

Case 4: $A > 0, B < 0$

For $j = -8$, $\gamma = 3$, and $\mu = -8$, the system again admits a single equilibrium point $(0, 0)$, which corresponds to a saddle point.

These results demonstrate that the qualitative structure of the phase space strongly depends on the parameters A and B, leading to different stability configurations and dynamical behaviors.

7.3. Chaotic behavior

To further explore the complex dynamics of the system, a periodic perturbation term is introduced into the dynamical model. The perturbed system can be written as

$$\begin{cases} \frac{d\Upsilon(\chi)}{d\chi} = \nu, \\ \frac{d\nu(\chi)}{d\chi} = A\Upsilon(\chi) - B\Upsilon^3(\chi) + \Omega\sin(\theta\chi), \end{cases} \tag{7.5}$$

where $\Omega \neq 0$ and $\theta \neq 0$ denote the amplitude and frequency of the external perturbation.

Figures 12–15 illustrate the dynamical behavior of system (7.5) for different parameter values through phase portraits, time series plots, and Poincaré maps.

For example, in Figure 12, the system is examined for the parameter set $j = 4$, $\gamma = -1.5$, $\mu = 0.25$, $\Omega = -5$, and $\theta = 5$. Under these conditions, the system exhibits quasi-periodic dynamics. When the perturbation parameter changes from $\Omega = -5$ to $\Omega = -3$, noticeable changes in the dynamical pattern occur, as illustrated in Figure 13.

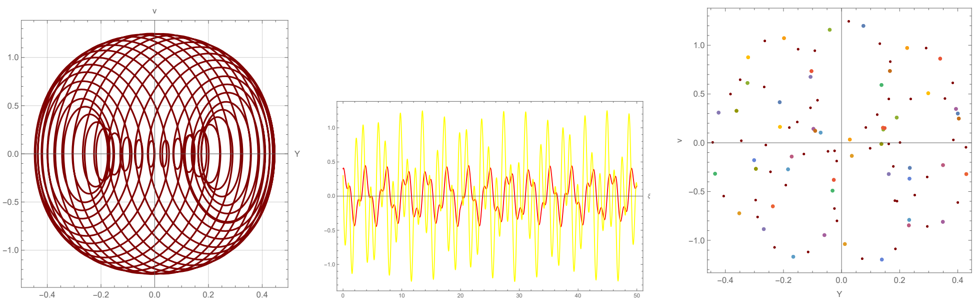
In Figure 14, for parameters $j = 5$, $\gamma = 1.5$, $\mu = 0.5$, $\Omega = 2.5$, and $\theta = 1.5$, the system displays weakly chaotic behavior. The trajectories become irregular and do not converge to fixed points, indicating non-periodic motion and complex phase-space evolution.

Finally, Figure 15 demonstrates the dynamical behavior for $j = 5$, $\gamma = 2.5$, $\mu = 5$, $\Omega = 0.05$, and $\theta = 2.5$. In this case, the system exhibits periodic behavior.

The presence of chaotic dynamics is supported by the irregular phase trajectories, non-periodic time series, and scattered distributions observed in the Poincaré maps. These features indicate sensitive dependence on system parameters and initial conditions, which are characteristic signatures of chaotic behavior in nonlinear dynamical systems.

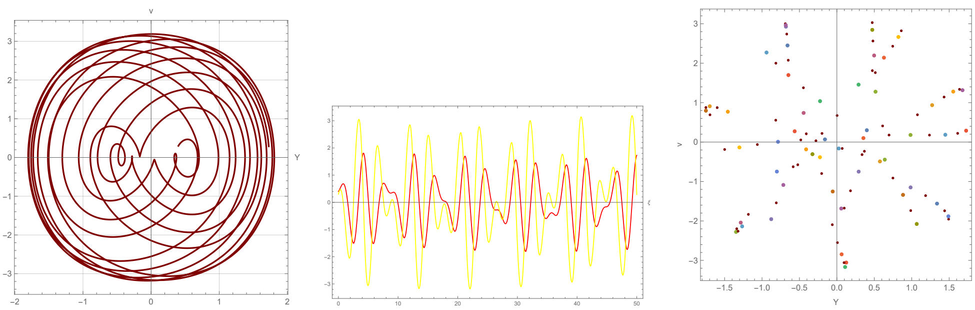
(a) (b) (c)

Figure 12. The depiction for the system (7.5) shows: (a) 2D phase diagram, (b) time series, and (c) Poincaré graph, where the perturbation term is $\Omega\sin(\theta\eta)$.



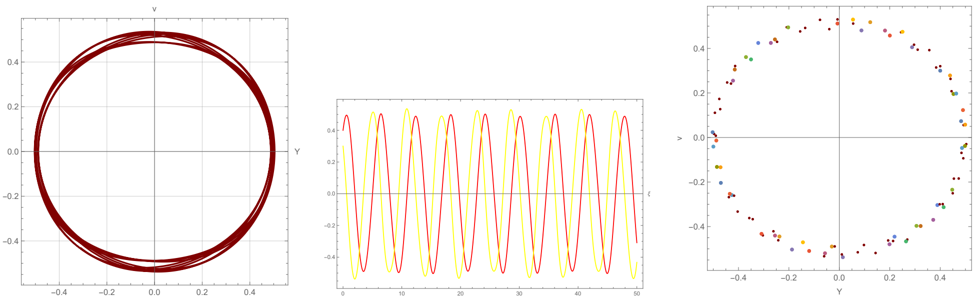
(a) (b) (c)

Figure 13. The depiction for the system (7.5) shows: (a) 2D phase diagram, (b) time series, and (c) Poincaré graph, where the perturbation term is $\Omega\sin(\theta\eta)$.



(a) (b) (c)

Figure 14. The depiction for the system (7.5) shows: (a) 2D phase diagram, (b) time series, and (c) Poincaré graph, where the perturbation term is $\Omega\sin(\theta\eta)$.



(a) (b) (c)

Figure 15. The depiction for the system (7.5) shows: (a) 2D phase diagram, (b) time series, and (c) Poincaré graph, where the perturbation term is $\Omega \sin(\theta\eta)$.

8. Conclusions

In this work, we have analytically investigated the nonlinear Klein-Gordon equation, an extension of the classical Klein-Gordon model that describes wave and particle dynamics at subatomic scales. This model captures a wide range of physical phenomena, including DNA dynamics, ferroelectric and ferromagnetic domain walls, and scalar fields in particle physics, condensed matter, and cosmology. We systematically obtained diverse analytical solutions, including solitary, periodic, and V-shaped waveforms, and classified them into distinct solution types. The stability and modulation instability of these solutions were examined, providing insight into the system's response to small perturbations.
Furthermore, bifurcation and qualitative chaos analyses illustrated the rich nonlinear dynamics under varying parameter regimes. While the employed methods are effective in generating and classifying multiple solution types, they remain primarily analytical and qualitative; quantitative chaos indicators could be explored in future work. The bifurcation diagrams and phase portraits demonstrate how variations in system parameters lead to transitions between different dynamical regimes, providing insight into the stability and evolution of the nonlinear waves. The present study lays a foundation for further investigations, and future research could apply new analytical or numerical techniques to explore additional soliton behaviors and their physical applications. Compared to existing literature, the current work offers a more unified framework by combining exact solution construction with stability analysis, modulation instability, and bifurcation analysis. This integrated approach distinguishes the current study from previous works and provides a more comprehensive understanding of the nonlinear dynamics of the governing equation. Overall, this study enhances the understanding of the dynamic properties and stability patterns of the nonlinear Klein-Gordon model and provides a framework for exploring its rich solution landscape in mathematical physics and related fields.

Author contributions

Conceptualization, supervision, resources, Kalim U. Tariq; visualization, methodology, Naif Almusallam; software, validation, Luai Abdulla Aldoghan; formal analysis, writing – review and editing, Sajawal A. Baloch; investigation, writing – original draft preparation, S. M. R. Kazmi; project administration, funding acquisition, Abdulaziz khalid Alsharidi; All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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