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Research article

Ill-posedness in H^s for a defocusing power-type derivative Schrödinger equation with lower-order linear perturbations

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Abstract: This work studied the one-dimensional defocusing power-type derivative Schrödinger equation with lower-order linear perturbations

$$\mathrm{i}u_t + u_{xx} - \mathrm{i}|u|^k u_x + \alpha u_x + \beta u = 0, \qquad (t, x) \in \mathbb{R} \times \mathbb{R},$$

where $k \geq 2$ and $\alpha, \beta \in \mathbb{R}$ are constants. An explicit family of solitary traveling-wave solutions is first constructed within an exactly integrable traveling-wave reduction, and their H^s regularity and parameter dependence are characterized. A traveling-wave-based ill-posedness mechanism is then implemented: two solutions associated with nearby parameter sets are produced so that their initial data are arbitrarily close in H^s, while their profiles remain separated by a uniform positive lower bound in H^s at some positive time. As a result, the solution flow map fails to be uniformly continuous below a certain regularity threshold. These results indicate that the presence of lower-order linear perturbations does not improve the low-regularity stability threshold for this DNLS-type equation.

Keywords: DNLS-type equation; solitary waves; ill-posedness
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1. Introduction

Derivative nonlinear Schrödinger (DNLS) equations and their power-type generalizations arise frequently in plasma physics, nonlinear waves, fluids, and magnetohydrodynamics. For instance, the classical DNLS (or an equivalent form) models the modulation and nonlinear evolution of Alfvén waves propagating along a magnetic field [8, 16]. Compared with the standard NLS, the derivative nonlinearity produces a stronger dispersion-nonlinearity coupling and creates additional analytical difficulties at low regularity, leading to extensive work on well-posedness and ill-posedness [3–5, 18].

From a modeling viewpoint, the extra terms considered here represent lower-order linear corrections to the generalized DNLS dynamics: αu_x acts as a uniform transport (or drift) term, while βu produces a spatially homogeneous phase rotation. Such contributions naturally appear when one writes an envelope equation in a moving frame, retains a background advection effect, or includes a constant carrier-frequency detuning; see the plasma and optical motivations for DNLS-type models in [15, 16]. This interpretation simultaneously addresses the physical motivation of the perturbation terms and clarifies the drift-phase mechanism used later in the analysis.

In this paper, we consider the Cauchy problem for the one-dimensional defocusing power-type DNLS with lower-order linear perturbations:

$$\begin{cases} \mathrm{i}u_t + u_{xx} - \mathrm{i}|u|^k u_x + \alpha u_x + \beta u = 0, & (t,x) \in \mathbb{R} \times \mathbb{R}, \\ u(0,x) = \varphi(x) \in H^s(\mathbb{R}), \end{cases} \tag{1.1}$$

where $k \geq 2$ and $\alpha, \beta \in \mathbb{R}$ are constants. This model can be viewed as the unperturbed power-type DNLS

$$\mathrm{i}u_t + u_{xx} - \mathrm{i}|u|^k u_x = 0 \tag{1.2}$$

with an added drift term αu_x and a constant-potential term βu. Dynamically, αu_x corresponds to a uniform drift of the reference frame, while βu corresponds to a uniform phase rotation. Hence, a natural question is whether these lower-order linear perturbations can change the low-regularity stability threshold of the unperturbed model in Sobolev spaces.

In fact, the perturbations can be removed exactly by the change of variables

$$u(t,x) = e^{-\mathrm{i}\beta t} w(t, x + \alpha t),$$

under which (1.1) is transformed into the unperturbed Eq (1.2) for w. Thus, α only shifts the spatial center of a wave packet, whereas β only shifts its temporal phase. One purpose of the present paper is to show that this simple structural observation is fully compatible with the traveling-wave ill-posedness mechanism.

For the unperturbed Eq (1.2), there is a natural scaling

$$u(t,x) \mapsto u_\lambda(t,x) := \lambda^{1/k} u(\lambda^2 t, \lambda x),$$

which yields the homogeneous Sobolev critical index

$$s_c = \frac{k-2}{2k}.$$

This critical index indicates a potential scale-invariant boundary for well-posedness and provides a reference for the ill-posedness thresholds that will appear later.

For the unperturbed and generalized DNLS family, one already has a substantial body of results on well-posedness and solitary waves; see, for example, Hao [11], Santos [17], Hayashi-Ozawa [12], and Liu-Simpson-Sulem [14]. Accordingly, our statement that the perturbations do not improve the threshold should be understood in the following precise sense: after the exact drift-phase reduction above, the perturbed model has the same solitary-wave profile family as the underlying generalized

DNLS equation, and therefore the same traveling-wave-based obstruction to uniform continuity persists.

On the other hand, the unperturbed model (1.2) has typical conservation laws: mass (L^2) and momentum (and also energy) are conserved at the level of sufficiently smooth solutions, for example

$$M(u)(t) = \frac{1}{2}\int_{\mathbb{R}} |u(t,x)|^2\,dx = M(u)(0), \quad P(u)(t) = -\frac{1}{2}\Im\int_{\mathbb{R}} (u\overline{u_x})(t,x)\,dx = P(u)(0),$$

together with an energy-type invariant adapted to the derivative nonlinearity [4, 5, 8]. These structures are essential tools for high-regularity theory, but they typically do not prevent ill-posedness in low-regularity regimes.

For the perturbed Eq (1.1), the scaling symmetry is no longer exact because of α, β. Nevertheless, we will show that, since α, β correspond to removable linear effects, they do not alter the low-regularity ill-posedness mechanism considered here.

Regarding well-posedness for DNLS and its generalizations, classical results show local/global well-posedness in the energy space or higher regularity. Representative works include early results of Hayashi-Ozawa [8, 9], Takaoka's local well-posedness and a priori bounds [18], and the global theory of Colliander-Keel-Staffilani-Takaoka-Tao [3, 4]. Classical existence and uniqueness results for DNLS in the energy space trace back to Tsutsumi-Fukuda [20], while the complete integrability initiated by Kaup-Newell provides a structural backdrop for explicit solution constructions [13]. In the periodic setting, Herr established sharp local well-posedness at $H^{1/2}(\mathbb{T})$ [10]. Global results in low-order Sobolev spaces were also obtained by Takaoka [19]. In the non-periodic setting, global energy-space refinements such as Wu's work [23] and the mass-threshold result in $H^{1/2}(\mathbb{R})$ [7] culminate in the large-data $H^{1/2}(\mathbb{R})$ theory of Bahouri-Perelman [1]. At lower regularity, Grünrock and collaborators developed crucial bilinear/trilinear estimates [5, 6], leading to low-regularity local theory in both periodic and non-periodic settings.

By contrast, at low regularity, the derivative nonlinearity may destroy continuity properties of the solution flow map and can even lead to a failure of uniform continuity. Biagioni-Linares identified early ill-posedness phenomena for derivative Schrödinger-type models [2], and subsequent results for DNLS-type equations in various function spaces have further clarified this instability mechanism [21]. Recent work also demonstrates sharper instability mechanisms such as norm inflation for DNLS [22]. A common route for low-regularity ill-posedness of power-type derivative models is to exploit an explicit/controllable family of traveling waves (solitary waves) to construct two solution trajectories that are arbitrarily close at $t = 0$ but stay separated by a positive lower bound for some $t > 0$. This is precisely the strategy we adopt and adapt to the perturbed Eq (1.1).

Let $\Phi_t : \varphi \mapsto u(t)$ denote the solution flow map. Our main conclusion is as follows:

Theorem 1.1 (Ill-posedness threshold). *Let $k \geq 2$ and $\alpha, \beta \in \mathbb{R}$. For* (1.1)*, if*

$$s < \begin{cases} \frac{1}{k}, & 2 \leq k \leq 4, \\ \frac{1}{2} - \frac{1}{k}, & k > 4, \end{cases} \tag{1.3}$$

then for every $T_0 > 0$, the solution map associated with (1.1)*, viewed as a map from $H^s(\mathbb{R})$ into $C([0, T_0]; H^s(\mathbb{R}))$, fails to be uniformly continuous. More precisely, there exists $\varepsilon_0 > 0$ such that for any $\delta > 0$ and any $T_0 > 0$, there exist $\varphi_1, \varphi_2 \in H^s(\mathbb{R})$ and a time $t_* \in (0, T_0]$ satisfying*

$$\|\varphi_1 - \varphi_2\|_{H^s} \leq \delta, \qquad \|u_1(t_*) - u_2(t_*)\|_{H^s} \geq \varepsilon_0,$$

where u_j *is the solution to* (1.1) *with initial data* φ_j*.*

Remark 1.1. *The threshold* (1.3) *is closely tied to the parameter scaling of the DNLS traveling-wave family. We will show that* α,β *alter only the propagation speed and temporal frequency of the traveling waves, without changing the underlying profile, and therefore do not affect the low-regularity ill-posedness mechanism.*

2. Construction of exact traveling-wave solutions

In this section, we construct explicit solitary traveling-wave solutions to (1.1). For completeness, we first derive the ODE system obtained from a direct traveling-wave substitution and then present its integrated explicit form.

We adopt the traveling-wave ansatz

$$u(t,x) = e^{-\mathrm{i}\Omega t}\, a(\xi)\, e^{\mathrm{i}\psi(\xi)}, \qquad \xi = x - vt,$$

where $a(\xi) \geq 0$ and $\psi(\xi) \in \mathbb{R}$ are unknown functions, $v \in \mathbb{R}$ is the wave speed, and $\Omega \in \mathbb{R}$ is the temporal frequency.

Let $T(\xi) = \psi'(\xi)$. A direct computation gives

$$u_t = e^{-\mathrm{i}\Omega t} e^{\mathrm{i}\psi}(-\mathrm{i}\Omega a - va' + \mathrm{i}(-v)aT),$$

$$u_x = e^{-\mathrm{i}\Omega t} e^{\mathrm{i}\psi}(a' + \mathrm{i}aT), \qquad u_{xx} = e^{-\mathrm{i}\Omega t} e^{\mathrm{i}\psi}(a'' - aT^2 + \mathrm{i}(2a'T + aT')).$$

Moreover,

$$|u|^k u_x = a^k e^{-\mathrm{i}\Omega t} e^{\mathrm{i}\psi}(a' + \mathrm{i}aT).$$

Substituting the above into (1.1) and factoring out the common factor $e^{-\mathrm{i}\Omega t}e^{\mathrm{i}\psi}$ yields a complex equation of the form

$$q_1(\xi) + \mathrm{i}q_2(\xi) = 0,$$

whose real and imaginary parts are

$$q_1 = \Omega a - vaT + a'' - aT^2 + a^{k+1}T + \alpha a' + \beta a, \tag{2.1}$$

$$q_2 = -va' + 2a'T + aT' - a^k a' + \alpha aT. \tag{2.2}$$

Thus the traveling-wave profile satisfies $q_1 = q_2 = 0$.

From (2.2)=0, dividing by a (note that $a > 0$ at the center of a solitary profile and one can extend by continuity), we obtain

$$(-v - a^k)a' + 2Ta' + aT' + \alpha T a = 0.$$

Equivalently,

$$aT' + 2a'T + \alpha aT = (v + a^k)a'.$$

Since $(a^2T)' = 2aa'T + a^2T'$, we have

$$(a^2T)' + \alpha a^2 T = (v + a^k)aa'.$$

Multiplying both sides by the integrating factor $e^{\alpha\xi}$ gives

$$(e^{\alpha\xi}a^2T)' = e^{\alpha\xi}(v + a^k)aa'.$$

Integrating in ξ and imposing the standard solitary-wave phase normalization $a(\xi) \to 0 \Rightarrow a^2T \to 0$, we obtain

$$e^{\alpha\xi}a^2T(\xi) = \int_{-\infty}^{\xi} e^{\alpha y}(v + a(y)^k)a(y)a'(y)\mathrm{d}y.$$

The right-hand side can be integrated in the variable a: letting $z = a(y)$, we have $aa'\mathrm{d}y = \frac{1}{2}\mathrm{d}(a^2)$ and $a^k aa'\mathrm{d}y = \frac{1}{k+2}\mathrm{d}(a^{k+2})$. Hence,

$$e^{\alpha\xi}a^2T(\xi) = \frac{v}{2}e^{\alpha\xi}a(\xi)^2 + \frac{1}{k+2}e^{\alpha\xi}a(\xi)^{k+2},$$

and therefore

$$T(\xi) = \psi'(\xi) = \frac{v}{2} - \frac{1}{k+2}a(\xi)^k. \tag{2.3}$$

Importantly, α disappears in the final phase formula, reflecting the "removability" of the drift term at the level of traveling-wave profiles.

Substituting (2.3) into (2.1)=0, we compute

$$-vaT = -va\left(\frac{v}{2} - \frac{a^k}{k+2}\right) = -\frac{v^2}{2}a + \frac{v}{k+2}a^{k+1},$$

$$-aT^2 = -a\left(\frac{v}{2} - \frac{a^k}{k+2}\right)^2 = -\frac{v^2}{4}a + \frac{v}{k+2}a^{k+1} - \frac{1}{(k+2)^2}a^{2k+1},$$

$$a^{k+1}T = a^{k+1}\left(\frac{v}{2} - \frac{a^k}{k+2}\right) = \frac{v}{2}a^{k+1} - \frac{1}{k+2}a^{2k+1}.$$

Substituting into (2.1) and collecting terms yields

$$a'' + (\Omega + \beta)a - \frac{3v^2}{4}a + \frac{v(k+4)}{2(k+2)}a^{k+1} - \frac{k+4}{(k+2)^2}a^{2k+1} + \alpha a' = 0. \tag{2.4}$$

Next we eliminate the $\alpha a'$ term by setting

$$a(\xi) = e^{-\frac{\alpha}{2}\xi}b(\xi).$$

Then

$$a' = e^{-\alpha\xi/2}(b' - \frac{\alpha}{2}b), \qquad a'' + \alpha a' = e^{-\alpha\xi/2}\left(b'' - \frac{\alpha^2}{4}b\right).$$

Substituting into (2.4) and multiplying by $e^{\alpha\xi/2}$ gives the equation for b:

$$b'' + \left(\Omega + \beta - \frac{3v^2}{4} - \frac{\alpha^2}{4}\right)b + \frac{v(k+4)}{2(k+2)}e^{-k\alpha\xi/2}b^{k+1} - \frac{k+4}{(k+2)^2}e^{-k\alpha\xi}b^{2k+1} = 0. \tag{2.5}$$

At this stage, we use the exact reduction (2.6) to place the perturbed problem in the same normal form as the standard generalized DNLS solitary-wave family. Hence, the formulas below are not introduced ad hoc: they are the usual generalized DNLS solitary waves written after removing the drift and phase corrections; see also [14].

Solving (2.5) yields explicit solitary-wave profiles. For later convenience, we summarize the obtained solution in the following form:

$$u(t,x) = e^{-\mathrm{i}\beta t}\, w(t, x+\alpha t). \tag{2.6}$$

Here (under the condition $4\omega > c^2$),

$$w_{c,\omega}(t,x) = G_{\omega,c}(-x-ct)\, e^{\mathrm{i}\omega t},$$

where

$$G_{\omega,c}(x) = \Psi_{\omega,c}(x) \exp\left\{-\frac{\mathrm{i}c}{2}x - \frac{\mathrm{i}}{k+2}\int_{-\infty}^{-x} \Psi_{\omega,c}(y)^k \,\mathrm{d}y\right\}, \tag{2.7}$$

$$\Psi_{\omega,c}(x) = \left(\frac{(k+2)(4\omega - c^2)}{4\sqrt{\omega}\cosh\!\left(\sigma\sqrt{4\omega - c^2}\, x\right) - c}\right)^{1/k}, \qquad \sigma = \frac{k}{k+2}. \tag{2.8}$$

By (2.6), we obtain an explicit solitary-wave solution to (1.1):

$$u_{c,\omega}(t,x) = e^{-\mathrm{i}\beta t}\, w_{c,\omega}(t, x+\alpha t) = G_{\omega,c}(-x-(c+\alpha)t)\, e^{\mathrm{i}(\omega-\beta)t}. \tag{2.9}$$

Based on the exact structure (2.9), we record the following.

Proposition. *For any $k \geq 2$ and any $\alpha, \beta \in \mathbb{R}$, if the parameters (c,ω) satisfy $4\omega > c^2$, then $u_{c,\omega}$ defined by (2.9) is a global smooth solution to (1.1), and for any $s \geq 0$,*

$$u_{c,\omega}(t,\cdot) \in H^s(\mathbb{R}), \qquad \forall t \in \mathbb{R}.$$

Moreover, the spatial profile depends only on (c,ω) and not on (α,β); α only shifts the traveling speed from c to $c+\alpha$, while β only shifts the temporal frequency from ω to $\omega-\beta$.

3. Ill-posedness in H^s

In this section, we use the explicit solitary-wave family (2.9) to construct two solution trajectories $u^{(1)}, u^{(2)}$ whose initial data are arbitrarily close in H^s, while at a later positive time they remain separated by a uniform positive amount in H^s. This yields the proof of Theorem 1.1.

For $f \in \mathcal{S}(\mathbb{R})$, we define the Fourier transform by

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi \mathrm{i} x\xi}\,\mathrm{d}x.$$

At time $t = 0$, the profile from (2.9) can be written as

$$\varphi_{c,\omega}(x) := u_{c,\omega}(0,x) = G_{\omega,c}(-x).$$

For this profile, there exists a function F (independent of N and controllable) and scale factors $d_4, d_5 > 0$ such that

$$\varphi_{c,\omega}(x) = d_5 e^{\mathrm{i}\frac{c}{2}x}\, F(d_4 x), \tag{3.1}$$

and

$$\widehat{\varphi_{c,\omega}}(\xi) = \frac{d_5}{d_4}\widehat{F}\left(\frac{\xi}{d_4} - \frac{c}{2d_4}\right). \tag{3.2}$$

The explicit form of F is determined by (2.7) and (2.8); the key point is that F and its derivative have sufficient square-integrability so that the estimates below hold.

Remark 3.1. *The identities* (3.1) *and* (3.2) *are central to the subsequent H^s distance estimates. They convert "profile changes induced by parameter variations" into "translations and dilations" in Fourier space, allowing us to decompose the difference into three terms I_1, I_2, I_3.*

Fix $s > 0$ satisfying the threshold condition (1.3). Take a large parameter $N \gg 1$. We now describe the parameter choice in the order in which it is used. First, fix an observation horizon $T_0 > 0$ and a small auxiliary exponent $\varepsilon_s > 0$. Next, choose N_1, N_2 around a common large scale N with gap $N_2 - N_1 = \delta N^{\varepsilon_s}$. Finally, define (c_j, ω_j) from these N_j so that the profile width is of order $d_{4,j}^{-1}$ and the later separation condition at time T_0 becomes automatic when N is large. This is the only role of the parameter hierarchy. Choose two parameter sets (c_1, ω_1) and (c_2, ω_2) (both satisfying $4\omega_j > c_j^2$) and set

$$c_j = N_j, \qquad N_1 < N_2, \qquad N_2 - N_1 = \delta N^{\varepsilon_s},$$

and choose $\omega_j = \omega(c_j)$ so that the scale factors $d_4(d), d_5(d)$ have the same asymptotic scaling as in the original argument as $N \to \infty$ (e.g., one may arrange $d_4 \sim N^{2\lambda s}$ and that d_5/d_4 obeys a power law; the exact choice of $\omega(c)$ is only required to ensure the power estimates below). Thus the construction is driven by two complementary requirements: the initial gap must be small enough to vanish in H^s, while the translated solitary waves at the prescribed observation time must be far enough apart compared with their width. The asymptotic rules for $d_{4,j}$ and $d_{5,j}$ are chosen precisely to make these two requirements compatible.

Let

$$\varphi_j := \varphi_{c_j,\omega_j}, \qquad u^{(j)}(t,x) := u_{c_j,\omega_j}(t,x), \qquad j = 1,2.$$

Then

$$\|\varphi_1 - \varphi_2\|_{H^s}^2 = \int_{\mathbb{R}} (1 + |\xi|^2)^s \left|\widehat{\varphi_1}(\xi) - \widehat{\varphi_2}(\xi)\right|^2 d\xi.$$

Using (3.2), scaling $\xi = d_{4,1}\eta$, and decomposing the difference into "dilation difference, translation difference, and amplitude difference", we obtain

$$\|\varphi_1 - \varphi_2\|_{H^s}^2 \le I_1 + I_2 + I_3, \tag{3.3}$$

where

$$I_1 := d_{4,1} \int_{\mathbb{R}} (1 + |d_{4,1}\eta|^2)^s \left| \frac{d_{5,1}}{d_{4,1}} \widehat{F}\left(\eta - \frac{c_1}{2d_{4,1}}\right) - \frac{d_{5,1}}{d_{4,1}} \widehat{F}\left(\frac{d_{4,1}}{d_{4,2}}\eta - \frac{c_1}{2d_{4,1}}\right) \right|^2 d\eta, \tag{3.4}$$

$$I_2 := d_{4,1} \int_{\mathbb{R}} (1 + |d_{4,1}\eta|^2)^s \left| \frac{d_{5,1}}{d_{4,1}} \widehat{F}\left(\frac{d_{4,1}}{d_{4,2}}\eta - \frac{c_1}{2d_{4,1}}\right) - \frac{d_{5,1}}{d_{4,1}} \widehat{F}\left(\frac{d_{4,1}}{d_{4,2}}\eta - \frac{c_2}{2d_{4,2}}\right) \right|^2 d\eta, \tag{3.5}$$

$$I_3 := d_{4,1} \int_{\mathbb{R}} (1 + |d_{4,1}\eta|^2)^s \left| \left(\frac{d_{5,1}}{d_{4,1}} - \frac{d_{5,2}}{d_{4,2}}\right) \widehat{F}\left(\frac{d_{4,1}}{d_{4,2}}\eta - \frac{c_2}{2d_{4,2}}\right) \right|^2 d\eta. \tag{3.6}$$

We next estimate I_1, I_2, I_3 separately. The key is to apply a first-order mean value formula to $\widehat{F}$ together with Cauchy-Schwarz, thereby reducing the differences to L^2 control of $\widehat{F'}$, and then to use the power-law scaling of $d_{4,j}, d_{5,j}$ to obtain decay in N.

Lemma 3.1. *There exist a constant $C > 0$ and an exponent $b = b(k,s) > 0$ (depending only on k, s and independent of N, δ) such that for N large enough,*

$$I_1 + I_2 + I_3 \leq C(N_2 - N_1)^2 N^{-b}.$$

Consequently, if we take $N_2 - N_1 = \delta N^{\varepsilon_s}$ and choose $2\varepsilon_s - b \leq 0$, then

$$\|\varphi_1 - \varphi_2\|_{H^s} \leq C\delta. \tag{3.7}$$

Proof of Lemma 3.1. We start from (3.3). To match the traveling-wave scaling, take a large parameter $N \gg 1$ and choose parameters as follows:

$$c_j = N_j \approx N, \qquad \omega_j = -\Big(N_j^{4\lambda s} + \frac{N_j^2}{4}\Big), \qquad j = 1, 2, \tag{3.8}$$

where $0 < \lambda < \frac{1}{2}$ will be fixed later by exponent conditions, and assume $N_1 < N_2$. The corresponding scale factors (write $d_{4j} = d_4(c_j, \omega_j)$ and $d_{5j} = d_5(c_j, \omega_j)$) satisfy

$$d_{4j} = 2N_j^{2\lambda s}, \qquad |d_{41} - d_{42}| = 2|N_1^{2\lambda s} - N_2^{2\lambda s}| \approx |N_1 - N_2|\, N^{2\lambda s - 1},$$

and

$$\frac{d_{5j}^2}{d_{4j}^2} = \frac{4^{\frac{2}{k}} - 1}{N_j^{4\lambda s}} \approx N_j^{(\frac{8}{k} - 4)\lambda s}, \qquad \left|\frac{d_{51}^2}{d_{41}^2} - \frac{d_{52}^2}{d_{42}^2}\right| \approx |N_1 - N_2|\, N^{(\frac{8}{k} - 4)\lambda s - 1}. \tag{3.9}$$

From (3.4) and using $d_{41} \sim d_{42} \sim N^{2\lambda s}$, for $\eta \in B_1(N^{1-2\lambda s})$, we have $(1 + |d_{41}\eta|^2)^s \approx (d_{41})^{2s}|\eta|^{2s}$, hence

$$\begin{aligned} I_1 &= (d_{41})^{2s+1}\frac{d_{51}^2}{d_{41}^2}\int_{\mathbb{R}}(1 + |\eta|^2)^s \Big|\widehat{F}\Big(\eta - \frac{c_1}{2d_{41}}\Big) - \widehat{F}\Big(\frac{d_{41}}{d_{42}}\eta - \frac{c_1}{2d_{41}}\Big)\Big|^2 d\eta \\ &\approx (d_{41})^{2s+1}\frac{d_{51}^2}{d_{41}^2} N^{2s(1-2\lambda s)} \int_{\mathbb{R}}\Big|\int_{\frac{d_{41}}{d_{42}}\eta - \frac{c_1}{2d_{41}}}^{\eta - \frac{c_1}{2d_{41}}} \widehat{F'}(\beta)\, d\beta\Big|^2 d\eta. \end{aligned}$$

By Cauchy-Schwarz,

$$\Big|\int_A^B \widehat{F'}(\beta)\, d\beta\Big|^2 \leq |B - A| \int_A^B |\widehat{F'}(\beta)|^2\, d\beta,$$

and $|B - A| = \big|1 - \frac{d_{41}}{d_{42}}\big||\eta|$. Therefore,

$$I_1 \lesssim (d_{41})^{2s+1}\frac{d_{51}^2}{d_{41}^2} N^{2s(1-2\lambda s)}\Big|1 - \frac{d_{41}}{d_{42}}\Big| \int_{\mathbb{R}} |\eta| \int_{\frac{d_{41}}{d_{42}}\eta - \frac{c_1}{2d_{41}}}^{\eta - \frac{c_1}{2d_{41}}} |\widehat{F'}(\beta)|^2\, d\beta\, d\eta. \tag{3.10}$$

Let

$$I_{11} := \int_0^\infty \eta \int_{\frac{d_{41}}{d_{42}}\eta - \frac{c_1}{2d_{41}}}^{\eta - \frac{c_1}{2d_{41}}} |\widehat{F'}(\alpha)|^2\, d\alpha\, d\eta,$$

$$I_{12} := \int_{-\infty}^{0} \eta \int_{\frac{d_{41}}{d_{42}}\eta - \frac{c_1}{2d_{41}}}^{\eta - \frac{c_1}{2d_{41}}} |\widehat{F}'(\alpha)|^2 \, d\alpha \, d\eta.$$

Then, (3.10) is equivalent to

$$I_1 \lesssim (d_{41})^{2s+1} \frac{d_{51}^2}{d_{41}^2} N^{2s(1-2\lambda s)} \Big| 1 - \frac{d_{41}}{d_{42}} \Big| (I_{11} - I_{12}). \tag{3.11}$$

Apply Fubini's theorem to I_{11}: for $\eta > 0$, the integration range of α is $\alpha \in [\frac{d_{41}}{d_{42}}\eta - \frac{c_1}{2d_{41}}, \eta - \frac{c_1}{2d_{41}}]$, hence

$$I_{11} = \int_{-\frac{c_1}{2d_{41}}}^{\infty} |\widehat{F}'(\alpha)|^2 \left(\int_{\alpha + \frac{c_1}{2d_{41}}}^{(\alpha + \frac{c_1}{2d_{41}})\frac{d_{42}}{d_{41}}} \eta \, d\eta \right) d\alpha = \frac{1}{2} \int_{-\frac{c_1}{2d_{41}}}^{\infty} |\widehat{F}'(\alpha)|^2 \Big(\alpha + \frac{c_1}{2d_{41}} \Big)^2 \Big[\Big(\frac{d_{42}}{d_{41}} \Big)^2 - 1 \Big] \, d\alpha. \tag{3.12}$$

Similarly,

$$I_{12} = \frac{1}{2} \int_{-\infty}^{-\frac{c_1}{2d_{41}}} |\widehat{F}'(\alpha)|^2 \Big(\alpha + \frac{c_1}{2d_{41}} \Big)^2 \Big[1 - \Big(\frac{d_{42}}{d_{41}} \Big)^2 \Big] \, d\alpha. \tag{3.13}$$

Combining (3.12) and (3.13) gives

$$I_{11} - I_{12} = \frac{1}{2} \int_{\mathbb{R}} |\widehat{F}'(\alpha)|^2 \Big(\alpha + \frac{c_1}{2d_{41}} \Big)^2 \Big[1 - \Big(\frac{d_{42}}{d_{41}} \Big)^2 \Big] \, d\alpha. \tag{3.14}$$

Note that

$$\frac{d_{41}^2 - d_{42}^2}{d_{41}^2} = \frac{4N_1^{4\lambda s} - 4N_2^{4\lambda s}}{4N_1^{4\lambda s}} \approx \frac{N_1 - N_2}{N}, \tag{3.15}$$

and substituting (3.14) and (3.15) into (3.11), together with $\int_{\mathbb{R}} |\widehat{F}'(\alpha)|^2 (\alpha + \frac{c_1}{2d_{41}})^2 \, d\alpha \approx N^{2(1-2\lambda s)} \|\widehat{F}'\|_2^2$ (by translation and scaling), yields

$$I_1 \lesssim N^{-2(2k\lambda s^2 + k\lambda s - 2ks - 4\lambda s + k)k^{-1}} |N_1 - N_2|^2 \, \|\widehat{F}'\|_2^2. \tag{3.16}$$

Similarly, from (3.5),

$$\begin{aligned} I_2 &= (d_{41})^{2s+1} \frac{d_{51}^2}{d_{41}^2} \int_{\mathbb{R}} (1 + |\eta|^2)^s \Big| \widehat{F}\Big(\eta \frac{d_{41}}{d_{42}} - \frac{c_1}{2d_{41}} \Big) - \widehat{F}\Big(\eta \frac{d_{41}}{d_{42}} - \frac{c_2}{2d_{42}} \Big) \Big|^2 \, d\eta \\ &\approx (d_{41})^{2s+1} \frac{d_{51}^2}{d_{41}^2} \Big(\frac{d_{42}}{d_{41}} \Big)^{2s+1} \int_{\mathbb{R}} |\eta|^{2s} \Big| \widehat{F}\Big(\eta - \frac{c_1}{2d_{41}} \Big) - \widehat{F}\Big(\eta - \frac{c_2}{2d_{42}} \Big) \Big|^2 \, d\eta. \end{aligned}$$

Using the mean value formula,

$$\widehat{F}\Big(\eta - \frac{c_1}{2d_{41}} \Big) - \widehat{F}\Big(\eta - \frac{c_2}{2d_{42}} \Big) = \int_{\eta - \frac{c_1}{2d_{41}}}^{\eta - \frac{c_2}{2d_{42}}} \widehat{F}'(\alpha) \, d\alpha,$$

and applying Cauchy-Schwarz, we obtain

$$I_2 \lesssim N^{-2s(2k\lambda - k - 4\lambda)k^{-1}} |N_2 - N_1|^2 \, \|\widehat{F}'\|_2^2. \tag{3.17}$$

From (3.6) and (3.9), we obtain

$$\begin{aligned} I_3 &= (d_{41})^{2s+1}\int_{\mathbb{R}}|\eta|^{2s}\left(\frac{d_{51}^2}{d_{41}^2}-\frac{d_{52}^2}{d_{42}^2}\right)^2\left|\widehat{F}\Big(\eta\frac{d_{41}}{d_{42}}-\frac{c_2}{2d_{42}}\Big)\right|^2 d\eta \\ &\approx (d_{42})^{2s+1}|N_1-N_2|^2N^{2(1-2\lambda s)}\Big(\frac{c_2}{2d_{42}}\Big)^{2s}\|\widehat{F}\|_2^2 \\ &\lesssim N^{-2(3k\lambda s-ks-8\lambda s+k)k^{-1}}(N_1-N_2)^2\,\|\widehat{F}\|_2^2. \end{aligned} \tag{3.18}$$

By Plancherel's theorem, there exists h such that $\widehat{F}=\mathcal{F}[h]$, hence $\|\widehat{F}\|_2=\|h\|_2$ and $\|\widehat{F}'\|_2=\|xh\|_2$, where

$$h(x)=2^{-\frac{1}{k}}\mathrm{e}^{\frac{x}{2}}\left(\mathrm{e}^{kx}+\frac{c}{2+k}\mathrm{e}^{\frac{kx}{2}}-\frac{\omega}{(2+k)^2}\right)^{-\frac{1}{k}}.$$

Using the explicit structure of $h(x)$, we obtain the bounds

$$\|\widehat{F}\|_2^2=\|h\|_2^2\lesssim\widetilde{C}_1+\widetilde{C}_2(-\omega)^{-2/k}\approx\widetilde{C}_1+\widetilde{C}_2N^{-k}, \tag{3.19}$$

and

$$\|\widehat{F}'\|_2^2=\|xh\|_2^2\lesssim\widetilde{C}_1+\widetilde{C}_2(-\omega)^{-2/k}\approx\widetilde{C}_1+\widetilde{C}_2N^{-k}. \tag{3.20}$$

(These estimates hold under our parameter choice (3.8).)

Substituting (3.20) into (3.16) and (3.17), and (3.19) into (3.18), we get

$$\begin{aligned} I_1 &\lesssim (N_1-N_2)^2\,N^{-2(2k\lambda s^2+k\lambda s-2ks-4\lambda s+k)k^{-1}}, & (3.21)\\ I_2 &\lesssim (N_1-N_2)^2\,N^{-2s(2k\lambda-k-4\lambda)k^{-1}}, \\ I_3 &\lesssim (N_1-N_2)^2\,N^{-2(3k\lambda s-ks-8\lambda s+k)k^{-1}}. & (3.22) \end{aligned}$$

Therefore, provided all exponents are positive,

$$2k\lambda s^2+k\lambda s-2ks-4\lambda s+k>0,\quad 2k\lambda-k-4\lambda>0,\quad 3k\lambda s-ks-8\lambda s+k>0,$$

it follows from (3.3), (3.21), and (3.22) that

$$\|\varphi_1-\varphi_2\|_{H^s}^2\le I_1+I_2+I_3\lesssim\frac{(N_1-N_2)^2}{N^b},$$

where

$$b=\min\Big\{2(2k\lambda s^2+k\lambda s-2ks-4\lambda s+k)k^{-1},\,2s(2k\lambda-k-4\lambda)k^{-1},\,2(3k\lambda s-ks-8\lambda s+k)k^{-1}\Big\}>0.$$

This proves the main inequality $I_1+I_2+I_3\lesssim(N_1-N_2)^2N^{-b}$.

Finally, taking $N_2-N_1=\delta N^{\varepsilon_s}$ and choosing $\varepsilon_s\le b/2$, we obtain

$$\|\varphi_1-\varphi_2\|_{H^s}\lesssim\delta,$$

which completes the proof. □

We now fix an arbitrary observation horizon $T_0 > 0$ and prove that the two traveling-wave solutions are already separated at some time $t_* \in (0, T_0]$ (in fact at $t_* = T_0$ for N sufficiently large). This is the formulation needed for failure of uniform continuity in the topology $C([0, T_0]; H^s)$.

By (2.9),

$$u^{(j)}(t, x) = G_{\omega_j, c_j}(-x - (c_j + \alpha)t)\, e^{\mathrm{i}(\omega_j - \beta)t}.$$

The phase factor does not affect Sobolev norms, hence the essential difference comes from the spatial translation of the profiles. Their relative displacement is

$$\Delta(t) := [(c_2 + \alpha) - (c_1 + \alpha)]t = (c_2 - c_1)t.$$

Since $T_0 > 0$ is fixed in advance and $c_2 - c_1 = \delta N^{\varepsilon_s}$ with $\varepsilon_s > 0$, we may choose N so large that the separation condition already holds at $t = T_0$. Thus, take N large enough so that

$$(c_2 - c_1)T_0 \gg \frac{1}{d_{4,1}} \sim N^{-2\lambda s}, \tag{3.23}$$

so that the two profiles are "almost disjoint" in physical space (the distance between centers is much larger than the typical width). By exponential decay and near-orthogonality, we obtain an L^2 lower bound

$$\|u^{(1)}(T_0) - u^{(2)}(T_0)\|_{L^2}^2 \geq \|u^{(1)}(T_0)\|_{L^2}^2 + \|u^{(2)}(T_0)\|_{L^2}^2 - o(1),$$

and therefore, there exists $\varepsilon_1 > 0$ such that for N large enough,

$$\|u^{(1)}(T_0) - u^{(2)}(T_0)\|_{L^2} \geq \varepsilon_1. \tag{3.24}$$

Moreover, using the frequency localization structure, one can lift the L^2 lower bound to an H^s lower bound: there exists $c_* > 0$ (independent of N) such that

$$\|u^{(1)}(T_0) - u^{(2)}(T_0)\|_{H^s} \geq c_*\, N^s\, \|u^{(1)}(T_0) - u^{(2)}(T_0)\|_{L^2} \geq \varepsilon_0, \tag{3.25}$$

where $\varepsilon_0 := c_* N^s \varepsilon_1$, and the growth in $c_* N^s$ can be balanced by the parameter normalization so that the final lower bound is a positive constant independent of N.

Lemma 3.2. *Under the threshold condition* (1.3), *for every $T_0 > 0$, there exists $\varepsilon_0 > 0$ such that for any given $\delta > 0$, one can choose N large enough (via the parameter construction in Lemma 3.1) so that*

$$\|\varphi_1 - \varphi_2\|_{H^s} \leq \delta, \qquad \|u^{(1)}(T_0) - u^{(2)}(T_0)\|_{H^s} \geq \varepsilon_0.$$

Proof. Lemma 3.1 yields the smallness of the initial gap; condition (3.23) ensures spatial separation at the arbitrarily prescribed time T_0, and then (3.24) and (3.25) give a positive lower bound in H^s. Here, α appears only in the common translation speed $c_j + \alpha$, so the relative displacement is still $(c_2 - c_1)T_0$, and β appears only as an overall phase, which does not affect the norm estimates. □

Proof of Theorem 1.1. Given any $\delta > 0$ and any $T_0 > 0$, choose s satisfying (1.3). Select parameters (c_j, ω_j) as in Lemma 3.1 and construct the two explicit solutions $u^{(j)} = u_{c_j, \omega_j}$. Then, (3.7) gives $\|\varphi_1 - \varphi_2\|_{H^s} \leq \delta$. By Lemma 3.2, the same prescribed T_0 satisfies $\|u^{(1)}(T_0) - u^{(2)}(T_0)\|_{H^s} \geq \varepsilon_0$. Hence, the flow map cannot be uniformly continuous in $C([0, T_0]; H^s(\mathbb{R}))$, and since T_0 is arbitrary, the asserted lack of uniform continuity follows. □

4. Conclusions

In this paper we established a traveling-wave-based ill-posedness mechanism for the one dimensional defocusing power-type derivative Schrödinger equation with lower-order linear perturbations. By combining the exact drift-phase reduction with an explicit solitary-wave family, we showed that the additional linear terms do not improve the low-regularity stability threshold in Sobolev spaces. More precisely, below the threshold in Theorem 1.1, the solution map fails to be uniformly continuous on every prescribed time interval. These results indicate that the instability mechanism is inherited from the underlying generalized DNLS structure rather than removed by lower-order transport or phase corrections.

Author contributions

Senyue Luo: Conceptualization, methodology, formal analysis, writing—original draft. Meilan Qiu: Validation, investigation, writing—review and editing. Fangfang Deng: Supervision, project administration. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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