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Research article

Progress on the Borodin–Kostochka conjecture: A structural approach via vertex partitions relative to a maximum clique

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Abstract: The Borodin-Kostochka conjecture states that for any graph G with $\Delta(G) \geq 9$, we have $\chi(G) \leq \max\{\Delta(G) - 1, \omega(G)\}$. In this paper, we study the structure of potential counterexamples by partitioning vertices according to the number of neighbors they have in a fixed maximum clique. This approach provides a sufficient condition for $\chi(G) \leq \Delta(G) - 1$. Consequently, we confirm the conjecture for any $\overline{K_{1,t}}$-free graph G with $t \geq 3$ and $\Delta(G) \geq 2t + 1$, strengthening and extending the recent work of Lan and Lin in 2024.

Keywords: Borodin-Kostochka conjecture; chromatic number; forbidden induced graphs; $\overline{K_{1,t}}$-free
Mathematics Subject Classification: 05C15

1. Introduction

All graphs considered in this paper are finite and simple. Given a positive integer k, a *proper k-coloring* of a given graph G is a mapping $\phi : V(G) \to \{1, 2, \ldots, k\}$ such that $\phi(u) \neq \phi(v)$ for every edge $uv \in E(G)$. If such a coloring exists, G is said to be *k-colorable*. The *chromatic number* of G, denoted by $\chi(G)$, is the smallest k for which G is k-colorable.

The *complement* of G, denoted as $\overline{G}$, shares the same vertex set $V(G)$. Two vertices in $\overline{G}$ are adjacent if and only if they are not adjacent in G. We denote as $\omega(G)$, $\Delta(G)$, and $\delta(G)$ the clique number, maximum degree, and minimum degree of G, respectively. Let H be a graph. We say that G is *H-free* if it does not contain an induced copy of H. Moreover, a $K_{1,t}$ represents a star graph.

In 1941, Brooks presented the foundational result that any graph G can be colored with at most $\Delta(G)$ colors, unless G is a complete graph or an odd cycle, in which case, $\chi(G) = \Delta(G) + 1$. A frequently used formulation of this theorem, which incorporates the clique number $\omega(G)$, is stated as follows.

Theorem 1.1. *[2] Let G be a graph with $\Delta(G) \geq 3$. Then $\chi(G) \leq \max\{\Delta(G), \omega(G)\}$.*

In 1977, Borodin and Kostochka [1] proved the following lemma, which gives a sufficient condition that any graph G can be colored with at most $\Delta(G)-1$ colors. In [1], they also proposed the influential conjecture that now bears their name.

Lemma 1.1. *[1] Let G be a graph. If* $\Delta(G) \geq 7$ *and* $\omega(G) \leq \lfloor \frac{\Delta(G)-1}{2} \rfloor$*, then* $\chi(G) \leq \Delta(G) - 1$.

Conjecture 1.1. *[1] Let G be a graph with* $\Delta(G) \geq 9$*. Then* $\chi(G) \leq \max\{\Delta(G) - 1, \omega(G)\}$.

The conjecture follows directly from Theorem 1.1 when $\omega(G) \geq \Delta(G)$; thus, the essential case is to prove that any graph G with $\Delta(G) \geq 9$ and $\omega(G) \leq \Delta(G) - 1$ satisfies $\chi(G) \leq \Delta(G) - 1$. Research toward this goal has developed along several lines, two of which have been especially prominent.

The first aims at relaxing the required bound on the clique number. Kostochka [15] showed that $\omega(G) \leq \Delta(G) - 29$ is sufficient. Mozhan [20] improved this to $\omega(G) \leq \lfloor \frac{2\Delta(G)+1}{3} \rfloor - 1$ for $\Delta(G) \geq 10$, and later proved in his PhD thesis that $\omega(G) \leq \Delta(G) - 4$ suffices when $\Delta(G) \geq 31$. Cranston and Rabern [8] subsequently lowered the degree condition to $\Delta(G) \geq 13$ when $\omega(G) \leq \Delta(G) - 4$. Subsequently, in 2020, MacDonald [19] demonstrated in his Master's thesis that any graph with $\chi(G) = \Delta(G) - 1$ and $\Delta(G) \geq 66$ contains a clique of size $\Delta(G) - 17$. In [13], Haxell and MacDonald further demonstrated that for any non-negative integer t, every graph G with a sufficiently large $\Delta(G)$ and $\chi(G) = \Delta(G) - t$ contains a clique of size $\Delta(G) - 2t^2 - 6t - 3$. Later, Xie [22] proved in his Master's thesis that every graph G with $\Delta(G) \geq 4t^2 + 11t + 7$ and $\chi(G) = \Delta(G) - t$ contains a clique of size $\Delta(G) - 2t^2 - 7t - 4$. Using probabilistic methods, Reed [21] obtained the current strongest general result, verifying the conjecture for all graphs with $\Delta(G) \geq 10^{14}$. Furthermore, he pointed out in the article that a more detailed analysis might allow for a potential weakening of this threshold to $\Delta(G) \geq 10^6$, and possibly even to $\Delta(G) \geq 10^3$. In addition to these positive results, it is known that the conjecture cannot be sharpened to require only $\Delta(G) \geq 8$, or $\omega(G) \leq \Delta(G) - 2$ [9].

A second and highly active direction establishes the conjecture for various hereditary graph classes, typically defined by forbidding one or a few induced subgraphs. The early work of Dhurandhar [10] proved it for graphs excluding $K_{1,3}$, $K_5 - e$ and a fixed graph H. Kierstead and Schmerl [14] later removed the extra condition on H. Cranston and Rabern confirmed it for claw-free graphs [6] and for $K_3 \vee \overline{K_6}$-free graphs [7]. More recently, the conjecture has been verified for an expanding family of classes including $\{P_5, C_4\}$-free graphs [11], $\{P_5, \text{gem}\}$-free graphs [9], hammer-free graphs [3], $\{P_2 \cup P_3, \text{house}\}$-free graphs [5], $\{P_2 \cup P_3, C_4\}$-free graphs [17], $\{P_2 \cup P_3, \text{banner}\}$-free graphs [18], odd-hole-free graphs [4] (recall that a *hole* is an induced cycle of length at least four, and an *odd hole* is a hole of odd length), and $\{P_5, C_5\}$-free graphs, $\{P_{10}, \text{paw}\}$-free graphs, and $\{P_6, C_4, N\}$-free graphs (where N is defined as either a bull or a diamond) in [12]. Figure 1 illustrates some of the forbidden configurations mentioned above.

In this paper, we develop a novel structural approach to analyze potential counterexamples to the Borodin-Kostochka conjecture. Unlike most previous work, which typically assumes a minimal counterexample and then derives a contradiction by analyzing the structure forced by a forbidden induced subgraph, we adopt a different perspective by directly investigating the structural properties that any potential counterexample must necessarily possess. Our main idea is to fix a maximum clique C in a given graph G and partition the remaining vertices $V(G) \setminus V(C)$ according to how many neighbors they have in C. This fine-grained partition enables us to systematically exploit the combinatorial constraints of vertex-critical graphs in our analysis, while also taking into account the restrictions arising from forbidding a specific induced subgraph. Although this line of inquiry does not yield an unconditional proof of the conjecture, it provides a versatile framework for establishing the conjecture under certain natural and verifiable conditions. Consequently, the lemmas we obtain constitute a powerful tool that not only recovers but also significantly extends a range of existing results.

As a demonstration of the strength and flexibility of this method, we apply it to the class of $\overline{K_{1,t}}$-free graphs. Our analysis yields the following theorem, which both strengthens and generalizes a recent result of Lan and Lin [16].

Theorem 1.2. *Let $t \geq 3$ be an integer and let G be a $\overline{K_{1,t}}$-free graph. If $\Delta(G) \geq 2t + 1$ and $\omega(G) \leq \Delta(G) - 1$, then $\chi(G) \leq \Delta(G) - 1$.*

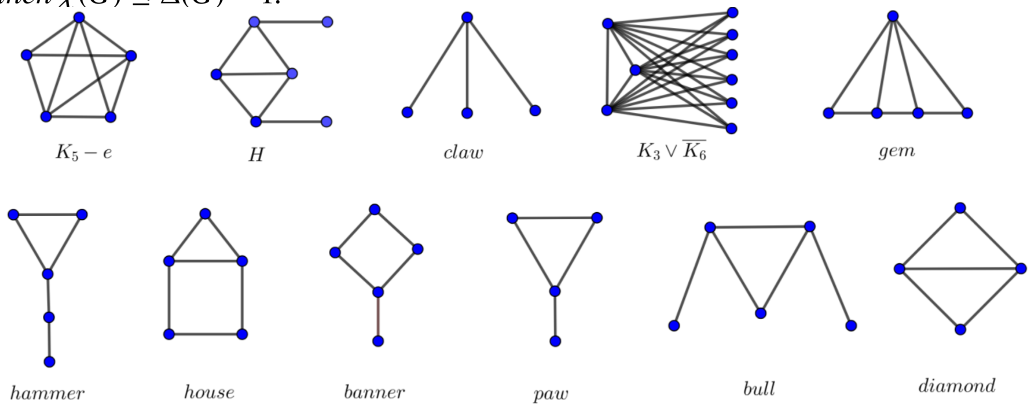

Figure 1. Illustration of some forbidden configurations.

2. Preliminary

For a graph G and a vertex set $X \subseteq V(G)$, we write $G[X]$ for the subgraph induced by X and $G \setminus X$ for the subgraph induced by $V(G) \setminus X$. When $X = \{x\}$, we abbreviate $G \setminus \{x\}$ to $G \setminus x$. For a vertex $x \in V(G)$, we denote $N_G(x)$ as its neighborhood, $N_G[x] = N_G(x) \cup \{x\}$ as its closed neighborhood, and $d_G(x) = |N_G(x)|$ as its degree; the order of G is $|G|$. We write $u \sim v$ when $uv \in E(G)$ (and $u \nsim v$ otherwise).

Let ϕ be a proper k-coloring of a graph G. If $\phi(x) = m$, we call x an *m-vertex*. For a vertex y, if there is exactly one m-vertex in $N_G(y)$, we say that the color m is a *unique color* in $N_G(y)$ and that the vertex is the *unique m-vertex* in $N_G(y)$. If there are at least two m-vertices in $N_G(y)$, then we say that the color m is a *repeated color* in $N_G(y)$; in this case, we also say that the vertex y has a repeated color m.

A graph G is called a *k-vertex-critical* graph for a positive integer k if $\chi(G) = k$ and $\chi(G \setminus v) \leq k - 1$ for every vertex v of G. For $k, j \in \mathbb{N}$, we denote $\mathcal{G}_{k,j}$ as the family of all k-vertex-critical graphs satisfying $\Delta(G) = k$ and $\omega(G) < k - j$; when $j = 0$, we write $\mathcal{G}_k := \mathcal{G}_{k,0}$. A graph class C is *hereditary* if every induced subgraph of a graph in C also belongs to C. Clearly, the graph class appearing in Theorem 1.2 is hereditary.

The following lemma, extracted from [23], provides a useful criterion for verifying the Borodin–Kostochka conjecture for hereditary classes.

Lemma 2.1. *[23] Let $k \geq 5$ be an integer, and let C be a hereditary graph class such that every $G \in C$ satisfies $\Delta(G) \geq k$ and $\omega(G) \leq \Delta(G) - 1$. If $C \cap \mathcal{G}_k = \emptyset$, then for any $G \in C$, $\chi(G) \leq \Delta(G) - 1$.*

In the remainder of this paper, we fix the following setting. Let $k \geq 3$ be an integer and let G be a connected k-vertex-critical graph with $\Delta(G) = k$ and $\omega(G) \leq k - 1$. Choose a vertex $u \in V(G)$ with degree k and let $N_G(u) := \{u_1, \ldots, u_{k-2}, x, y\}$. Consider any proper $(k-1)$-coloring ϕ of $G \setminus u$ such that

$\phi(u_i) = i$ for $i = 1, \ldots, k-2$, and $\phi(x) = \phi(y) = k-1$. We will use the shorthand $[n] := \{1, \ldots, n\}$ and $[u_n] := \{u_1, \ldots, u_n\}$.

Lemma 2.2. *[23] Let $k \geq 3$ be an integer, and let G be a connected k-vertex-critical graph with $\Delta(G) = k$ and $\omega(G) \leq k-1$. Choose a vertex $u \in V(G)$ with degree k. Then, for every proper $(k-1)$-coloring ϕ of $G \setminus u$, the following statements hold.*

(1) $\phi(N_G(u)) = [k-1]$ and there are exactly two vertices x, y in $N_G(u)$ satisfying $\phi(x) = \phi(y)$.
(2) For any vertex $a \in [u_{k-2}]$, $\phi(N_{G\setminus u}[a]) = [k-1]$, and there are at most two vertices in $N_{G\setminus u}(a)$ assigned the same color.
(3) Either $\phi(N_{G\setminus u}[x]) = [k-1]$ or $\phi(N_{G\setminus u}[y]) = [k-1]$.
(4) Let $a \in [u_{k-2}]$. If $c \in N_G(a) \setminus N_G[u]$ is the unique $\phi(c)$-vertex in $N_{G\setminus u}(a)$, then $\phi(N_{G\setminus u}[c]) = [k-1]$, and $\phi(a)$ is a repeated color in $N_{G\setminus u}(c)$.
(5) For any two distinct vertices $a, b \in [u_{k-2}]$, if $a \nsim b$, then the following statements hold.
(5.1) No vertex $c \in N_G(u)$ is the unique $\phi(c)$-vertex in both $N_{G\setminus u}(a)$ and $N_{G\setminus u}(b)$.
(5.2) $|N_G(a) \cap N_G(b) \cap N_G(u)| \leq 2$.

Lemma 2.2(3) guarantees that either $\phi(N_{G\setminus u}[x]) = [k-1]$ or $\phi(N_{G\setminus u}[y]) = [k-1]$. For simplicity, we fix the convention throughout the paper that

$$\phi(N_{G\setminus u}[x]) = [k-1].$$

We next introduce the vertex partitioning scheme that underlies our structural analysis. Let C be a clique in G with $|C| = \omega(G) = j$. The vertex set $V(G) \setminus V(C)$ is partitioned according to the number of neighbors each vertex has in C as follows:

$$\begin{aligned}
A_0 &:= \{v \in V(G) \setminus V(C) : |N_G(v) \cap V(C)| = 0\},\\
A_1 &:= \{v \in V(G) \setminus V(C) : |N_G(v) \cap V(C)| = 1\},\\
A_2 &:= \{v \in V(G) \setminus V(C) : |N_G(v) \cap V(C)| = 2\},\\
&\vdots\\
A_{j-1} &:= \{v \in V(G) \setminus V(C) : |N_G(v) \cap V(C)| = j-1\}.
\end{aligned}$$

An illustration of this partition for the case $j = 4$ is given in Figure 2.

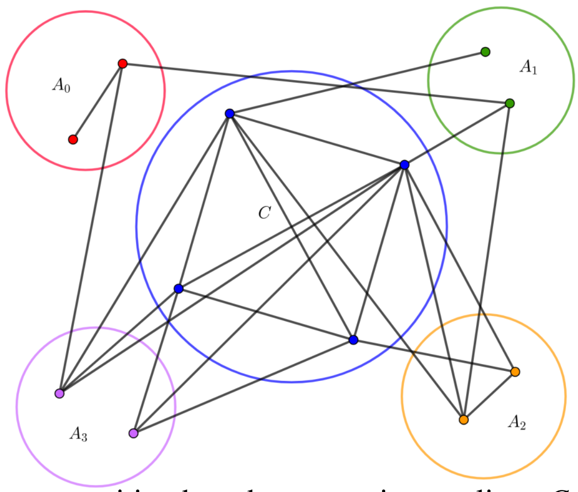

Figure 2. Vertex partition based on a maximum clique C with $|C| = 4$.

The next three lemmas exploit this vertex partition and provide the technical machinery for the proofs that follow.

We first analyze the structure of a potential counterexample in which the maximum clique C contains no vertex of maximum degree k. This investigation culminates in Lemma 2.3, which establishes that under the condition $A_0 = \emptyset$, the adjacency between any vertex u of degree k and the clique C is highly constrained; specifically, we obtain $|N_G(u) \cap V(C)| = 1$.

Lemma 2.3. *Let $k \geq 4$ be an integer and suppose that there is a connected k-vertex-critical graph G with $\Delta(G) = k$ and $3 \leq \omega(G) = j \leq k-1$. Furthermore, assume that G contains a clique C of size j, in which there is no vertex of degree k. Additionally, the vertex partition $A_0, A_1, \ldots, A_{j-1}$ of $V(G) \setminus V(C)$ satisfies $A_0 = \emptyset$. Then for any vertex $u \in V(G)$ with $d_G(u) = k$, we have $|N_G(u) \cap V(C)| = 1$.*

Proof of Lemma 2.3. We prove Lemma 2.3 by contradiction. Let $k \geq 4$ be an integer, and let G be a connected k-vertex-critical graph with $\Delta(G) = k$ and $3 \leq \omega(G) = j \leq k-1$. Suppose that there is such a graph G which contains a clique C of size j where there is no vertex of degree k. Additionally, the vertex partition $A_0, A_1, \ldots, A_{j-1}$ of $V(G) \setminus V(C)$ satisfies $A_0 = \emptyset$. Then a vertex $u \in V(G)$ with $d_G(u) = k$ exists such that

$$2 \leq |N_G(u) \cap V(C)| \leq j-1.$$

We will demonstrate that no such graph G can exist.

Note that $3 \leq |C| = j \leq k-1$. We consider any proper $(k-1)$-coloring ϕ of $G \setminus u$. Given that $|N_G(u) \cap V(C)| \geq 2$, the intersection $N_G(u) \cap V(C)$ must contain at least one vertex from the set $\{u_1, u_2, \ldots, u_{k-2}\}$. For any vertex $c \in N_G(u) \cap V(C) \cap \{u_1, \ldots, u_{k-2}\}$, Lemma 2.2(2) implies that $\phi(N_{G\setminus u}[c]) = [k-1]$. Furthermore, since no vertex in C has degree k, it follows that $d_G(c) = k-1$. Consequently, given that $\phi(N_{G\setminus u}[c]) = [k-1]$ and $d_G(c) = k-1$, there is no repeated color in $N_{G\setminus u}(c)$.

Claim 2.1. *For every vertex $c' \in V(C) \setminus \{c\}$, $\phi(N_{G\setminus u}[c']) = [k-1]$.*

Subproof. Suppose that there is a vertex $c' \in V(C) \setminus \{c\}$ and there is no r-vertex in $N_{G\setminus u}(c')$ for some $r \in [k-1] \setminus \{\phi(c')\}$. In this case, we can recolor c' with color r. Since there is no repeated color in $N_{G\setminus u}(c)$, the vertex c' is the unique $\phi(c')$-vertex in $N_{G\setminus u}(c)$. Consequently, we can further recolor c with $\phi(c')$ and u with $\phi(c)$, obtaining a proper $(k-1)$-coloring of G. This contradicts the fact that $\chi(G) = k$.

Thus, for every vertex $c'' \in V(C)$, $\phi(N_{G\setminus u}[c'']) = [k-1]$, and $d_G(c'') = k-1$.

We now distinguish two cases, depending on whether a $(k-1)$-vertex belongs to $N_G(u) \cap V(C)$.

Case 2.1. *There is a $(k-1)$-vertex in $N_G(u) \cap V(C)$.*

Given that $2 \leq |N_G(u) \cap V(C)| \leq j-1$ and that there is a $(k-1)$-vertex in $N_G(u) \cap V(C)$, we can assume, without loss of generality, that the vertices u_1 and x belong to $N_G(u) \cap V(C)$.

Concerning vertex u_1, there is no repeated color in $N_{G\setminus u}(u_1)$. Consequently, for any vertex $s \in V(C) \setminus (N_G(u) \cap V(C))$, Lemma 2.2(4) implies the existence of a 1-vertex in $N_{G\setminus u}(s)$ distinct from u_1.

We now consider the vertex y. The condition $A_0 = \emptyset$ necessitates that y is adjacent to at least one vertex in C. Since x is not adjacent to y, we will examine two possibilities.

First, let us assume that y is adjacent to a vertex $s' \in (N_G(u) \cap V(C)) \setminus \{x\}$. In this case, the adjacencies $x \sim s'$ and $y \sim s'$ would result in color $k-1$ being a repeated color in $N_{G\setminus u}(s')$. This situation contradicts the fact that there is no repeated color in $N_{G\setminus u}(c)$, where $c \in N_G(u) \cap V(C) \cap \{u_1, \ldots, u_{k-2}\}$.

Second, let us assume that y is adjacent to a vertex $s'' \in V(C) \setminus (N_G(u) \cap V(C))$. For this vertex s'', we have $\phi(N_{G\setminus u}[s'']) = [k-1]$ and $d_G(s'') = k-1$. Consequently, there can neither be two repeated colors nor three r-vertices for any $r \in [k-1] \setminus \{\phi(s'')\}$ in $N_{G\setminus u}(s'')$. However, since $y \sim s''$, $x \sim s''$, and $N_{G\setminus u}(s'')$ contains a 1-vertex distinct from u_1, there are two repeated colors, $k-1$ and 1, in $N_{G\setminus u}(s'')$, leading to a contradiction.

Therefore, the assumption that $N_G(u) \cap V(C)$ contains a $(k-1)$-vertex is untenable.

Case 2.2. *No $(k-1)$-vertex in $N_G(u) \cap V(C)$ exists.*

Given that $2 \leq |N_G(u) \cap V(C)| \leq j-1$ and there is no $(k-1)$-vertex in $N_G(u) \cap V(C)$, we can assume, without loss of generality, that the vertices u_1 and u_2 belong to $N_G(u) \cap V(C)$.

We now consider any vertex $s \in V(C) \setminus (N_G(u) \cap V(C))$. For this vertex, we have $\phi(N_{G\setminus u}[s]) = [k-1]$ and $d_G(s) = k-1$. Consequently, there can neither be two repeated colors nor three r-vertices for any $r \in [k-1] \setminus \{\phi(s)\}$ in $N_{G\setminus u}(s)$. Furthermore, since there is no repeated color in $N_{G\setminus u}(u_1)$ or $N_{G\setminus u}(u_2)$, Lemma 2.2(4) implies the existence of a 1-vertex distinct from u_1 and a 2-vertex distinct from u_2 in $N_{G\setminus u}(s)$. This situation results in the presence of two repeated colors, 1 and 2, in $N_{G\setminus u}(s)$, leading to a contradiction.

Therefore, the assumption that $N_G(u) \cap V(C)$ contains no $(k-1)$-vertex is untenable.

Having derived contradictions in all cases, we conclude that the assumed graph G cannot exist. □

Having resolved the case where C contains no vertex of maximum degree k, we now shift our focus to the complementary—and structurally more intricate—situation in which the maximum clique C contains a vertex u of degree k. Under the additional hypotheses $A_0 = A_1 = \emptyset$, Lemma 2.4 yields strong structural constraints on the clique C. In particular, it forces the equality $\omega(G) = k-1$ and, moreover, characterizes the color classes of its vertices under any proper $(k-1)$-coloring of $G \setminus u$.

Lemma 2.4. *Let $k \geq 4$ be an integer, and suppose that there exists a connected k-vertex-critical graph G exists with $\Delta(G) = k$ and $3 \leq \omega(G) = j \leq k-1$. Furthermore, assume that G contains a clique C of size j, in which there is a vertex u with degree k. Additionally, the vertex partition $A_0, A_1, \ldots, A_{j-1}$ of $V(G) \setminus V(C)$ satisfies $A_0 = A_1 = \emptyset$. Under these conditions, the following statements hold.*

(1) $\omega(G) = j = k-1$.

(2) For any proper $(k-1)$-coloring ϕ of $G \setminus u$, the vertex set of the clique C is exactly $\{u, u_1, u_2, \ldots, u_{k-2}\}$.

Proof of Lemma 2.4. We begin by proving Lemma 2.4(1) through contradiction. Let $k \geq 4$ be an integer, and let G be a connected k-vertex-critical graph with $\Delta(G) = k$ and $3 \leq \omega(G) = j \leq k-1$. Suppose that there is such a graph G which contains a clique C of size j where there is a vertex u with degree k. Additionally, the vertex partition $A_0, A_1, \ldots, A_{j-1}$ of $V(G) \setminus V(C)$ satisfies $A_0 = A_1 = \emptyset$. We assume that $\omega(G) = j \neq k-1$, which implies

$$3 \leq \omega(G) = j \leq k-2.$$

We will show that this assumption leads to a contradiction.

First, we consider the case when $k = 4$, where the condition $3 \leq j \leq k-2$ implies $3 \leq j \leq 2$, which is a contradiction.

Next, we consider the case when $k \geq 5$. Given that $u \in V(C)$ and $d_G(u) = k$, we consider any proper $(k-1)$-coloring ϕ of $G \setminus u$. The remainder of the proof is divided into two cases, depending on whether a $(k-1)$-vertex belongs to C.

Case 2.3. *No $(k-1)$-vertex in $V(C)$ exists.*

Given that there is no $(k-1)$-vertex in $V(C)$, we can assume, without loss of generality, that $V(C) \setminus \{u\} = \{u_1, u_2, \ldots, u_{j-1}\}$. Since $j \leq k-2$, it follows that $j-1 \leq k-3 < k-2$. Therefore, there is a vertex $u_{k-2} \in N_G(u)$ that does not belong to $V(C)$. Given that $|N_G(u_{k-2}) \cap V(C)| \leq j-1$, there is at least one vertex in the set $\{u_1, u_2, \ldots, u_{j-1}\}$ that is not adjacent to u_{k-2}. We may assume, without loss of generality, that u_1 is not adjacent to u_{k-2}. By Lemma 2.2(2), the condition $u_1 \nsim u_{k-2}$ guarantees the existence of a 1-vertex c_1 in $N_{G\setminus u}(u_{k-2})$.

Since $A_0 = A_1 = \emptyset$, the vertex u_{k-2} must be adjacent to at least two vertices in C. Given that $u_{k-2} \sim u$, it follows that u_{k-2} must also be adjacent to at least one vertex in $V(C) \setminus \{u\}$. Therefore, we may assume, without loss of generality, that $u_{j-1} \in \{u_2, \ldots, u_{j-1}\}$ satisfies $u_{j-1} \sim u_{k-2}$.

Currently, we have $u_1 \nsim u_{k-2}$, $u_1 \sim u_{j-1}$, and $u_{k-2} \sim u_{j-1}$. According to Lemma 2.2(5.2), at most one vertex in the set $\{u_2, \ldots, u_{j-2}\}$ can be adjacent to u_{k-2}. Furthermore, Lemma 2.2(5.1) implies the existence of a $(k-1)$-vertex that is distinct from u_{j-1} in either $N_{G\setminus u}(u_1)$ or $N_{G\setminus u}(u_{k-2})$.

We now consider two possibilities based on the existence of a vertex in the set $\{u_2, \ldots, u_{j-2}\}$ that is adjacent to u_{k-2}.

Subcase 2.3.1. *There is a vertex in the set $\{u_2, \ldots, u_{j-2}\}$ that is adjacent to u_{k-2}.*

Without loss of generality, we assume that u_{j-2} is adjacent to u_{k-2}. Given that $u_1 \nsim u_{k-2}$, $u_1 \sim u_{j-2}$, and $u_{k-2} \sim u_{j-2}$, Lemma 2.2(5.1) guarantees the existence of a $(j-2)$-vertex, distinct from u_{j-2}, in either $N_{G\setminus u}(u_1)$ or $N_{G\setminus u}(u_{k-2})$. Furthermore, since there is a $(k-1)$-vertex that is distinct from u_{j-1} in either $N_{G\setminus u}(u_1)$ or $N_{G\setminus u}(u_{k-2})$, and Lemma 2.2(2) indicates that both u_1 and u_{k-2} can have at most one repeated color, we may assume, without loss of generality, that there is a $(j-1)$-vertex c_{j-1} in $N_{G\setminus u}(u_1)$ and a $(j-2)$-vertex c_{j-2} in $N_{G\setminus u}(u_{k-2})$.

Since $A_0 = A_1 = \emptyset$ and $d_G(u) = k$, each of the vertices c_1, c_{j-1}, and c_{j-2} must be adjacent to at least two vertices in the set $V(C) \setminus \{u\} = \{u_1, u_2, \ldots, u_{j-1}\}$. Furthermore, according to Lemma 2.2(2), each vertex in $V(C) \setminus \{u\}$ can have at most one repeated color. Consequently, if such a graph G exists, it necessitates that $j-1 \geq 6$ or, equivalently, $j \geq 7$.

Consider any vertex $c \in \{u_2, \ldots, u_{j-3}\}$. Since u_{k-2} is not adjacent to c, Lemma 2.2(2) guarantees the existence of a $\phi(c)$-vertex in $N_{G\setminus u}(u_{k-2})$. The conditions $A_0 = A_1 = \emptyset$ and $d_G(u) = k$ necessitate that this $\phi(c)$-vertex is adjacent to at least two vertices in the set $V(C) \setminus \{u\}$. Given that each vertex in $V(C) \setminus \{u\}$ can have at most one repeated color, the vertex u_{k-2} cannot be adjacent to at most $\left\lfloor \frac{j-7}{2} \right\rfloor$ vertices in $\{u_2, \ldots, u_{j-3}\}$. However, since $|\{u_2, \ldots, u_{j-3}\}| = j-4$, it follows that $j-4 > \left\lfloor \frac{j-7}{2} \right\rfloor$ for $j \geq 7$, leading to a contradiction.

Subcase 2.3.2. *Each vertex in the set $\{u_2, \ldots, u_{j-2}\}$ is not adjacent to u_{k-2}.*

Given that $A_0 = A_1 = \emptyset$ and $d_G(u) = k$, both the vertex c_1 in $N_{G\setminus u}(u_{k-2})$ and the $(j-1)$-vertex, which is distinct from u_{j-1} in either $N_{G\setminus u}(u_1)$ or $N_{G\setminus u}(u_{k-2})$, must each be adjacent to at least two vertices in the set $V(C) \setminus \{u\} = \{u_1, u_2, \ldots, u_{j-1}\}$. Consequently, if such a graph G exists, it requires that $j-1 \geq 4$ or, equivalently, $j \geq 5$.

For any vertex $c \in \{u_2, \ldots, u_{j-2}\}$, given that $u_{k-2} \nsim c$, Lemma 2.2(2) guarantees the existence of a $\phi(c)$-vertex in $N_{G\setminus u}(u_{k-2})$. Furthermore, the conditions $A_0 = A_1 = \emptyset$ and $d_G(u) = k$ require this $\phi(c)$-vertex to be adjacent to at least two vertices in the set $V(C) \setminus \{u\}$. Since each vertex in $V(C) \setminus \{u\}$ can have at most one repeated color, it follows that the vertex u_{k-2} cannot be adjacent to at most $\lfloor \frac{j-5}{2} \rfloor$ vertices in $\{u_2, \ldots, u_{j-2}\}$. However, since $|\{u_2, \ldots, u_{j-2}\}| = j - 3$, it follows that $j - 3 > \lfloor \frac{j-5}{2} \rfloor$ for $j \geq 5$, leading to a contradiction.

Both subcases lead to contradictions; therefore, the assumption that no $(k-1)$-vertex exists in $V(C)$ cannot be upheld.

Case 2.4. *A $(k-1)$-vertex in $V(C)$ exists.*

Given the existence of a $(k-1)$-vertex c in $V(C)$, we may assume, without loss of generality, that $V(C) \setminus \{u\} = \{u_1, u_2, \ldots, u_{j-2}, c\}$. We distinguish between two subcases based on the adjacency relations between the vertices $u_1, \ldots, u_{j-2}$ and the vertices u_{k-3} and u_{k-2}.

Subcase 2.4.1. *Each vertex in $\{u_1, u_2, \ldots, u_{j-2}\}$ is adjacent to both u_{k-3} and u_{k-2}.*

Claim 2.2. *In this subcase, $\phi(N_{G\setminus u}[c]) = [k-1]$.*

Subproof. To establish a contradiction, we assume that $\phi(N_{G\setminus u}[c]) \neq [k-1]$. Without loss of generality, we can let c be the vertex y (thus, $\phi(y) = k-1$ and $\phi(N_{G\setminus u}[y]) \neq [k-1]$). We assert that there is a $(k-1)$-vertex c^i_{k-1} in $N_{G\setminus u}(u_i)$ for any $1 \leq i \leq j-2$ such that $\phi(N_{G\setminus u}[c^i_{k-1}]) = [k-1]$, and any two vertices in $\{c^1_{k-1}, c^2_{k-1}, \ldots, c^{j-2}_{k-1}\}$ may be identical. If this is not the case, we can assert, without loss of generality, that there is no $(k-1)$-vertex c^1_{k-1} in $N_{G\setminus u}(u_1)$. By Lemma 2.2(2), there can be at most two $(k-1)$-vertices in $N_{G\setminus u}(u_1)$. If there is exactly one $(k-1)$-vertex y in $N_{G\setminus u}(u_1)$ and no r-vertex in $N_{G\setminus u}(y)$ for some $r \in [k-2]$, we can recolor u_1 with $k-1$, y with r, and u with 1, thereby achieving a proper $(k-1)$-coloring of G, which leads to a contradiction. If there are two $(k-1)$-vertices c_{k-1} and y in $N_{G\setminus u}(u_1)$, and there are the colors $r', r \in [k-2]$ (possibly $r' = r$) such that there is no r'-vertex in $N_{G\setminus u}(c_{k-1})$ and no r-vertex in $N_{G\setminus u}(y)$, we can recolor u with 1, u_1 with $k-1$, c_{k-1} with r', and y with r, again yielding a proper $(k-1)$-coloring of G, which results in a contradiction. Therefore, such a $(k-1)$-vertex c^i_{k-1} must exist; consequently, each vertex in $\{u_1, u_2, \ldots, u_{j-2}\}$ must have a repeated color $k-1$.

We next demonstrate that the vertex x cannot be the $(k-1)$-vertex c^i_{k-1} in $N_{G\setminus u}(u_i)$ for any $1 \leq i \leq j-2$. Without loss of generality, we assume the contrary, namely that x is the $(k-1)$-vertex c^1_{k-1} in $N_{G\setminus u}(u_1)$. Consequently, u_1 has a repeated color $k-1$. We assert that any vertex $s \in \{u_2, \ldots, u_{j-2}\}$ must be adjacent to x. If $s \nsim x$, since $\phi(N_{G\setminus u}[x]) = [k-1]$, there is a $\phi(s)$-vertex c_s in $N_{G\setminus u}(x)$. The conditions that $A_0 = A_1 = \emptyset$, $d_G(u) = k$, and u_1 has a repeated color $k-1$ necessitate that c_s is adjacent to at least two vertices in $\{u_2, \ldots, u_{j-2}, y\} \setminus \{s\}$, and in particular, to at least one vertex in $\{u_2, \ldots, u_{j-2}\} \setminus \{s\}$. Therefore, there must be at least one vertex in $\{u_2, \ldots, u_{j-2}\} \setminus \{s\}$ that has two repeated colors, $k-1$ and $\phi(s)$, which contradicts Lemma 2.2(2). Hence, each vertex $s \in \{u_2, \ldots, u_{j-2}\}$ is adjacent to x and, consequently, $G[\{u, u_1, \ldots, u_{j-2}, x\}]$ forms a clique of size j. Note that both $G[\{u, u_1, \ldots, u_{j-2}, x\}]$ and $G[\{u, u_1, \ldots, u_{j-2}, u_{k-2}\}]$ are cliques of size j; consequently, $x \nsim u_{k-2}$. By Lemma 2.2(2), there is a $(k-2)$-vertex c_{k-2} in $N_{G\setminus u}(x)$. Furthermore, the conditions $A_0 = A_1 = \emptyset$ and $d_G(u) = k$ imply that c_{k-2} is adjacent to at least two vertices in $\{u_1, \ldots, u_{j-2}, y\}$ and, in particular, to at least one vertex in $\{u_1, \ldots, u_{j-2}\}$. Therefore, there must be at least one vertex in $\{u_1, \ldots, u_{j-2}\}$ that has two repeated colors,

$k-1$ and $k-2$, which contradicts Lemma 2.2(2). Thus, x cannot be the $(k-1)$-vertex c^i_{k-1} in $N_{G\setminus u}(u_i)$ for any $1 \leq i \leq j-2$.

Given that $A_0 = A_1 = \emptyset$ and $d_G(u) = k$, the vertex x must be adjacent to at least one vertex in $\{u_1, \ldots, u_{j-2}\}$. This implies the existence of a vertex u_i in $\{u_1, \ldots, u_{j-2}\}$ such that $1 \leq i \leq j-2$ that is adjacent to at least three distinct $(k-1)$-vertices, namely x, y, and c^i_{k-1}, which contradicts Lemma 2.2(2). Therefore, our initial supposition is false, leading us to conclude that $\phi(N_{G\setminus u}[c]) = [k-1]$.

Since both $G[\{u, u_1, \ldots, u_{j-2}, c\}]$ and $G[\{u, u_1, \ldots, u_{j-2}, u_{k-2}\}]$ are cliques of size j, it follows that $u_{k-2} \nsim c$. Similarly, there must be a $(k-1)$-vertex c_{k-1} in $N_{G\setminus u}(u_{k-2})$ with $\phi(N_{G\setminus u}[c_{k-1}]) = [k-1]$.

Claim 2.3. *Each vertex in $\{u_1, \ldots, u_{j-2}\}$ has a repeated color distinct from $k-2$.*

Subproof. Consider any vertex $s \in \{u_1, \ldots, u_{j-2}\}$. If $s \sim c_{k-1}$, then s has a repeated color $k-1$. Conversely, if $s \nsim c_{k-1}$, since $\phi(N_{G\setminus u}[c_{k-1}]) = [k-1]$, there is a $\phi(s)$-vertex c_s in $N_{G\setminus u}(c_{k-1})$. The conditions $A_0 = A_1 = \emptyset$ and $d_G(u) = k$ necessitate that c_s is adjacent to at least two vertices in $\{u_1, \ldots, u_{j-2}, c\} \setminus \{s\}$ and, in particular, to at least one vertex in $\{u_1, \ldots, u_{j-2}\} \setminus \{s\}$. Consequently, there must be at least one vertex in $\{u_1, \ldots, u_{j-2}\} \setminus \{s\}$ that has a repeated color $\phi(s)$. By considering all vertices in $\{u_1, u_2, ..., u_{j-2}\}$, it is evident that if the graph G exists, then each vertex in $\{u_1, \ldots, u_{j-2}\}$ has a repeated color distinct from $k-2$.

Finally, since $c \nsim u_{k-2}$ and $\phi(N_{G\setminus u}[c]) = [k-1]$, Lemma 2.2(2) guarantees the existence of a $(k-2)$-vertex c_{k-2} in $N_{G\setminus u}(c)$. Furthermore, the conditions $A_0 = A_1 = \emptyset$ and $d_G(u) = k$ imply that c_{k-2} is adjacent to at least two vertices in $\{u_1, \ldots, u_{j-2}, c\}$. Consequently, by Claim 2.3, there must be some vertex in $\{u_1, \ldots, u_{j-2}\}$ that has two repeated colors, which contradicts Lemma 2.2(2). This completes the proof of this subcase.

Subcase 2.4.2. *There is a vertex in $\{u_1, u_2, \ldots, u_{j-2}\}$ that is not adjacent to u_{k-3} or u_{k-2}.*

Without loss of generality, we assume that $u_1 \nsim u_{k-3}$. By Lemma 2.2(2), a 1-vertex c_1 in $N_{G\setminus u}(u_{k-3})$ exists. Given that $A_0 = A_1 = \emptyset$ and $d_G(u) = k$, the vertex c_1 must be adjacent to at least two vertices in the set $\{u_1, \ldots, u_{j-2}, c\}$. Since $c_1 \nsim u_1$, if the graph exists, it follows that $j \geq 4$, and c_1 must be adjacent to at least one vertex in $\{u_2, \ldots, u_{j-2}\}$. Consequently, there must be at least one vertex in $\{u_2, \ldots, u_{j-2}\}$ that has a repeated color 1.

Claim 2.4. *Each vertex in $\{u_1, \ldots, u_{j-2}\}$ has a repeated color distinct from $k-1$.*

Subproof. We consider any vertex $s \in \{u_2, \ldots, u_{j-2}\}$ and distinguish two situations based on whether s is adjacent to u_{k-3}.

First, if $s \sim u_{k-3}$, given that $u_1 \nsim u_{k-3}$, $u_1 \sim s$, and $s \sim u_{k-3}$, Lemma 2.2(5.1) guarantees the existence of a $\phi(s)$-vertex c_s that is distinct from s in either $N_{G\setminus u}(u_1)$ or $N_{G\setminus u}(u_{k-3})$. If $c_s \in N_{G\setminus u}(u_{k-3})$, the conditions $A_0 = A_1 = \emptyset$ and $d_G(u) = k$ necessitate that c_s is adjacent to at least two vertices in the set $\{u_1, \ldots, u_{j-2}, c\} \setminus \{s\}$ and, in particular, to at least one vertex in $\{u_1, \ldots, u_{j-2}\} \setminus \{s\}$. Consequently, there must be at least one vertex in $\{u_1, \ldots, u_{j-2}\} \setminus \{s\}$ that has a repeated color $\phi(s)$. If $s \in N_G(u_1)$, then u_1 itself has a repeated color $\phi(s)$. By considering all vertices in $\{u_2, \ldots, u_{j-2}\}$, it is evident that if the graph G exists, then each vertex in $\{u_1, \ldots, u_{j-2}\}$ has a repeated color distinct from $k-1$.

Second, if $s \nsim u_{k-3}$, then Lemma 2.2(2) guarantees the existence of a $\phi(s)$-vertex c_s in $N_{G\setminus u}(u_{k-3})$. Furthermore, the conditions $A_0 = A_1 = \emptyset$ and $d_G(u) = k$ necessitate that c_s is adjacent to at least one vertex in $\{u_1, \ldots, u_{j-2}\} \setminus \{s\}$. Consequently, there must be at least one vertex in $\{u_1, \ldots, u_{j-2}\} \setminus \{s\}$ that has a repeated color $\phi(s)$. By considering all vertices in $\{u_2, ..., u_{j-2}\}$, it is evident that if the graph G exists, each vertex in $\{u_1, \ldots, u_{j-2}\}$ has a repeated color distinct from $k-1$.

By considering all possibilities, it is evident that each vertex in $\{u_1, \ldots, u_{j-2}\}$ has a repeated color distinct from $k-1$.

Consider another $(k-1)$-vertex $c' \in N_G(u)$. Given that $A_0 = A_1 = \emptyset$ and $d_G(u) = k$, the vertex c' must be adjacent to at least one vertex in the set $\{u_1, \ldots, u_{j-2}\}$. Consequently, there must be at least one vertex in $\{u_1, \ldots, u_{j-2}\}$ that has two repeated colors, which contradicts Lemma 2.2(2). This completes the proof of this subcase.

After thoroughly examining all cases, we conclude that, under the assumptions of Lemma 2.4, it follows that $\omega(G) = j = k-1$. This completes the proof of Lemma 2.4(1).

We now proceed to prove Lemma 2.4(2). For the sake of contradiction, assume that there is a proper $(k-1)$-coloring ϕ of $G \setminus u$ such that the vertex set of the clique C is not $\{u, u_1, \ldots, u_{k-2}\}$. Without loss of generality, we can assume that $V(C) = \{u, u_1, \ldots, u_{k-3}, c\}$, where $c \in \{x, y\}$. We will again consider two possibilities.

First, we assume that every vertex in the set $\{u_1, u_2, \ldots, u_{k-3}\}$ is adjacent to u_{k-2}. Under this assumption, the reasoning applied in Subcase 2.4.1 can be utilized to demonstrate that at least one vertex in the set $\{u_1, \ldots, u_{k-3}\}$ has two repeated colors, which contradicts Lemma 2.2(2).

Second, we assume that there is a vertex in the set $\{u_1, u_2, \ldots, u_{k-3}\}$ that is not adjacent to u_{k-2}. In this situation, the reasoning applied in Subcase 2.4.2 can be utilized to demonstrate that at least one vertex in the set $\{u_1, \ldots, u_{k-3}\}$ has two repeated colors, leading to a contradiction.

Therefore, the vertex set of C must be exactly $\{u, u_1, \ldots, u_{k-2}\}$. This completes the proof of Lemma 2.4(2). □

By synthesizing the structural information obtained in Lemmas 2.3 and 2.4, we are now in a position to derive a unified and general sufficient condition for a graph to be $(\Delta(G)-1)$-colorable. Lemma 2.5 asserts that if the "lower" part of the vertex partition, namely all sets A_i with $i \leq \lfloor \frac{j}{3} \rfloor$, are empty, then the graph cannot serve as a counterexample to the conjecture. This lemma constitutes the core technical ingredient in the proof of our main theorem.

Lemma 2.5. *Let G be a graph with $\Delta(G) = k \geq 3$ and $\omega(G) = j \leq \Delta(G) - 1$. If G contains a clique C of size j such that the vertex partition $A_0, A_1, \ldots, A_{j-1}$ of $V(G) \setminus V(C)$ satisfies $A_i = \emptyset$ for every $i \leq \lfloor \frac{j}{3} \rfloor$, then $\chi(G) \leq \Delta(G) - 1$.*

Proof of Lemma 2.5. We prove Lemma 2.5 by contradiction. Let G be a counterexample to Lemma 2.5 with $|G|$ minimal. For this graph G, we have $\Delta(G) = k \geq 3$ and $\omega(G) = j \leq \Delta(G) - 1$. Moreover, there is a clique C in G with $|C| = \omega(G)$ such that the vertex partition $A_0, A_1, \ldots, A_{j-1}$ of $V(G) \setminus V(C)$ satisfies $A_0 = A_1 = \cdots = A_{\lfloor \frac{j}{3} \rfloor} = \emptyset$. Since G is a counterexample graph, we have $\chi(G) \geq \Delta(G)$. Given that $\omega(G) \leq \Delta(G) - 1$, Theorem 1.1 implies $\chi(G) = \Delta(G) = k$. The minimality of $|G|$ ensures that G is connected and k-vertex-critical. We will derive a contradiction by considering two cases, depending on whether the clique C contains a vertex with degree k.

Case 2.5. *There is no vertex with degree k in C.*

We will examine the cases for $k = 3$ and $k \geq 4$ separately.

Subcase 2.5.1. $k = 3$.

For the graph G, we see that G is a connected 3-vertex-critical graph with $\omega(G) = j = 2$. Since the clique C satisfies $|C| = j = 2$, it follows that $A_0 = \emptyset$. Given that there is no vertex of degree 3 in C and $\Delta(G) = 3$, a vertex $u \in V(G) \setminus V(C)$ such that $d_G(u) = 3$ exists. Since $\omega(G) = 2$ and $A_0 = \emptyset$, it follows that $|N_G(u) \cap V(C)| = 1$. Without loss of generality, assume $V(C) = \{v_1, v_2\}$ and $N_G(u) \cap V(C) = \{v_1\}$. Given that $\omega(G) = 2$, the remaining two vertices in $N_G(u)$ must be adjacent to v_2, which implies that $d_G(v_2) = 3$, contradicting the assumption that there is no vertex of degree 3 in C.

Subcase 2.5.2. $k \geq 4$.

We now demonstrate that the graph G does not exist when $k \geq 4$. Based on the proof for $k = 3$, it is evident that if $k \geq 4$ and $j \neq 2$, then the graph G cannot exist. When $k \geq 4$ and $j \geq 3$, for the clique C, we have the sets $A_0 = A_1 = \emptyset$, which contradicts Lemma 2.3.

Case 2.6. *There is a vertex u with degree k in C.*

We also examine the cases for $k = 3$ and $k \geq 4$ separately.

Subcase 2.6.1. $k = 3$.

For the graph G, we have that G is a connected 3-vertex-critical graph with $\omega(G) = j = 2$. Given that there is a vertex u with degree 3 in C and $\omega(G) = 2$, for every proper 2-coloring ϕ of $G \setminus u$, it follows that $u_1 \nsim x$ and $u_1 \nsim y$. According to Lemma 2.2(2) and $\phi(N_{G\setminus u}[x]) = [2]$, there is a $\phi(x)$-vertex c_x in $N_{G\setminus u}(u_1)$ and a 1-vertex c_1 in $N_{G\setminus u}(x)$.

First, we assume that $V(C) = \{u_1, u\}$. Since $A_0 = \emptyset$ and $d_G(u) = 3$, the vertex c_1 must be adjacent to at least one vertex in C. Given that $c_1 \nsim u$, we conclude that $c_1 \sim u_1$, which is a contradiction.

Next, we assume that $V(C) = \{x, u\}$. Since $A_0 = \emptyset$ and $d_G(u) = 3$, the vertex c_x must be adjacent to at least one vertex in C. Given that $c_x \nsim u$, we conclude that $c_x \sim x$, which is a contradiction.

Finally, we assume that $V(C) = \{y, u\}$. Since $A_0 = \emptyset$ and $d_G(u) = 3$, the vertex c_x must be adjacent to at least one vertex in C. Given that $c_x \nsim u$, we conclude that $c_x \sim y$, which is a contradiction.

Subcase 2.6.2. $k \geq 4$.

We next demonstrate that the graph G does not exist when $k \geq 4$. Based on the proof for $k = 3$, it is clear that when $k \geq 4$ and $j \neq 2$, the graph G cannot exist. When $k \geq 4$ and $j \geq 3$, for the clique C, we have the sets $A_0 = A_1 = \emptyset$. Furthermore, we consider any proper $(k-1)$-coloring ϕ of $G \setminus u$; Lemma 2.4 guarantees that $j = k - 1$ and $V(C) = \{u, u_1, \ldots, u_{k-2}\}$.

Next, we examine the adjacency between x and the vertices in $V(C) \setminus \{u\} = \{u_1, \ldots, u_{k-2}\}$.

Claim 2.5. *The vertex x is not adjacent to at most two vertices in $\{u_1, \ldots, u_{k-2}\}$.*

Subproof. Let S be the set of vertices in $\{u_1, \ldots, u_{k-2}\}$ that are not adjacent to x, and let $m = |S|$. For each $s \in S$, since $s \nsim x$ and $\phi(N_{G\setminus u}[x]) = [k-1]$, a $\phi(s)$-vertex c_s in $N_{G\setminus u}(x)$ exists. The conditions $A_0 = A_1 = \cdots = A_{\lfloor \frac{j}{3} \rfloor} = \emptyset$ and $d_G(u) = k$ necessitate that c_s is adjacent to at least $1 + \lfloor \frac{j}{3} \rfloor$ vertices in $\{u_1, \ldots, u_{k-2}\}$. Moreover, Lemma 2.2(2) implies each vertex in $\{u_1, \ldots, u_{k-2}\}$ can have at most one repeated color. Consequently, we obtain the inequality

$$m \leq \frac{j-1}{1 + \lfloor \frac{j}{3} \rfloor} < 3.$$

Given that $\omega(G) = j$ and C is a clique of size j, vertex x must not be adjacent to at least one vertex in $\{u_1, \dots, u_{k-2}\}$. Moreover, Claim 2.5 indicates that x is not adjacent to either one or two vertices in $\{u_1, \dots, u_{k-2}\}$. We also divide our proof into two situations, depending on the number of vertices that x is not adjacent to in $\{u_1, \dots, u_{k-2}\}$.

Situation 1. *x is not adjacent to two vertices in* $\{u_1, \dots, u_{k-2}\}$.

Without loss of generality, we assume that x is not adjacent to u_1 and u_2. Since $\phi(N_{G\setminus u}[x]) = [k-1]$, a 1-vertex c_1 and a 2-vertex c_2 in $N_{G\setminus u}(x)$ exist. Given that $A_0 = A_1 = \emptyset$ and $d_G(u) = k$, the vertices c_1 and c_2 must each be adjacent to at least two vertices in $\{u_1, u_2, \dots, u_{k-2}\}$. Moreover, Lemma 2.2(2) implies that each vertex in $\{u_1, u_2, \dots, u_{k-2}\}$ can have at most one repeated color. Therefore, if the graph G exists, then $j = k - 1 \geq 5$. Since u_1 is not adjacent to x, Lemma 2.2(2) guarantees the existence of a $(k-1)$-vertex c_{k-1} in $N_{G\setminus u}(u_1)$ such that $\phi(N_{G\setminus u}[c_{k-1}]) = [k-1]$.

Claim 2.6. *c_{k-1} is adjacent to any vertex in* $\{u_3, u_4, \dots, u_{k-2}\}$.

Subproof. Suppose that a vertex $s \in \{u_3, \dots, u_{k-2}\}$ exists such that s is not adjacent to c_{k-1}. Since $\phi(N_{G\setminus u}[c_{k-1}]) = [k-1]$, there must be a $\phi(s)$-vertex c_s in $N_{G\setminus u}(c_{k-1})$. Given that the vertices c_1, c_2, and c_s must each be adjacent to at least $1+\lfloor \frac{j}{3} \rfloor$ vertices in $\{u_1, \dots, u_{k-2}\}$, and that Lemma 2.2(2) indicates any vertex in $\{u_1, u_2, \dots, u_{k-2}\}$ can have at most one repeated color, it follows that $3(\lfloor \frac{j}{3} \rfloor + 1) = 3(\lfloor \frac{k-1}{3} \rfloor + 1) > k-2$ for $k \geq 6$, leading to a contradiction. Consequently, c_{k-1} is adjacent to each vertex in $\{u_3, \dots, u_{k-2}\}$, and each vertex in $\{u_3, \dots, u_{k-2}\}$ has a repeated color $k-1$.

Since $A_0 = A_1 = \emptyset$, $c_1 \nsim u_1$, and $d_G(u) = k$, the vertex c_1 must be adjacent to at least two vertices in $\{u_2, \dots, u_{k-2}\}$ and, in particular, to at least one vertex in $\{u_3, \dots, u_{k-2}\}$. Consequently, there must be at least one vertex in $\{u_3, \dots, u_{k-2}\}$ that has two repeated colors, $k-1$ and 1, contradicting Lemma 2.2(2).

Situation 2. *x is not adjacent to exactly one vertex in* $\{u_1, \dots, u_{k-2}\}$.

Without loss of generality, we assume that x is not adjacent to u_1. Given that $\phi(N_{G\setminus u}[x]) = [k-1]$, a 1-vertex c_1 in $N_{G\setminus u}(x)$ exists. Since $A_0 = A_1 = \emptyset$, $c_1 \nsim u_1$, and $d_G(u) = k$, the vertex c_1 must be adjacent to at least two vertices in the set $\{u_2, \dots, u_{k-2}\}$. This implies that if the graph G exists, then $j = k-1 \geq 4$. Furthermore, since x is not adjacent to u_1, by Lemma 2.2(2), there must be a $(k-1)$-vertex c_{k-1} in $N_{G\setminus u}(u_1)$ such that $\phi(N_{G\setminus u}[c_{k-1}]) = [k-1]$.

It is evident that c_{k-1} is not adjacent to at most one vertex in the set $\{u_2, \dots, u_{k-2}\}$.

When c_{k-1} is adjacent to each vertex in $\{u_2, \dots, u_{k-2}\}$, it follows that each vertex in this set has a repeated color $(k-1)$. Given that $A_0 = A_1 = \emptyset$ and $d_G(u) = k$, vertex c_1 must be adjacent to at least two vertices in $\{u_1, \dots, u_{k-2}\}$. Consequently, there must be at least one vertex in $\{u_2, \dots, u_{k-2}\}$ that has two repeated colors 1 and $(k-1)$, which contradicts Lemma 2.2(2).

When c_{k-1} is not adjacent to one vertex in $\{u_2, \dots, u_{k-2}\}$, we can assume, without loss of generality, that c_{k-1} is not adjacent to u_2. Since $\phi(N_{G\setminus u}[c_{k-1}]) = [k-1]$, a 2-vertex c_2 in $N_{G\setminus u}(c_{k-1})$ exists. Given that each vertex in the set $\{u_3, \dots, u_{k-2}\}$ has a repeated color $(k-1)$, and considering the conditions $A_0 = A_1 = \emptyset$, $c_1 \nsim u_1$ and $d_G(u) = k$, the vertex c_1 must be adjacent to at least two vertices in $\{u_2, \dots, u_{k-2}\}$. Consequently, there must be at least one vertex in $\{u_3, \dots, u_{k-2}\}$ that has two repeated colors 1 and $(k-1)$, which contradicts Lemma 2.2(2).

Both situations lead to a contradiction; therefore, this completes the proof of this subcase.

Having derived contradictions in all cases, we conclude that no counterexample graph G exists. Therefore, Lemma 2.5 holds. □

3. Proof of Theorem 1.2

Proof of Theorem 1.2. Let $t \geq 3$ be an integer, and let C denote the class of all $\overline{K_{1,t}}$-free graphs. For any graph $G \in C$, it follows that $\Delta(G) \geq 2t+1$ and $\omega(G) \leq \Delta(G) - 1$. According to Lemma 2.1, to prove Theorem 1.2, it is sufficient to demonstrate that $C \cap \mathcal{G}_{2t+1} = \emptyset$.

Suppose, to the contrary, that $C \cap \mathcal{G}_{2t+1} \neq \emptyset$. Let $G \in C \cap \mathcal{G}_{2t+1}$ be a counterexample graph with $|G|$ minimal. For the graph G, we have $\Delta(G) = 2t+1$, and the minimal $|G|$ guarantees that G is connected and $(2t+1)$-vertex-critical. Furthermore, Lemma 1.1 indicates that $t+1 \leq \omega(G) = j \leq 2t$.

We aim to demonstrate that such a graph G does not exist. We choose any clique C in G such that $|C| = \omega(G) = j$ and partition the vertex set $V(G) \setminus V(C)$ into sets $A_0, A_1, \ldots, A_{j-1}$.

Claim 3.1. *The sets A_0 and A_1 are both $\emptyset$.*

Subproof. We prove Claim 3.1 by contradiction. We begin by assuming that the set A_0 is not empty and there is a vertex $u \in A_0$. Given that $j \geq t+1 > t$, there are t vertices $v_1, v_2, \ldots, v_t$ in $V(C)$ that are not adjacent to u. Consequently, $G[\{u, v_1, v_2, \ldots, v_t\}]$ is a $\overline{K_{1,t}}$, which is a contradiction.

Next, we assume that the set A_1 is not empty and that a vertex $u \in A_1$ exists. Given that $j \geq t+1 > t$, there must be t vertices $v_1, v_2, \ldots, v_t$ in $V(C)$ that are not adjacent to u. Consequently, $G[\{u, v_1, v_2, \ldots, v_t\}]$ is a $\overline{K_{1,t}}$, which leads to a contradiction.

Thus, the sets A_0 and A_1 are both $\emptyset$.

Then, for the graph G, we see that G is a connected $(2t+1)$-vertex-critical graph with $\Delta(G) = 2t+1$ and $t+1 \leq \omega(G) = j \leq 2t$, where $t \geq 3$. Furthermore, in the graph G, there is a clique C with $|C| = \omega(G)$ for which the vertex partition $A_0, A_1, \ldots, A_{j-1}$ of $V(G) \setminus V(C)$ satisfies $A_0 = A_1 = \emptyset$. If there is no vertex with degree $2t+1$ in C, then, according to Lemma 2.3, for any vertex $u \in V(G)$ with degree $2t+1$, it follows that $|N_G(u) \cap V(C)| = 1$. This contradicts the condition $A_1 = \emptyset$. Consequently, there must be a vertex with degree $2t+1$ in C. Thus, according to Lemma 2.4(1), we conclude that $\omega(G) = j = 2t$.

Claim 3.2. *The sets $A_0 = A_1 = \cdots = A_{\lfloor \frac{j}{3} \rfloor} = \emptyset$.*

Subproof. We prove Claim 3.2 by contradiction. We assume that $A_k \neq \emptyset$ for some $0 \leq k \leq \lfloor \frac{j}{3} \rfloor$ and that a vertex $u \in A_k$ exists. Since $2t - k > t$ for any $0 \leq k \leq \lfloor \frac{j}{3} \rfloor$, where $j = 2t$ and $t \geq 3$, there must be t vertices $v_1, v_2, \ldots, v_t$ in C that are not adjacent to u. Consequently, $G[\{u, v_1, v_2, \ldots, v_t\}]$ forms a $\overline{K_{1,t}}$, leading to a contradiction.

Consequently, for the clique C in G, there must be a vertex with degree $2t+1$ in C, and the sets $A_0 = A_1 = \cdots = A_{\lfloor \frac{j}{3} \rfloor} = \emptyset$. According to Lemma 2.5, we conclude that $\chi(G) \leq \Delta(G) - 1 = 2t$, which contradicts $\chi(G) = \Delta(G) = 2t+1$.

In summary, we have demonstrated that the counterexample graph G does not exist, thereby completing the proof of Theorem 1.2. □

4. Conclusions

In this paper, we introduced a structural approach to the Borodin-Kostochka conjecture based on partitioning the vertices of a graph relative to a fixed maximum clique. By analyzing the properties of the sets A_i in potential k-vertex-critical counterexamples, we derived a series of lemmas (Lemmas 2.3–2.5) that provide sufficient conditions for a graph to satisfy $\chi(G) \leq \Delta(G) - 1$.

The applicability of this method was demonstrated by verifying the conjecture for the class of $\overline{K_{1,t}}$-free graphs. We proved that for any integer $t \geq 3$, every $\overline{K_{1,t}}$-free graph G with $\Delta(G) \geq 2t + 1$ and $\omega(G) \leq \Delta(G) - 1$ satisfies $\chi(G) \leq \Delta(G) - 1$. This result not only generalizes the previous work of Lan and Lin [16] from $t = 3$ to all $t \geq 3$, but also establishes a degree condition that grows linearly with the parameter t.

Our structural lemmas provide a versatile framework for tackling the conjecture in other hereditary graph classes. In the future, it would be interesting to apply this partition method to classes defined by forbidding more complex subgraphs, such as the unions of paths or graphs with small cliques. Another potential direction is to refine the analysis of the sets A_i to determine whether the condition $A_i = \emptyset$ for $i \leq \lfloor \frac{j}{3} \rfloor$ in Lemma 2.5 can be weakened or further exploited. We believe that this vertex partitioning perspective may contribute to a deeper understanding of the structural constraints that force a graph to be $(\Delta(G) - 1)$-colorable, and ultimately to a full resolution of the Borodin-Kostochka conjecture.

Use of AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares that he has no conflict of interest.

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