AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(3): 8467–8491.
DOI: 10.3934/math.2026348
Received: 20 January 2026
Revised: 05 March 2026
Accepted: 19 March 2026
Published: 30 March 2026

Research article

Generalized Hermite-Hadamard and Ostrowski inequalities involving tempered fractional integrals

Abdelghani Lakhdari[1,2,3], **Thabet Abdeljawad**[4,5,6,7,*], **Manar A. Alqudah**[8], **Nabil Mlaiki**[4]

[1] Department CPST, National Higher School of Technology and Engineering, Annaba 23005, Algeria
[2] Department of Mathematics, Faculty of Science and Arts, Kocaeli University, Kocaeli, Türkiye
[3] Department of Mathematics, Saveetha School of Engineering, SIMATS, Saveetha University, Chennai 602105, Tamil Nadu, India
[4] Department of Mathematics and Sciences, Prince Sultan University, P.O. Box 66833, Riyadh 11586, Saudi Arabia
[5] Department of Fundamental Sciences, Faculty of Engineering and Architecture, Istanbul Gelisim University, Avcılar-Istanbul, 34310, Türkiye
[6] Department of Medical Research, China Medical University, Taichung 40402, Taiwan
[7] Department of Mathematics, Kyung Hee University, 26 Kyungheedae-ro, Dongdaemun-gu, 02447 Seoul, Korea
[8] Department of Mathematical Sciences, College of Science, Princess Nourah bint Abdulrahman University, P.O. Box 84424, Riyadh 11671, Saudi Arabia

* **Correspondence:** Email: tabdeljawad@gelisim.edu.tr, tabdeljawad@psu.edu.sa.

Abstract: In this work, we explored the use of tempered fractional integrals in the development of novel inequalities for differentiable functions that satisfy the s-convexity condition. Drawing upon developments in fractional calculus, we extended several classical results to this generalized setting. We first derived a new version of the Hermite-Hadamard inequality tailored to tempered fractional integrals. Subsequently, we introduced a new integral identity, which serves as a fundamental tool for deriving Ostrowski-type inequalities within the same framework. These contributions enhance the theoretical understanding of fractional calculus and highlight its relevance in modern mathematical analysis. A selection of examples and corollaries is also presented to illustrate the applicability and impact of the proposed results.

Keywords: Hermite-Hadamard inequality; Ostrowski's inequality; tempered fractional integrals; s-convex functions
Mathematics Subject Classification: 25A51, 25D10, 25D15

1. Background on classical integral inequalities

Convexity is a fundamental concept that significantly enhances the derivation of error bounds in quadrature formulas, offering a structured approach to refining numerical integration accuracy. Its inherent properties, such as predictable behavior under interpolation and the ability to define precise curvature, enable the formulation of sharper error estimates. These estimates are critical in numerical analysis, where the effectiveness of quadrature rules hinges on minimizing approximation errors. By integrating convexity into the analysis, researchers can exploit optimization techniques and functional inequalities to design algorithms with improved stability and convergence. Consequently, convexity not only strengthens the theoretical underpinnings of quadrature methods but also drives innovations in their practical implementation, ensuring more reliable computational outcomes.

Definition 1.1. *[1] A function $\mathscr{V} : \mathscr{D} \to \mathbb{R}$ is expressed as convex, if the inequality*

$$\mathscr{V}\left(\ell\mathfrak{y}_1 + (1-\ell)\,\mathfrak{y}_2\right) \leq \ell\mathscr{V}\left(\mathfrak{y}_1\right) + (1-\ell)\,\mathscr{V}(\mathfrak{y}_2)$$

holds for all $\ell \in [0,1]$ and $\mathfrak{y}_1, \mathfrak{y}_2 \in \mathscr{D}$.

Among the fundamental results related to the concept of convexity, one of the most notable is the Hermite-Hadamard inequality, which plays a pivotal role in theory and applications. The inequality in question characterizes the convexity of a function. Specifically, if $\mathscr{V}$ is a convex function defined on the interval $[\mathfrak{c}_1, \mathfrak{c}_2]$, then the following holds:

$$\mathscr{V}\left(\frac{\mathfrak{c}_1 + \mathfrak{c}_2}{2}\right) \leq \frac{1}{\mathfrak{c}_2 - \mathfrak{c}_1}\int\limits_{\mathfrak{c}_1}^{\mathfrak{c}_2} \mathscr{V}(\mathfrak{l})\, d\mathfrak{l} \leq \frac{\mathscr{V}(\mathfrak{c}_1) + \mathscr{V}(\mathfrak{c}_2)}{2}.$$

We refer interested readers to [2,3] for results involving Hermite-Hadamard inequalities and related literature.

Several generalizations of the concept of convexity have been introduced with the aim of broadening the class of functions that exhibit properties similar to those of convex functions. Among these extensions, we highlight the notion of s-convexity in the second sense, defined by Hudzik and Maligranda [4] in the following manner:

Definition 1.2. *[4] A function $\mathscr{V} : \mathscr{D} \subset [0,\infty) \to \mathbb{R}^+$ is called s-convex in the second sense, for a fixed $s \in (0,1]$, if the inequality*

$$\mathscr{V}(\ell\mathfrak{y}_1 + (1-\ell)\mathfrak{y}_2) \ \leq \ell^s\mathscr{V}(\mathfrak{y}_1) + (1-\ell)^s\mathscr{V}(\mathfrak{y}_2)$$

holds for all $\mathfrak{y}_1, \mathfrak{y}_2 \in \mathscr{D}$ and $\ell \in [0,1]$.

Hermite-Hadamard inequality for s-convex functions was provided by Dragomir and Fitzpatrick [5] in the following manner:

$$2^{s-1}\mathscr{V}\left(\frac{\mathfrak{c}_1 + \mathfrak{c}_2}{2}\right) \leq \frac{1}{\mathfrak{c}_2 - \mathfrak{c}_1}\int\limits_{\mathfrak{c}_1}^{\mathfrak{c}_2} \mathscr{V}(\mathfrak{l})\, d\mathfrak{l} \leq \frac{\mathscr{V}(\mathfrak{c}_1) + \mathscr{V}(\mathfrak{c}_2)}{s+1}.$$

Among the integral inequalities, one of the most celebrated is the Ostrowski inequality, which is stated as follows:

Theorem 1.1. *[6] Let* $\mathscr{V} : [\mathfrak{c}_1, \mathfrak{c}_2] \to \mathbb{R}$ *be a mapping that is differentiable on* $(\mathfrak{c}_1, \mathfrak{c}_2)$ *with* $\mathscr{V}' \in L([\mathfrak{c}_1, \mathfrak{c}_2])$*. Then, for all* $\varkappa \in [\mathfrak{c}_1, \mathfrak{c}_2]$*, we have*

$$\left| \mathscr{V}(\varkappa) - \frac{1}{\mathfrak{c}_2 - \mathfrak{c}_1} \int\limits_{\mathfrak{c}_1}^{\mathfrak{c}_2} \mathscr{V}(\mathfrak{l})\, d\mathfrak{l} \right| \leq (\mathfrak{c}_2 - \mathfrak{c}_1) \left[\frac{1}{4} + \frac{\left(\varkappa - \frac{\mathfrak{c}_1 + \mathfrak{c}_2}{2}\right)^2}{(\mathfrak{c}_2 - \mathfrak{c}_1)^2} \right] \|\mathscr{V}'\|_\infty,$$

where $\|\mathscr{V}'\|_\infty = \sup\limits_{t \in [\mathfrak{c}_1, \mathfrak{c}_2]} |\mathscr{V}'(t)|$.

This inequality constitutes a foundational result in mathematical analysis, establishing an upper bound on the deviation of a function from its integral mean over a specified interval. Since its introduction by Alexander Ostrowski in 1938, this inequality has become a cornerstone in approximation theory, numerical integration, and probability theory, offering valuable insights into the behavior of functions and their integrals. Its significance lies in its ability to quantify the error in approximating a function's integral using its values at specific points, making it particularly useful in the development of quadrature rules and error estimations. Over the years, researchers have devoted considerable attention to generalizing and extending the Ostrowski inequality to various settings, including higher dimensions, fractional calculus, and time scales. These extensions not only enrich the theoretical framework of mathematical inequalities but also broaden their applicability to diverse fields such as optimization, signal processing, and stochastic processes. As a result, the Ostrowski inequality continues to be a subject of active research, with ongoing efforts to explore its connections with other inequalities and its potential applications in emerging areas of science and engineering.

In [7], Alomari and Darus studied Ostrowski's inequality for the class of differentiable convex functions and provided the following result:

Theorem 1.2. *[7] Let* $\mathscr{V} : [\mathfrak{c}_1, \mathfrak{c}_2] \to \mathbb{R}$ *be a mapping that is differentiable on* $(\mathfrak{c}_1, \mathfrak{c}_2)$ *with* $\mathscr{V}' \in L([\mathfrak{c}_1, \mathfrak{c}_2])$*. If* $|\mathscr{V}'|$ *is convex, then for all* $\varkappa \in [\mathfrak{c}_1, \mathfrak{c}_2]$*, we have*

$$\left| \mathscr{V}(\varkappa) - \frac{1}{\mathfrak{c}_2 - \mathfrak{c}_1} \int\limits_{\mathfrak{c}_1}^{\mathfrak{c}_2} \mathscr{V}(\mathfrak{l})\, d\mathfrak{l} \right| \leq \left[\frac{1}{6} + \frac{1}{3} \left(\frac{\mathfrak{c}_2 - \varkappa}{\mathfrak{c}_2 - \mathfrak{c}_1} \right)^3 \right] |\mathscr{V}'(\mathfrak{c}_1)| + \left[\frac{1}{6} + \frac{1}{3} \left(\frac{\varkappa - \mathfrak{c}_1}{\mathfrak{c}_2 - \mathfrak{c}_1} \right)^3 \right] |\mathscr{V}'(\mathfrak{c}_2)| .$$

Alomari et al. [8] established Ostrowski-type inequalities for quasi-convex functions and, subsequently, for functions whose derivatives are s-convex in the second sense [9]. In [10], Meftah derived Ostrowski inequalities for functions possessing logarithmically preinvex first derivatives, while Meftah and Azaizia [11] obtained fractional Ostrowski-type inequalities for functions whose first derivatives are MT-preinvex. For further works on Ostrowski-type inequalities, the reader is referred to [12–14], where generalizations and applications of these inequalities have been extensively studied.

Set et al. [15] provided the following Ostrowski's inequality for differentiable s-convex functions:

Theorem 1.3. *[15] Let* $\mathscr{V} : [\mathfrak{c}_1, \mathfrak{c}_2] \to \mathbb{R}$ *be a mapping that is differentiable on* $(\mathfrak{c}_1, \mathfrak{c}_2)$ *with* $\mathscr{V}' \in L([\mathfrak{c}_1, \mathfrak{c}_2])$*. If* $|\mathscr{V}'|$ *is s-convex, then for all* $\varkappa \in [\mathfrak{c}_1, \mathfrak{c}_2]$*, we have*

$$\left| \mathscr{V}(\varkappa) - \frac{1}{\mathfrak{c}_2 - \mathfrak{c}_1} \int\limits_{\mathfrak{c}_1}^{\mathfrak{c}_2} \mathscr{V}(\mathfrak{l})\, d\mathfrak{l} \right| \leq \frac{\mathfrak{c}_2 - \mathfrak{c}_1}{(s+1)(s+2)} \left[\left(2(s+1) \left(\frac{\mathfrak{c}_2 - \varkappa}{\mathfrak{c}_2 - \mathfrak{c}_1} \right)^{s+2} - (s+2) \left(\frac{\mathfrak{c}_2 - \varkappa}{\mathfrak{c}_2 - \mathfrak{c}_1} \right)^{s+1} + 1 \right) |\mathscr{V}'(\mathfrak{c}_1)| \right.$$

$$+\left(2(s+1)\left(\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}\right)^{s+2}-(s+2)\left(\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}\right)^{s+1}+1\right)|\mathscr{V}'(\mathfrak{c}_2)|\Bigg].$$

Fractional calculus is a generalization of classical calculus that extends the concepts of differentiation and integration to non-integer orders. Unlike integer-order derivatives and integrals, which are local and global operators, respectively, fractional calculus offers a mathematical framework in which differentiation and integration can be extended to non-integer orders, making it a valuable tool for modeling systems that exhibit memory effects. In recent decades, this field has attracted growing interest due to its applicability across domains such as physics, engineering, and economics, where processes often display behaviors like anomalous diffusion, long-range correlations, or fractal structures. The most commonly used definitions of fractional operators include the Riemann-Liouville, Caputo, and Hadamard formulations, among others, each with its own advantages and areas of application. Fractional calculus not only enriches mathematical theory but also provides a more accurate and flexible description of real-world processes that cannot be adequately captured by traditional integer-order models.

In [16], Yıldız et al. explored Ostrowski's inequality within the context of Riemann-Liouville fractional integrals and derived the following result:

Theorem 1.4. *[16] Let* $\mathscr{V}:[\mathfrak{c}_1,\mathfrak{c}_2]\to\mathbb{R}$ *be a mapping that is differentiable on* $(\mathfrak{c}_1,\mathfrak{c}_2)$ *with* $\mathscr{V}'\in L([\mathfrak{c}_1,\mathfrak{c}_2])$. *If* $|\mathscr{V}'|$ *is convex, then for all* $\varkappa\in[\mathfrak{c}_1,\mathfrak{c}_2]$, *the following holds:*

$$\begin{aligned}&\left|\frac{(\varkappa-\mathfrak{c}_1)^{\alpha}+(\mathfrak{c}_2-\varkappa)^{\alpha}}{(\mathfrak{c}_2-\mathfrak{c}_1)^{\alpha}}\mathscr{V}(\varkappa)-\frac{\Gamma(\alpha+1)}{(\mathfrak{c}_2-\mathfrak{c}_1)^{\alpha}}\left(\mathscr{J}^{\alpha}_{\varkappa^-}[\mathscr{V}](\mathfrak{c}_1)+\mathscr{J}^{\alpha}_{\varkappa^+}[\mathscr{V}](\mathfrak{c}_2)\right)\right|\\ &\leq(\mathfrak{c}_2-\mathfrak{c}_1)\left[\left(\frac{1}{\alpha+1}\left(\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}\right)^{\alpha+1}-\frac{1}{\alpha+2}\left(\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}\right)^{\alpha+2}+\frac{1}{\alpha+2}\left(\frac{\mathfrak{c}_2-\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}\right)^{\alpha+2}\right)|\mathscr{V}'(\mathfrak{c}_1)|\right.\\ &\left.+\left(\frac{1}{\alpha+2}\left(\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}\right)^{\alpha+2}+\frac{1}{\alpha+1}\left(\frac{\mathfrak{c}_2-\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}\right)^{\alpha+1}-\frac{1}{\alpha+2}\left(\frac{\mathfrak{c}_2-\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}\right)^{\alpha+2}\right)|\mathscr{V}'(\mathfrak{c}_2)|\right],\end{aligned}$$

where $\mathscr{J}^{\alpha}_{\varkappa^+}$ *and* $\mathscr{J}^{\alpha}_{\varkappa^-}$ *denote the left and right Riemann-Liouville fractional integrals of order* $\alpha>0$, *respectively.*

Tempered fractional integrals represent an important extension of classical fractional calculus, incorporating an exponential tempering factor to model phenomena with long-range memory effects that decay at a faster rate. By introducing parameter $\delta>0$, these integrals provide a balance between the power-law behavior characteristic of standard fractional operators and the exponential decay observed in many physical and financial systems. This makes them particularly suitable for applications in anomalous diffusion processes, stochastic modeling, and complex systems where memory effects are significant but not unbounded. The mathematical framework of tempered fractional calculus not only enriches our understanding of dynamical systems but also offers a versatile tool for describing real-world phenomena with enhanced accuracy.

In the study of tempered fractional integral inequalities, various contributions have been made. For instance, Mohammed et al. [17] developed Hermite-Hadamard-type inequalities for convex functions by incorporating tempered fractional integrals. Moreover, Cao et al. [18] explored parametrized inequalities using these integrals, and Peng and Du [19] extended this work with a multiparameter analysis of tempered fractional inequalities. Haider and coauthors studied Milne-type inequality in [20, 21], while Lakhdari et al. investigated Bullen's inequality in [22]. For additional insights and related studies, readers are encouraged to consult [23, 24].

Incorporating tempered fractional integrals, Mohammed and coauthors [17] established the following Hermite-Hadamard inequality:

Theorem 1.5. *Let* $\mathscr{V} : [\mathfrak{c}_1, \mathfrak{c}_2] \to \mathbb{R}$ *be a positive function such that* $\mathscr{V} \in L([\mathfrak{c}_1, \mathfrak{c}_2])$ *with* $\mathfrak{c}_1 < \mathfrak{c}_2$. *Then, we have*

$$\mathscr{V}\left(\frac{\mathfrak{c}_1+\mathfrak{c}_2}{2}\right) \leq \frac{2^{\alpha-1}\Gamma(\alpha)}{\gamma_{\delta\left(\frac{\mathfrak{c}_2-\mathfrak{c}_1}{2}\right)}(\alpha,1)(\mathfrak{c}_2-\mathfrak{c}_1)^{\alpha}}\left(\mathscr{T}^{\alpha,\delta}_{\left(\frac{\mathfrak{c}_1+\mathfrak{c}_2}{2}\right)^-}\mathscr{V}(\mathfrak{c}_1) + \mathscr{T}^{\alpha,\delta}_{\left(\frac{\mathfrak{c}_1+\mathfrak{c}_2}{2}\right)^+}\mathscr{V}(\mathfrak{c}_2)\right) \leq \frac{\mathscr{V}(\mathfrak{c}_1)+\mathscr{V}(\mathfrak{c}_2)}{2},$$

where $\mathscr{V}$ *is convex on* $[\mathfrak{c}_1, \mathfrak{c}_2]$, *and* $\mathscr{T}^{\alpha,\delta}_{\mathfrak{c}_1^+}$ *and* $\mathscr{T}^{\alpha,\delta}_{\mathfrak{c}_2^-}$ *are the left and right tempered fractional integrals, and* $\gamma_\lambda(.)$ *denotes the* λ*-incomplete Gamma function.*

In the same article, the authors provided the following midpoint-type inequality involving tempered fractional integrals.

Theorem 1.6. *Let* $\mathscr{V} : \mathscr{D} \to \mathbb{R}$ *be a differentiable and* $\mathscr{V}'$ *be integrable on* $[\mathfrak{c}_1, \mathfrak{c}_2]$, *where* $\mathfrak{c}_1, \mathfrak{c}_2 \in \mathcal{D}^\circ$ *with* $\mathfrak{c}_1 < \mathfrak{c}_2$. *If* $|\mathscr{V}'|$ *is convex, then we have*

$$\begin{aligned}&\left|\mathscr{V}\left(\frac{\mathfrak{c}_1+\mathfrak{c}_2}{2}\right) - \frac{2^{\alpha-1}\Gamma(\alpha)}{\gamma_{\frac{\delta}{2}}(\alpha, \mathfrak{c}_2-\mathfrak{c}_1)}\left(\mathscr{T}^{\alpha,\delta}_{\left(\frac{\mathfrak{c}_1+\mathfrak{c}_2}{2}\right)^-}\mathscr{V}(\mathfrak{c}_1) + \mathscr{T}^{\alpha,\delta}_{\left(\frac{\mathfrak{c}_1+\mathfrak{c}_2}{2}\right)^+}\mathscr{V}(\mathfrak{c}_2)\right)\right| \\ \leq&\frac{(\mathfrak{c}_2-\mathfrak{c}_1)^{\alpha+1}}{4\gamma_{\frac{\delta}{2}}(\alpha, \mathfrak{c}_2-\mathfrak{c}_1)}\left[\frac{\gamma_{\frac{\delta}{2}}(\alpha, \mathfrak{c}_2-\mathfrak{c}_1)}{(\mathfrak{c}_2-\mathfrak{c}_1)^{\alpha}} - \frac{\gamma_{\frac{\delta}{2}}(\alpha+1, \mathfrak{c}_2-\mathfrak{c}_1)}{(\mathfrak{c}_2-\mathfrak{c}_1)^{\alpha+1}}\right]\left(|\mathscr{V}'(\mathfrak{c}_1)| + |\mathscr{V}'(\mathfrak{c}_2)|\right).\end{aligned}$$

In [25], Sarikaya and Kiriş provided the following Ostrowski's inequality via tempered fractional integrals:

Theorem 1.7. *Let* $\mathscr{V} : [\mathfrak{c}_1, \mathfrak{c}_2] \to \mathbb{R}$ *be a differentiable mapping on* $(\mathfrak{c}_1, \mathfrak{c}_2)$ *with* $\mathfrak{c}_1 < \mathfrak{c}_2$ *such that* $\mathscr{V}' \in L([\mathfrak{c}_1, \mathfrak{c}_2])$. *If* $|\mathscr{V}'| \leq M$, $\varkappa \in [\mathfrak{c}_1, \mathfrak{c}_2]$, *then the following inequality holds:*

$$\begin{aligned}&\left|\left[\frac{\gamma(\alpha, \delta(\varkappa-\mathfrak{c}_1)) + \gamma(\alpha, \delta(\mathfrak{c}_2-\varkappa))}{\delta^{\alpha}}\right]\mathscr{V}(\varkappa) - \Gamma(\alpha)\left(\mathscr{T}^{\alpha,\delta}_{\mathfrak{c}_1^+}[\mathscr{V}](\varkappa) + \mathscr{T}^{\alpha,\delta}_{\mathfrak{c}_2^-}[\mathscr{V}](\varkappa)\right)\right| \\ \leq& M\left[\frac{\gamma(\alpha+1, \delta(\varkappa-\mathfrak{c}_1))}{\delta^{\alpha+1}} + \frac{\gamma(\alpha+1, \delta(\mathfrak{c}_2-\varkappa))}{\delta^{\alpha+1}}\right],\end{aligned}$$

where $\mathscr{T}^{\alpha,\delta}_{\mathfrak{c}_1^+}$ *and* $\mathscr{T}^{\alpha,\delta}_{\mathfrak{c}_2^-}$ *are the left and right tempered fractional integrals, respectively, and* $\gamma(.,.)$ *is the incomplete gamma function.*

Motivated by the aforementioned studies, we generalize Ostrowski's inequality in this work to the framework of tempered fractional integrals. To accomplish this, we begin by establishing a new integral identity, which plays a central role in deriving novel forms of Ostrowski-type inequalities involving tempered fractional integrals for differentiable functions that are s-convex.

The paper is structured as follows: In Section 2, we provide a concise overview of the essential definitions and mathematical background required for our analysis, including fundamental concepts related to fractional calculus. In Section 3, we present an extended version of the Hermite-Hadamard inequality tailored to s-convex functions within the tempered fractional calculus setting. In Section 4, we begin by introducing a key tempered fractional integral identity, which serves as a foundational tool for developing new Ostrowski-type inequality in the context of tempered fractional integrals under s-convexity assumptions on the derivatives. In Section 5, we further explore additional Ostrowski-type inequalities by incorporating Hölder's inequality and the power mean inequality. Some applications are provided in Section 6. Finally, in Section 7, we conclude the study with a summary of the key findings and perspectives for future research.

2. Basic notions from fractional calculus

In this section, we recall some fundamental tools from fractional calculus.

Definition 2.1. *[26] Let* $\mathscr{V} \in L[\mathfrak{c}_1, \mathfrak{c}_2]$. *The fractional Riemann-Liouville integrals* $\mathscr{J}^{\alpha}_{\mathfrak{c}_1^+}\mathscr{V}$ *and* $\mathscr{J}^{\alpha}_{\mathfrak{c}_2^-}\mathscr{V}$ *of order* $\alpha > 0$ *are defined by*

$$\mathscr{J}^{\alpha}_{\mathfrak{c}_1^+}\mathscr{V}(\varkappa) = \frac{1}{\Gamma(\alpha)}\int\limits_{\mathfrak{c}_1}^{\varkappa} (\varkappa - u)^{\alpha-1}\, \mathscr{V}(u)du, \quad \varkappa > \mathfrak{c}_1,$$

$$\mathscr{J}^{\alpha}_{\mathfrak{c}_2^-}\mathscr{V}(\varkappa) = \frac{1}{\Gamma(\alpha)}\int\limits_{\varkappa}^{\mathfrak{c}_2} (u - \varkappa)^{\alpha-1}\, \mathscr{V}(u)du, \quad \mathfrak{c}_2 > \varkappa,$$

respectively, where $\Gamma(\ell)$ *is the Gamma function and* $\mathscr{J}^{0}_{\mathfrak{c}_1^+}\mathscr{V}(\varkappa) = \mathscr{J}^{0}_{\mathfrak{c}_2^-}\mathscr{V}(\varkappa) = \mathscr{V}(\varkappa)$.

Definition 2.2. *[27] Let* $\mathscr{V} : \mathscr{D} \to \mathbb{R}$. *The left-sided fractional tempered integrals* $\mathscr{T}^{\alpha,\delta}_{\mathfrak{c}_1^+}$ *and the right-sided one denoted by* $\mathscr{T}^{\alpha,\delta}_{\mathfrak{c}_2^-}$ *of order* $\alpha > 0$ *and* $\delta \geq 0$ *are defined by*

$$\mathscr{T}^{\alpha,\delta}_{\mathfrak{c}_1^+}\left[\mathscr{V}\right](\varkappa) = \frac{1}{\Gamma(\alpha)}\int\limits_{\mathfrak{c}_1}^{\varkappa} (\varkappa - \ell)^{\alpha-1}\, \mathfrak{e}^{-\delta(\varkappa-\ell)}\mathscr{V}(\ell)d\ell, \ \varkappa > \mathfrak{c}_1,$$

$$\mathscr{T}^{\alpha,\delta}_{\mathfrak{c}_2^-}\left[\mathscr{V}\right](\varkappa) = \frac{1}{\Gamma(\alpha)}\int\limits_{\varkappa}^{\mathfrak{c}_2} (\ell - \varkappa)^{\alpha-1}\, \mathfrak{e}^{-\delta(\ell-\varkappa)}\mathscr{V}(\ell)d\ell, \ \mathfrak{c}_2 > \varkappa.$$

Remark 2.1. *If we choose* $\delta = 0$, $\mathscr{T}^{\alpha,\delta}_{\mathfrak{c}_1^+}\left[\mathscr{V}\right](\varkappa) = \mathscr{R}^{\alpha}_{\mathfrak{c}_1^+}\left[\mathscr{V}\right](\varkappa)$ *and* $\mathscr{T}^{\alpha,\delta}_{\mathfrak{c}_2^-}\left[\mathscr{V}\right](\varkappa) = \mathscr{R}^{\alpha}_{\mathfrak{c}_2^-}\left[\mathscr{V}\right](\varkappa)$.

Definition 2.3. *[17] For* $\alpha > 0$ *and* $\lambda \geq 0$, *the incomplete Gamma and* λ*-incomplete Gamma functions are respectively given by*

$$\gamma(\alpha, \ell) = \int\limits_{0}^{\ell} u^{\alpha-1}\mathfrak{e}^{-u}du \ \text{ and } \ \gamma_{\lambda}(\alpha, \ell) = \int\limits_{0}^{\ell} u^{\alpha-1}\mathfrak{e}^{-\lambda u}du.$$

Definition 2.4. *[26] The incomplete beta function is defined for any complex numbers* $\mathfrak{v}_1, \mathfrak{v}_2$ *such that* $Re(\mathfrak{v}_1) > 0$ *and* $Re(\mathfrak{v}_2) > 0$ *by*

$$B_z(\mathfrak{v}_1, \mathfrak{v}_2) = \int\limits_{0}^{z} u^{\mathfrak{v}_1-1}(1-u)^{\mathfrak{v}_2-1}du, \ \ 0 < z \leq 1.$$

It is worth noting that when $z = 1$, the classical beta function is recovered.

3. Tempered fractional Hermite-Hadamard inequality

In this section, we establish a new version of the Hermite-Hadamard inequality using tempered fractional integrals for s-convex functions.

Theorem 3.1. *Let $\mathscr{V} : \mathscr{D} \subset [0, +\infty) \to \mathbb{R}$ be a mapping that is differentiable on $[\mathfrak{c}_1, \mathfrak{c}_2]$, where $\mathfrak{c}_1, \mathfrak{c}_2 \in \mathscr{D}^\circ$ with $\mathfrak{c}_1 < \mathfrak{c}_2$. If $\mathscr{V}$ is s-convex, then the inequality*

$$\begin{aligned} 2^{s-1}\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}(\alpha, 1)\,\mathscr{V}\left(\frac{\mathfrak{c}_1+\mathfrak{c}_2}{2}\right) &\leq \frac{\Gamma(\alpha)}{2(\mathfrak{c}_2-\mathfrak{c}_1)^\alpha}\left(\mathscr{T}^{\alpha,\delta}_{\mathfrak{c}_2^-}[\mathscr{V}](\mathfrak{c}_1) + \mathscr{T}^{\alpha,\delta}_{\mathfrak{c}_1^+}[\mathscr{V}](\mathfrak{c}_2)\right) \\ &\leq \left(B(\alpha, s+1) + \frac{1}{\alpha+s}\right)\frac{\mathscr{V}(\mathfrak{c}_1)+\mathscr{V}(\mathfrak{c}_2)}{2} \end{aligned}$$

holds.

Proof. From the fact that $\mathscr{V}$ is s-convex, we have

$$\mathscr{V}\left(\frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}\right) \leq \frac{1}{2^s}\mathscr{V}(\mathfrak{y}_1) + \frac{1}{2^s}\mathscr{V}(\mathfrak{y}_2). \tag{3.1}$$

For $\mathfrak{y}_1 = (1-\ell)\,\mathfrak{c}_1 + \ell\mathfrak{c}_2$ and $\mathfrak{y}_2 = \ell\mathfrak{c}_1 + (1-\ell)\,\mathfrak{c}_2$ with $\ell \in [0, 1]$, (3.1) gives

$$2^s\mathscr{V}\left(\frac{\mathfrak{c}_1+\mathfrak{c}_2}{2}\right) \leq \mathscr{V}((1-\ell)\,\mathfrak{c}_1 + \ell\mathfrak{c}_2) + \mathscr{V}(\ell\mathfrak{c}_1 + (1-\ell)\,\mathfrak{c}_2). \tag{3.2}$$

Multiplying both sides of (3.2) by $\ell^{\alpha-1}\mathfrak{e}^{-\delta(\mathfrak{c}_2-\mathfrak{c}_1)\ell}$, and then integrating the resulting inequality with respect to ℓ over $[0, 1]$ yields

$$\begin{aligned} &2^s\mathscr{V}\left(\frac{\mathfrak{c}_1+\mathfrak{c}_2}{2}\right)\int_0^1 \ell^{\alpha-1}\mathfrak{e}^{-\delta(\mathfrak{c}_2-\mathfrak{c}_1)\ell}\,d\ell \\ &= 2^s\mathscr{V}\left(\frac{\mathfrak{c}_1+\mathfrak{c}_2}{2}\right)\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}(\alpha, 1) \\ &\leq \int_0^1 \ell^{\alpha-1}\mathfrak{e}^{-\delta(\mathfrak{c}_2-\mathfrak{c}_1)\ell}\mathscr{V}((1-\ell)\,\mathfrak{c}_1 + \ell\mathfrak{c}_2)\,d\ell + \int_0^1 \ell^{\alpha-1}\mathfrak{e}^{-\delta(\mathfrak{c}_2-\mathfrak{c}_1)\ell}\mathscr{V}(\ell\mathfrak{c}_1 + (1-\ell)\,\mathfrak{c}_2)\,d\ell \\ &= \frac{1}{(\mathfrak{c}_2-\mathfrak{c}_1)^\alpha}\int_{\mathfrak{c}_1}^{\mathfrak{c}_2}(r-\mathfrak{c}_1)^{\alpha-1}\,\mathfrak{e}^{-\delta(r-\mathfrak{c}_1)}\mathscr{V}(r)\,dr + \tfrac{1}{(\mathfrak{c}_2-\mathfrak{c}_1)^\alpha}\int_{\mathfrak{c}_1}^{\mathfrak{c}_2}(\mathfrak{c}_2-r)^{\alpha-1}\,\mathfrak{e}^{-\delta(\mathfrak{c}_2-r)}\mathscr{V}(r)\,dr \\ &= \frac{\Gamma(\alpha)}{(\mathfrak{c}_2-\mathfrak{c}_1)^\alpha}\left(\frac{1}{\Gamma(\alpha)}\int_{\mathfrak{c}_1}^{\mathfrak{c}_2}(r-\mathfrak{c}_1)^{\alpha-1}\,\mathfrak{e}^{-\delta(r-\mathfrak{c}_1)}\mathscr{V}(r)\,dr + \frac{1}{\Gamma(\alpha)}\int_{\mathfrak{c}_1}^{\mathfrak{c}_1}(\mathfrak{c}_2-r)^{\alpha-1}\,\mathfrak{e}^{-\delta(\mathfrak{c}_2-r)}\mathscr{V}(r)\,dr\right) \\ &= \frac{\Gamma(\alpha)}{(\mathfrak{c}_2-\mathfrak{c}_1)^\alpha}\left(\mathscr{T}^{\alpha,\delta}_{\mathfrak{c}_2^-}[\mathscr{V}](\mathfrak{c}_1) + \mathscr{T}^{\alpha,\delta}_{\mathfrak{c}_1^+}[\mathscr{V}](\mathfrak{c}_2)\right). \end{aligned} \tag{3.3}$$

Thus, we have

$$2^s \gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}(\alpha,1)\,\mathscr{V}\left(\tfrac{\mathfrak{c}_1+\mathfrak{c}_2}{2}\right) \le \tfrac{\Gamma(\alpha)}{(\mathfrak{c}_2-\mathfrak{c}_1)^\alpha}\left(\mathscr{T}^{\alpha,\delta}_{\mathfrak{c}_2^-}[\mathscr{V}](\mathfrak{c}_1) + \mathscr{T}^{\alpha,\delta}_{\mathfrak{c}_1^+}[\mathscr{V}](\mathfrak{c}_2)\right). \tag{3.4}$$

On the other hand, from the fact that $\mathscr{V}$ is s-convex, we have

$$\begin{aligned}
&\mathscr{V}((1-\ell)\mathfrak{c}_1 + \ell\mathfrak{c}_2) + \mathscr{V}(\ell\mathfrak{c}_1 + (1-\ell)\mathfrak{c}_2)\\
&\le (1-\ell)^s\mathscr{V}(\mathfrak{c}_1) + \ell^s\mathscr{V}(\mathfrak{c}_2) + \ell^s\mathscr{V}(\mathfrak{c}_1) + (1-\ell)^s\mathscr{V}(\mathfrak{c}_2)\\
&= ((1-\ell)^s + \ell^s)(\mathscr{V}(\mathfrak{c}_1) + \mathscr{V}(\mathfrak{c}_2)).
\end{aligned} \tag{3.5}$$

Multiplying both sides of (3.5) by $\ell^{\alpha-1}\mathfrak{e}^{-\delta(\mathfrak{c}_2-\mathfrak{c}_1)\ell}$, and then integrating the resulting inequality with respect to ℓ over $[0,1]$, we get

$$\begin{aligned}
&\tfrac{\Gamma(\alpha)}{(\mathfrak{c}_2-\mathfrak{c}_1)^\alpha}\left(\mathscr{T}^{\alpha,\delta}_{\mathfrak{c}_2^-}[\mathscr{V}](\mathfrak{c}_1) + \mathscr{T}^{\alpha,\delta}_{\mathfrak{c}_1^+}[\mathscr{V}](\mathfrak{c}_2)\right)\\
&= \int_0^1 \ell^{\alpha-1}\mathfrak{e}^{-\delta(\mathfrak{c}_2-\mathfrak{c}_1)\ell}\mathscr{V}((1-\ell)\mathfrak{c}_1 + \ell\mathfrak{c}_2)\,d\ell + \int_0^1 \ell^{\alpha-1}\mathfrak{e}^{-\delta(\mathfrak{c}_2-\mathfrak{c}_1)\ell}\mathscr{V}(\ell\mathfrak{c}_1 + (1-\ell)\mathfrak{c}_2)\,d\ell\\
&\le \left(\int_0^1 \ell^{\alpha-1}\mathfrak{e}^{-\delta(\mathfrak{c}_2-\mathfrak{c}_1)\ell}((1-\ell)^s + \ell^s)\,d\ell\right)(\mathscr{V}(\mathfrak{c}_1) + \mathscr{V}(\mathfrak{c}_2))\\
&\le \left(\int_0^1 \ell^{\alpha-1}((1-\ell)^s + \ell^s)\,d\ell\right)(\mathscr{V}(\mathfrak{c}_1) + \mathscr{V}(\mathfrak{c}_2))\\
&= \left(\int_0^1 \left(\ell^{\alpha-1}(1-\ell)^s + \ell^{\alpha+s-1}\right)d\ell\right)(\mathscr{V}(\mathfrak{c}_1) + \mathscr{V}(\mathfrak{c}_2))\\
&= \left(B(\alpha,s+1) + \frac{1}{\alpha+s}\right)(\mathscr{V}(\mathfrak{c}_1) + \mathscr{V}(\mathfrak{c}_2)),
\end{aligned} \tag{3.6}$$

where we have used the fact that $\mathfrak{e}^{-\delta(\mathfrak{c}_2-\mathfrak{c}_1)\ell} \le 1$. Thus, from (3.4) and (3.6), we have

$$\begin{aligned}
2^s \gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}(\alpha,1)\,\mathscr{V}\left(\frac{\mathfrak{c}_1+\mathfrak{c}_2}{2}\right) &\le \frac{\Gamma(\alpha)}{(\mathfrak{c}_2-\mathfrak{c}_1)^\alpha}\left(\mathscr{T}^{\alpha,\delta}_{\mathfrak{c}_2^-}[\mathscr{V}](\mathfrak{c}_1) + \mathscr{T}^{\alpha,\delta}_{\mathfrak{c}_1^+}[\mathscr{V}](\mathfrak{c}_2)\right)\\
&\le \left(B(\alpha,s+1) + \frac{1}{\alpha+s}\right)(\mathscr{V}(\mathfrak{c}_1) + \mathscr{V}(\mathfrak{c}_2)).
\end{aligned} \tag{3.7}$$

Multiplying both sides of (3.7) by $\frac{1}{2}$, we get the desired result.

Corollary 3.1. *By setting $s = 1$, Theorem 3.1 yields*

$$2^{s-1}\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}(\alpha,1)\,\mathscr{V}\left(\frac{\mathfrak{c}_1+\mathfrak{c}_2}{2}\right) \le \frac{\Gamma(\alpha)}{2(\mathfrak{c}_2-\mathfrak{c}_1)^\alpha}\left(\mathscr{T}^{\alpha,\delta}_{\mathfrak{c}_2^-}[\mathscr{V}](\mathfrak{c}_1) + \mathscr{T}^{\alpha,\delta}_{\mathfrak{c}_1^+}[\mathscr{V}](\mathfrak{c}_2)\right) \le \frac{\mathscr{V}(\mathfrak{c}_1) + \mathscr{V}(\mathfrak{c}_2)}{2\alpha}.$$

Corollary 3.2. *By setting $\delta = 0$, Theorem 3.1 yields*

$$2^{s-1}\mathscr{V}\left(\frac{\mathfrak{c}_1+\mathfrak{c}_2}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(\mathfrak{c}_2-\mathfrak{c}_1)^{\alpha}}\left(\mathscr{J}^{\alpha}_{\mathfrak{c}_2^-}[\mathscr{V}](\mathfrak{c}_1) + \mathscr{J}^{\alpha}_{\mathfrak{c}_1^+}[\mathscr{V}](\mathfrak{c}_2)\right)$$
$$\leq \alpha\left(B(\alpha, s+1) + \frac{1}{\alpha+s}\right)\left(\frac{\mathscr{V}(\mathfrak{c}_1)+\mathscr{V}(\mathfrak{c}_2)}{2}\right).$$

Remark 3.1. *The result obtained in Theorem 3.1 represents a generalization of several results in the literature. Specifically, it encompasses the following special cases:*

1. *For $\delta = 0$ and $s = 1$, our result aligns with the findings in [28, Theorem 2].*
2. *In the absence of fractional operators ($\alpha = 1$ and $\delta = 0$), the inequality reduces to the result by Dragomir and Fitzpatrick in [5, Theorem 2.1].*

This highlights the versatility and generality of Theorem 3.1, which unifies and extends these particular cases into a broader framework.

To confirm the correctness of Theorem 3.1, we present the following illustrative example:

Example 3.1. Consider the function $\mathscr{V}$ defined on $[\mathfrak{c}_1, \mathfrak{c}_2] = [0, 1]$ by $\mathscr{V}(\mathfrak{l}) = \mathfrak{l}^s$.

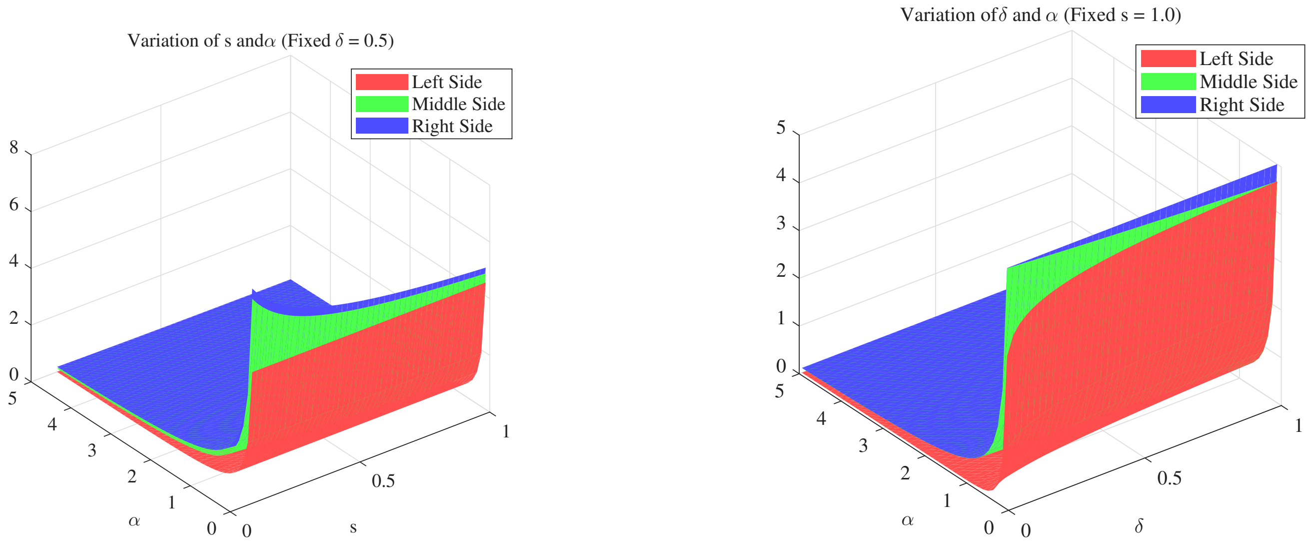
By invoking Theorem 3.1, we obtain

$$2^{s-1}\left(\frac{1}{2}\right)^s \leq \frac{1}{2\gamma_\delta(\alpha,1)}\left[\frac{\gamma(\alpha+s,\delta)}{\delta^{\alpha+s}} + B(\alpha,s+1)\,{}_1F_1(\alpha;\alpha+s+1;-\delta)\right] \leq \frac{B(\alpha,s+1)+\frac{1}{\alpha+s}}{2\gamma_\delta(\alpha,1)}, \tag{3.8}$$

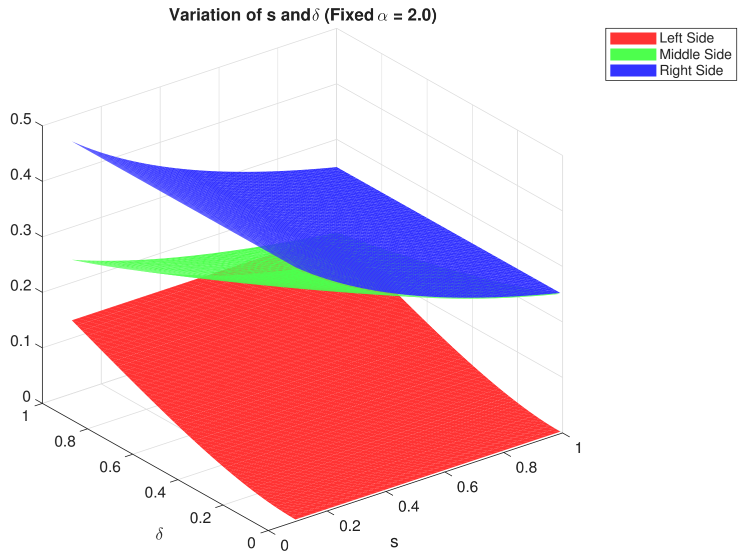
where $B(\cdot,\cdot)$ and ${}_1F_1(\cdot,\cdot,\cdot)$ represent the Beta and the Kummer's confluent hypergeometric function, respectively.

To provide a comprehensive numerical validation of inequality (3.8), we present three-dimensional graphical representations of the three members of the inequality across parameter domains. In all figures, the red surface represents the left-hand side (lower bound), the green surface illustrates the middle member involving tempered fractional integrals, and the blue surface depicts the right-hand side (upper bound).

- Figure 1(a) illustrates the behavior of the inequality as a function of the power $s \in (0, 1]$ and the fractional order $\alpha \in (0, 5]$, while keeping the tempering parameter fixed at $\delta = 0.5$. It is observed that the functional values increase as α grows, with the middle term consistently remaining between the two bounds.
- Figure 1(b) explores the relationship between the tempering parameter $\delta \in [0, 1]$ and the fractional order $\alpha \in (0, 5]$ for a fixed value of $s = 1$. This visualization highlights how the "tempering" effect influences the growth of the fractional integrals, showing a smooth transition as the decay factor varies.
- Figure 1(c) depicts the inequality by varying $s \in (0, 1]$ and $\delta \in [0, 1]$ at a constant fractional order $\alpha = 2$. This perspective confirms the stability of the bounds regardless of the specific combination of the exponent and the tempering scale.



(a) Fixed $\delta = 0.5$ **(b)** Fixed $s = 1.0$



(c) Fixed $\alpha = 2.0$

Figure 1. 3D numerical representation of the Hermite-Hadamard inequality (3.8).

Based on these graphical representations, we observe that the green surface is strictly confined between the red and blue surfaces across all tested intervals. Therefore, these numerical simulations confirm the correctness and validity of the results presented in Theorem 3.1.

4. Tempered Ostrowski-type inequalities

In this section, we establish a new version of Ostrowski-type inequality involving tempered fractional integrals for differentiable s-convex functions.

We begin by introducing a key tool in the form of an identity involving tempered fractional integrals, which is essential for establishing our major results.

Lemma 4.1. *Let $\mathscr{V}:\mathscr{D}\to\mathbb{R}$ be a differentiable function. If $\mathscr{V}'$ is integrable on $[\mathfrak{c}_1,\mathfrak{c}_2]$, where $\mathfrak{c}_1,\mathfrak{c}_2\in\mathscr{D}^\circ$ with $\mathfrak{c}_1<\mathfrak{c}_2$, then the following equality holds:*

$$\left[\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}\left(\alpha,\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}\right)+\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}\left(\alpha,\frac{\mathfrak{c}_2-\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}\right)\right]\mathscr{V}(\varkappa)-\frac{\Gamma(\alpha)}{(\mathfrak{c}_2-\mathfrak{c}_1)^{\alpha}}\left(\mathscr{T}^{\alpha,\delta}_{\varkappa^-}[\mathscr{V}](\mathfrak{c}_1)+\mathscr{T}^{\alpha,\delta}_{\varkappa^+}[\mathscr{V}](\mathfrak{c}_2)\right)$$

$$=(\mathfrak{c}_2-\mathfrak{c}_1)\left(\int_0^{\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}}\left(\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}(\alpha,\ell)\right)\mathscr{V}'((1-\ell)\,\mathfrak{c}_1+\ell\mathfrak{c}_2)\,d\ell-\int_{\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}}^{1}\left(\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}(\alpha,1-\ell)\right)\mathscr{V}'((1-\ell)\,\mathfrak{c}_1+\ell\mathfrak{c}_2)\,d\ell\right).$$

Proof. Let

$$\mathscr{I}_1=\int_0^{\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}}\left(\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}(\alpha,\ell)\right)\mathscr{V}'((1-\ell)\,\mathfrak{c}_1+\ell\mathfrak{c}_2)\,d\ell,$$

$$\mathscr{I}_2=\int_{\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}}^{1}\left(\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}(\alpha,1-\ell)\right)\mathscr{V}'((1-\ell)\,\mathfrak{c}_1+\ell\mathfrak{c}_2)\,d\ell.$$

Integrating by parts $\mathscr{I}_1$, we get

$$\begin{aligned}\mathscr{I}_1&=\frac{1}{\mathfrak{c}_2-\mathfrak{c}_1}\left(\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}(\alpha,\ell)\right)\mathscr{V}((1-\ell)\,\mathfrak{c}_1+\ell\mathfrak{c}_2)\Big|_0^{\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}}-\frac{1}{\mathfrak{c}_2-\mathfrak{c}_1}\int_0^{\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}}\ell^{\alpha-1}\mathrm{e}^{-\delta(\mathfrak{c}_2-\mathfrak{c}_1)\ell}\mathscr{V}((1-\ell)\,\mathfrak{c}_1+\ell\mathfrak{c}_2)\,d\ell\\&=\frac{1}{\mathfrak{c}_2-\mathfrak{c}_1}\left(\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}\left(\alpha,\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}\right)\right)\mathscr{V}(\varkappa)-\frac{1}{(\mathfrak{c}_2-\mathfrak{c}_1)^{\alpha+1}}\int_{\mathfrak{c}_1}^{\varkappa}(u-\mathfrak{c}_1)^{\alpha-1}\,\mathrm{e}^{-\delta(u-\mathfrak{c}_1)}\mathscr{V}(u)\,du\\&=\frac{1}{\mathfrak{c}_2-\mathfrak{c}_1}\left(\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}\left(\alpha,\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}\right)\right)\mathscr{V}(\varkappa)-\frac{\Gamma(\alpha)}{(\mathfrak{c}_2-\mathfrak{c}_1)^{\alpha+1}}\mathscr{T}^{\alpha,\delta}_{\varkappa^-}[\mathscr{V}](\mathfrak{c}_1).\end{aligned}\tag{4.1}$$

Similarly, we obtain

$$\begin{aligned}\mathscr{I}_2=&\frac{1}{\mathfrak{c}_2-\mathfrak{c}_1}\left(\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}(\alpha,1-\ell)\right)\mathscr{V}((1-\ell)\,\mathfrak{c}_1+\ell\mathfrak{c}_2)\Big|_{\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}}^{1}\\&+\frac{1}{\mathfrak{c}_2-\mathfrak{c}_1}\int_{\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}}^{1}(1-\ell)^{\alpha-1}\,\mathrm{e}^{-\delta(\mathfrak{c}_2-\mathfrak{c}_1)(1-\ell)}\mathscr{V}((1-\ell)\,\mathfrak{c}_1+\ell\mathfrak{c}_2)\,d\ell\\=&-\frac{1}{\mathfrak{c}_2-\mathfrak{c}_1}\left(\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}\left(\alpha,\frac{\mathfrak{c}_2-\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}\right)\right)\mathscr{V}(\varkappa)+\frac{1}{(\mathfrak{c}_2-\mathfrak{c}_1)^{\alpha+1}}\int_{\varkappa}^{\mathfrak{c}_2}(\mathfrak{c}_2-u)^{\alpha-1}\,\mathrm{e}^{-\delta(\mathfrak{c}_2-u)}\mathscr{V}(u)\,du\\=&-\frac{1}{\mathfrak{c}_2-\mathfrak{c}_1}\left(\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}\left(\alpha,\frac{\mathfrak{c}_2-\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}\right)\right)\mathscr{V}(\varkappa)+\frac{\Gamma(\alpha)}{(\mathfrak{c}_2-\mathfrak{c}_1)^{\alpha+1}}\mathscr{T}^{\alpha,\delta}_{\varkappa^+}[\mathscr{V}](\mathfrak{c}_2).\end{aligned}\tag{4.2}$$

By subtracting equality (4.2) from (4.1), and then multiplying the resulting equality by $(\mathfrak{c}_2 - \mathfrak{c}_1)$, the desired outcome is obtained.

Theorem 4.1. *Let* $\mathscr{V} : \mathscr{D} \subset [0, +\infty) \to \mathbb{R}$ *be a differentiable mapping and* $\mathscr{V}'$ *is integrable on* $[\mathfrak{c}_1, \mathfrak{c}_2]$*, where* $\mathfrak{c}_1, \mathfrak{c}_2 \in \mathscr{D}^\circ$ *with* $\mathfrak{c}_1 < \mathfrak{c}_2$*. If* $|\mathscr{V}'|$ *is s-convex, then the following inequality holds:*

$$\begin{aligned}
&\left|\left[\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}\left(\alpha, \frac{\varkappa - \mathfrak{c}_1}{\mathfrak{c}_2 - \mathfrak{c}_1}\right) + \gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}\left(\alpha, \frac{\mathfrak{c}_2 - \varkappa}{\mathfrak{c}_2 - \mathfrak{c}_1}\right)\right] \mathscr{V}(\varkappa) - \frac{\Gamma(\alpha)}{(\mathfrak{c}_2 - \mathfrak{c}_1)^\alpha} \left(\mathscr{T}^{\alpha,\delta}_{\varkappa^-}[\mathscr{V}](\mathfrak{c}_1) + \mathscr{T}^{\alpha,\delta}_{\varkappa^+}[\mathscr{V}](\mathfrak{c}_2)\right)\right| \\
&\leq (\mathfrak{c}_2 - \mathfrak{c}_1)\left[\left(\Upsilon_1\left(\alpha, \delta, s, \frac{\varkappa - \mathfrak{c}_1}{\mathfrak{c}_2 - \mathfrak{c}_1}\right) + \Upsilon_2\left(\alpha, \delta, s, \frac{\mathfrak{c}_2 - \varkappa}{\mathfrak{c}_2 - \mathfrak{c}_1}\right)\right)|\mathscr{V}'(\mathfrak{c}_1)|\right. \\
&\left. + \left(\Upsilon_2\left(\alpha, \delta, s, \frac{\varkappa - \mathfrak{c}_1}{\mathfrak{c}_2 - \mathfrak{c}_1}\right) + \Upsilon_1\left(\alpha, \delta, s, \frac{\mathfrak{c}_2 - \varkappa}{\mathfrak{c}_2 - \mathfrak{c}_1}\right)\right)|\mathscr{V}'(\mathfrak{c}_2)|\right],
\end{aligned}$$

where Υ_1 *and* Υ_2 *are defined as in* (4.3) *and* (4.4)*, respectively.*

Proof. Using Lemma 4.1, and the fact that $|\mathscr{V}'|$ is s-convex, we deduce

$$\begin{aligned}
&\left|\left[\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}\left(\alpha, \frac{\varkappa - \mathfrak{c}_1}{\mathfrak{c}_2 - \mathfrak{c}_1}\right) + \gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}\left(\alpha, \frac{\mathfrak{c}_2 - \varkappa}{\mathfrak{c}_2 - \mathfrak{c}_1}\right)\right] \mathscr{V}(\varkappa) - \frac{\Gamma(\alpha)}{(\mathfrak{c}_2 - \mathfrak{c}_1)^\alpha} \left(\mathscr{T}^{\alpha,\delta}_{\varkappa^-}[\mathscr{V}](\mathfrak{c}_1) + \mathscr{T}^{\alpha,\delta}_{\varkappa^+}[\mathscr{V}](\mathfrak{c}_2)\right)\right| \\
&\leq (\mathfrak{c}_2 - \mathfrak{c}_1)\left[\int_0^{\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}} \left|\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}(\alpha, \ell)\right| |\mathscr{V}'((1-\ell)\mathfrak{c}_1 + \ell\mathfrak{c}_2)|\, d\ell\right. \\
&\left. + \int_{\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}}^1 \left|\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}(\alpha, 1-\ell)\right| |\mathscr{V}'((1-\ell)\mathfrak{c}_1 + \ell\mathfrak{c}_2)|\, d\ell\right] \\
&\leq (\mathfrak{c}_2 - \mathfrak{c}_1)\left[\int_0^{\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}} \left|\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}(\alpha, \ell)\right| \left((1-\ell)^s |\mathscr{V}'(\mathfrak{c}_1)| + \ell^s |\mathscr{V}'(\mathfrak{c}_2)|\right) d\ell\right. \\
&\left. + \int_{\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}}^1 \left|\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}(\alpha, 1-\ell)\right| \left((1-\ell)^s |\mathscr{V}'(\mathfrak{c}_1)| + \ell^s |\mathscr{V}'(\mathfrak{c}_2)|\right) d\ell\right] \\
&= (\mathfrak{c}_2 - \mathfrak{c}_1)\left[|\mathscr{V}'(\mathfrak{c}_1)| \int_0^{\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}} \left|\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}(\alpha, \ell)\right| (1-\ell)^s d\ell + |\mathscr{V}'(\mathfrak{c}_2)| \int_0^{\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}} \left|\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}(\alpha, \ell)\right| \ell^s d\ell\right. \\
&\left. + |\mathscr{V}'(\mathfrak{c}_1)| \int_{\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}}^1 \left|\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}(\alpha, 1-\ell)\right| (1-\ell)^s d\ell + |\mathscr{V}'(\mathfrak{c}_2)| \int_{\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}}^1 \left|\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}(\alpha, 1-\ell)\right| \ell^s d\ell\right]
\end{aligned}$$

$$
\begin{aligned}
&= (\mathfrak{c}_2 - \mathfrak{c}_1)\Bigg[|\mathscr{V}'(\mathfrak{c}_1)|\left(\int_0^{\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}} \left|\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}(\alpha,\ell)\right|(1-\ell)^s\,d\ell + \int_0^{\frac{\mathfrak{c}_2-\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}} \left|\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}(\alpha,\ell)\right|\ell^s d\ell\right) \\
&\quad + |\mathscr{V}'(\mathfrak{c}_2)|\left(\int_0^{\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}} \left|\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}(\alpha,\ell)\right|\ell^s d\ell + \int_0^{\frac{\mathfrak{c}_2-\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}} \left|\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}(\alpha,\ell)\right|(1-\ell)^s\,d\ell\right)\Bigg] \\
&= (\mathfrak{c}_2 - \mathfrak{c}_1)\Bigg[\left(\Upsilon_1\left(\alpha,\delta,s,\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}\right) + \Upsilon_2\left(\alpha,\delta,s,\frac{\mathfrak{c}_2-\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}\right)\right)|\mathscr{V}'(\mathfrak{c}_1)| \\
&\quad + \left(\Upsilon_2\left(\alpha,\delta,s,\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}\right) + \Upsilon_1\left(\alpha,\delta,s,\frac{\mathfrak{c}_2-\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}\right)\right)|\mathscr{V}'(\mathfrak{c}_2)|\Bigg],
\end{aligned}
$$

where

$$
\Upsilon_1(\alpha,\delta,s,\tau) = \int_0^{\tau} \left|\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}(\alpha,\ell)\right|(1-\ell)^s\,d\ell \tag{4.3}
$$

and

$$
\Upsilon_2(\alpha,\delta,s,\tau) = \int_0^{\tau} \left|\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}(\alpha,\ell)\right|\ell^s d\ell. \tag{4.4}
$$

This completes the proof.

Corollary 4.1. *By setting $\varkappa = \frac{\mathfrak{c}_1+\mathfrak{c}_2}{2}$, Theorem 4.1 yields the following midpoint inequality via tempered fractional integrals:*

$$
\begin{aligned}
&\left|\mathscr{V}\left(\frac{\mathfrak{c}_1+\mathfrak{c}_2}{2}\right) - \frac{2^{\alpha-1}\Gamma(\alpha)}{\gamma_{\frac{\delta}{2}}(\alpha,\mathfrak{c}_2-\mathfrak{c}_1)}\left(\mathscr{T}^{\alpha,\delta}_{\left(\frac{\mathfrak{c}_1+\mathfrak{c}_2}{2}\right)^-}\mathscr{V}(\mathfrak{c}_1) + \mathscr{T}^{\alpha,\delta}_{\left(\frac{\mathfrak{c}_1+\mathfrak{c}_2}{2}\right)^+}\mathscr{V}(\mathfrak{c}_2)\right)\right| \\
&\leq \frac{2^{\alpha-1}(\mathfrak{c}_2-\mathfrak{c}_1)^{\alpha+1}}{\gamma_{\frac{\delta}{2}}(\alpha,\mathfrak{c}_2-\mathfrak{c}_1)}\left[\Upsilon_1\left(\alpha,\delta,s,\tfrac{1}{2}\right) + \Upsilon_2\left(\alpha,\delta,s,\tfrac{1}{2}\right)\right]\left(|\mathscr{V}'(\mathfrak{c}_1)| + |\mathscr{V}'(\mathfrak{c}_2)|\right),
\end{aligned} \tag{4.5}
$$

where Υ_1 and Υ_2 are defined as in (4.3) *and* (4.4)*, respectively, and we have used the fact that*

$$
\gamma_\delta(\alpha,\beta) = \beta^\alpha\gamma_{\delta\beta}(\alpha,1), \quad for\,\beta > 0.
$$

Moreover, by setting $s = 1$, one can recover Theorem 1.6 as provided by Mohammed et al. in [17].

Furthermore, when $\delta = 0$ and $s = 1$, inequality (4.5) *leads to the classical result found in [29, Theorem 5] (with $q = 1$).*

Additionally, choosing $\delta = 0$, $\alpha = 1$, and $s = 1$ yields the classical midpoint inequality as stated in [30, Theorem 2.2].

Corollary 4.2. *By setting $\delta = 0$ in Theorem 4.1, we obtain*

$$
\left|\frac{(\varkappa-\mathfrak{c}_1)^\alpha + (\mathfrak{c}_2-\varkappa)^\alpha}{(\mathfrak{c}_2-\mathfrak{c}_1)^\alpha}\mathscr{V}(\varkappa) - \frac{\Gamma(\alpha+1)}{(\mathfrak{c}_2-\mathfrak{c}_1)^\alpha}\left(\mathscr{J}^\alpha_{\varkappa^-}[\mathscr{V}](\mathfrak{c}_1) + \mathscr{J}^\alpha_{\varkappa^+}[\mathscr{V}](\mathfrak{c}_2)\right)\right|
$$

$$\leq (\mathfrak{c}_2-\mathfrak{c}_1)\left[\left(B_{\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}}(\alpha+1,s+1)+\frac{1}{\alpha+s+1}\left(\frac{\mathfrak{c}_2-\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}\right)^{\alpha+s+1}\right)|\mathscr{V}'(\mathfrak{c}_1)|\right.$$
$$\left.+\left(\frac{1}{\alpha+s+1}\left(\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}\right)^{\alpha+s+1}+B_{\frac{\mathfrak{c}_2-\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}}(\alpha+1,s+1)\right)|\mathscr{V}'(\mathfrak{c}_2)|\right],$$

where $B_z(.,.)$ is the incomplete beta function.

Corollary 4.3. *By setting $s=1$ in Theorem 4.1, we obtain*

$$\left|\left[\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}\left(\alpha,\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}\right)+\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}\left(\alpha,\frac{\mathfrak{c}_2-\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}\right)\right]\mathscr{V}(\varkappa)-\frac{\Gamma(\alpha)}{(\mathfrak{c}_2-\mathfrak{c}_1)^{\alpha}}\left(\mathscr{T}^{\alpha,\delta}_{\varkappa^-}[\mathscr{V}](\mathfrak{c}_1)+\mathscr{T}^{\alpha,\delta}_{\varkappa^+}[\mathscr{V}](\mathfrak{c}_2)\right)\right|$$
$$\leq (\mathfrak{c}_2-\mathfrak{c}_1)\left[\xi\left(\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}\right)|\mathscr{V}'(\mathfrak{c}_1)|+\xi\left(\frac{\mathfrak{c}_2-\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}\right)|\mathscr{V}'(\mathfrak{c}_2)|\right],$$

where

$$\xi(z)=\frac{2z-z^2}{2}\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}(\alpha,z)-\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}(\alpha+1,z)+\frac{1}{2}\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}(\alpha+2,z)$$
$$+\frac{1}{2}(1-z)^2\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}(\alpha,1-z)-\frac{1}{2}\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}(\alpha+2,1-z).$$

Remark 4.1. *Theorem 4.1 generalizes several results in the literature, as detailed below:*

(1) When the tempered fractional integral reduces to the classical Riemann-Liouville fractional integral ($\delta=0$) and by considering classical convexity ($s=1$), Theorem 4.1 aligns with the findings presented by Yıldız and coauthors in [16, Theorem 2.1].
(2) For $\delta=0$ and $\alpha=1$, Theorem 4.1 will be reduced to Theorem 7 from [15].
(3) By setting $\delta=0$ and $\alpha=s=1$, Theorem 4.1 coincides with Corollary 3.3 from [31].

5. Further results

In this section, we provide further interpretations and extensions of the results presented in Section 4.

Theorem 5.1. *Let $\mathscr{V}:\mathscr{D}\subset[0,+\infty)\to\mathbb{R}$ be a differentiable mapping and $\mathscr{V}'$ is integrable on $[\mathfrak{c}_1,\mathfrak{c}_2]$, where $\mathfrak{c}_1,\mathfrak{c}_2\in\mathscr{D}^{\circ}$ with $\mathfrak{c}_1<\mathfrak{c}_2$. If $|\mathscr{V}'|^q$ is s-convex for $q>1$ with $pq=p+q$, then the following inequality holds:*

$$\left|\left[\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}\left(\alpha,\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}\right)+\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}\left(\alpha,\frac{\mathfrak{c}_2-\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}\right)\right]\mathscr{V}(\varkappa)-\frac{\Gamma(\alpha)}{(\mathfrak{c}_2-\mathfrak{c}_1)^{\alpha}}\left(\mathscr{T}^{\alpha,\delta}_{\varkappa^-}[\mathscr{V}](\mathfrak{c}_1)+\mathscr{T}^{\alpha,\delta}_{\varkappa^+}[\mathscr{V}](\mathfrak{c}_2)\right)\right|$$
$$\leq\frac{\mathfrak{c}_2-\mathfrak{c}_1}{(s+1)^{\frac{1}{q}}}\left[\left(\int_0^{\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}}\left|\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}(\alpha,\ell)\right|^p d\ell\right)^{\frac{1}{p}}\left(\left(1-\left(\frac{\mathfrak{c}_2-\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}\right)^{s+1}\right)|\mathscr{V}'(\mathfrak{c}_1)|^q+\left(\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}\right)^{s+1}|\mathscr{V}'(\mathfrak{c}_2)|^q\right)^{\frac{1}{q}}\right.$$

$$+\left(\int_0^{\frac{\mathfrak{c}_2-\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}} \left|\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}(\alpha,\ell)\right|^p d\ell\right)^{\frac{1}{p}}\left(\left(\frac{\mathfrak{c}_2-\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}\right)^{s+1}|\mathscr{V}'(\mathfrak{c}_1)|^q+\left(1-\left(\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}\right)^{s+1}\right)|\mathscr{V}'(\mathfrak{c}_2)|^q\right)^{\frac{1}{q}}\Bigg].$$

Proof. Using Lemma 4.1, and the fact that $|\mathscr{V}'|^q$ is s-convex along with Hölder's inequality, we deduce

$$\begin{aligned}
&\left|\left[\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}\left(\alpha,\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}\right)+\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}\left(\alpha,\frac{\mathfrak{c}_2-\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}\right)\right]\mathscr{V}(\varkappa)-\frac{\Gamma(\alpha)}{(\mathfrak{c}_2-\mathfrak{c}_1)^{\alpha}}\left(\mathscr{T}^{\alpha,\delta}_{\varkappa^-}[\mathscr{V}](\mathfrak{c}_1)+\mathscr{T}^{\alpha,\delta}_{\varkappa^+}[\mathscr{V}](\mathfrak{c}_2)\right)\right|\\
&\leq(\mathfrak{c}_2-\mathfrak{c}_1)\left[\left(\int_0^{\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}}\left|\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}(\alpha,\ell)\right|^p d\ell\right)^{\frac{1}{p}}\left(\int_0^{\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}}|\mathscr{V}'((1-\ell)\mathfrak{c}_1+\ell\mathfrak{c}_2)|^q d\ell\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\left(\int_{\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}}^1\left|\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}(\alpha,1-\ell)\right|^p d\ell\right)^{\frac{1}{p}}\left(\int_{\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}}^1|\mathscr{V}'((1-\ell)\mathfrak{c}_1+\ell\mathfrak{c}_2)|^q d\ell\right)^{\frac{1}{q}}\right]\\
&\leq(\mathfrak{c}_2-\mathfrak{c}_1)\left[\left(\int_0^{\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}}\left|\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}(\alpha,\ell)\right|^p d\ell\right)^{\frac{1}{p}}\left(\int_0^{\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}}\left((1-\ell)^s|\mathscr{V}'(\mathfrak{c}_1)|^q+\ell^s|\mathscr{V}'(\mathfrak{c}_2)|^q\right)d\ell\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\left(\int_{\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}}^1\left|\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}(\alpha,1-\ell)\right|^p d\ell\right)^{\frac{1}{p}}\left(\int_{\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}}^1\left((1-\ell)^s|\mathscr{V}'(\mathfrak{c}_1)|^q+\ell^s|\mathscr{V}'(\mathfrak{c}_2)|^q\right)d\ell\right)^{\frac{1}{q}}\right]\\
&=\frac{\mathfrak{c}_2-\mathfrak{c}_1}{(s+1)^{\frac{1}{q}}}\left[\left(\int_0^{\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}}\left|\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}(\alpha,\ell)\right|^p d\ell\right)^{\frac{1}{p}}\left(\left(1-\left(\frac{\mathfrak{c}_2-\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}\right)^{s+1}\right)|\mathscr{V}'(\mathfrak{c}_1)|^q+\left(\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}\right)^{s+1}|\mathscr{V}'(\mathfrak{c}_2)|^q\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\left(\int_0^{\frac{\mathfrak{c}_2-\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}}\left|\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}(\alpha,\ell)\right|^p d\ell\right)^{\frac{1}{p}}\left(\left(\frac{\mathfrak{c}_2-\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}\right)^{s+1}|\mathscr{V}'(\mathfrak{c}_1)|^q+\left(1-\left(\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}\right)^{s+1}\right)|\mathscr{V}'(\mathfrak{c}_2)|^q\right)^{\frac{1}{q}}\right].
\end{aligned}$$

The proof is completed.

Corollary 5.1. *In Theorem 5.1, if we attempt to take* $\delta = 0$, *we obtain*

$$\left|\frac{(\varkappa-\mathfrak{c}_1)^{\alpha}+(\mathfrak{c}_2-\varkappa)^{\alpha}}{(\mathfrak{c}_2-\mathfrak{c}_1)^{\alpha}}\mathscr{V}(\varkappa)-\frac{\Gamma(\alpha+1)}{(\mathfrak{c}_2-\mathfrak{c}_1)^{\alpha}}\left(\mathscr{J}^{\alpha}_{\varkappa^-}[\mathscr{V}](\mathfrak{c}_1)+\mathscr{J}^{\alpha}_{\varkappa^+}[\mathscr{V}](\mathfrak{c}_2)\right)\right|$$

$$\leq(\mathfrak{c}_2-\mathfrak{c}_1)\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left[\left(\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}\right)^{\alpha+\frac{1}{p}}\left(\left(1-\left(\frac{\mathfrak{c}_2-\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}\right)^{s+1}\right)|\mathscr{V}'(\mathfrak{c}_1)|^q+\left(\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}\right)^{s+1}|\mathscr{V}'(\mathfrak{c}_2)|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{\mathfrak{c}_2-\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}\right)^{\alpha+\frac{1}{p}}\left(\left(\frac{\mathfrak{c}_2-\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}\right)^{s+1}|\mathscr{V}'(\mathfrak{c}_1)|^q+\left(1-\left(\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}\right)^{s+1}\right)|\mathscr{V}'(\mathfrak{c}_2)|^q\right)^{\frac{1}{q}}\right].$$

Corollary 5.2. *If we attempt to take* $\alpha = 1$ *and* $\delta = 0$, *Theorem 5.1 yields*

$$\left|\mathscr{V}(\varkappa)-\frac{1}{\mathfrak{c}_2-\mathfrak{c}_1}\int_{\mathfrak{c}_1}^{\mathfrak{c}_2}\mathscr{V}(\mathfrak{l})\,d\mathfrak{l}\right|$$

$$\leq(\mathfrak{c}_2-\mathfrak{c}_1)\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}\right)^{1+\frac{1}{p}}\left(\left(1-\left(\frac{\mathfrak{c}_2-\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}\right)^{s+1}\right)|\mathscr{V}'(\mathfrak{c}_1)|^q+\left(\tfrac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}\right)^{s+1}|\mathscr{V}'(\mathfrak{c}_2)|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{\mathfrak{c}_2-\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}\right)^{1+\frac{1}{p}}\left(\left(\frac{\mathfrak{c}_2-\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}\right)^{s+1}|\mathscr{V}'(\mathfrak{c}_1)|^q+\left(1-\left(\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}\right)^{s+1}\right)|\mathscr{V}'(\mathfrak{c}_2)|^q\right)^{\frac{1}{q}}\right].$$

Corollary 5.3. *If we attempt to take* $s = 1$, *Theorem 5.1 gives*

$$\left|\left[\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}\left(\alpha,\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}\right)+\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}\left(\alpha,\frac{\mathfrak{c}_2-\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}\right)\right]\mathscr{V}(\varkappa)-\frac{\Gamma(\alpha)}{(\mathfrak{c}_2-\mathfrak{c}_1)^{\alpha}}\left(\mathscr{T}^{\alpha,\delta}_{\varkappa^-}[\mathscr{V}]\,\mathfrak{c}_1)+\mathscr{T}^{\alpha,\delta}_{\varkappa^+}[\mathscr{V}](\mathfrak{c}_2)\right)\right|$$

$$\leq\frac{\mathfrak{c}_2-\mathfrak{c}_1}{(s+1)^{\frac{1}{q}}}\left[\left(\int_0^{\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}}\left|\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}(\alpha,\ell)\right|^p d\ell\right)^{\frac{1}{p}}\left(\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}\right)^{\frac{1}{q}}\left(\frac{2\mathfrak{c}_2-\mathfrak{c}_1-\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}|\mathscr{V}'(\mathfrak{c}_1)|^q+\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}|\mathscr{V}'(\mathfrak{c}_2)|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^{\frac{\mathfrak{c}_2-\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}}\left|\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}(\alpha,\ell)\right|^p d\ell\right)^{\frac{1}{p}}\left(\frac{\mathfrak{c}_2-\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}\right)^{\frac{1}{q}}\left(\frac{\mathfrak{c}_2-\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}|\mathscr{V}'(\mathfrak{c}_1)|^q+\frac{\mathfrak{c}_2-2\mathfrak{c}_1+\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}|\mathscr{V}'(\mathfrak{c}_2)|^q\right)^{\frac{1}{q}}\right].$$

Corollary 5.4. *By taking* $\delta = 0$ *and* $s = 1$, *Theorem 5.1 yields*

$$\left|\frac{(\varkappa-\mathfrak{c}_1)^{\alpha}+(\mathfrak{c}_2-\varkappa)^{\alpha}}{(\mathfrak{c}_2-\mathfrak{c}_1)^{\alpha}}\mathscr{V}(\varkappa)-\frac{\Gamma(\alpha+1)}{(\mathfrak{c}_2-\mathfrak{c}_1)^{\alpha}}\left(\mathscr{J}^{\alpha}_{\varkappa^-}[\mathscr{V}](\mathfrak{c}_1)+\mathscr{J}^{\alpha}_{\varkappa^+}[\mathscr{V}](\mathfrak{c}_2)\right)\right|$$

$$\leq\frac{\mathfrak{c}_2-\mathfrak{c}_1}{2^{\frac{1}{q}}}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left[\left(\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}\right)^{\alpha+1}\left(\frac{2\mathfrak{c}_2-\mathfrak{c}_1-\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}|\mathscr{V}'(\mathfrak{c}_1)|^q+\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}|\mathscr{V}'(\mathfrak{c}_2)|^q\right)^{\frac{1}{q}}\right.$$

$$+\left(\frac{\mathfrak{c}_2-\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}\right)^{\alpha+1}\left(\frac{\mathfrak{c}_2-\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}|\mathscr{V}'(\mathfrak{c}_1)|^q+\frac{\mathfrak{c}_2-2\mathfrak{c}_1+\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}|\mathscr{V}'(\mathfrak{c}_2)|^q\right)^{\frac{1}{q}}\Bigg].$$

Corollary 5.5. *By taking $\delta = 0$ and $s = \alpha = 1$, Theorem 5.1 gives*

$$\left|\mathscr{V}(\varkappa)-\frac{1}{\mathfrak{c}_2-\mathfrak{c}_1}\int_{\mathfrak{c}_1}^{\mathfrak{c}_2}\mathscr{V}(\mathfrak{l})\,d\mathfrak{l}\right|$$

$$\leq \frac{\mathfrak{c}_2-\mathfrak{c}_1}{2^{\frac{1}{q}}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}\right)^2\left(\frac{2\mathfrak{c}_2-\mathfrak{c}_1-\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}|\mathscr{V}'(\mathfrak{c}_1)|^q+\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}|\mathscr{V}'(\mathfrak{c}_2)|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{\mathfrak{c}_2-\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}\right)^2\left(\frac{\mathfrak{c}_2-\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}|\mathscr{V}'(\mathfrak{c}_1)|^q+\frac{\mathfrak{c}_2-2\mathfrak{c}_1+\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}|\mathscr{V}'(\mathfrak{c}_2)|^q\right)^{\frac{1}{q}}\right],$$

which is the same result provided in [31, Corollary 3.9].

Theorem 5.2. *Let $\mathscr{V}:\mathscr{D}\subset[0,+\infty)\to\mathbb{R}$ be a differentiable mapping and $\mathscr{V}'$ is integrable on $[\mathfrak{c}_1,\mathfrak{c}_2]$, where $\mathfrak{c}_1,\mathfrak{c}_2\in\mathscr{D}^\circ$ with $\mathfrak{c}_1<\mathfrak{c}_2$. If $|\mathscr{V}'|^q$ is s-convex for $q>1$, then the following inequality holds:*

$$\left|\left[\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}\left(\alpha,\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}\right)+\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}\left(\alpha,\frac{\mathfrak{c}_2-\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}\right)\right]\mathscr{V}(\varkappa)-\frac{\Gamma(\alpha)}{(\mathfrak{c}_2-\mathfrak{c}_1)^\alpha}\left(\mathscr{T}^{\alpha,\delta}_{\varkappa^-}[\mathscr{V}](\mathfrak{c}_1)+\mathscr{T}^{\alpha,\delta}_{\varkappa^+}[\mathscr{V}](\mathfrak{c}_2)\right)\right|$$

$$\leq(\mathfrak{c}_2-\mathfrak{c}_1)\left[\left(\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}\left(\alpha,\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}\right)-\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}\left(\alpha+1,\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}\right)\right)^{1-\frac{1}{q}}\right.$$

$$\times\left(\Upsilon_1\left(\alpha,\delta,s,\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}\right)|\mathscr{V}'(\mathfrak{c}_1)|^q+\Upsilon_2\left(\alpha,\delta,s,\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}\right)|\mathscr{V}'(\mathfrak{c}_2)|^q\right)^{\frac{1}{q}}$$

$$+\left(\frac{\mathfrak{c}_2-\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}\left(\alpha,\frac{\mathfrak{c}_2-\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}\right)-\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}\left(\alpha+1,\frac{\mathfrak{c}_2-\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}\right)\right)^{1-\frac{1}{q}}$$

$$\left.\times\left(\Upsilon_2\left(\alpha,\delta,s,\frac{\mathfrak{c}_2-\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}\right)|\mathscr{V}'(\mathfrak{c}_1)|^q+\Upsilon_1\left(\alpha,s,\delta,\frac{\mathfrak{c}_2-\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}\right)|\mathscr{V}'(\mathfrak{c}_2)|^q\right)^{\frac{1}{q}}\right],$$

where Υ_1 and Υ_2 are defined as in (4.3) *and* (4.4)*, respectively.*

Proof. Using Lemma 4.1, and the fact that $|\mathscr{V}'|^q$ is s-convex along with power mean inequality, we deduce

$$\left|\left[\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}\left(\alpha,\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}\right)+\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}\left(\alpha,\frac{\mathfrak{c}_2-\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}\right)\right]\mathscr{V}(\varkappa)-\frac{\Gamma(\alpha)}{(\mathfrak{c}_2-\mathfrak{c}_1)^\alpha}\left(\mathscr{T}^{\alpha,\delta}_{\varkappa^-}[\mathscr{V}](\mathfrak{c}_1)+\mathscr{T}^{\alpha,\delta}_{\varkappa^+}[\mathscr{V}](\mathfrak{c}_2)\right)\right|$$

$$\leq(\mathfrak{c}_2-\mathfrak{c}_1)\left[\left(\int_0^{\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}}\left|\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}(\alpha,\ell)\right|d\ell\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}}\left|\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}(\alpha,\ell)\right||\mathscr{V}'((1-\ell)\mathfrak{c}_1+\ell\mathfrak{c}_2)|^q\,d\ell\right)^{\frac{1}{q}}\right.$$

$$+\left(\int\limits_{\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}}^{1}\left|\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}(\alpha,1-\ell)\right|d\ell\right)^{1-\frac{1}{q}}\left(\int\limits_{\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}}^{1}\left|\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}(\alpha,1-\ell)\right|\left|\mathscr{V}'\left((1-\ell)\,\mathfrak{c}_1+\ell\mathfrak{c}_2\right)\right|^q d\ell\right)^{\frac{1}{q}}\Bigg]$$

$$\leq(\mathfrak{c}_2-\mathfrak{c}_1)\Bigg[\left(\int\limits_{0}^{\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}}\left|\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}(\alpha,\ell)\right|d\ell\right)^{1-\frac{1}{q}}\left(\int\limits_{0}^{\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}}\left|\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}(\alpha,\ell)\right|\left((1-\ell)^s\left|\mathscr{V}'(\mathfrak{c}_1)\right|^q+\ell^s\left|\mathscr{V}'(\mathfrak{c}_2)\right|^q\right)d\ell\right)^{\frac{1}{q}}$$

$$+\left(\int\limits_{\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}}^{1}\left|\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}(\alpha,1-\ell)\right|d\ell\right)^{1-\frac{1}{q}}\left(\int\limits_{\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}}^{1}\left|\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}(\alpha,1-\ell)\right|\left((1-\ell)^s\left|\mathscr{V}'(\mathfrak{c}_1)\right|^q+\ell^s\left|\mathscr{V}'(\mathfrak{c}_2)\right|^q\right)d\ell\right)^{\frac{1}{q}}\Bigg]$$

$$=(\mathfrak{c}_2-\mathfrak{c}_1)\Bigg[\left(\int\limits_{0}^{\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}}\left|\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}(\alpha,\ell)\right|d\ell\right)^{1-\frac{1}{q}}$$

$$\times\left(\left|\mathscr{V}'(\mathfrak{c}_1)\right|^q\int\limits_{0}^{\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}}\left|\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}(\alpha,\ell)\right|(1-\ell)^s\,d\ell+\left|\mathscr{V}'(\mathfrak{c}_2)\right|^q\int\limits_{0}^{\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}}\left|\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}(\alpha,\ell)\right|\ell^s d\ell\right)^{\frac{1}{q}}$$

$$\times\left(\int\limits_{0}^{\frac{\mathfrak{c}_2-\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}}\left|\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}(\alpha,\ell)\right|d\ell\right)^{1-\frac{1}{q}}$$

$$\times\left(\left|\mathscr{V}'(\mathfrak{c}_1)\right|^q\int\limits_{0}^{\frac{\mathfrak{c}_2-\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}}\left|\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}(\alpha,\ell)\right|\ell^s d\ell+\left|\mathscr{V}'(\mathfrak{c}_2)\right|^q\int\limits_{0}^{\frac{\mathfrak{c}_2-\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}}\left|\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}(\alpha,\ell)\right|(1-\ell)^s\,d\ell\right)^{\frac{1}{q}}\Bigg]$$

$$=(\mathfrak{c}_2-\mathfrak{c}_1)\Bigg[\left(\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}\left(\alpha,\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}\right)-\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}\left(\alpha+1,\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}\right)\right)^{1-\frac{1}{q}}$$

$$\times\left(\Upsilon_1\left(\alpha,\delta,s,\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}\right)\left|\mathscr{V}'(\mathfrak{c}_1)\right|^q+\Upsilon_2\left(\alpha,\delta,s,\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}\right)\left|\mathscr{V}'(\mathfrak{c}_2)\right|^q\right)^{\frac{1}{q}}$$

$$+\left(\frac{\mathfrak{c}_2-\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}\left(\alpha,\frac{\mathfrak{c}_2-\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}\right)-\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}\left(\alpha+1,\frac{\mathfrak{c}_2-\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}\right)\right)^{1-\frac{1}{q}}$$

$$\times\left(\Upsilon_2\left(\alpha,\delta,s,\frac{\mathfrak{c}_2-\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}\right)|\mathscr{V}'(\mathfrak{c}_1)|^q+\Upsilon_1\left(\alpha,\delta,s,\frac{\mathfrak{c}_2-\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}\right)|\mathscr{V}'(\mathfrak{c}_2)|^q\right)^{\frac{1}{q}}\right],$$

where we have used (4.3) and (4.4),

$$\begin{aligned}\int_0^{\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}}\left|\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}(\alpha,\ell)\right|d\ell &=\int_0^{\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}}\left(\int_0^t u^{\alpha-1}\mathfrak{e}^{-\delta(\mathfrak{c}_2-\mathfrak{c}_1)u}du\right)d\ell=\int_0^{\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}}\left(\int_u^{\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}}u^{\alpha-1}\mathfrak{e}^{-\delta(\mathfrak{c}_2-\mathfrak{c}_1)u}d\ell\right)du\\
&=\int_0^{\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}}\left(u^{\alpha-1}\mathfrak{e}^{-\delta(\mathfrak{c}_2-\mathfrak{c}_1)u}\left(\tfrac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}-u\right)\right)du\\
&=\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}\int_0^{\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}}\left(u^{\alpha-1}\mathfrak{e}^{-\delta(\mathfrak{c}_2-\mathfrak{c}_1)u}\right)du-\int_0^{\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}}\left(u^{\alpha+1-1}\mathfrak{e}^{-\delta(\mathfrak{c}_2-\mathfrak{c}_1)u}\right)du\\
&=\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}\left(\alpha,\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}\right)-\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}\left(\alpha+1,\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}\right)\end{aligned}$$

and

$$\int_0^{\frac{\mathfrak{c}_2-\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}}\left|\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}(\alpha,\ell)\right|d\ell=\frac{\mathfrak{c}_2-\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}\left(\alpha,\frac{\mathfrak{c}_2-\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}\right)-\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}\left(\alpha+1,\frac{\mathfrak{c}_2-\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}\right).$$

The proof is completed.

Corollary 5.6. *By setting $\delta=0$ in Theorem 5.2, we obtain*

$$\begin{aligned}&\left|\frac{(\varkappa-\mathfrak{c}_1)^\alpha+(\mathfrak{c}_2-\varkappa)^\alpha}{(\mathfrak{c}_2-\mathfrak{c}_1)^\alpha}\mathscr{V}(\varkappa)-\frac{\Gamma(\alpha+1)}{(\mathfrak{c}_2-\mathfrak{c}_1)^\alpha}\left(\mathscr{J}^{\alpha}_{\varkappa^-}[\mathscr{V}](\mathfrak{c}_1)+\mathscr{J}^{\alpha}_{\varkappa^+}[\mathscr{V}](\mathfrak{c}_2)\right)\right|\\
\le&\frac{(\mathfrak{c}_2-\mathfrak{c}_1)}{(\alpha+1)^{1-\frac{1}{q}}}\left[\left(\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}\right)^{(\alpha+1)\left(1-\frac{1}{q}\right)}\left(B_{\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}}(\alpha+1,s+1)|\mathscr{V}'(\mathfrak{c}_1)|^q+\frac{1}{\alpha+s+1}\left(\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}\right)^{\alpha+s+1}|\mathscr{V}'(\mathfrak{c}_2)|^q\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\frac{\mathfrak{c}_2-\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}\right)^{(\alpha+1)\left(1-\frac{1}{q}\right)}\left(\frac{1}{\alpha+s+1}\left(\frac{\mathfrak{c}_2-\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}\right)^{\alpha+s+1}|\mathscr{V}'(\mathfrak{c}_1)|^q+B_{\frac{\mathfrak{c}_2-\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}}(\alpha+1,s+1)|\mathscr{V}'(\mathfrak{c}_2)|^q\right)^{\frac{1}{q}}\right].\end{aligned}$$

Corollary 5.7. *If we attempt to take $\alpha=1$ and $\delta=0$ in Theorem 5.2, we obtain*

$$\begin{aligned}&\left|\mathscr{V}(\varkappa)-\frac{1}{\mathfrak{c}_2-\mathfrak{c}_1}\int_{\mathfrak{c}_1}^{\mathfrak{c}_2}\mathscr{V}(\mathfrak{l})\,d\mathfrak{l}\right|\\
\le&\frac{(\mathfrak{c}_2-\mathfrak{c}_1)}{(s+1)(s+2)2^{1-\frac{1}{q}}}\left[\left(\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}\right)^{2\left(1-\frac{1}{q}\right)}\left\{\left(1-(s+2)\left(\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}\right)\left(\frac{\mathfrak{c}_2-\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}\right)^{s+1}-\left(\frac{\mathfrak{c}_2-\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}\right)^{s+2}\right)|\mathscr{V}'(\mathfrak{c}_1)|^q\right.\right.\end{aligned}$$

$$+ (s+1)\left(\frac{\varkappa - \mathfrak{c}_1}{\mathfrak{c}_2 - \mathfrak{c}_1}\right)^{s+2} |\mathscr{V}'(\mathfrak{c}_2)|^q \Bigg\}^{\frac{1}{q}} + \left(\frac{\mathfrak{c}_2 - \varkappa}{\mathfrak{c}_2 - \mathfrak{c}_1}\right)^{2\left(1-\frac{1}{q}\right)} \Bigg\{ (s+1)\left(\frac{\mathfrak{c}_2 - \varkappa}{\mathfrak{c}_2 - \mathfrak{c}_1}\right)^{s+2} |\mathscr{V}'(\mathfrak{c}_1)|^q$$

$$+ \left(1 - (s+2)\left(\frac{\mathfrak{c}_2 - \varkappa}{\mathfrak{c}_2 - \mathfrak{c}_1}\right)\left(\frac{\varkappa - \mathfrak{c}_1}{\mathfrak{c}_2 - \mathfrak{c}_1}\right)^{s+1} - \left(\frac{\varkappa - \mathfrak{c}_1}{\mathfrak{c}_2 - \mathfrak{c}_1}\right)^{s+2}\right) |\mathscr{V}'(\mathfrak{c}_2)|^q \Bigg\}^{\frac{1}{q}} \Bigg],$$

where $B_z(.,.)$ is the incomplete beta function.

Corollary 5.8. *By setting $s = 1$, Theorem 5.2 yields*

$$\left| \left[\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}\left(\alpha, \frac{\varkappa - \mathfrak{c}_1}{\mathfrak{c}_2 - \mathfrak{c}_1}\right) + \gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}\left(\alpha, \frac{\mathfrak{c}_2 - \varkappa}{\mathfrak{c}_2 - \mathfrak{c}_1}\right)\right] \mathscr{V}(\varkappa) - \frac{\Gamma(\alpha)}{(\mathfrak{c}_2 - \mathfrak{c}_1)^\alpha}\left(\mathscr{T}^{\alpha,\delta}_{\varkappa^-}[\mathscr{V}](\mathfrak{c}_1) + \mathscr{T}^{\alpha,\delta}_{\varkappa^+}[\mathscr{V}](\mathfrak{c}_2)\right)\right|$$

$$\leq (\mathfrak{c}_2 - \mathfrak{c}_1)\left[\left(\frac{\varkappa - \mathfrak{c}_1}{\mathfrak{c}_2 - \mathfrak{c}_1}\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}\left(\alpha, \frac{\varkappa - \mathfrak{c}_1}{\mathfrak{c}_2 - \mathfrak{c}_1}\right) - \gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}\left(\alpha + 1, \frac{\varkappa - \mathfrak{c}_1}{\mathfrak{c}_2 - \mathfrak{c}_1}\right)\right)^{1-\frac{1}{q}} \left(\mathscr{L}_1 |\mathscr{V}'(\mathfrak{c}_1)|^q + \mathscr{L}_2 |\mathscr{V}'(\mathfrak{c}_2)|^q\right)^{\frac{1}{q}}\right.$$

$$\left. + \left(\frac{\mathfrak{c}_2 - \varkappa}{\mathfrak{c}_2 - \mathfrak{c}_1}\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}\left(\alpha, \frac{\mathfrak{c}_2 - \varkappa}{\mathfrak{c}_2 - \mathfrak{c}_1}\right) - \gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}\left(\alpha + 1, \frac{\mathfrak{c}_2 - \varkappa}{\mathfrak{c}_2 - \mathfrak{c}_1}\right)\right)^{1-\frac{1}{q}} \left(\mathscr{L}_3 |\mathscr{V}'(\mathfrak{c}_1)|^q + \mathscr{L}_4 |\mathscr{V}'(\mathfrak{c}_2)|^q\right)^{\frac{1}{q}}\right],$$

where $\mathscr{L}_1$, $\mathscr{L}_2$, $\mathscr{L}_3$ and $\mathscr{L}_4$ are given by

$$\mathscr{L}_1 = \int_0^{\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}} \left|\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}(\alpha, \ell)\right| (1 - \ell)\, d\ell = \int_0^{\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}} \left(\int_u^{\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}} u^{\alpha-1}\mathfrak{e}^{-\delta(\mathfrak{c}_2-\mathfrak{c}_1)u}(1 - \ell)\, d\ell\right) du$$

$$= \int_0^{\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}} \left(u^{\alpha-1}\mathfrak{e}^{-\delta(\mathfrak{c}_2-\mathfrak{c}_1)u}\left(\left(\frac{\varkappa - \mathfrak{c}_1}{\mathfrak{c}_2 - \mathfrak{c}_1} - \frac{1}{2}\left(\tfrac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}\right)^2\right) - \left(u - \frac{1}{2}u^2\right)\right)\right) du$$

$$= \left(\frac{\varkappa - \mathfrak{c}_1}{\mathfrak{c}_2 - \mathfrak{c}_1} - \frac{1}{2}\left(\frac{\varkappa - \mathfrak{c}_1}{\mathfrak{c}_2 - \mathfrak{c}_1}\right)^2\right) \int_0^{\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}} u^{\alpha-1}\mathfrak{e}^{-\delta(\mathfrak{c}_2-\mathfrak{c}_1)u} du - \int_0^{\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}} u^{\alpha+1-1}\mathfrak{e}^{-\delta(\mathfrak{c}_2-\mathfrak{c}_1)u} du + \frac{1}{2}\int_0^{\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}} u^{\alpha+2-1}\mathfrak{e}^{-\delta(\mathfrak{c}_2-\mathfrak{c}_1)u} du$$

$$= \left(\frac{\varkappa - \mathfrak{c}_1}{\mathfrak{c}_2 - \mathfrak{c}_1} - \frac{1}{2}\left(\frac{\varkappa - \mathfrak{c}_1}{\mathfrak{c}_2 - \mathfrak{c}_1}\right)^2\right)\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}\left(\alpha, \frac{\varkappa - \mathfrak{c}_1}{\mathfrak{c}_2 - \mathfrak{c}_1}\right) - \gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}\left(\alpha + 1, \frac{\varkappa - \mathfrak{c}_1}{\mathfrak{c}_2 - \mathfrak{c}_1}\right) + \frac{1}{2}\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}\left(\alpha + 2, \frac{\varkappa - \mathfrak{c}_1}{\mathfrak{c}_2 - \mathfrak{c}_1}\right),$$

$$\mathscr{L}_2 = \int_0^{\frac{\mathfrak{c}_2-\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}} \left|\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}(\alpha, \ell)\right| \ell\, d\ell = \int_0^{\frac{\mathfrak{c}_2-\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}} \left(\int_u^{\frac{\mathfrak{c}_2-\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}} u^{\alpha-1}\mathfrak{e}^{-\delta(\mathfrak{c}_2-\mathfrak{c}_1)u}\ell\, d\ell\right) du$$

$$= \frac{1}{2}\left(\frac{\mathfrak{c}_2 - \varkappa}{\mathfrak{c}_2 - \mathfrak{c}_1}\right)^2 \gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}\left(\alpha, \frac{\mathfrak{c}_2 - \varkappa}{\mathfrak{c}_2 - \mathfrak{c}_1}\right) - \frac{1}{2}\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}\left(\alpha + 2, \frac{\mathfrak{c}_2 - \varkappa}{\mathfrak{c}_2 - \mathfrak{c}_1}\right),$$

$$\mathscr{L}_3=\int_0^{\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}}\left|\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}(\alpha,\ell)\right|\ell d\ell=\int_0^{\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}}\left(\int_u^{\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}}u^{\alpha-1}\mathrm{e}^{-\delta(\mathfrak{c}_2-\mathfrak{c}_1)u}\ell d\ell\right)du$$

$$=\frac{1}{2}\left(\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}\right)^2\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}\left(\alpha,\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}\right)-\frac{1}{2}\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}\left(\alpha+2,\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}\right)$$

and

$$\mathscr{L}_4=\int_0^{\frac{\mathfrak{c}_2-\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}}\left|\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}(\alpha,\ell)\right|(1-\ell)\,d\ell=\int_0^{\frac{\mathfrak{c}_2-\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}}\left(\int_u^{\frac{\mathfrak{c}_2-\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}}u^{\alpha-1}\mathrm{e}^{-\delta(\mathfrak{c}_2-\mathfrak{c}_1)u}(1-\ell)\,d\ell\right)du$$

$$=\left(\left(\frac{\mathfrak{c}_2-\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}\right)-\frac{1}{2}\left(\tfrac{\mathfrak{c}_2-\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}\right)^2\right)\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}\left(\alpha,\frac{\mathfrak{c}_2-\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}\right)-\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}\left(\alpha+1,\frac{\mathfrak{c}_2-\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}\right)+\frac{1}{2}\gamma_{\delta(\mathfrak{c}_2-\mathfrak{c}_1)}\left(\alpha+2,\frac{\mathfrak{c}_2-\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}\right).$$

Corollary 5.9. *In Theorem 5.2, if we take $\delta=0$ and $s=1$, we obtain*

$$\left|\frac{(\varkappa-\mathfrak{c}_1)^\alpha+(\mathfrak{c}_2-\varkappa)^\alpha}{(\mathfrak{c}_2-\mathfrak{c}_1)^\alpha}\mathscr{V}(\varkappa)-\frac{\Gamma(\alpha+1)}{(\mathfrak{c}_2-\mathfrak{c}_1)^\alpha}\left(\mathscr{J}^\alpha_{\varkappa^-}[\mathscr{V}](\mathfrak{c}_1)+\mathscr{J}^\alpha_{\varkappa^+}[\mathscr{V}](\mathfrak{c}_2)\right)\right|$$

$$\leq\frac{\mathfrak{c}_2-\mathfrak{c}_1}{\alpha+1}\left[\left(\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}\right)^{\alpha+1}\left(\frac{(\alpha+2)(\mathfrak{c}_2-\mathfrak{c}_1)-(\alpha+1)(\varkappa-\mathfrak{c}_1)}{(\alpha+2)(\mathfrak{c}_2-\mathfrak{c}_1)}|\mathscr{V}'(\mathfrak{c}_1)|^q+\frac{(\alpha+1)(\varkappa-\mathfrak{c}_1)}{(\alpha+2)(\mathfrak{c}_2-\mathfrak{c}_1)}|\mathscr{V}'(\mathfrak{c}_2)|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{\mathfrak{c}_2-\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}\right)^{\alpha+1}\left(\frac{(\alpha+1)(\mathfrak{c}_2-\varkappa)}{(\alpha+2)(\mathfrak{c}_2-\mathfrak{c}_1)}|\mathscr{V}'(\mathfrak{c}_1)|^q+\frac{(\alpha+2)(\mathfrak{c}_2-\mathfrak{c}_1)-(\alpha+1)(\mathfrak{c}_2-\varkappa)}{(\alpha+2)(\mathfrak{c}_2-\mathfrak{c}_1)}|\mathscr{V}'(\mathfrak{c}_2)|^q\right)^{\frac{1}{q}}\right].$$

Corollary 5.10. *By choosing $\delta=0$ and $s=\alpha=1$, Theorem 5.2 gives*

$$\left|\mathscr{V}(\varkappa)-\frac{1}{\mathfrak{c}_2-\mathfrak{c}_1}\int_{\mathfrak{c}_1}^{\mathfrak{c}_2}\mathscr{V}(\mathfrak{l})\,d\mathfrak{l}\right|\leq\frac{\mathfrak{c}_2-\mathfrak{c}_1}{2}\left[\left(\frac{\varkappa-\mathfrak{c}_1}{\mathfrak{c}_2-\mathfrak{c}_1}\right)^2\left(\frac{3\mathfrak{c}_2-\mathfrak{c}_1-2\varkappa}{3(\mathfrak{c}_2-\mathfrak{c}_1)}|\mathscr{V}'(\mathfrak{c}_1)|^q+\frac{2(\varkappa-\mathfrak{c}_1)}{3(\mathfrak{c}_2-\mathfrak{c}_1)}|\mathscr{V}'(\mathfrak{c}_2)|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{\mathfrak{c}_2-\varkappa}{\mathfrak{c}_2-\mathfrak{c}_1}\right)^2\left(\frac{2(\mathfrak{c}_2-\varkappa)}{3(\mathfrak{c}_2-\mathfrak{c}_1)}|\mathscr{V}'(\mathfrak{c}_1)|^q+\frac{\mathfrak{c}_2-3\mathfrak{c}_1+2\varkappa}{3(\mathfrak{c}_2-\mathfrak{c}_1)}|\mathscr{V}'(\mathfrak{c}_2)|^q\right)^{\frac{1}{q}}\right],$$

which is the same result provided in [31, Corollary 3.16].

6. Applications to special means

In this section, we apply the obtained inequality to the study of special means. We recall that for $0<\mathfrak{c}_1<\mathfrak{c}_2$, the arithmetic mean is defined by $A(\mathfrak{c}_1,\mathfrak{c}_2)=\frac{\mathfrak{c}_1+\mathfrak{c}_2}{2}$, the geometric mean $G(\mathfrak{c}_1,\mathfrak{c}_2)=\sqrt{\mathfrak{c}_1\mathfrak{c}_2}$, and the generalized logarithmic mean by $L_p(\mathfrak{c}_1,\mathfrak{c}_2)=\left[\frac{\mathfrak{c}_2^{p+1}-\mathfrak{c}_1^{p+1}}{(p+1)(\mathfrak{c}_2-\mathfrak{c}_1)}\right]^{1/p}$ for $p\in\mathbb{R}\setminus\{-1,0\}$.

Proposition 6.1. *Let* $\mathfrak{c}_1, \mathfrak{c}_2 \in \mathbb{R}$ *with* $0 < \mathfrak{c}_1 < \mathfrak{c}_2$ *and* $0 < s < 1$. *Then, the following inequality holds:*

$$\begin{aligned}&\left|(s+2)(s+3)\left(A(\mathfrak{c}_1^2, \mathfrak{c}_2^2) - G^2(\mathfrak{c}_1, \mathfrak{c}_2)\right)A^{s+1}(\mathfrak{c}_1, \mathfrak{c}_2) + 4A^{s+3}(\mathfrak{c}_1, \mathfrak{c}_2) - 4A(\mathfrak{c}_1^{s+3}, \mathfrak{c}_2^{s+3})\right|\\ &\leq \frac{2^{s+2} - s - 3}{2^{s+1}}(\mathfrak{c}_2 - \mathfrak{c}_1)^3\left(\mathfrak{c}_1^s + \mathfrak{c}_2^s\right).\end{aligned}$$

Proof. The assertion follows from Corollary 4.1 with $\alpha = 2$ and $\delta = 0$ applied to the function $\mathscr{V}(\mathfrak{t}) = \mathfrak{t}^{s+1}$.

Proposition 6.2. *Let* $\mathfrak{c}_1, \mathfrak{c}_2 \in \mathbb{R}$ *with* $0 < \mathfrak{c}_1 < \mathfrak{c}_2$ *and* $0 < s < 1$. *Then, the following inequality holds:*

$$\left|(\mathfrak{c}_2 - \mathfrak{c}_1)\mathfrak{c}_1^{s+1} - 2\left(\mathfrak{c}_2 L_{s+1}^{s+1}(\mathfrak{c}_1, \mathfrak{c}_2) + L_{s+2}^{s+2}(\mathfrak{c}_1, \mathfrak{c}_2)\right)\right| \leq (\mathfrak{c}_2 - \mathfrak{c}_1)^2 \frac{(s+1)(s+2)\mathfrak{c}_1^s + \mathfrak{c}_2^s}{(s+2)(s+3)}.$$

Proof. The assertion follows from Corollary 4.2 with $\alpha = 2$ applied to the function $\mathscr{V}(\mathfrak{t}) = \mathfrak{t}^{s+1}$.

7. Conclusions

In conclusion, we have successfully explored the application of tempered fractional integrals in developing novel inequalities for differentiable s-convex functions. By extending classical results to a generalized fractional framework, we have provided a new version of Hermite-Hadamard inequality via tempered fractional integrals. Subsequently, we have introduced a new integral identity that serves as a cornerstone for deriving Ostrowski-type inequalities. This approach not only deepens our understanding of the properties of s-convex functions but also offers a powerful and flexible tool for analyzing systems with exponentially decaying memory effects. The findings contribute significantly to the theoretical advancement of fractional calculus and broaden its scope in mathematical analysis. Moreover, the practical relevance of these results is highlighted through several illustrative examples and applications. Researchers could further investigate the extension of these results to other generalized classes of convexity, such as (η, h)-convexity or harmonic s-convexity, which may yield even more versatile bounds. Additionally, exploring these inequalities within the framework of multivariate tempered fractional calculus or applying them to stochastic processes could open new avenues in optimization and error analysis.

Author contributions

A. Lakhdari, T. Abdeljawad, M. A. Alqudah and N. Mlaiki: Conceptualization, Analysis, Writing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgements

The author Manar A. Alqudah extends the appreciation to Princess Nourah bint Abdulrahman University Researchers Supporting Project number (PNURSP2026R14), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia.

The authors T. Abdeljawad and N. Mlaiki would like to thank the Prince Sultan University for paying the APC and for the support through the TAS research lab.

Conflict of interest

The authors declare no conflicts of interest.

References

1. C. P. Niculescu, Convexity according to the geometric mean, *Math. Inequal. Appl.*, **3** (2000), 155–167. https://doi.org/10.7153/mia-03-19
2. A. Lakhdari, B. Bin-Mohsin, F. Jarad, H. Xu, B. Meftah, A parametrized approach to generalized fractional integral inequalities: Hermite-Hadamard and Maclaurin variants, *J. King Saud Univ. Sci.*, **36** (2024), 103523.
3. A. Lakhdari, H. Budak, N. Mlaiki, B. Meftah, T. Abdeljawad, New insights on fractal-fractional integral inequalities: Hermite-Hadamard and Milne estimates, *Chaos Solitons Fract.*, **193** (2025), 116087. https://doi.org/10.1016/j.chaos.2025.116087
4. H. Hudzik, L. Maligranda, Some remarks on s-convex functions, *Aequationes Math.*, **48** (1994), 100–111. https://doi.org/10.1007/BF01837981
5. S. S. Dragomir, S. Fitzpatrick, The Hadamard inequalities for s-convex functions in the second sense, *Demonstratio Math.*, **32** (1999), 687–696. https://doi.org/10.1515/dema-1999-0403
6. A. Ostrowski, Über die absolutabweichung einer differentiierbaren funktion von ihrem integralmittelwert, *Comment. Math. Helv.*, **10** (1937), 226–227. https://doi.org/10.1007/BF01214290
7. M. Alomari, M. Darus, Some Ostrowski type inequalities for convex functions with applications, *RGMIA*, **13** (2010), 3.
8. M. Alomari, M. Darus, Some Ostrowski type inequalities for quasi-convex functions with applications to special means, *RGMIA Res. Rep. Coll*, **13** (2010), 6.
9. M. Alomari, M. Darus, S. S. Dragomir, P. Cerone, Ostrowski type inequalities for functions whose derivatives are s-convex in the second sense, *Appl. Math. Lett.*, **23** (2010), 1071–1076. https://doi.org/10.1016/j.aml.2010.04.038
10. B. Meftah, Ostrowski inequalities for functions whose first derivatives are logarithmically preinvex, *Chin. J. Math.*, **2016** (2016), 1–10. https://doi.org/10.1155/2016/5292603
11. B. Meftah, A. Azaizia, Fractional Ostrowski type inequalities for functions whose first derivatives are MT-preinvex, *Rev. MATUA*, **6** (2019), 33–43.
12. P. Cerone, S. S. Dragomir, Ostrowski type inequalities for functions whose derivatives satisfy certain convexity assumptions, *Demonstratio Math.*, **37** (2004), 299–308. https://doi.org/10.1515/dema-2004-0208
13. M. Z. Sarikaya, On the Ostrowski type integral inequality, *Acta Math. Univ. Comenianae*, **LXXIX** (2010), 129–134.

14. E. Set, New inequalities of Ostrowski type for mappings whose derivatives are s-convex in the second sense via fractional integrals, *Comput. Math. Appl.*, **63** (2012), 1147–1154. https://doi.org/10.1016/j.camwa.2011.12.023

15. E. Set, M. E. Özdemir, M. Z. Sarıkaya, New inequalities of Ostrowski's type for s-convex functions in the second sense with applications, *Facta Univ. Ser. Math. Inform.*, **27** (2012), 67–82.

16. C. Yıldız, M. E. Özdemir, M. Z. Sarikaya, New generalizations of Ostrowski-like type inequalities for fractional integrals, *Kyungpook Math. J.*, **56** (2016), 161–172. https://doi.org/10.5666/KMJ.2016.56.1.161

17. P. O. Mohammed, M. Z. Sarikaya, D. Baleanu, On the generalized Hermite-Hadamard inequalities via the tempered fractional integrals, *Symmetry*, **12** (2020), 595. https://doi.org/10.3390/sym12040595

18. Y. Cao, J. F. Cao, P. Z. Tan, T. S. Du, Some parameterized inequalities arising from the tempered fractional integrals involving the (μ, η)-incomplete gamma functions, *J. Math. Inequal.*, **16** (2022), 1091–1121. https://dx.doi.org/10.7153/jmi-2022-16-73

19. P. Z. Tan, T. S. Du, On the multi-parameterized inequalities involving the tempered fractional integral operators, *Filomat*, **37** (2023), 4919–4941. https://doi.org/10.2298/FIL2315919T

20. W. Haider, H. Budak, A. Shehzadi, F. Hezenci, H. B. Chen, A comprehensive study on Milne-type inequalities with tempered fractional integrals, *Bound. Value Probl.*, **2024** (2024), 53. https://doi.org/10.1186/s13661-024-01855-1

21. W. Haider, H. Budak, A. Shehzadi, F. Hezenci, H. B. Chen, Analysing Milne-type inequalities by using tempered fractional integrals, *Anal. Math. Phys.*, **14** (2024), 101. https://doi.org/10.1007/s13324-024-00958-3

22. A. Lakhdari, M. Z. Sarikaya, H. Budak, B. Meftah, On tempered fractional Bullen-type inequalities: analysis in different function settings, *Kuwait J. Sci.*, **53** (2026), 100515. https://doi.org/10.1016/j.kjs.2025.100515

23. F. Hezenci, H. Budak, Midpoint-type inequalities via twice-differentiable functions on tempered fractional integrals, *J. Inequal. Appl.*, **2023** (2023), 150. https://doi.org/10.1186/s13660-023-03064-3

24. G. Rahman, K. S. Nisar, T. Abdeljawad, Tempered fractional integral inequalities for convex functions, *Mathematics*, **8** (2020), 500. https://doi.org/10.3390/math8040500

25. M. Z. Sarikaya, M. E. Kiriş, On the generalized Ostrowski type inequalities via tempered fractional integrals, *TJMM*, **12** (2020), 129–134.

26. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier, 2006.

27. F. Sabzikar, M. M. Meerschaert, J. H. Chen, Tempered fractional calculus, *J. Comput. Phys.*, **293** (2015), 14–28. https://doi.org/10.1016/j.jcp.2014.04.024

28. M. Z. Sarikaya, E. Set, H. Yaldiz, N. Başak, Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities, *Math. Comput. Model.*, **57** (2013), 2403–2407. https://doi.org/10.1016/j.mcm.2011.12.048

29. M. Z. Sarikaya, H. Yıldırım, On Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals, *Miskolc Math. Notes*, **17** (2016), 1049–1059. https://doi.org/10.18514/MMN.2017.1197
30. U. S. Kırmacı, Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula, *Appl. Math. Comput.*, **147** (2004), 137–146. https://doi.org/10.1016/S0096-3003(02)00657-4
31. B. Meftah, Ostrowski's inequality for functions whose first derivatives are s-preinvex in the second sense, *Khayyam J. Math.*, **3** (2017), 61–80.

AIMS Press