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Research article

Approximate controllability of higher-order Sobolev-type stochastic AB-fractional differential inclusions with impulses and Clarke sub-differentials

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Abstract: In this paper, we investigated the existence of mild solutions and the approximate controllability of a novel class of Sobolev-type stochastic impulsive differential inclusions driven by a higher-order Atangana–Baleanu fractional derivative in the Caputo sense. The system is formulated in an infinite-dimensional framework and incorporates Brownian motion processes, impulsive effects, and non-smooth multi-valued nonlinearities described via Clarke's generalized sub-differential. By employing methods from fractional evolution theory, stochastic analysis, and multi-valued fixed-point theory, we established sufficient conditions for solvability and approximate controllability. The results extended classical controllability frameworks to systems exhibiting memory, randomness, and impulsive dynamics. An illustrative example is provided to demonstrate the applicability of the theoretical findings.

Keywords: fractional derivatives; Sobolev-type; stochastic differential inclusions; impulsive systems; approximate controllability; Clarke sub-differential; nonlocal conditions; brownian motion process; mild solutions
Mathematics Subject Classification: 26A33, 34A60, 34K45, 60H15, 93B05, 93E03

Abbreviations:
DE ⇒ Differential equation
IDE ⇒ Impulsive differential equation
$\mathbb{SSDI}$ ⇒ Sobolev-type stochastic differential inclusion
bd ⇒ Bounded
cl ⇒ Closed
cv ⇒ Convex
$\mathbb{MV}$ ⇒ Multi-valued
s.t. ⇒ Such that
$\mathbb{MEAS}$ ⇒ Measurable
$\mathbb{USC}$ ⇒ Upper semi-continuous
$\mathbb{LO}$ ⇒ Linear operator
C.C. ⇒ Completely continuous
r.c. ⇒ Relatively compact
AB ⇒ Atangana–Baleanu
HF ⇒ Hilfer fractional
$\mathbb{BM}$ ⇒ Brownian motion
RL ⇒ Riemann Liouville
C-FD ⇒ Caputo fractional derivative
AB-FD ⇒ Atangana–Baleanu fractional derivative
ML ⇒ Mittag Leffler
FDI ⇒ Fractional differential inclusion
FC ⇒ Fractional calculus
FO ⇒ Fractional operator
$\mathbb{FI}$ ⇒ Fractional integral
$\mathbb{ACO}$ ⇒ Approximate controllability
$\mathbb{MS}$ ⇒ Mild solution
FPT ⇒ Fixed point theorem
$\mathbb{LDCT}$ ⇒ Lebesgue dominated convergence theorem
BS ⇒ Banach space
HS ⇒ Hilbert space

1. Introduction

During the past three decades, FC has emerged as a robust analytical paradigm for modeling complex dynamical systems with memory and hereditary effects. Its intrinsic ability to incorporate nonlocal temporal behavior has made it a valuable tool in diverse areas such as nonlinear dynamics, viscoelasticity, electrical circuit theory, and biophysical processes [1–3]. In particular, FC has proven effective in capturing anomalous diffusion phenomena and stochastic dynamics arising in physical and engineering systems. This has led to significant developments in theoretical and applied directions, including fractional stochastic models and impulsive systems [4–6]. Moreover, qualitative analysis and bifurcation structures in nonlinear systems continue to attract increasing attention, further enriching the interplay between fractional operators and complex dynamical behaviors [7]. Motivated by the demand for more accurate and realistic mathematical descriptions of natural phenomena, researchers have focused on extending classical fractional operators to incorporate singular and non-singular memory kernels [8–10].

The classical fractional derivatives introduced by RL and Caputo are characterized by singular convolution kernels and have served as fundamental tools in the development of fractional dynamical models. Nevertheless, the singular nature of these kernels may complicate the analytical treatment and the physical interpretation of certain models. In response to these challenges, several generalized FOs with non-singular kernels have been proposed in recent years. Among them, the nonsingular formulations introduced by Caputo and Fabrizio and by Atangana and Baleanu have received particular attention. These operators are based on exponential-type and ML memory kernels, respectively, thereby offering more flexible descriptions of memory-dependent processes (see [1, 2]). Furthermore, Abdeljawad [11] extended the AB-FO to the higher-order setting, thereby broadening its applicability in the study of fractional dynamical systems.

The selection of the AB-FD in the Caputo sense in this work is motivated by several analytical and modeling advantages. In contrast to classical fractional derivatives, the AB operator involves a non-singular and non-local ML kernel, which enables it to capture memory effects in a more realistic manner while avoiding the singular behavior associated with traditional power-law kernels. As a consequence, this operator has proved to be particularly suitable for describing dynamical systems in which the influence of past states decays according to ML-type memory laws. From a modeling standpoint, the incorporation of the AB operator enables the considered stochastic impulsive system to integrate hereditary memory effects together with stochastic perturbations and impulsive phenomena within a unified mathematical framework. Such a formulation provides a flexible and robust structure for the analysis of complex dynamical processes arising in physics, engineering, and applied sciences where nonlocal temporal interactions play a significant role. Nevertheless, it should be emphasized that the AB derivative also introduces certain analytical challenges. Compared with the classical Caputo or RL operators, the mathematical theory associated with the AB derivative is developing, and many analytical techniques must be adapted to accommodate the presence of the non-singular ML kernel. Consequently, the study of existence, stability, and controllability properties for systems governed by the AB operator remains an active and evolving area of research.

From a stochastic standpoint, $\mathbb{BM}$ remains a fundamental model for Gaussian perturbations characterized by independent and stationary increments. Its mathematical formulation and wide applicability in stochastic analysis have been extensively developed in the literature [12–14]. Moreover, the inherent path-wise irregularity of Brownian trajectories makes such processes particularly suitable for modeling random fluctuations in complex dynamical systems. This feature has been effectively utilized in the study of stochastic impulsive and integro-differential systems [15].

Controllability constitutes a central pillar in modern control theory, particularly in the study of dynamical systems, where it plays a crucial role in qualitative analysis and control design. In recent years, considerable attention has been directed toward controllability properties of nonlinear systems in infinite-dimensional spaces, investigated via diverse analytical methodologies. Within this framework, the concepts of exact controllability and $\mathbb{ACO}$ are inherently different. The former requires steering the system exactly to a desired terminal state, whereas the latter only ensures that the system trajectory can be driven arbitrarily close to the target. From a practical perspective, $\mathbb{ACO}$ is often more relevant, as it better reflects realistic operational constraints and tolerances. Furthermore, establishing exact controllability results for infinite-dimensional systems is typically highly nontrivial, which enhances the significance of $\mathbb{ACO}$ in such settings. In this context, a number of researchers have addressed $\mathbb{ACO}$ for nonlinear evolution equations and related systems [16–18]. Additional contributions have further expanded the applicability of controllability theory to fractional and integro-differential systems, highlighting its effectiveness in a wide range of applications [19].

Impulsive differential equations (IDE) and inclusion frameworks provide a powerful mathematical setting for describing systems subject to abrupt changes and discontinuities arising in real-world processes. These models have stimulated substantial interest in establishing existence results for classical and mild solutions of fractional impulsive systems in abstract spaces [20–22]. Furthermore, the incorporation of non-instantaneous impulses and stability considerations has enriched the qualitative theory of such systems, leading to significant advances in the analysis of dynamical behavior under impulsive effects [23, 24].

The significance of differential inclusions governed by the Clarke sub-differential stems from their intrinsic linkage to the sub-differential calculus of locally Lipschitz and cv functions (see [25]). This framework exerts substantial influence on modeling and analysis, particularly in advanced mechanical and engineering settings where non-convex energy landscapes arise in variational representations of virtual work inequalities. Such formulations naturally lead to inequality or unilateral problems, whose treatment demands methodologies fundamentally distinct from those employed for classical variational inequalities or bilateral equality-based systems. Consequently, the systematic investigation of differential inclusions of Clarke type becomes indispensable in these contexts.

In this vein, Wang et al. [26] established $\mathbb{ACO}$ results for higher-order fractional (HF) evolution hemi-variational inequalities. Ahmed and Ragusa [27] established the nonlocal controllability of Sobolev-type conformable fractional stochastic evolution inclusions with Clarke subdifferential. Ahmed and Zhu [28] studied the nonlocal controllability for Hilfer fractional differential inclusions with Clarke subdifferential and nonlinear noise. Kavitha and Vijayakumar [29] analyzed $\mathbb{ACO}$ for HF neutral evolution inclusions involving generalized Clarke sub-differentials. Furthermore, Hussain et al. [30] employed measures of non-compactness, sine and cosine operator families, $\mathbb{MV}$-mappings, and fixed point principles to derive existence criteria for $\mathbb{MS}$s of Caputo fractional nonlocal control systems of order $1 < \mu < 2$ with Clarke-type sub-differential terms. Moreover, Johnson and Vijayakumar [31] examined optimal control problems for second-order differential inclusions incorporating Sobolev-type structures, delay effects, $\mathbb{MV}$-dynamics, and Clarke sub-differentials.

In contemporary scholarly discourse, a growing body of work has been devoted to the analytical treatment of fractional evolution systems with orders that lie within the interval $\mu \in (1, 2)$, leading to substantial progress in the establishment of existence and controllability properties. These developments have been driven by the deployment of sine and cosine operator families, Laplace transform techniques, and a spectrum of FP methodologies. Among the early contributions, Shu et al. [32] investigated the existence and $\mathbb{ACO}$ for RL fractional stochastic differential systems of order $1 < \mu < 2$ under nonlocal constraints. Subsequently, Zhou and He [33] derived existence and controllability results for Caputo fractional evolution equations within the same order range. In the setting of the higher-order fractional derivative, Zhou and He [34] addressed the Cauchy problem for fractional evolution equations of order $1 < \mu < 2$ posed on semi-infinite intervals. Furthermore, Pradeesh and Vijayakumar [35] established existence results for Sobolev-type HF stochastic evolution inclusions of order $1 < \mu < 2$ subject to non-local conditions.

Stimulated by these outstanding theoretical deficiencies, this contribution is devoted to the analysis of the $\mathbb{ACO}$ for an unprecedented class of $\mathbb{SSIDI}$s driven by a higher-order AB-FD interpreted in the Caputo sense. The proposed evolution system is formulated within an infinite-dimensional HS and is influenced by Gaussian fluctuations described via $\mathbb{BM}$ processes. In addition, the dynamics encapsulate non-local hereditary mechanisms together with non-smooth, $\mathbb{MV}$ non-linearities modeled through Clarke-type sub-differentials. This setting yields a genuinely novel amalgamation of higher-order fractional memory effects, stochastic perturbations, impulsive phenomena, and non-convex variational structures, thereby broadening the theoretical horizon of fractional stochastic control to encompass a substantially richer family of infinite-dimensional dynamical systems.

$$\begin{aligned} {}^{\mathfrak{ABC}}D^{\hbar}_{0+}\bar{Y}x(\tau) &\in Zx(\tau) + \mho U(\tau) + \mathfrak{F}(\tau, x_\tau, \int_0^\tau \mathbb{G}(\tau,\bar{s},x_{\bar{s}})\,d\bar{s}) + \zeta(\tau, x_\tau, \int_0^\tau \mathbb{H}(\tau,\bar{s},x_{\bar{s}})\,d\bar{s})\frac{d\bar{W}(\tau)}{d\tau} \\ &\quad + \partial\mathcal{S}(\tau, x(\tau)), \quad \tau \in \mathfrak{J}^* := (0,\rho], \quad \tau \neq \tau_\kappa, \\ \delta x\,|_{\tau=\tau_\kappa} &= \mathcal{I}_\kappa(x(\tau_\kappa)),\ \delta x'\,|_{\tau=\tau_\kappa} = \widehat{\mathcal{I}}_\kappa(x(\tau_\kappa)) \quad \kappa = 1,2,3,\cdots,\gamma, \\ x(0) + \vartheta(x) &= x_0 = \varphi \in \mathbb{B},\ x'(0) + \beta(x) = x_1 \in \mathbb{L}^2_0(\Omega, \mathbb{X}). \end{aligned} \tag{1.1}$$

The symbol ${}^{\mathfrak{ABC}}D^{\hbar}_{0+}$ designates the AB-FD of order $1 < \hbar < 2$ interpreted in the Caputo framework. The state trajectory $x(\cdot)$ evolves in a separable HS $\mathbb{X}$ endowed with the inner product $\langle\cdot,\cdot\rangle$ and the associated norm $\|\cdot\|$. Let $\mathbb{Y}$ denote another separable HS equipped with the inner product $\langle\cdot,\cdot\rangle_{\mathbb{Y}}$ and norm $\|\cdot\|_{\mathbb{Y}}$. Random perturbations are introduced through an $\mathbb{Y}$-valued $\mathbb{BM}$ process $\{\bar{W}(\tau)\}_{\tau\geq 0}$ with a non-negative, symmetric covariance operator $\mathcal{Q}$. Let $0 = \tau_0 < \tau_1 < \cdots < \tau_\gamma < \tau_{\gamma+1} = \rho$ be the prescribed impulsive instants. The corresponding jump discontinuity at $\tau = \tau_\kappa$ is defined by $\delta x\big|_{\tau=\tau_\kappa} = x(\tau_\kappa^+) - x(\tau_\kappa^-)$. The operators Z and $\bar{Y}$ act linearly on $\mathbb{X}$, whereas $\partial\mathcal{S}(\tau, x(\tau))$ denotes the Clarke sub-differential of a locally Lipschitz functional $\mathcal{S}(\tau, x(\tau))$. The stochastic setting is described by a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_\tau\}_{\tau\geq 0}, \mathbb{P})$ satisfying the usual hypotheses, namely, the filtration $\{\mathcal{F}_\tau\}_{\tau\geq 0}$ is right-continuous, non-decreasing, and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. The mappings $\vartheta(\cdot)$ and $\beta(\cdot)$ are defined on the space of continuous functions over the interval $\mathfrak{J} := [0,\rho]$. The incorporation of non-local initial constraints enhances the descriptive accuracy of the model by encoding information distributed over a time interval rather than relying exclusively on point-wise specifications. For each $\tau \in \mathfrak{J}$, the history segment $x_\tau : (-\infty, 0] \to \mathbb{X}$, defined by $x_\tau(\theta) = x(\tau+\theta)$ for $\theta \leq 0$, is assumed to belong to an abstract phase space $\mathbb{B}$. The initial function $\varphi = \{\varphi(\tau) : \tau \in (-\infty, 0]\}$ is taken as an $\mathcal{F}_0$-$\mathbb{MEAS}$, $\mathbb{B}$-valued random variable, independent of the $\mathbb{BM}$ process $\bar{W}(\tau)$, and possessing finite second-order moments. The non-linear mappings $\mathfrak{F} : \mathfrak{J}\times\mathbb{B}\times\mathbb{X} \to \mathbb{X}$ and $\zeta : \mathfrak{J}\times\mathbb{B}\times\mathbb{X} \to \mathbb{L}(\mathbb{Y}, \mathbb{X})$, together with the hereditary kernels $\mathbb{G}, \mathbb{H} : \mathcal{D} \times \mathbb{B} \to \mathbb{X}$ and the impulsive operators $\mathcal{I}_\kappa, \widehat{\mathcal{I}}_\kappa : \mathbb{X} \to \mathbb{X}$, are assumed to satisfy a collection of structural hypotheses stated later. Here, $\mathcal{D} = \{(\tau, \bar{s}) \in \mathfrak{J}\times\mathfrak{J} : \bar{s} \leq \tau\}$, and $\mathbb{L}(\mathbb{Y}, \mathbb{X})$ signifies the BS of bounded $\mathbb{LO}$s mapping $\mathbb{Y}$ into $\mathbb{X}$. Throughout this exposition, $U(\cdot)$ designates the admissible control affiliated with (1.1). We postulate that $U \in \mathbb{L}_2(\mathfrak{J}, \mathcal{U})$, where $\mathcal{U}$ is a separable HS of control actuation, and $\mathbb{L}_2(\mathfrak{J}, \mathcal{U})$ encapsulates the collection of square-integrable control functions defined over $\mathfrak{J}$. The control enters the dynamics through a bd-$\mathbb{LO}$ $\mho : \mathcal{U} \to \mathbb{X}$, thereby furnishing a mechanism for steering the system trajectories toward prescribed target configurations.

Remark 1.1. *The control function $U(\cdot)$ represents an external input applied to the system in order to influence the evolution of the state trajectory. Its effect on the dynamics is introduced through the bounded linear operator $\mho : \mathcal{U} \to \mathbb{X}$ appearing in the term $\mho U(\tau)$ in system* (1.1). *This* *operator determines how the control signal acts on the state space $\mathbb{X}$ and thus describes the mechanism through which the system can be manipulated. From a control-theoretic viewpoint, the presence of the term $\mho U(\tau)$ provides the means to steer the system trajectories despite the presence of fractional memory effects, stochastic perturbations, and impulsive disturbances. The main objective of the control process is therefore to select an admissible control $U \in \mathbb{L}_2(\mathfrak{J}, \mathcal{U})$ such that the corresponding $\mathbb{MS}$ of system* (1.1) *can be driven arbitrarily close to a prescribed target state at the terminal time ρ, which corresponds to the notion of $\mathbb{ACO}$ investigated in this study.*

Recent advances in stochastic fractional dynamics, impulsive systems, and $\mathbb{MV}$-differential inclusions have significantly broadened the analytical landscape. For instance, several researchers have addressed the existence and controllability of fractional stochastic evolution equations and inclusions of order $1 < \mu < 2$ under nonlocal constraints [36, 37]. Furthermore, investigations into Sobolev-type structures, hereditary nonlinearities, and Clarke sub-differentials have provided refined analytical tools for modeling systems with impulsive and non-smooth interactions [38]. By incorporating these contemporary insights, this work positions itself at the forefront of current research, bridging gaps in higher-order AB fractional stochastic impulsive systems and extending $\mathbb{ACO}$ results to a broader class of $\mathbb{MV}$, non-smooth dynamical models. Compared with classical controllability results for FDEs or inclusions, we extend the analytical framework by simultaneously accommodating higher-order AB fractional memory, stochastic perturbations via Q-Brownian motion processes, impulsive discontinuities, and non-smooth multi-valued dynamics. While prior works, such as those in [32, 35, 36], have investigated $\mathbb{ACO}$ for fractional stochastic or Sobolev-type systems, they typically treat either memory effects, stochastic influences, or impulsive/non-smooth features in isolation. In contrast, the proposed framework unifies these mechanisms within a single infinite-dimensional setting, thereby providing broader applicability and offering a more comprehensive understanding of controllability in systems where hereditary effects, random fluctuations, and abrupt state changes coexist. This comparison highlights the novelty and the practical relevance of our approach relative to the literature.

To the best of our knowledge, this study constitutes one of the earliest efforts to coherently integrate multiple sources of analytical irregularity, namely higher-order AB fractional memory, impulsive effects, stochastic perturbations, and non-smooth $\mathbb{MV}$ dynamics, within a unified and rigorous mathematical framework.

Many real-world systems exhibit abrupt structural changes, random fluctuations, and non-smooth interactions that cannot be adequately represented by smooth, single-valued nonlinearities. The inclusion of impulsive effects, stochastic perturbations driven by Brownian motion processes, and multi-valued nonlinearities described via Clarke's generalized sub-differential thus leads to a substantially richer class of dynamical models capable of capturing these phenomena realistically.

It is noteworthy that stochastic differential inclusions ($\mathbb{SDI}$s) generalize classical stochastic differential equations ($\mathbb{SDE}$s) by permitting multi-valued right-hand sides. This extension is particularly significant for modeling systems with abrupt changes, non-smooth interactions, or non-convex dynamics. While the existence and $\mathbb{ACO}$ results could, in principle, be specialized to the single-valued case of $\mathbb{SDE}$s, the inclusion-based formulation provides a more flexible and realistic framework. By incorporating multi-valued mappings through Clarke's generalized sub-differential, this analysis captures a wider range of phenomena, including hereditary memory effects, stochastic perturbations, and impulsive discontinuities, thereby offering broader applicability and deeper insight into the control of complex stochastic systems.

The motivation for investigating system (1.1) stems from the increasing need to analyze dynamical processes in which memory effects, stochastic perturbations, impulsive phenomena, and non-smooth interactions coexist within a unified mathematical framework. Classical integer-order models often fail to capture hereditary characteristics naturally arising in complex physical, biological, and engineering systems. Fractional-order operators provide a powerful mechanism for incorporating such memory effects. In particular, the AB-FD with a non-singular ML kernel offers a robust modeling tool as it accounts for non-local temporal interactions while avoiding the singular kernel behavior present in many classical fractional derivatives. Motivated by these considerations, we investigate the solvability and $\mathbb{ACO}$ of a class of Sobolev-type stochastic impulsive differential inclusions governed by a higher-order AB-FD formulated in the Caputo sense. Unlike earlier studies that typically treat fractional dynamics, stochastic perturbations, impulsive effects, or non-smooth inclusions separately, the proposed framework integrates all these mechanisms simultaneously within an infinite-dimensional HS setting. Consequently, the analysis extends several existing controllability theories to a broader class of fractional stochastic systems with memory, impulsive effects, and non-smooth dynamics.

Our major objectives of this study are twofold: First, to establish the existence of mild solutions ($\mathbb{MS}$s) for the considered system, and second, to investigate the $\mathbb{ACO}$ of an unexplored class of Sobolev-type stochastic impulsive differential inclusions ($\mathbb{SSIDI}$s). The dynamics of this class are governed by the interaction of several analytically challenging components, specifically:

1. Higher-order nonlocal fractional dynamics driven by the AB-FD in the Caputo sense.
2. Stochastic perturbations generated by Q-Brownian motion processes.
3. Impulsive variations in the system trajectory occurring at finitely many prescribed time instants.
4. Non-smooth and potentially non-convex interactions represented through Clarke's generalized sub-differential.

The development of this unified analytical framework is motivated by the observation that a wide range of physical, engineering, and socio-economic systems simultaneously exhibit hereditary memory effects, abrupt state transitions, and stochastic perturbations. Accurately capturing these intertwined features necessitates advanced mathematical models capable of integrating such mechanisms within a single coherent structure. Several recent contributions have addressed these aspects in a partial manner, particularly in the study of stochastic fractional systems with impulsive effects, delays, and multivalued dynamics [39–41]. In addition, further investigations have incorporated Clarke-type sub-differential inclusions, nonlocal controllability, and stochastic influences such as Poisson jumps and non-Gaussian noise into fractional frameworks [38, 42]. However, a comprehensive theory that unifies higher-order AB fractional dynamics, impulsive behavior, and non-smooth $\mathbb{MV}$ inclusions within a stochastic infinite-dimensional setting remains largely unexplored. In this work, we aim to bridge this gap by developing a systematic analytical framework capable of capturing these coupled phenomena within a unified formulation.

The results obtained herein not only deepen the theoretical understanding of fractional stochastic evolution systems but also provide verifiable conditions ensuring the existence of solutions and $\mathbb{ACO}$ in settings where classical deterministic or lower-order fractional models prove inadequate. Consequently, the proposed approach offers a mathematically rigorous and practically relevant methodology for analyzing and controlling systems governed by memory-dependent effects, stochastic perturbations, and impulsive dynamics.

From a methodological perspective, the analysis combines tools from infinite-dimensional stochastic analysis, fractional evolution equations, non-smooth variational techniques, and $\mathbb{MV}$-FP theory in BSs. By extending classical controllability concepts to accommodate higher-order fractional memory, impulsive disturbances, and stochastic effects, this work establishes a robust foundation for further investigations into complex dynamical systems beyond the scope of traditional smooth deterministic models.

The organization of this work is arranged as follows. In Section 2, we consolidate the foundational apparatus, encompassing the functional framework, symbolic conventions, and a collection of auxiliary lemmas that serve as the analytical bedrock for the ensuing developments. In Section 3, we establish rigorous existence results for $\mathbb{MS}$s under appropriate structural assumptions and subsequently derive $\mathbb{ACO}$ properties through the deployment of FP-methodologies in conjunction with $\mathbb{MEAS}$ selection techniques. Section 4 is devoted to a carefully engineered exquisite example that corroborates the abstract hypotheses and elucidates the tangible applicability of the theoretical conclusions. We conclude in Section 5 with concluding observations and a discussion of plausible trajectories for future investigation.

2. Preliminaries

In this section, we consolidate the principal notions of FC, along with fundamental elements from semi-group theory and cosine operator frameworks, and collect a set of auxiliary analytical results that underpin the arguments developed in the subsequent sections.
Throughout the paper, the following notation and functional spaces will be employed:

- The spaces $\mathbb{X}$ and $\mathbb{Y}$ are real separable Hilbert structures equipped with canonical inner products $\langle \cdot, \cdot \rangle$, which induce the associated Hilbertian norms denoted by $\|\cdot\|$.
- A Q–$\mathbb{BM}$ process $\{\bar{W}(\tau) : \tau \geq 0\}$ is considered, where Q is linear, bd, and of trace class, that is, $\mathrm{Tr}(Q) < \infty$.
- Assume that $\{e_m\}_{m\geq 1}$ constitutes a complete orthonormal basis of $\mathbb{Y}$, and let $\{h_m\}_{m\geq 1}$ be a bd sequence of non-negative real numbers **s.t.** $Qe_m = h_m e_m$, $\forall\, m \in \mathbf{N}$. Let $\{B_m\}_{m\geq 1}$ be a sequence of mutually independent real-valued $\mathbb{BM}$s. Then the $\mathbb{BM}$ process admits the representation

$$\langle \bar{W}(\tau), \mathfrak{f} \rangle = \sum_{m=1}^{\infty} \sqrt{h_m}\, \langle e_m, \mathfrak{f} \rangle\, B_m(\tau), \qquad \mathfrak{f} \in \mathbb{X},\ \tau \in \mathfrak{J}^*.$$

- We denote by $\mathbb{L}_2(\Omega, \mathbb{X})$ the BS comprising all $\mathbb{X}$-valued random variables that are strongly $\mathbb{MEAS}$ and second-order integrable in the mean-square sense. Its norm is prescribed by

$$\|x(\cdot)\|_{\mathbb{L}_2(\Omega,\mathbb{X})} := \left(\mathbb{E}\,\|x(\cdot,\chi)\|^2\right)^{\frac{1}{2}},$$

  where

$$\mathbb{E}(x) = \int_\Omega x(\chi)\, d\mathbb{P}.$$

- Let $\mathbb{L}_2^0 := \mathbb{L}_2(Q^{1/2}\mathbb{Y}, \mathbb{X})$ designate the Hilbert–Schmidt operator class consisting of all linear mappings from the weighted HS $Q^{1/2}\mathbb{Y}$ into $\mathbb{X}$. This space is endowed with the canonical

Hilbert–Schmidt topology induced by the norm

$$\|\phi\|^2_{\mathbb{L}_2^0} = \mathrm{Tr}\,(\phi\, Q\, \phi^*) < \infty, \qquad \phi \in \mathbb{L}_2^0,$$

where ϕ^* denotes the adjoint of ϕ and $\mathrm{Tr}(\cdot)$ stands for the trace operator.

Lemma 2.1. *[43] Let $\phi : \mathfrak{J} \to \mathbb{L}_0^2$ be a strongly $\mathbb{MEAS}$ process satisfying*

$$\int_0^\rho \mathbb{E}\left\|\phi(\bar{s})\right\|^2_{\mathbb{L}_0^2} d\bar{s} < \infty.$$

Consequently, the stochastic integral taken with respect to the Q–$\mathbb{BM}$ process is rigorously well posed and satisfies the subsequent quadratic moment bound:

$$\mathbb{E}\left\|\int_0^\rho \phi(\bar{s})\, d\bar{W}(\bar{s})\right\|^2 \leq \mathrm{Tr}(Q) \int_0^\rho \mathbb{E}\left\|\phi(\bar{s})\right\|^2_{\mathbb{L}_0^2} d\bar{s}.$$

- Let $\mathfrak{R} = (-\infty, \rho]$. The BS $\mathbb{C}(\mathfrak{R}; \mathbb{L}_2(\Omega; \mathbb{X}))$ comprises continuous mappings $x(\cdot) : \mathfrak{R} \to \mathbb{L}_2(\Omega; \mathbb{X})$ endowed with the norm:
$$\|x\|_{\mathbb{C}} := \sup_{\tau \in \mathfrak{R}} \left(\mathbb{E}\|x(\tau)\|^2\right)^{1/2}.$$
- To rigorously accommodate impulsive neutral stochastic fractional dynamical systems, we exhibit a space indicated by $\mathbb{B}$. Let $\mathfrak{G} : \mathfrak{R} \to (0, \infty)$ be a continuous weighting kernel satisfying
$$k = \int_{-\infty}^0 \mathfrak{G}(\tau)\, d\tau < \infty.$$
$\forall$ fixed $\iota > 0$, we stipulate
$$\mathbb{B} \;:=\; \Big\{\phi : \mathfrak{R} \to \mathbb{X} \;:\; (\mathbb{E}\|\phi(\tau)\|^2)^{1/2} \text{ is bd and } \mathbb{MEAS} \text{ on } [-\iota, 0],$$
$$\int_{-\infty}^0 \mathfrak{G}(\bar{s}) \sup_{\tau \in [\bar{s}, 0]} (\mathbb{E}\|\phi(\tau)\|^2)^{1/2}\, d\bar{s} < \infty\Big\}.$$
It can be readily verified that $\mathbb{B}$, with
$$\|\phi\|_{\mathbb{B}} = \int_{-\infty}^0 \mathfrak{G}(\bar{s}) \sup_{\bar{s} \leq \tau \leq 0} (\mathbb{E}\|\phi(\tau)\|^2)^{1/2}\, d\bar{s}, \qquad \phi \in \mathbb{B}, \tag{2.1}$$
constitutes a BS.
We next introduce the trajectory space
$$\mathbb{B}_\rho = \Big\{x : \mathfrak{R} \to \mathbb{L}_2(\Omega, \mathbb{X}) \;:\; x|_{\mathfrak{J}_\kappa} \in \mathbb{C}(\mathfrak{J}_\kappa, \mathbb{L}_2(\Omega, \mathbb{X})),\ \exists\, x(\tau_\kappa^-) = x(\tau_\kappa),\ x(\tau_\kappa^+),\ \varphi \in \mathbb{B},\ \kappa = 0, 1, \ldots, \gamma\Big\},$$
together with its differentiable counterpart
$$\bar{\mathbb{B}}_\rho = \big\{x : \mathfrak{R} \to \mathbb{L}_2(\Omega, \mathbb{X}) \;:\; x|_{\mathfrak{J}_\kappa} \in \mathbb{C}^1(\mathfrak{J}_\kappa, \mathbb{L}_2(\Omega, \mathbb{X})),\ \exists\, x'(\tau_\kappa^-) = x'(\tau_\kappa),\ x'(\tau_\kappa^+),\ \varphi \in \mathbb{B},\ \kappa = 0, 1, \ldots, \gamma\big\},$$
where $x|_{\mathfrak{J}_\kappa}$ signifies the canonical truncation of the process x to the temporal slice $\mathfrak{J}_\kappa = (\tau_\kappa, \tau_{\kappa+1}]$, for $\kappa = 0, 1, \ldots, \gamma$, and $\mathbb{C}(\mathfrak{J}_\kappa, \mathbb{L}_2(\Omega, \mathbb{X}))$ denotes the function space comprising mean-square continuous $\mathbb{X}$-valued stochastic trajectories defined over $\mathfrak{J}_\kappa$.

Finally, we equip $\mathbb{B}_\rho$ with the seminorm

$$\|x\|_\rho = \|\varphi\|_{\mathbb{B}} + \sup_{\bar{s}\in\mathfrak{J}} \left(\mathbb{E}\|x(\bar{s})\|^2\right)^{1/2}, \qquad x \in \mathbb{B}_\rho, \tag{2.2}$$

which is instrumental in the subsequent existence and controllability analysis.

The following lemma provides an essential estimate [44].

Lemma 2.2. *If $x \in \mathbb{B}_\rho$, then the associated history segment $x_\tau \in \mathbb{B}$, $\forall\ \tau \in \mathfrak{J}$. Furthermore, the following two–sided estimate holds:*

$$k\left(\mathbb{E}\|x(\tau)\|^2\right)^{1/2} \ \le\ \|x_\tau\|_{\mathbb{B}} \ \le\ k \sup_{\bar{s}\in[0,\tau]} \left(\mathbb{E}\|x(\bar{s})\|^2\right)^{1/2} + \|\varphi\|_{\mathbb{B}},$$

which quantifies the continuous embedding of the trajectory histories into $\mathbb{B}$ in terms of the present state and the prescribed initial datum.

Definition 2.1. *[45] An operator Z is termed sectorial provided that it fulfills the subsequent structural and resolvent-based criteria:*

1. *Z is a cl and $\mathbb{LO}$.*
2. *$\exists$ constants $\mathfrak{x} > 0$, $\phi \in \mathbb{R}$, and $\iota \in [\frac{\pi}{2}, \pi]$* ***s.t.****:*
   - *(a) The sector*

     $$\widehat{\Xi}(\iota, \phi) = \left\{\mathfrak{z} \in \mathbb{C} : \mathfrak{z} \neq \phi,\ |\arg(\mathfrak{z} - \phi)| < \iota\right\} \subset \varrho(Z),$$

     where $\varrho(Z)$ denotes the resolvent set of Z.
   - *(b) The resolvent operator satisfies the estimate*

     $$\|\mathcal{R}(\mathfrak{z}, Z)\| \le \frac{\mathfrak{x}}{|\mathfrak{z} - \phi|}, \qquad \mathfrak{z} \in \widehat{\Xi}(\mathfrak{x}, \phi),$$

     where $\mathcal{R}(\mathfrak{z}, Z) = (\mathfrak{z}I - Z)^{-1}$ denotes the resolvent of Z.

Moreover, for every $\mathfrak{z} \in \varrho(Z)$, the identity

$$Z\,\mathcal{R}(\mathfrak{z}, Z) = \mathfrak{z}\,\mathcal{R}(\mathfrak{z}, Z) - I$$

holds.

The formulation of AB-FDs relies crucially on structural properties of the ML family of functions (see [46]). In particular, the one-parameter ML function is defined by

$$\mathcal{E}_\hbar(\omega) = \sum_{j=0}^{\infty} \frac{\omega^j}{\Gamma(\hbar j + 1)}, \qquad \Re(\hbar) > 0,\ \omega \in \mathbb{C},$$

while its two-parameter generalization takes the form

$$\mathcal{E}_{\hbar,\hbar_1}(\omega) = \sum_{j=0}^{\infty} \frac{\omega^j}{\Gamma(\hbar j + \hbar_1)}, \qquad \Re(\hbar) > 0,\ \Re(\hbar_1) > 0,\ \omega \in \mathbb{C}.$$

It is immediate from the definitions that the classical ML function is recovered as a special case, namely $\mathcal{E}_{\hbar,1}(\omega) = \mathcal{E}_\hbar(\omega)$.

Definition 2.2. *[6] Let $x : \mathfrak{J} \to \mathbb{R}$ be a real-valued function. The RL-$\mathbb{FI}$ of order $\hbar > 0$ is introduced through the convolution-type operator*

$$(\mathcal{R}_{0+}^{\hbar} x)(\tau) = \frac{1}{\Gamma(\hbar)} \int_0^{\tau} (\tau - \bar{s})^{\hbar-1} x(\bar{s})\, d\bar{s}, \qquad \tau > 0,$$

whenever the above integral is well defined in the Lebesgue sense.

Definition 2.3. *[6] Let $n - 1 < \hbar < n$, $n = \lceil \hbar \rceil$. The RL-FD of order $\hbar$ is defined by*

$$D_{0+}^{\hbar} x(\tau) = \frac{d^n}{d\tau^n} \mathcal{R}_{0+}^{n-\hbar} x(\tau) = \frac{1}{\Gamma(n-\tau)} \frac{d^n}{d\tau^n} \int_0^{\tau} (\tau - \bar{s})^{n-\hbar-1} x(\bar{s})\, d\bar{s}.$$

Definition 2.4. *[6] Let $x : \mathfrak{J} \to \mathbb{R}$ be a function whose n-th classical derivative exists and belongs to $L^1(\mathfrak{J})$. For a fractional order $\hbar \in (n-1, n)$, the C-FD is specified by*

$$({}^{C}D_{0+}^{\hbar} x)(\tau) = \frac{1}{\Gamma(n-\hbar)} \int_0^{\tau} (\tau - \bar{s})^{n-\hbar-1}\, x^{(n)}(\bar{s})\, d\bar{s}, \qquad \tau > 0,$$

provided that the above integral is finite.

Definition 2.5. *(Higher-order AB-FD in Caputo sense) [11] Assume that $x^{(n)} \in \mathcal{H}^1(0, \rho)$ and let the fractional order satisfy $\hbar \in (n, n+1]$, where $n = 0, 1, 2, \ldots$ and $\vartheta = \hbar - n$.*

The higher-order AB-FD in the Caputo sense is then rigorously defined as

$$ {}^{\mathfrak{ABC}}D_{0+}^{\hbar} x(\tau) = \frac{\mathfrak{M}(\hbar - n)}{1 - \hbar + n} \int_0^{\tau} (\tau - \bar{s})^{n-\hbar} \mathcal{E}_{\hbar-n}\Big[-\theta(\tau - \bar{s})^{\hbar-n} \Big] x^{(n+1)}(\bar{s})\, d\bar{s}, \quad \tau \in (0, \rho], \tag{2.3}$$

where the kernel parameter is $\theta = \frac{\hbar-n}{1-\hbar+n}$, and $\mathcal{E}_{\hbar-n}(\cdot)$ denotes the one-parameter ML function. The normalization factor $\mathfrak{M}(\cdot)$ satisfies

$$\mathfrak{M}(0) = \mathfrak{M}(1) = 1,$$

ensuring the consistency of the fractional operator with classical integer-order derivatives.

Correspondingly, the AB fractional integral of order $\hbar$ is formulated as

$$ {}^{\mathfrak{AB}}\mathcal{R}_{0+}^{\hbar} x(\tau) = \frac{n + 1 - \hbar}{\mathfrak{M}(\hbar - n)} (\mathcal{R}_{0+}^{n} x)(\tau) + \frac{\hbar - n}{\mathfrak{M}(\hbar - n)} (\mathcal{R}_{0+}^{\hbar} x)(\tau), \tag{2.4}$$

which properly blends integer-order and fractional-order RL integrals within a non-singular, nonlocal memory framework.

Let $\mathbb{I}(\mathbb{X})$ represent the ensemble of all non-empty subsets of the HS $\mathbb{X}$. For notational precision and subsequent analytical deployment, we delineate the following subfamilies of $\mathbb{I}(\mathbb{X})$ distinguished by topological and geometric properties:

$$\begin{aligned} \mathbb{I}_{\mathrm{cl}}(\mathbb{X}) &:= \{ \mathfrak{X} \in \mathbb{I}(\mathbb{X}) : \mathfrak{X} \text{ cl} \}, \quad \mathbb{I}_{\mathrm{bd}}(\mathbb{X}) := \{ \mathfrak{X} \in \mathbb{I}(\mathbb{X}) : \mathfrak{X} \text{ bd} \}, \\ \mathbb{I}_{\mathrm{cp}}(\mathbb{X}) &:= \{ \mathfrak{X} \in \mathbb{I}(\mathbb{X}) : \mathfrak{X} \text{ compact} \}, \quad \mathbb{I}_{\mathrm{cv}}(\mathbb{X}) := \{ \mathfrak{X} \in \mathbb{I}(\mathbb{X}) : \mathfrak{X} \text{ cv} \}. \end{aligned}$$

Definition 2.6. *A $\mathbb{MV}$-function $\mathcal{T} : \mathfrak{J}^* \times \mathbb{X} \to \mathbb{I}_{bd,cl,cv}(\mathbb{X})$ is termed an* $\mathbb{L}_2$-Carathéodory multi-map *if:*

1. $\forall\, x \in \mathbb{X}$, *the map* $\tau \mapsto \mathcal{T}(\tau, x)$ *is* $\mathbb{MEAS}$*;*
2. *for almost every* $\tau \in \mathfrak{J}^*$*, the map* $x \mapsto \mathcal{T}(\tau, x)$ *is* $\mathbb{USC}$*;*
3. *for every radius* $\jmath > 0$, $\exists$ *a function* $\ell_\jmath \in \mathbb{L}_1(\mathfrak{J}^*; \mathbb{R}_+)$ ***s.t.***

$$\sup_{f \in \mathcal{D}(\tau,x)} \mathbb{E}\|f\|^2 \leq \ell_\jmath(\tau), \quad \forall\, \|x\|^2_{\mathbb{X}} \leq r,\ a.e.\ \tau \in \mathfrak{J}^*.$$

Lemma 2.3. *[47] Let* $\mathfrak{J}^*$ *be a compact interval in* $\mathbb{R}$*, and let* $\mathbb{U}$ *be a HS. Suppose* $\mathcal{T} : \mathfrak{J}^* \times \mathbb{X} \to \mathbb{I}_{bd,cl,cv}(\mathbb{L}(\mathbb{Y}, \mathbb{X}))$ *is an* $\mathbb{L}_2$*-Carathéodory* $\mathbb{MV}$*-function. For each* $x \in \mathcal{W}$*, define the set*

$$\bar{\mathbb{S}}_{\mathcal{T},x} := \{f \in \mathbb{L}_2(\mathfrak{J}^*, \mathbb{L}(\mathbb{Y}, \mathbb{X})) : f(\tau) \in \mathcal{T}(\tau, \mathfrak{Z}(\tau)),\ a.e.\ \tau \in \mathfrak{J}^*\}.$$

Let $\psi : \mathbb{L}_2(\mathfrak{J}^*, \mathbb{X}) \to \mathbb{X}$ *be a linear and continuous operator. Then the composition*

$$\psi \circ \bar{\mathbb{S}}_{\mathcal{T},x} : \mathbb{X} \to \mathbb{I}_{cp,cv}(\mathbb{X}), \quad x \mapsto \psi(\bar{\mathbb{S}}_{\mathcal{T},x}),$$

is a $\mathbb{MV}$*-operator with a cl graph in* $\mathbb{C}(\mathfrak{J}, \mathbb{X}) \times \mathbb{C}(\mathfrak{J}, \mathbb{X})$*.*

We subsequently define

$$\mathbb{N} : \mathbb{L}_2(\mathfrak{J}, \mathbb{X}) \longrightarrow 2^{\mathbb{L}_2(\mathfrak{J},\mathbb{X})},$$

by

$$\mathbb{N}(x) := \Big\{\mathfrak{g} \in \mathbb{L}_2(\mathfrak{J}, \mathbb{X}) \ :\ \mathfrak{g}(\tau) \in \partial\mathcal{S}(\tau, x(\tau)) \text{ for a.e. } \tau \in \mathfrak{J},\ \forall x \in \mathbb{L}_2(\mathfrak{J}, \mathbb{X})\Big\}.$$

Assume now two cl-$\mathbb{LO}$s $\bar{Y} : \mathbb{D}(\bar{Y}) \subset \mathbb{X} \to \mathbb{X}$ and $Z : \mathbb{D}(Z) \subset \mathbb{X} \to \mathbb{X}$ satisfy:

(i) $\bar{Y}$ and Z are cl.
(ii) $\mathbb{D}(\bar{Y}) \subset \mathbb{D}(Z)$ and Z is bijective.
(iii) $\bar{Y}^{-1} : \mathbb{X} \to \mathbb{D}(\bar{Y}) \subset \mathbb{X}$ is continuous. Denote $\left\|\bar{Y}\right\| = \aleph$ and $\left\|\bar{Y}^{-1}\right\| = \widehat{\aleph}$.
(iv) For $\xi \in \varrho(Z\bar{Y}^{-1})$, the resolvent $\mathcal{R}(\xi, Z\bar{Y}^{-1})$ is compact.

Definition 2.7. *[48] Let* $\mathbb{A}$ *be a BS,* $\mathbb{A}^*$ *its dual, and* $\mathcal{S} : \mathbb{A} \to \mathbb{R}$ *a locally Lipschitz functional. Then:*

- *The Clarke directional derivative of* $\mathcal{S}$ *at* $x \in \mathbb{A}$ *in the direction* $v \in \mathbb{A}$ *is:*

$$\mathcal{S}^0(x; v) = \lim_{\mathfrak{l}\to 0^+} \sup_{y\to x} \frac{\mathcal{S}(y + \mathfrak{l}v) - \mathcal{S}(y)}{\mathfrak{l}}.$$

- *The Clarke generalized gradient is:*

$$\partial\mathcal{S}(x) := \Big\{x^* \in \mathbb{A}^* : \mathcal{S}^0(x; v) \geq \langle x^*, v\rangle \text{ for all } v \in \mathbb{A}\Big\}.$$

Definition 2.8. *[10] An* $\mathcal{F}_\tau$*-adapted process* $x \in \mathbb{X}$ *is called a* $\mathbb{MS}$ *of problem (1.1), if* $x \in \mathbb{B}_\rho$*,* $U(\cdot) \in \mathbb{L}_2(\mathfrak{J}, \mathcal{U})$ *and* $\mathfrak{g} \in \mathbb{L}_2(\mathfrak{J}, \mathbb{X})$ ***s.t.*** $\mathfrak{g}(\tau) \in \partial\mathcal{S}(\tau, x(\tau))$*,* $x_0 \in \mathbb{B}$*,* $x_1 \in \mathbb{L}^2_0(\Omega, \mathbb{X})$ *and*

1. *the process* $x(\tau)$ *is* $\mathbb{MEAS}$ *and* $\mathcal{F}_\tau$*–adapted for every* $\tau \in (-\infty, \rho]$*, and it possesses almost surely càdlàg trajectories on the interval* $\mathfrak{J}$*.*
2. $\forall\ \tau \in (-\infty, \rho]$*, the random variable* $x(\tau)$ *takes values in the phase space* $\mathbb{B}$*. Moreover, for every impulsive sub-interval* $(\tau_\kappa, \tau_{\kappa+1}]$*,* $\kappa = 1, \ldots, \gamma$*, the restriction of* $x(\cdot)$ *to* $(\tau_\kappa, \tau_{\kappa+1}]$ *is continuous.*

3. $\forall\ \tau \geq 0$, *the state trajectory* $x(\tau)$ *admits the subsequent integral formulation:*

$$x(\tau) = \begin{cases} \varphi(\tau), & \tau \in (-\infty, 0], \\ \widehat{\Upsilon}_1, & \tau \in [0, \tau_1], \\ \widehat{\Upsilon}_2, & \tau \in (\tau_\kappa, \tau_{\kappa+1}],\ \kappa = 1, 2, \cdots, \gamma, \end{cases}$$

where

$$\begin{aligned}
\widehat{\Upsilon}_1 \ =\ & \bar{Y}^{-1}\mathfrak{R}\mathfrak{V}_\hbar(\tau)\bar{Y}\Big\{\phi(0) - \vartheta(x)\Big\} + \bar{Y}^{-1}\mathfrak{R}\mathfrak{K}_\hbar(\tau)\bar{Y}\Big\{x_1 - \beta(x)\Big\} \\
& + \frac{\wp\mathfrak{R}(2-\hbar)}{\mathfrak{M}(\hbar-1)\Gamma(\hbar)} \int_0^\tau \bar{Y}^{-1}(\tau - \bar{s})^{\hbar-1}\Big\{\mho U(\bar{s}) + \mathfrak{g}(\bar{s})\Big\} d\bar{s} \\
& + \frac{\wp\mathfrak{R}(2-\hbar)}{\mathfrak{M}(\hbar-1)\Gamma(\hbar)} \int_0^\tau \bar{Y}^{-1}(\tau - \bar{s})^{\hbar-1}\zeta\Big(\bar{s}, x_{\bar{s}}, \int_0^{\bar{s}} \mathbb{H}(\bar{s}, \mathfrak{U}, x_\mathfrak{U})\, d\mathfrak{U}\Big) d\bar{W}(\bar{s}) \\
& + \frac{\wp\mathfrak{R}(2-\hbar)}{\mathfrak{M}(\hbar-1)\Gamma(\hbar)} \int_0^\tau \bar{Y}^{-1}(\tau - \bar{s})^{\hbar-1}\mathfrak{F}\Big(\bar{s}, x_{\bar{s}}, \int_0^{\bar{s}} \mathbb{G}(\bar{s}, \mathfrak{U}, x_\mathfrak{U})\, d\mathfrak{U}\Big) d\bar{s} \\
& + \frac{(\hbar-1)\mathfrak{R}^2}{\mathfrak{M}(\hbar-1)} \int_0^\tau \bar{Y}^{-1}\mathfrak{Q}_\hbar(\tau - \bar{s})\Big\{\mho U(\bar{s}) + \mathfrak{g}(\bar{s})\Big\} d\bar{s} \\
& + \frac{(\hbar-1)\mathfrak{R}^2}{\mathfrak{M}(\hbar-1)} \int_0^\tau \bar{Y}^{-1}\mathfrak{Q}_\hbar(\tau - \bar{s})\zeta\Big(\bar{s}, x_{\bar{s}}, \int_0^{\bar{s}} \mathbb{H}(\bar{s}, \mathfrak{U}, x_\mathfrak{U})\, d\mathfrak{U}\Big) d\bar{W}(\bar{s}) \\
& + \frac{(\hbar-1)\mathfrak{R}^2}{\mathfrak{M}(\hbar-1)} \int_0^\tau \bar{Y}^{-1}\mathfrak{Q}_\hbar(\tau - \bar{s})\mathfrak{F}\Big(\bar{s}, x_{\bar{s}}, \int_0^{\bar{s}} \mathbb{G}(\bar{s}, \mathfrak{U}, x_\mathfrak{U})\, d\mathfrak{U}\Big) d\bar{s},
\end{aligned}$$

$$\begin{aligned}
\widehat{\Upsilon}_2 \ =\ & \bar{Y}^{-1}\mathfrak{R}\mathfrak{V}_\hbar(\tau)\bar{Y}\Big\{\phi(0) - \vartheta(x)\Big\} + \bar{Y}^{-1}\mathfrak{R}\mathfrak{K}_\hbar(\tau)\bar{Y}\Big\{x_1 - \beta(x)\Big\} \\
& + \frac{\wp\mathfrak{R}(2-\hbar)}{\mathfrak{M}(\hbar-1)\Gamma(\hbar)} \int_0^\tau \bar{Y}^{-1}(\tau - \bar{s})^{\hbar-1}\Big\{\mho U(\bar{s}) + \mathfrak{g}(\bar{s})\Big\} d\bar{s} \\
& + \frac{\wp\mathfrak{R}(2-\hbar)}{\mathfrak{M}(\hbar-1)\Gamma(\hbar)} \int_0^\tau \bar{Y}^{-1}(\tau - \bar{s})^{\hbar-1}\zeta\Big(\bar{s}, x_{\bar{s}}, \int_0^{\bar{s}} \mathbb{H}(\bar{s}, \mathfrak{U}, x_\mathfrak{U})\, d\mathfrak{U}\Big) d\bar{W}(\bar{s}) \\
& + \frac{\wp\mathfrak{R}(2-\hbar)}{\mathfrak{M}(\hbar-1)\Gamma(\hbar)} \int_0^\tau \bar{Y}^{-1}(\tau - \bar{s})^{\hbar-1}\mathfrak{F}\Big(\bar{s}, x_{\bar{s}}, \int_0^{\bar{s}} \mathbb{G}(\bar{s}, \mathfrak{U}, x_\mathfrak{U})\, d\mathfrak{U}\Big) d\bar{s} \\
& + \frac{(\hbar-1)\mathfrak{R}^2}{\mathfrak{M}(\hbar-1)} \int_0^\tau \bar{Y}^{-1}\mathfrak{Q}_\hbar(\tau - \bar{s})\Big\{\mho U(\bar{s}) + \mathfrak{g}(\bar{s})\Big\} d\bar{s} \\
& + \frac{(\hbar-1)\mathfrak{R}^2}{\mathfrak{M}(\hbar-1)} \int_0^\tau \bar{Y}^{-1}\mathfrak{Q}_\hbar(\tau - \bar{s})\zeta\Big(\bar{s}, x_{\bar{s}}, \int_0^{\bar{s}} \mathbb{H}(\bar{s}, \mathfrak{U}, x_\mathfrak{U})\, d\mathfrak{U}\Big) d\bar{W}(\bar{s}) \\
& + \frac{(\hbar-1)\mathfrak{R}^2}{\mathfrak{M}(\hbar-1)} \int_0^\tau \bar{Y}^{-1}\mathfrak{Q}_\hbar(\tau - \bar{s})\mathfrak{F}\Big(\bar{s}, x_{\bar{s}}, \int_0^{\bar{s}} \mathbb{G}(\bar{s}, \mathfrak{U}, x_\mathfrak{U})\, d\mathfrak{U}\Big) d\bar{s} \\
& + \sum_{0<\tau_\kappa<\tau} \bar{Y}^{-1}\mathfrak{R}\mathfrak{V}_\hbar(\tau - \tau_\kappa)\bar{Y}\mathcal{I}_\kappa(x(\tau_\kappa)) + \sum_{0<\tau_\kappa<\tau} \bar{Y}^{-1}\mathfrak{R}\mathfrak{K}_\hbar(\tau - \tau_\kappa)\bar{Y}\widehat{\mathcal{I}}_\kappa(x(\tau_\kappa)).
\end{aligned}$$

Here, $\mathfrak{R} = \Delta^*(\Delta^* I - Z\bar{Y}^{-1})^{-1}$, $\wp = -\mu^* Z\bar{Y}^{-1}(\Delta^* I - Z\bar{Y}^{-1})^{-1}$, *with* $\Delta^* = \frac{\mathfrak{M}(\hbar-1)}{2-\hbar}$, $\quad \mu^* = \frac{\hbar-1}{2-\hbar}$, *and*

$$\mathfrak{V}_\hbar(\tau) \ =\ \frac{1}{2\pi i} \int_\chi e^{\xi\tau} \xi^{\hbar-1} (\xi^\hbar I - \wp)^{-1} d\xi,$$

$$\mathfrak{K}_{\hbar}(\tau) = \frac{1}{2\pi i}\int_{\chi} e^{\xi\tau}\xi^{\hbar-2}(\xi^{\hbar}I - \wp)^{-1}d\xi,$$
$$\mathfrak{Q}_{\hbar}(\tau) = \frac{1}{2\pi i}\int_{\chi} e^{\xi\tau}(\xi^{\hbar}I - \wp)^{-1}d\xi.$$

Remark 2.1. *We emphasize that the displayed representation of the* $\mathbb{MS}$ *must be interpreted in the sense of an integral inclusion rather than a pointwise identity. More precisely, a stochastic process* $x(\cdot)$ *is said to be a* $\mathbb{MS}$ *of system* (1.1) *provided that* x *is* $\mathcal{F}_\tau$*-adapted, satisfying the prescribed impulsive jump circumstances, and* $\exists$ *an* $\mathcal{F}_\tau$*-* $\mathbb{MEAS}$

$$\mathfrak{g}(\cdot) : \tau \mapsto \mathfrak{g}(\tau) \in \partial\mathcal{S}(\tau, x(\tau)) \quad \textit{for a.e. } \tau \in \mathfrak{J},$$

s.t. *the corresponding variation-of-constants formula holds upon substituting the* $\mathbb{MV}$*-mapping* $\partial\mathcal{S}(\tau, x(\tau))$ *by its single-valued realization* $\mathfrak{g}(\tau)$*. Equivalently, the set-valued term is to be understood via the theory of* $\mathbb{MEAS}$ *selections (see, for instance, [49]), ensuring that each occurrence of the* $\mathbb{MV}$ *expression yields an admissible* $\mathbb{MEAS}$ *integrand. Under the imposed measurability and growth hypotheses on* $\partial\mathcal{S}$*, the stochastic processes* ζ *and* $\mathfrak{F}$ *are well-defined, while the impulsive summation precisely captures the discontinuities generated by the impulse operator* $\mathcal{I}_\kappa$*. Consequently, the notion of* $\mathbb{MS}$ *employed in this work is fully compatible with the underlying stochastic differential inclusion* (1.1) *and aligns with the standard analytical framework for* $\mathbb{MV}$ *stochastic evolution systems.*

Remark 2.2. *(Derivation of the integral representation) For completeness, we briefly outline the derivation connecting the abstract system* (1.1) *to the displayed integral inclusion. Formally, the representation is obtained by applying Laplace transform techniques (see, [1]) to the linear part of the system and exploiting the resolvent properties of the associated operator families. The stochastic multi-valued terms* ζ *and* $\mathfrak{F}$ *are incorporated through measurable selections in the sense of* $\mathbb{MEAS}$ *theory, while the impulsive summation captures the discontinuities generated by* $\mathcal{I}_\kappa$*. This construction ensures that the variation-of-constants formula in the integral representation is fully consistent with the abstract* $\mathbb{SSDI}$ *and preserves the rigorous definition of a* $\mathbb{MS}$ *in the sense described in Remark 2.1.*

To investigate the $\mathbb{ACO}$ properties of the nonlinear stochastic control system (1.1), we begin by scrutinizing the associated linearized dynamics, which serves as a fundamental analytical benchmark for the subsequent controllability analysis.

$$\begin{cases} {}^{\mathfrak{ABC}}D^{\hbar}_{0+}\bar{Y}x(\tau) \in Zx(\tau) + \mho U(\tau), & \tau \in \mathfrak{J}^* := (0, \rho], \\ x(0) + \vartheta(x) = x_0 = \varphi \in \mathbb{B},\ x'(0) + \beta(x) = x_1 \in \mathbb{L}^2_0(\Omega, \mathbb{X}). \end{cases} \tag{2.5}$$

We introduce the controllability operator corresponding to system (2.5), which encapsulates the influence of admissible controls on the state evolution and plays a central role in the ensuing controllability analysis.

$$\mathfrak{Q}^{\rho}_0 = \int_0^{\rho} \bar{Y}^{-1}\mathfrak{Q}_{\hbar}(\rho - \bar{s})\mho\bar{Y}^{*-1}\mho^*\mathfrak{Q}^*_{\hbar}(\rho - \bar{s})\, d\bar{s},$$

and let $\mathcal{R}(\xi, \mathfrak{Q}^{\rho}_0) = (\xi I + \mathfrak{Q}^{\rho}_0)^{-1}$ for $\xi > 0$, where $\mho^*$, $\bar{Y}^{*-1}$, and $\mathfrak{Q}^*_{\hbar}(\tau)$ denote the adjoint of $\mho$, $\bar{Y}^{-1}$, and $\mathfrak{Q}_{\hbar}(\tau)$, respectively. It follows that $\mathfrak{Q}^{\rho}_0$ is a bd-$\mathbb{LO}$ on $\mathbb{X}$.

Lemma 2.4. *[50] The linear dynamical system* (2.5) *is said to possess the property of* $\mathbb{ACO}$ *on* $\mathfrak{J}$ *iff the resolvent family generated by the associated controllability operator satisfies the asymptotic vanishing condition*

$$\xi\,(\xi I + \mathfrak{Q}_0^\rho)^{-1} \longrightarrow 0 \quad as\ \xi \to 0^+,$$

where the convergence is interpreted in the sense of the strong operator topology.

Let $x(\tau; x_0, x_1, U)$ represent the state trajectory of system (1.1) generated by an admissible control $U(\cdot)$, and let $x_\rho(x_0, x_1, U)$ denote the corresponding state attained at the terminal instant ρ. The set of all states reachable at time ρ is defined as

$$\mathfrak{L}(\rho; x_0, x_1) = \{\, x_\rho(x_0, x_1, U) \ : \ U(\cdot) \in \mathbb{L}_2(\mathfrak{J}, \mathcal{U})\,\}.$$

We further write $\overline{\mathfrak{L}}(\rho; x_0, x_1)$ for the closure of this set in the HS $\mathbb{L}_2(\Omega; \mathbb{X})$.

Definition 2.9. *The system* (1.1) *is said to possess the property of* $\mathbb{ACO}$ *on* $\mathfrak{J}$ *whenever the closure of its reachable set satisfies*

$$\overline{\mathfrak{L}}(\rho; x_0, x_1) = \mathbb{L}_2(\Omega, \mathcal{F}_\rho; \mathbb{X}),$$

that is, every $\mathcal{F}_\rho$*-*$\mathbb{MEAS}$ *square–integrable* $\mathbb{X}$*-valued random variable can be approximated arbitrarily well by terminal states attainable through admissible controls.*

Lemma 2.5. *[50] For any terminal state* $x_\rho \in \mathbb{L}_2(\Omega, \mathcal{F}_\rho; \mathbb{X})$*,* $\exists$ *a stochastic process* $\widehat{\varrho}(\cdot) \in \mathbb{L}_2(\Omega; \mathbb{L}_2(\mathfrak{J}; \mathbb{L}(\mathbb{Y}, \mathbb{X})))$ ***s.t.***

$$x_\rho = \mathbb{E}[x_\rho] + \int_0^\rho \widehat{\varrho}(\bar{s})\, d\bar{W}(\bar{s}).$$

Moreover, for each $\xi > 0$ *and* $x_\rho \in \mathbb{L}_2(\Omega, \mathcal{F}_\rho; \mathbb{X})$*, the control function*

$$U(\tau) = \bar{Y}^{*-1}\mho^{*}\mathfrak{Q}_\hbar^{*}(\rho - \tau)\mathcal{R}(\xi, \mathfrak{Q}_0^\rho)\Theta(x(\cdot)),$$

is admissible, here

$$\Theta(x(\cdot)) = \begin{cases} \Upsilon_1, & \tau \in [0, \tau_1], \\ \Upsilon_1, & \tau \in [\tau_\kappa, \tau_{\kappa+1}], \end{cases}$$

where

$$\begin{aligned}
\Upsilon_1 \ = \ & \mathbb{E}[x_\rho] + \int_0^\rho \widehat{\varrho}(\bar{s})\, d\bar{W}(\bar{s}) - \bar{Y}^{-1}\mathfrak{R}\mathfrak{V}_\hbar(\rho)\bar{Y}\Big\{\phi(0) - \vartheta(x)\Big\} - \bar{Y}^{-1}\mathfrak{R}\mathfrak{K}_\hbar(\rho)\bar{Y}\Big\{x_1 - \beta(x)\Big\} \\
& - \frac{\wp\mathfrak{R}(2-\hbar)}{\mathfrak{M}(\hbar-1)\Gamma(\hbar)} \int_0^\rho \bar{Y}^{-1}(\rho - \bar{s})^{\hbar-1}\mathfrak{g}(\bar{s})d\bar{s} \\
& - \frac{\wp\mathfrak{R}(2-\hbar)}{\mathfrak{M}(\hbar-1)\Gamma(\hbar)} \int_0^\rho \bar{Y}^{-1}(\rho - \bar{s})^{\hbar-1}\zeta\Big(\bar{s}, x_{\bar{s}}, \int_0^{\bar{s}} \mathbb{H}(\bar{s}, \mathfrak{U}, x_{\mathfrak{U}})\, d\mathfrak{U}\Big)d\bar{W}(\bar{s}) \\
& - \frac{\wp\mathfrak{R}(2-\hbar)}{\mathfrak{M}(\hbar-1)\Gamma(\hbar)} \int_0^\rho \bar{Y}^{-1}(\rho - \bar{s})^{\hbar-1}\mathfrak{F}\Big(\bar{s}, x_{\bar{s}}, \int_0^{\bar{s}} \mathbb{G}(\bar{s}, \mathfrak{U}, x_{\mathfrak{U}})\, d\mathfrak{U}\Big)d\bar{s} \\
& - \frac{(\hbar-1)\mathfrak{R}^2}{\mathfrak{M}(\hbar-1)} \int_0^\rho \bar{Y}^{-1}\mathfrak{Q}_\hbar(\rho - \bar{s})\mathfrak{g}(\bar{s})d\bar{s}
\end{aligned}$$

$$
\begin{aligned}
&- \frac{(\hbar-1)\mathfrak{R}^2}{\mathfrak{M}(\hbar-1)} \int_0^{\rho} \bar{Y}^{-1}\mathfrak{Q}_{\hbar}(\rho-\bar{s})\zeta\Big(\bar{s}, x_{\bar{s}}, \int_0^{\bar{s}} \mathbb{H}(\bar{s}, \mathfrak{U}, x_{\mathfrak{U}})\, d\mathfrak{U}\Big) d\bar{W}(\bar{s}) \\
&- \frac{(\hbar-1)\mathfrak{R}^2}{\mathfrak{M}(\hbar-1)} \int_0^{\rho} \bar{Y}^{-1}\mathfrak{Q}_{\hbar}(\rho-\bar{s})\mathfrak{F}\Big(\bar{s}, x_{\bar{s}}, \int_0^{\bar{s}} \mathbb{G}(\bar{s}, \mathfrak{U}, x_{\mathfrak{U}})\, d\mathfrak{U}\Big) d\bar{s},
\end{aligned}
$$

$$
\begin{aligned}
\Upsilon_2 \;=\; & \mathbb{E}[x_{\rho}] + \int_0^{\rho} \widehat{\varrho}(\bar{s})\, d\bar{W}(\bar{s}) - \bar{Y}^{-1}\mathfrak{R}\mathfrak{V}_{\hbar}(\rho)\bar{Y}\Big\{\phi(0) - \vartheta(x)\Big\} - \bar{Y}^{-1}\mathfrak{R}\mathfrak{K}_{\hbar}(\rho)\bar{Y}\Big\{x_1 - \beta(x)\Big\} \\
&- \frac{\wp\mathfrak{R}(2-\hbar)}{\mathfrak{M}(\hbar-1)\Gamma(\hbar)} \int_0^{\rho} \bar{Y}^{-1}(\rho-\bar{s})^{\hbar-1}\mathfrak{g}(\bar{s})d\bar{s} \\
&- \frac{\wp\mathfrak{R}(2-\hbar)}{\mathfrak{M}(\hbar-1)\Gamma(\hbar)} \int_0^{\rho} \bar{Y}^{-1}(\rho-\bar{s})^{\hbar-1}\zeta\Big(\bar{s}, x_{\bar{s}}, \int_0^{\bar{s}} \mathbb{H}(\bar{s}, \mathfrak{U}, x_{\mathfrak{U}})\, d\mathfrak{U}\Big) d\bar{W}(\bar{s}) \\
&- \frac{\wp\mathfrak{R}(2-\hbar)}{\mathfrak{M}(\hbar-1)\Gamma(\hbar)} \int_0^{\rho} \bar{Y}^{-1}(\rho-\bar{s})^{\hbar-1}\mathfrak{F}\Big(\bar{s}, x_{\bar{s}}, \int_0^{\bar{s}} \mathbb{G}(\bar{s}, \mathfrak{U}, x_{\mathfrak{U}})\, d\mathfrak{U}\Big) d\bar{s} \\
&- \frac{(\hbar-1)\mathfrak{R}^2}{\mathfrak{M}(\hbar-1)} \int_0^{\rho} \bar{Y}^{-1}\mathfrak{Q}_{\hbar}(\rho-\bar{s})\mathfrak{g}(\bar{s})d\bar{s} \\
&- \frac{(\hbar-1)\mathfrak{R}^2}{\mathfrak{M}(\hbar-1)} \int_0^{\rho} \bar{Y}^{-1}\mathfrak{Q}_{\hbar}(\rho-\bar{s})\zeta\Big(\bar{s}, x_{\bar{s}}, \int_0^{\bar{s}} \mathbb{H}(\bar{s}, \mathfrak{U}, x_{\mathfrak{U}})\, d\mathfrak{U}\Big) d\bar{W}(\bar{s}) \\
&- \frac{(\hbar-1)\mathfrak{R}^2}{\mathfrak{M}(\hbar-1)} \int_0^{\rho} \bar{Y}^{-1}\mathfrak{Q}_{\hbar}(\rho-\bar{s})\mathfrak{F}\Big(\bar{s}, x_{\bar{s}}, \int_0^{\bar{s}} \mathbb{G}(\bar{s}, \mathfrak{U}, x_{\mathfrak{U}})\, d\mathfrak{U}\Big) d\bar{s} \\
&- \sum_{\kappa=1}^{\gamma} \bar{Y}^{-1}\mathfrak{R}\mathfrak{V}_{\hbar}(\rho-\tau_{\kappa})\bar{Y}\mathcal{I}_{\kappa}(x(\tau_{\kappa})) - \sum_{\kappa=1}^{\gamma} \bar{Y}^{-1}\mathfrak{R}\mathfrak{K}_{\hbar}(\rho-\tau_{\kappa})\bar{Y}\widehat{\mathcal{I}}_{\kappa}(x(\tau_{\kappa})).
\end{aligned}
$$

To establish the existence of solutions for the stochastic system under consideration, we employ a fixed point approach based on a $\mathbb{MV}$ nonlinear alternative. In particular, the analysis relies on a well-known extension of the classical Leray–Schauder principle for set-valued mappings, commonly referred to as Martelli's fixed point theorem. This result provides the existence of a fixed point for $\mathbb{USC}$-$\mathbb{MV}$ operators with nonempty, cv, and compact values under suitable compactness and boundedness conditions. For completeness, we recall the theorem below.

Theorem 2.1. *(Martelli fixed point theorem) [51] Let $\mathbb{K}$ be a BS endowed with the locally cv topology and let $\mathbb{T} : \mathbb{K} \to 2^{\mathbb{K}}$ be an $\mathbb{USC}$ set-valued mapping with nonempty, cv, and compact values. Suppose that there exists a closed neighborhood $\mathcal{T}$ of the origin in $\mathbb{K}$ such that $\mathbb{T}(\mathcal{T})$ is relatively compact in $\mathbb{K}$.*

If the set

$$\Lambda = \Big\{y \in \mathbb{K} :\ \varepsilon y \in \mathbb{T}(y)\, \text{for some}\ \varepsilon > 1\Big\}$$

is bd in $\mathbb{K}$, then $\mathbb{T}$ possesses at least one FP in $\mathbb{K}$; that is, there exists $y^* \in \mathbb{K}$ such that

$$y^* \in \mathbb{T}(y^*).$$

3. Existence

We now impose the following standing hypotheses.

($H1$) In the interest of notational conciseness and with no abatement of generality, we impose the following standing assumption. The operator families $\mathfrak{V}_\hbar(\tau)$, $\mathfrak{K}_\hbar(\tau)$, and $\mathfrak{Q}_\hbar(\tau)$ generated by $Z\bar{Y}^{-1}$ are compact on $\bar{\mathbb{D}}(Z\bar{Y}^{-1})$ for all $\tau \geq 0$, and satisfy the uniform bounds

$$\sup_{\tau\in\mathfrak{J}} \|\mathfrak{V}_\hbar(\tau)\| \leq \widehat{M}, \qquad \sup_{\tau\in\mathfrak{J}} \|\mathfrak{K}_\hbar(\tau)\| \leq \widehat{M}, \qquad \sup_{\tau\in\mathfrak{J}} \|\mathfrak{Q}_\hbar(\tau)\| \leq \widehat{M}.$$

For an in-depth elaboration, we consign the reader to [52]. In addition, we postulate the existence of $\theta > 0$ and $\psi > 0$ **s.t.**

$$\|\mathfrak{R}\| \leq \theta, \qquad \|\wp\| \leq \psi,$$

and the resolvent-type estimate

$$\left\|\xi\,\mathcal{R}(\xi, \mathfrak{Q}_0^\rho)\right\| \leq 1, \qquad \text{for every } \ \xi > 0,$$

is satisfied.

($H2$) The functional $\mathcal{S} : \mathfrak{J} \times \mathbb{X} \to \mathbb{R}$ is postulated to adhere to the ensuing stipulations:

1. $\forall$ fixed $x \in \mathbb{X}$, the mapping $\tau \mapsto \mathcal{S}(\tau, x)$ is Lebesgue $\mathbb{MEAS}$ on $\mathfrak{J}$.
2. For almost every $\tau \in \mathfrak{J}$, the function $x \mapsto \mathcal{S}(\tau, x)$ is locally Lipschitz continuous on $\mathbb{X}$.
3. $\exists\ \mathfrak{I}_\mathfrak{g} \in \mathbb{L}_1(\mathfrak{J}, \mathbb{R}^+)$ and a constant $k_\mathfrak{g} > 0$ **s.t.**

$$\mathbb{E}\left\|\partial\mathcal{S}(\tau, x(\tau))\right\|^2 = \sup\left\{\|\mathfrak{g}\|^2 : \mathfrak{g} \in \partial\mathcal{S}(\tau, x(\tau))\right\} \leq \mathfrak{I}_\mathfrak{g}(\tau) + k_\mathfrak{g}\|x\|^2,$$

$\forall\ x \in \mathbb{X}$ and for almost every $\tau \in \mathfrak{J}$.

($H3$) (1) The nonlinear operator $\mathfrak{F} : \mathfrak{J} \times \mathbb{B} \times \mathbb{X} \to \mathbb{X}$ fulfills the subsequent requirements:

(i) $\forall$ fixed pair $(x_1, x_2) \in \mathbb{B} \times \mathbb{X}$, the mapping $\tau \longmapsto \mathfrak{F}(\tau, x_1, x_2)$ is $\mathbb{MEAS}$ on $\mathfrak{J}$.

(ii) For almost every $\tau \in \mathfrak{J}$, the function $(x_1, x_2) \longmapsto \mathfrak{F}(\tau, x_1, x_2)$ is continuous on $\mathbb{B} \times \mathbb{X}$.

(iii) $\exists$ a continuous weight function $\mathfrak{b}_\mathfrak{F} : [0,\infty) \to [0,\infty)$ and a continuous, monotone increasing function $\Psi_\mathfrak{F} : [0,\infty) \to [0,\infty)$ **s.t.**

$$\mathbb{E}\left\|\mathfrak{F}(\tau, x_1, x_2)\right\|_\mathbb{X}^2 \leq \mathfrak{b}_\mathfrak{F}(\tau)\,\Psi_\mathfrak{F}\left(\|x_1\|_\mathbb{B}^2 + \mathbb{E}\|x_2\|_\mathbb{X}^2\right),$$

$\forall\ (x_1, x_2) \in \mathbb{B} \times \mathbb{X}$ and for every $\tau \in \mathfrak{J}$.

(2) $\forall\ (\tau, \bar{s}) \in \mathcal{D}$, the mapping $\mathbb{G}(\tau, \bar{s}, \cdot) : \mathbb{B} \to \mathbb{X}$ is continuous, whereas for every $x \in \mathbb{B}$ the function $(\tau, \bar{s}) \mapsto \mathbb{G}(\tau, \bar{s}, x)$ is $\mathbb{MEAS}$ on $\mathcal{D}$. Furthermore, $\exists$ a constant $\mathfrak{w}_\mathbb{G} > 0$ **s.t.**

$$\mathbb{E}\left\|\mathbb{G}(\tau, \bar{s}, x)\right\|_\mathbb{X}^2 \leq \mathfrak{w}_\mathbb{G}\left(1 + \|x\|_\mathbb{B}^2\right), \quad \forall\, x \in \mathbb{B}.$$

($H4$) (1) The stochastic perturbation operator $\zeta : \mathfrak{J}\times\mathbb{B}\times\mathbb{X} \to \mathbb{L}_0^2$ satisfies the Carathéodory framework. Moreover, there exist a locally integrable function $\mathfrak{b}_\zeta : \mathfrak{J} \to \mathbb{R}^+$ and a monotone non-decreasing function $\Psi_\zeta : [0,\infty) \to [0,\infty)$ **s.t.**

$$\mathbb{E}\left\|\zeta(\tau, x_1, x_2)\right\|_{\mathbb{L}_0^2}^2 \leq \mathfrak{b}_\zeta(\tau)\,\Psi_\zeta\left(\|x_1\|_\mathbb{B}^2 + \mathbb{E}\|x_2\|_\mathbb{X}^2\right),$$

$\forall\ (x_1, x_2) \in \mathbb{B} \times \mathbb{X}$ and every $\tau \in \mathfrak{J}$.

(2) $\forall\ (\tau, \bar{s}) \in \mathcal{D}$, the mapping $\mathbb{H}(\tau, \bar{s}, \cdot) : \mathbb{B} \to \mathbb{X}$ is continuous, whereas for any fixed $x \in \mathbb{B}$ the function $(\tau, \bar{s}) \mapsto \mathbb{H}(\tau, \bar{s}, x)$ is $\mathbb{MEAS}$ on $\mathcal{D}$. Furthermore, $\exists$ a constant $\mathfrak{w}_{\mathbb{H}} > 0$ **s.t.**

$$\mathbb{E}\left\|\mathbb{H}(\tau, \bar{s}, x)\right\|_{\mathbb{X}}^2 \leq \mathfrak{w}_{\mathbb{H}} \left(1 + \|x\|_{\mathbb{B}}^2\right), \quad \forall\, x \in \mathbb{B}.$$

$(H5)$ The impulsive effects are assumed to satisfy the following properties:

1. For each $\kappa \in \{1, \dots, \gamma\}$, the impulse mappings $\mathcal{I}_\kappa, \widehat{\mathcal{I}}_\kappa : \mathbb{X} \to \mathbb{X}$ are C.C. In addition, $\exists$ constants $d_1^\kappa > 0$ and $d_2^\kappa > 0$ **s.t.** the uniform moment bounds

$$\mathbb{E}\left\|\mathcal{I}_\kappa(x)\right\|_{\mathbb{X}}^2 \leq d_1^\kappa, \qquad \mathbb{E}\left\|\widehat{\mathcal{I}}_\kappa(x)\right\|_{\mathbb{X}}^2 \leq d_2^\kappa, \quad \forall\, x \in \mathbb{X},$$

hold.

2. $\exists$ constants $\iota_1^*, \iota_2^* > 0$ **s.t.**, $\forall\ x, \bar{x} \in \mathbb{X}$ and each $\kappa = 1, 2, \dots, \gamma$, the Lipschitz-type estimates

$$\mathbb{E}\left\|\mathcal{I}_\kappa(x) - \mathcal{I}_\kappa(\bar{x})\right\|^2 \leq \iota_1^*\, \mathbb{E}\|x - \bar{x}\|^2, \qquad \mathbb{E}\left\|\widehat{\mathcal{I}}_\kappa(x) - \widehat{\mathcal{I}}_\kappa(\bar{x})\right\|^2 \leq \iota_2^*\, \mathbb{E}\|x - \bar{x}\|^2,$$

are satisfied.

Remark 3.1. *Although hypothesis $(H5)$ imposes complete continuity together with mean–square Lipschitz regularity on the impulsive operators $\mathcal{I}_\kappa$ and $\widehat{\mathcal{I}}_\kappa$, this framework accommodates a broad spectrum of nonlinear operators, including integral transforms and Nemytskii-type mappings that frequently emerge in control-theoretic formulations and physical system modeling. An explicit realization substantiating the admissibility of $(H5)$ is furnished in Section 4.*

$(H6)$ The mappings $\vartheta, \beta : \mathbb{B} \to \mathbb{X}$ are continuous and satisfy the following growth and Lipschitz-type conditions: $\exists$ constants $\mathfrak{w}_\vartheta, \mathfrak{w}_\beta, \mathfrak{w}_1^\vartheta, \mathfrak{w}_1^\beta > 0$ **s.t.**

$$\mathbb{E}\|\vartheta(x_1) - \vartheta(x_2)\|_{\mathbb{X}}^2 \leq \mathfrak{w}_\vartheta \|x_1 - x_2\|_{\mathbb{B}}^2, \qquad \mathbb{E}\|\vartheta(x)\|_{\mathbb{X}}^2 \leq \mathfrak{w}_1^\vartheta\Big(1 + \|x\|_{\mathbb{B}}^2\Big),$$

$$\mathbb{E}\|\beta(x_1) - \beta(x_2)\|_{\mathbb{X}}^2 \leq \mathfrak{w}_\beta \|x_1 - x_2\|_{\mathbb{B}}^2, \qquad \mathbb{E}\|\beta(x)\|_{\mathbb{X}}^2 \leq \mathfrak{w}_1^\beta\Big(1 + \|x\|_{\mathbb{B}}^2\Big),$$

$\forall\ x, x_1, x_2 \in \mathbb{B}$.

Lemma 3.1. *[53] Pretend that $(H1)$ and $(H2)$ are genuine. Then, $\forall\ x \in \mathbb{L}_2(\mathfrak{J}, \mathbb{X})$, the set-valued mapping $\mathbb{N}(x)$ is non-empty, cv, and weakly compact. Moreover, if $x_n \to x$ strongly in $\mathbb{L}_2(\mathfrak{J}, \mathbb{X})$ and $x_n \in \mathbb{N}(x_n)$ with $x_n \rightharpoonup x$ weakly in $\mathbb{L}_2(\mathfrak{J}, \mathbb{X})$, then it is undoubtedly that $x \in \mathbb{N}(x)$.*

Theorem 3.1. *Under assumptions $(H1)$–$(H6)$, the dynamical system (1.1) admits at least one $\mathbb{MS}$ on the interval $\mathfrak{J}$ corresponding to every admissible control function $U(\cdot) \in \mathbb{L}_2(\mathfrak{J}, \mathcal{U})$, provided that the smallness condition*

$$2\gamma\left(\widehat{\aleph}\aleph\theta\widehat{M}\right)^2 \cdot (\iota_1^* + \iota_2^*) < 1$$

is fulfilled.

Proof. We begin by introducing the mapping $\Lambda : \mathbb{B}_\rho \to \mathbb{B}_\rho$, which is specified as follows:

$$\Lambda x(\tau) = \begin{cases} \varphi(\tau), & \tau \in (-\infty, 0], \\ \widehat{\Upsilon}_2, & \tau \in \mathfrak{J}. \end{cases}$$

Our objective is to demonstrate that the mapping Λ admits at least one FP in the BS $\mathbb{B}_\rho$; such a FP coincides with a $\mathbb{MS}$ of the system (1.1). Let $\varphi \in \mathbb{B}$ be arbitrary. We stipulate the auxiliary trajectory $y : (-\infty, \rho] \to \mathbb{X}$ via the prescription

$$y(\tau) = \begin{cases} \varphi(\tau), & \tau \in (-\infty, 0], \\ \bar{Y}^{-1}\mathfrak{R}\mathfrak{V}_\hbar(\tau)\bar{Y}\varphi(0), & \tau \in \mathfrak{J}. \end{cases}$$

By virtue of its construction, it is immediate that $y \in \mathbb{B}_\rho$. Moreover, for an arbitrary $\bar{x} \in \mathbb{C}(\mathfrak{J}, \mathbb{X})$, we demonstrate the lifted operator $\widehat{x} : (-\infty, \rho] \to \mathbb{X}$ categorized as

$$\widehat{x}(\tau) = \begin{cases} 0, & \tau \in (-\infty, 0], \\ \bar{x}(\tau), & \tau \in \mathfrak{J}, \end{cases}$$

and set

$$x(\tau) = y(\tau) + \widehat{x}(\tau), \qquad \tau \in \mathfrak{J}.$$

It is straightforward to affirm that a function x solves system (1.1) iff the associated auxiliary function $\bar{x}$ fulfills the null-history requirement $\bar{x}_0(\tau) = 0, \ \ \forall\, \tau \in (-\infty, 0]$, and, moreover, satisfies the corresponding evolution relation for every $\tau \in \mathfrak{J}$.

$$\begin{aligned}
\bar{x}(\tau) \ = \ & \bar{Y}^{-1}\mathfrak{R}\mathfrak{V}_\hbar(\tau)\bar{Y}\Big\{-\vartheta(y+\widehat{x})\Big\} + \bar{Y}^{-1}\mathfrak{R}\mathfrak{K}_\hbar(\tau)\bar{Y}\Big\{x_1 - \beta(y+\widehat{x})\Big\} \\
& + \frac{\wp\mathfrak{R}(2-\hbar)}{\mathfrak{M}(\hbar-1)\Gamma(\hbar)}\int_0^\tau \bar{Y}^{-1}(\tau-\bar{s})^{\hbar-1}\Big\{\mho U(\bar{s}) + \mathfrak{g}(\bar{s})\Big\}d\bar{s} \\
& + \frac{\wp\mathfrak{R}(2-\hbar)}{\mathfrak{M}(\hbar-1)\Gamma(\hbar)}\int_0^\tau \bar{Y}^{-1}(\tau-\bar{s})^{\hbar-1}\zeta\Big(\bar{s}, y_{\bar{s}} + \widehat{x}_{\bar{s}}, \int_0^{\bar{s}} \mathbb{H}(\bar{s}, \mathfrak{U}, y_{\mathfrak{U}} + \widehat{x}_{\mathfrak{U}})\, d\mathfrak{U}\Big)d\bar{W}(\bar{s}) \\
& + \frac{\wp\mathfrak{R}(2-\hbar)}{\mathfrak{M}(\hbar-1)\Gamma(\hbar)}\int_0^\tau \bar{Y}^{-1}(\tau-\bar{s})^{\hbar-1}\mathfrak{F}\Big(\bar{s}, y_{\bar{s}} + \widehat{x}_{\bar{s}}, \int_0^{\bar{s}} \mathbb{G}(\bar{s}, \mathfrak{U}, y_{\mathfrak{U}} + \widehat{x}_{\mathfrak{U}})\, d\mathfrak{U}\Big)d\bar{s} \\
& + \frac{(\hbar-1)\mathfrak{R}^2}{\mathfrak{M}(\hbar-1)}\int_0^\tau \bar{Y}^{-1}\mathfrak{Q}_\hbar(\tau-\bar{s})\Big\{\mho U(\bar{s}) + \mathfrak{g}(\bar{s})\Big\}d\bar{s} \\
& + \frac{(\hbar-1)\mathfrak{R}^2}{\mathfrak{M}(\hbar-1)}\int_0^\tau \bar{Y}^{-1}\mathfrak{Q}_\hbar(\tau-\bar{s})\zeta\Big(\bar{s}, y_{\bar{s}} + \widehat{x}_{\bar{s}}, \int_0^{\bar{s}} \mathbb{H}(\bar{s}, \mathfrak{U}, y_{\mathfrak{U}} + \widehat{x}_{\mathfrak{U}})\, d\mathfrak{U}\Big)d\bar{W}(\bar{s}) \\
& + \frac{(\hbar-1)\mathfrak{R}^2}{\mathfrak{M}(\hbar-1)}\int_0^\tau \bar{Y}^{-1}\mathfrak{Q}_\hbar(\tau-\bar{s})\mathfrak{F}\Big(\bar{s}, y_{\bar{s}} + \widehat{x}_{\bar{s}}, \int_0^{\bar{s}} \mathbb{G}(\bar{s}, \mathfrak{U}, y_{\mathfrak{U}} + \widehat{x}_{\mathfrak{U}})\, d\mathfrak{U}\Big)d\bar{s} \\
& + \sum_{0<\tau_\kappa<\tau} \bar{Y}^{-1}\mathfrak{R}\mathfrak{V}_\hbar(\tau-\tau_\kappa)\bar{Y}\mathcal{I}_\kappa(y(\tau_\kappa) + \widehat{x}(\tau_\kappa)) + \sum_{0<\tau_\kappa<\tau} \bar{Y}^{-1}\mathfrak{R}\mathfrak{K}_\hbar(\tau-\tau_\kappa)\bar{Y}\widehat{\mathcal{I}}_\kappa(y(\tau_\kappa) + \widehat{x}(\tau_\kappa)).
\end{aligned}$$

We now introduce the weighted subspace

$$\widehat{\mathbb{B}}_\rho := \Big\{\bar{x} \in \mathbb{B}_\rho : \ \bar{x}_0 = 0 \in \mathbb{B}\Big\}.$$

For any $\bar{x} \in \widehat{\mathbb{B}}_\rho$, the induced norm reduces to

$$\|\bar{x}\|_{\widehat{\mathbb{B}}_\rho} = \sup_{\tau\in\mathfrak{J}}\Big(\mathbb{E}\|\bar{x}(\tau)\|^2\Big)^{1/2}.$$

It follows by standard completeness arguments that the normed space $(\widehat{\mathbb{B}}_\rho, \|\cdot\|_{\widehat{\mathbb{B}}_\rho})$ is a BS. We next introduce the operator

$$\widehat{\Lambda} : \widehat{\mathbb{B}}_\rho \longrightarrow \widehat{\mathbb{B}}_\rho,$$

which is specified as follows.

$$\widehat{\Lambda}\bar{x}(\tau) = \begin{cases} 0, & \tau \in (-\infty, 0], \\ \bar{x}(\tau), & \tau \in \mathfrak{J}. \end{cases}$$

We hereby delineate the set

$$\mathbb{B}_\epsilon := \Big\{ y \in \widehat{\mathbb{B}}_\rho : \mathbb{E}\|y\|^2_{\widehat{\mathbb{B}}_\rho} \le \epsilon,\ \epsilon > 0 \Big\}.$$

Evidently, $\mathbb{B}_\epsilon$ constitutes a bd, cl, and cv subset of $\widehat{\mathbb{B}}_\rho$. For $x \in \mathbb{B}_\epsilon$ and Lemma 2.2, we can obtain that

$$\begin{aligned} \|y_\tau + \widehat{x_\tau}\|^2_{\mathbb{B}} &\le 2\Big(\|y_\tau\|^2_{\mathbb{B}} + \|\widehat{x_\tau}\|^2_{\mathbb{B}}\Big), \\ &\le 4\Big(k^2 \sup_{\bar{s}\in[0,\tau]} \mathbb{E}\|y(\bar{s})\|^2_{\mathbb{X}} + \|y_0\|^2_{\mathbb{B}}\Big) + 4\Big(k^2 \sup_{\bar{s}\in[0,\tau]} \mathbb{E}\|\widehat{x}(\bar{s})\|^2_{\mathbb{X}} + \|\widehat{x_0}\|^2_{\mathbb{B}}\Big), \\ &\le 4\Big(\|\varphi\|^2_{\mathbb{B}} + k^2\Big\{\epsilon + \theta^2\widehat{M}^2\mathbb{E}\|\varphi(0)\|^2_{\mathbb{X}}\Big\}\Big) = \epsilon^*. \end{aligned}$$

For clarity of exposition, the argument is decomposed into five distinct stages, which are detailed below.
Step 1. $\forall\ \bar{x} \in \widehat{\mathbb{B}}_\rho$, the image set $\widehat{\Lambda}(\bar{x})$ is non-empty, cv, and weakly compact. Indeed, invoking Lemma 3.1 yields that $\widehat{\Lambda}(\bar{x}) \subset \mathbb{N}(\bar{x})$ is non-void and possesses weakly compact values. Furthermore, since the $\mathbb{MV}$-mapping $\mathbb{N}(\bar{x})$ admits cv values, it follows that for any $\mathfrak{g}_1, \mathfrak{g}_2 \in \mathbb{N}(\bar{x})$ and any $\sigma \in (0,1)$, the cv combination $\sigma\mathfrak{g}_1 + (1-\sigma)\mathfrak{g}_2$ also belongs to $\mathbb{N}(\bar{x})$. Consequently, the set $\widehat{\Lambda}(\bar{x})$ inherits convexity.
Step 2. The operator $\widehat{\Lambda}(\bar{x})$ transmutes bd subsets of $\mathbb{B}_\epsilon$ into bd subsets within the same space.
In light of the characterization of the set $\mathbb{B}_\epsilon$ within $\widehat{\mathbb{B}}_\rho$, our objective is to establish the existence of a constant $\epsilon > 0$ **s.t.** for every $\Xi \in \widehat{\Lambda}(\bar{x})$ with $\bar{x} \in \mathbb{B}_\epsilon$, the following estimate holds:

$$\mathbb{E}\|\Xi(\tau)\|^2 \le \epsilon.$$

Suppose that $\Xi \in \widehat{\Lambda}(\bar{x})$. Then, $\exists$ a selection $\mathfrak{g} \in \mathbb{N}(\bar{x})$ satisfies

$$\begin{aligned} \Xi(\tau) &= \bar{Y}^{-1}\mathfrak{R}\mathfrak{V}_\hbar(\tau)\bar{Y}\Big\{-\vartheta(y+\widehat{x})\Big\} + \bar{Y}^{-1}\mathfrak{R}\mathfrak{K}_\hbar(\tau)\bar{Y}\Big\{x_1 - \beta(y+\widehat{x})\Big\} \\ &+ \frac{\wp\mathfrak{R}(2-\hbar)}{\mathfrak{M}(\hbar-1)\Gamma(\hbar)} \int_0^\tau \bar{Y}^{-1}(\tau-\bar{s})^{\hbar-1}\Big\{\mho U(\bar{s}) + \mathfrak{g}(\bar{s})\Big\} d\bar{s} \\ &+ \frac{\wp\mathfrak{R}(2-\hbar)}{\mathfrak{M}(\hbar-1)\Gamma(\hbar)} \int_0^\tau \bar{Y}^{-1}(\tau-\bar{s})^{\hbar-1}\zeta\Big(\bar{s}, y_{\bar{s}} + \widehat{x}_{\bar{s}}, \int_0^{\bar{s}} \mathbb{H}(\bar{s}, \mathfrak{U}, y_{\mathfrak{U}} + \widehat{x}_{\mathfrak{U}})\, d\mathfrak{U}\Big) d\bar{W}(\bar{s}) \\ &+ \frac{\wp\mathfrak{R}(2-\hbar)}{\mathfrak{M}(\hbar-1)\Gamma(\hbar)} \int_0^\tau \bar{Y}^{-1}(\tau-\bar{s})^{\hbar-1}\mathfrak{F}\Big(\bar{s}, y_{\bar{s}} + \widehat{x}_{\bar{s}}, \int_0^{\bar{s}} \mathbb{G}(\bar{s}, \mathfrak{U}, y_{\mathfrak{U}} + \widehat{x}_{\mathfrak{U}})\, d\mathfrak{U}\Big) d\bar{s} \\ &+ \frac{(\hbar-1)\mathfrak{R}^2}{\mathfrak{M}(\hbar-1)} \int_0^\tau \bar{Y}^{-1}\mathfrak{Q}_\hbar(\tau-\bar{s})\Big\{\mho U(\bar{s}) + \mathfrak{g}(\bar{s})\Big\} d\bar{s} \\ &+ \frac{(\hbar-1)\mathfrak{R}^2}{\mathfrak{M}(\hbar-1)} \int_0^\tau \bar{Y}^{-1}\mathfrak{Q}_\hbar(\tau-\bar{s})\zeta\Big(\bar{s}, y_{\bar{s}} + \widehat{x}_{\bar{s}}, \int_0^{\bar{s}} \mathbb{H}(\bar{s}, \mathfrak{U}, y_{\mathfrak{U}} + \widehat{x}_{\mathfrak{U}})\, d\mathfrak{U}\Big) d\bar{W}(\bar{s}) \end{aligned}$$

$$
\begin{aligned}
&+\frac{(\hbar-1)\Re^2}{\mathfrak{M}(\hbar-1)}\int_0^\tau \bar{Y}^{-1}\mathfrak{Q}_\hbar(\tau-\bar{s})\mathfrak{F}\Big(\bar{s},y_{\bar{s}}+\widehat{x_{\bar{s}}},\int_0^{\bar{s}}\mathbb{G}(\bar{s},\mathfrak{U},y_{\mathfrak{U}}+\widehat{x_{\mathfrak{U}}})\,d\mathfrak{U}\Big)d\bar{s}\\
&+\sum_{0<\tau_\kappa<\tau}\bar{Y}^{-1}\Re\mathfrak{V}_\hbar(\tau-\tau_\kappa)\bar{Y}\mathcal{I}_\kappa(y(\tau_\kappa)+\widehat{x}(\tau_\kappa))\\
&+\sum_{0<\tau_\kappa<\tau}\bar{Y}^{-1}\Re\mathfrak{K}_\hbar(\tau-\tau_\kappa)\bar{Y}\widehat{\mathcal{I}_\kappa}(y(\tau_\kappa)+\widehat{x}(\tau_\kappa)),\quad \tau\in\mathfrak{J}. \qquad (3.1)
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\mathbb{E}\|\Xi(\tau)\|^2 \le\;& 12\mathbb{E}\Big\|\bar{Y}^{-1}\Re\mathfrak{V}_\hbar(\tau)\bar{Y}\Big\{-\vartheta(y+\widehat{x})\Big\}\Big\|^2+12\mathbb{E}\Big\|\bar{Y}^{-1}\Re\mathfrak{K}_\hbar(\tau)\bar{Y}\Big\{x_1-\beta(y+\widehat{x})\Big\}\Big\|^2\\
&+12\mathbb{E}\left\|\frac{\wp\Re(2-\hbar)}{\mathfrak{M}(\hbar-1)\Gamma(\hbar)}\int_0^\tau\bar{Y}^{-1}(\tau-\bar{s})^{\hbar-1}\mho U(\bar{s})d\bar{s}\right\|^2\\
&+12\mathbb{E}\left\|\frac{\wp\Re(2-\hbar)}{\mathfrak{M}(\hbar-1)\Gamma(\hbar)}\int_0^\tau\bar{Y}^{-1}(\tau-\bar{s})^{\hbar-1}\mathfrak{g}(\bar{s})d\bar{s}\right\|^2\\
&+12\mathbb{E}\left\|\frac{\wp\Re(2-\hbar)}{\mathfrak{M}(\hbar-1)\Gamma(\hbar)}\int_0^\tau\bar{Y}^{-1}(\tau-\bar{s})^{\hbar-1}\zeta\Big(\bar{s},y_{\bar{s}}+\widehat{x_{\bar{s}}},\int_0^{\bar{s}}\mathbb{H}(\bar{s},\mathfrak{U},y_{\mathfrak{U}}+\widehat{x_{\mathfrak{U}}})\,d\mathfrak{U}\Big)d\bar{W}(\bar{s})\right\|^2\\
&+12\mathbb{E}\left\|\frac{\wp\Re(2-\hbar)}{\mathfrak{M}(\hbar-1)\Gamma(\hbar)}\int_0^\tau\bar{Y}^{-1}(\tau-\bar{s})^{\hbar-1}\mathfrak{F}\Big(\bar{s},y_{\bar{s}}+\widehat{x_{\bar{s}}},\int_0^{\bar{s}}\mathbb{G}(\bar{s},\mathfrak{U},y_{\mathfrak{U}}+\widehat{x_{\mathfrak{U}}})\,d\mathfrak{U}\Big)d\bar{s}\right\|^2\\
&+12\mathbb{E}\left\|\frac{(\hbar-1)\Re^2}{\mathfrak{M}(\hbar-1)}\int_0^\tau\bar{Y}^{-1}\mathfrak{Q}_\hbar(\tau-\bar{s})\mho U(\bar{s})d\bar{s}\right\|^2\\
&+12\mathbb{E}\left\|\frac{(\hbar-1)\Re^2}{\mathfrak{M}(\hbar-1)}\int_0^\tau\bar{Y}^{-1}\mathfrak{Q}_\hbar(\tau-\bar{s})\mathfrak{g}(\bar{s})d\bar{s}\right\|^2\\
&+12\mathbb{E}\left\|\frac{(\hbar-1)\Re^2}{\mathfrak{M}(\hbar-1)}\int_0^\tau\bar{Y}^{-1}\mathfrak{Q}_\hbar(\tau-\bar{s})\zeta\Big(\bar{s},y_{\bar{s}}+\widehat{x_{\bar{s}}},\int_0^{\bar{s}}\mathbb{H}(\bar{s},\mathfrak{U},y_{\mathfrak{U}}+\widehat{x_{\mathfrak{U}}})\,d\mathfrak{U}\Big)d\bar{W}(\bar{s})\right\|^2\\
&+12\mathbb{E}\left\|\frac{(\hbar-1)\Re^2}{\mathfrak{M}(\hbar-1)}\int_0^\tau\bar{Y}^{-1}\mathfrak{Q}_\hbar(\tau-\bar{s})\mathfrak{F}\Big(\bar{s},y_{\bar{s}}+\widehat{x_{\bar{s}}},\int_0^{\bar{s}}\mathbb{G}(\bar{s},\mathfrak{U},y_{\mathfrak{U}}+\widehat{x_{\mathfrak{U}}})\,d\mathfrak{U}\Big)d\bar{s}\right\|^2\\
&+12\mathbb{E}\left\|\sum_{0<\tau_\kappa<\tau}\bar{Y}^{-1}\Re\mathfrak{V}_\hbar(\tau-\tau_\kappa)\bar{Y}\mathcal{I}_\kappa(y(\tau_\kappa)+\widehat{x}(\tau_\kappa))\right\|^2\\
&+12\mathbb{E}\left\|\sum_{0<\tau_\kappa<\tau}\bar{Y}^{-1}\Re\mathfrak{K}_\hbar(\tau-\tau_\kappa)\bar{Y}\widehat{\mathcal{I}_\kappa}(y(\tau_\kappa)+\widehat{x}(\tau_\kappa))\right\|^2\\
\le\;& 12\aleph^2\widehat{\aleph}^2\theta^2\widehat{M}^2\mathfrak{w}_1^\vartheta(1+\epsilon^*)+24\aleph^2\widehat{\aleph}^2\theta^2\widehat{M}^2\Big[\|x_1\|^2+\mathfrak{w}_2^\beta(1+\epsilon^*)\Big]+12\aleph^2\widehat{\aleph}^2\theta^2\widehat{M}^2\gamma\sum_{\kappa=1}^{\gamma}d_1^\kappa\\
&+12\aleph^2\widehat{\aleph}^2\theta^2\widehat{M}^2\gamma\sum_{\kappa=1}^{\gamma}d_2^\kappa+\frac{12\widehat{\aleph}^2\psi^2\theta^2(2-\hbar)^2\widehat{M}^2\|\mho\|^2\rho^{4\hbar-3}}{\xi^2\mathfrak{M}^2(\hbar-1)\Gamma^2(\hbar)(4\hbar-3)}\mathcal{K}\\
&+\frac{12\psi^2\theta^2(2-\hbar)^2\widehat{\aleph}^2\rho^{2\hbar-1}}{\mathfrak{M}^2(\hbar-1)\Gamma^2(\hbar)(2\hbar-1)}\Big[\big\|\mathfrak{I}_{\mathfrak{g}}\big\|+k_{\mathfrak{g}}\rho\epsilon^*\Big]
\end{aligned}
$$

$$+\frac{12\psi^2\theta^2(2-\hbar)^2\widehat{\aleph}^2\,\mathrm{Tr}(Q)\rho^{2\hbar}}{\mathfrak{M}^2(\hbar-1)\Gamma^2(\hbar)(2\hbar-1)}\Psi_\zeta\Big(\epsilon^*+\mathfrak{w}_{\mathbb{H}}(1+\epsilon^*)\rho\Big)\sup_{\bar{s}\in\mathfrak{J}}\mathfrak{b}_\zeta(\bar{s})$$
$$+\frac{12\psi^2\theta^2(2-\hbar)^2\widehat{\aleph}^2\rho^{2\hbar}}{\mathfrak{M}^2(\hbar-1)\Gamma^2(\hbar)(2\hbar-1)}\Psi_{\mathfrak{F}}\Big(\epsilon^*+\mathfrak{w}_{\mathbb{G}}(1+\epsilon^*)\rho\Big)\sup_{\bar{s}\in\mathfrak{J}}\mathfrak{b}_{\mathfrak{F}}(\bar{s})$$
$$+\frac{12\widehat{\aleph}^2\theta^4(\hbar-1)^2\widehat{M}^2\,\|\mho\|^2\rho}{\xi^2\mathfrak{M}^2(\hbar-1)}\mathcal{K}+\frac{12\theta^4(\hbar-1)^2\widehat{\aleph}^2\widehat{M}^2\rho}{\mathfrak{M}^2(\hbar-1)}\Big[\,\big\|\mathfrak{I}_{\mathfrak{g}}\big\|+k_{\mathfrak{g}}\rho\epsilon^*\Big]$$
$$+\frac{12\theta^4(\hbar-1)^2\widehat{\aleph}^2\widehat{M}^2\,\mathrm{Tr}(Q)\rho}{\mathfrak{M}^2(\hbar-1)}\Psi_\zeta\Big(\epsilon^*+\mathfrak{w}_{\mathbb{H}}(1+\epsilon^*)\rho\Big)\sup_{\bar{s}\in\mathfrak{J}}\mathfrak{b}_\zeta(\bar{s})$$
$$+\frac{12\theta^4(\hbar-1)^2\widehat{\aleph}^2\widehat{M}^2\rho}{\mathfrak{M}^2(\hbar-1)}\Psi_{\mathfrak{F}}\Big(\epsilon^*+\mathfrak{w}_{\mathbb{G}}(1+\epsilon^*)\rho\Big)\sup_{\bar{s}\in\mathfrak{J}}\mathfrak{b}_{\mathfrak{F}}(\bar{s}),$$

where

$$\begin{aligned}\mathcal{K} &= 12\mathbb{E}\left\|y_\rho+\widehat{x}_\rho\right\|^2+12\int_0^\rho\mathbb{E}\left\|\widehat{\varrho}(\bar{s})\right\|^2\,d\bar{s}+12\aleph^2\widehat{\aleph}^2\theta^2\widehat{M}^2\mathfrak{w}_1^{\vartheta}(1+\epsilon^*)\\
&\quad+24\aleph^2\widehat{\aleph}^2\theta^2\widehat{M}^2\Big[\,\|x_1\|^2+\mathfrak{w}_2^{\beta}(1+\epsilon^*)\Big]+12\aleph^2\widehat{\aleph}^2\theta^2\widehat{M}^2\gamma\sum_{\kappa=1}^{\gamma}d_1^{\kappa}\\
&\quad+\frac{12\psi^2\theta^2(2-\hbar)^2\widehat{\aleph}^2\rho^{2\hbar-1}}{\mathfrak{M}^2(\hbar-1)\Gamma^2(\hbar)(2\hbar-1)}\Big[\,\big\|\mathfrak{I}_{\mathfrak{g}}\big\|+k_{\mathfrak{g}}\rho\epsilon^*\Big]\\
&\quad+\frac{12\psi^2\theta^2(2-\hbar)^2\widehat{\aleph}^2\,\mathrm{Tr}(Q)\rho^{2\hbar}}{\mathfrak{M}^2(\hbar-1)\Gamma^2(\hbar)(2\hbar-1)}\Psi_\zeta\Big(\epsilon^*+\mathfrak{w}_{\mathbb{H}}(1+\epsilon^*)\rho\Big)\sup_{\bar{s}\in\mathfrak{J}}\mathfrak{b}_\zeta(\bar{s})\\
&\quad+\frac{12\psi^2\theta^2(2-\hbar)^2\widehat{\aleph}^2\rho^{2\hbar}}{\mathfrak{M}^2(\hbar-1)\Gamma^2(\hbar)(2\hbar-1)}\Psi_{\mathfrak{F}}\Big(\epsilon^*+\mathfrak{w}_{\mathbb{G}}(1+\epsilon^*)\rho\Big)\sup_{\bar{s}\in\mathfrak{J}}\mathfrak{b}_{\mathfrak{F}}(\bar{s})\\
&\quad+\frac{12\theta^4(\hbar-1)^2\widehat{\aleph}^2\widehat{M}^2\rho}{\mathfrak{M}^2(\hbar-1)}\Big[\,\big\|\mathfrak{I}_{\mathfrak{g}}\big\|+k_{\mathfrak{g}}\rho\epsilon^*\Big]\\
&\quad+\frac{12\theta^4(\hbar-1)^2\widehat{\aleph}^2\widehat{M}^2\,\mathrm{Tr}(Q)\rho}{\mathfrak{M}^2(\hbar-1)}\Psi_\zeta\Big(\epsilon^*+\mathfrak{w}_{\mathbb{H}}(1+\epsilon^*)\rho\Big)\sup_{\bar{s}\in\mathfrak{J}}\mathfrak{b}_\zeta(\bar{s})\\
&\quad+\frac{12\theta^4(\hbar-1)^2\widehat{\aleph}^2\widehat{M}^2\rho}{\mathfrak{M}^2(\hbar-1)}\Psi_{\mathfrak{F}}\Big(\epsilon^*+\mathfrak{w}_{\mathbb{G}}(1+\epsilon^*)\rho\Big)\sup_{\bar{s}\in\mathfrak{J}}\mathfrak{b}_{\mathfrak{F}}(\bar{s}).\end{aligned}$$

By dividing both sides by ϵ and subsequently letting $\epsilon\to\infty$, one infers that the operator $\widehat{\Lambda}$ conveys bd subsets of $\mathbb{B}_\epsilon$ into bd subsets of the same space.

Step 3. The operator $\widehat{\Lambda}$ is cl for every $x\in\widehat{\mathbb{B}}_\rho$.

Let $\{\Xi_n(\tau)\}_{n\geq 0}\subset\widehat{\Lambda}(\bar{x})$ be a sequence **s.t.** $\Xi_n\to\Xi$ in $\widehat{\mathbb{B}}_\rho$ as $n\to\infty$. Then $\exists\ \mathfrak{g}_n\in\mathbb{N}(\bar{x})$ satisfies, for every $\tau\in(\tau_\kappa,\tau_{\kappa+1}]$,

$$\begin{aligned}\Xi_n(\tau) &= \bar{Y}^{-1}\mathfrak{R}\mathfrak{V}_\hbar(\tau)\bar{Y}\Big\{-\vartheta(y+\widehat{x})\Big\}+\bar{Y}^{-1}\mathfrak{R}\mathfrak{K}_\hbar(\tau)\bar{Y}\Big\{x_1-\beta(y+\widehat{x})\Big\}\\
&\quad+\frac{\wp\mathfrak{R}(2-\hbar)}{\mathfrak{M}(\hbar-1)\Gamma(\hbar)}\int_0^\tau\bar{Y}^{-1}(\tau-\bar{s})^{\hbar-1}\Big\{\mho\bar{Y}^{*-1}\mho^*\mathfrak{Q}_\hbar^*(\rho-\tau)\mathcal{R}(\xi,\mathfrak{Q}_0^\rho)\bar{Y}_n(\bar{x}(\cdot))\Big\}d\bar{s}\end{aligned}$$

$$
\begin{aligned}
&+\frac{\wp\Re(2-\hbar)}{\mathfrak{M}(\hbar-1)\Gamma(\hbar)}\int_0^\tau \bar{Y}^{-1}(\tau-\bar{s})^{\hbar-1}\mathfrak{g}_n(\bar{s})d\bar{s}\\
&+\frac{\wp\Re(2-\hbar)}{\mathfrak{M}(\hbar-1)\Gamma(\hbar)}\int_0^\tau \bar{Y}^{-1}(\tau-\bar{s})^{\hbar-1}\zeta\Big(\bar{s},y_{\bar{s}}+\widehat{x_{\bar{s}}},\int_0^{\bar{s}}\mathbb{H}(\bar{s},\mathfrak{U},y_{\mathfrak{U}}+\widehat{x_{\mathfrak{U}}})\,d\mathfrak{U}\Big)d\bar{W}(\bar{s})\\
&+\frac{\wp\Re(2-\hbar)}{\mathfrak{M}(\hbar-1)\Gamma(\hbar)}\int_0^\tau \bar{Y}^{-1}(\tau-\bar{s})^{\hbar-1}\mathfrak{F}\Big(\bar{s},y_{\bar{s}}+\widehat{x_{\bar{s}}},\int_0^{\bar{s}}\mathbb{G}(\bar{s},\mathfrak{U},y_{\mathfrak{U}}+\widehat{x_{\mathfrak{U}}})\,d\mathfrak{U}\Big)d\bar{s}\\
&+\frac{(\hbar-1)\Re^2}{\mathfrak{M}(\hbar-1)}\int_0^\tau \bar{Y}^{-1}\mathfrak{Q}_\hbar(\tau-\bar{s})\Big\{\mho\bar{Y}^{*-1}\mho^*\mathfrak{Q}^*_\hbar(\rho-\tau)\mathcal{R}(\xi,\mathfrak{Q}_0^\rho)\Theta_n(\bar{x}(\cdot))\Big\}d\bar{s}\\
&+\frac{(\hbar-1)\Re^2}{\mathfrak{M}(\hbar-1)}\int_0^\tau \bar{Y}^{-1}\mathfrak{Q}_\hbar(\tau-\bar{s})\mathfrak{g}_n(\bar{s})d\bar{s}\\
&+\frac{(\hbar-1)\Re^2}{\mathfrak{M}(\hbar-1)}\int_0^\tau \bar{Y}^{-1}\mathfrak{Q}_\hbar(\tau-\bar{s})\zeta\Big(\bar{s},y_{\bar{s}}+\widehat{x_{\bar{s}}},\int_0^{\bar{s}}\mathbb{H}(\bar{s},\mathfrak{U},y_{\mathfrak{U}}+\widehat{x_{\mathfrak{U}}})\,d\mathfrak{U}\Big)d\bar{W}(\bar{s})\\
&+\frac{(\hbar-1)\Re^2}{\mathfrak{M}(\hbar-1)}\int_0^\tau \bar{Y}^{-1}\mathfrak{Q}_\hbar(\tau-\bar{s})\mathfrak{F}\Big(\bar{s},y_{\bar{s}}+\widehat{x_{\bar{s}}},\int_0^{\bar{s}}\mathbb{G}(\bar{s},\mathfrak{U},y_{\mathfrak{U}}+\widehat{x_{\mathfrak{U}}})\,d\mathfrak{U}\Big)d\bar{s}\\
&+\sum_{0<\tau_\kappa<\tau}\bar{Y}^{-1}\Re\mathfrak{V}_\hbar(\tau-\tau_\kappa)\bar{Y}\mathcal{I}_\kappa(y(\tau_\kappa)+\widehat{x}(\tau_\kappa))\\
&+\sum_{0<\tau_\kappa<\tau}\bar{Y}^{-1}\Re\mathfrak{K}_\hbar(\tau-\tau_\kappa)\bar{Y}\widehat{\mathcal{I}}_\kappa(y(\tau_\kappa)+\widehat{x}(\tau_\kappa)),
\end{aligned}
$$

where

$$
\begin{aligned}
\Theta_n(\bar{x}(\cdot)) \;=\; &\mathbb{E}[y_\rho+\widehat{x_\rho}]+\int_0^\rho \widehat{\varrho}(\bar{s})\,d\bar{W}(\bar{s})-\bar{Y}^{-1}\Re\mathfrak{V}_\hbar(\rho)\bar{Y}\Big\{\phi(0)-\vartheta(y+\widehat{x})\Big\}\\
&-\bar{Y}^{-1}\Re\mathfrak{K}_\hbar(\rho)\bar{Y}\Big\{x_1-\beta(y+\widehat{x})\Big\}-\frac{\wp\Re(2-\hbar)}{\mathfrak{M}(\hbar-1)\Gamma(\hbar)}\int_0^\rho \bar{Y}^{-1}(\rho-\bar{s})^{\hbar-1}\mathfrak{g}_n(\bar{s})d\bar{s}\\
&-\frac{\wp\Re(2-\hbar)}{\mathfrak{M}(\hbar-1)\Gamma(\hbar)}\int_0^\rho \bar{Y}^{-1}(\rho-\bar{s})^{\hbar-1}\zeta\Big(\bar{s},y_{\bar{s}}+\widehat{x_{\bar{s}}},\int_0^{\bar{s}}\mathbb{H}(\bar{s},\mathfrak{U},y_{\mathfrak{U}}+\widehat{x_{\mathfrak{U}}})\,d\mathfrak{U}\Big)d\bar{W}(\bar{s})\\
&-\frac{\wp\Re(2-\hbar)}{\mathfrak{M}(\hbar-1)\Gamma(\hbar)}\int_0^\rho \bar{Y}^{-1}(\rho-\bar{s})^{\hbar-1}\mathfrak{F}\Big(\bar{s},y_{\bar{s}}+\widehat{x_{\bar{s}}},\int_0^{\bar{s}}\mathbb{G}(\bar{s},\mathfrak{U},y_{\mathfrak{U}}+\widehat{x_{\mathfrak{U}}})\,d\mathfrak{U}\Big)d\bar{s}\\
&-\frac{(\hbar-1)\Re^2}{\mathfrak{M}(\hbar-1)}\int_0^\rho \bar{Y}^{-1}\mathfrak{Q}_\hbar(\rho-\bar{s})\mathfrak{g}_n(\bar{s})d\bar{s}\\
&-\frac{(\hbar-1)\Re^2}{\mathfrak{M}(\hbar-1)}\int_0^\rho \bar{Y}^{-1}\mathfrak{Q}_\hbar(\rho-\bar{s})\zeta\Big(\bar{s},y_{\bar{s}}+\widehat{x_{\bar{s}}},\int_0^{\bar{s}}\mathbb{H}(\bar{s},\mathfrak{U},y_{\mathfrak{U}}+\widehat{x_{\mathfrak{U}}})\,d\mathfrak{U}\Big)d\bar{W}(\bar{s})\\
&-\frac{(\hbar-1)\Re^2}{\mathfrak{M}(\hbar-1)}\int_0^\rho \bar{Y}^{-1}\mathfrak{Q}_\hbar(\rho-\bar{s})\mathfrak{F}\Big(\bar{s},y_{\bar{s}}+\widehat{x_{\bar{s}}},\int_0^{\bar{s}}\mathbb{G}(\bar{s},\mathfrak{U},y_{\mathfrak{U}}+\widehat{x_{\mathfrak{U}}})\,d\mathfrak{U}\Big)d\bar{s}\\
&-\sum_{\kappa=1}^{\gamma}\bar{Y}^{-1}\Re\mathfrak{V}_\hbar(\rho-\tau_\kappa)\bar{Y}\mathcal{I}_\kappa(x(\tau_\kappa))-\sum_{\kappa=1}^{\gamma}\bar{Y}^{-1}\Re\mathfrak{K}_\hbar(\rho-\tau_\kappa)\bar{Y}\widehat{\mathcal{I}}_\kappa(x(\tau_\kappa)).
\end{aligned}
$$

Owing to the weak compactness of $\mathbb{N}(\bar{x})$, the sequence $\{\mathfrak{g}_n\}$ possesses a weakly convergent subsequence, converging to some $\mathfrak{g}\in\mathbb{N}(\bar{x})$. As a direct consequence of this weak convergence, one infers that

$$\Xi_n(\tau)\longrightarrow\Xi(\tau),\qquad \text{as } n\to\infty.$$

Step 4. The $\mathbb{MV}$-operator $\widehat{\Lambda}(\bar{x})$ is $\mathbb{USC}$ and possesses the condensing property.
We split the operator $\widehat{\Lambda}$ into the sum of two components, namely

$$\widehat{\Lambda} = \widehat{\Lambda}_1 + \widehat{\Lambda}_2,$$

where the operators $\widehat{\Lambda}_1$ and $\widehat{\Lambda}_2$ are specified as follows:

$$\begin{aligned}
\widehat{\Lambda}_1\bar{x}(\tau) &= \sum_{\kappa=1}^{\gamma} \bar{Y}^{-1}\mathfrak{R}\mathfrak{V}_{\hbar}(\tau-\tau_\kappa)\bar{Y}\mathcal{I}_\kappa(y(\tau_\kappa)+\widehat{x}(\tau_\kappa)) + \sum_{\kappa=1}^{\gamma} \bar{Y}^{-1}\mathfrak{R}\mathfrak{K}_{\hbar}(\tau-\tau_\kappa)\bar{Y}\widehat{\mathcal{I}}_\kappa(y(\tau_\kappa)+\widehat{x}(\tau_\kappa)),\\
\widehat{\Lambda}_2\bar{x}(\tau) &= \bar{Y}^{-1}\mathfrak{R}\mathfrak{V}_{\hbar}(\tau)\bar{Y}\Big\{-\vartheta(y+\widehat{x})\Big\} + \bar{Y}^{-1}\mathfrak{R}\mathfrak{K}_{\hbar}(\tau)\bar{Y}\Big\{x_1-\beta(y+\widehat{x})\Big\}\\
&\quad + \frac{\wp\mathfrak{R}(2-\hbar)}{\mathfrak{M}(\hbar-1)\Gamma(\hbar)}\int_0^\tau \bar{Y}^{-1}(\tau-\bar{s})^{\hbar-1}\Big\{\mho U(\bar{s})+\mathfrak{g}(\bar{s})\Big\}d\bar{s}\\
&\quad + \frac{\wp\mathfrak{R}(2-\hbar)}{\mathfrak{M}(\hbar-1)\Gamma(\hbar)}\int_0^\tau \bar{Y}^{-1}(\tau-\bar{s})^{\hbar-1}\zeta\Big(\bar{s},y_{\bar{s}}+\widehat{x}_{\bar{s}},\int_0^{\bar{s}}\mathbb{H}(\bar{s},\mathfrak{U},y_{\mathfrak{U}}+\widehat{x}_{\mathfrak{U}})\,d\mathfrak{U}\Big)d\bar{W}(\bar{s})\\
&\quad + \frac{\wp\mathfrak{R}(2-\hbar)}{\mathfrak{M}(\hbar-1)\Gamma(\hbar)}\int_0^\tau \bar{Y}^{-1}(\tau-\bar{s})^{\hbar-1}\mathfrak{F}\Big(\bar{s},y_{\bar{s}}+\widehat{x}_{\bar{s}},\int_0^{\bar{s}}\mathbb{G}(\bar{s},\mathfrak{U},y_{\mathfrak{U}}+\widehat{x}_{\mathfrak{U}})\,d\mathfrak{U}\Big)d\bar{s}\\
&\quad + \frac{(\hbar-1)\mathfrak{R}^2}{\mathfrak{M}(\hbar-1)}\int_0^\tau \bar{Y}^{-1}\mathfrak{Q}_{\hbar}(\tau-\bar{s})\Big\{\mho U(\bar{s})+\mathfrak{g}(\bar{s})\Big\}d\bar{s}\\
&\quad + \frac{(\hbar-1)\mathfrak{R}^2}{\mathfrak{M}(\hbar-1)}\int_0^\tau \bar{Y}^{-1}\mathfrak{Q}_{\hbar}(\tau-\bar{s})\zeta\Big(\bar{s},y_{\bar{s}}+\widehat{x}_{\bar{s}},\int_0^{\bar{s}}\mathbb{H}(\bar{s},\mathfrak{U},y_{\mathfrak{U}}+\widehat{x}_{\mathfrak{U}})\,d\mathfrak{U}\Big)d\bar{W}(\bar{s})\\
&\quad + \frac{(\hbar-1)\mathfrak{R}^2}{\mathfrak{M}(\hbar-1)}\int_0^\tau \bar{Y}^{-1}\mathfrak{Q}_{\hbar}(\tau-\bar{s})\mathfrak{F}\Big(\bar{s},y_{\bar{s}}+\widehat{x}_{\bar{s}},\int_0^{\bar{s}}\mathbb{G}(\bar{s},\mathfrak{U},y_{\mathfrak{U}}+\widehat{x}_{\mathfrak{U}})\,d\mathfrak{U}\Big)d\bar{s}.
\end{aligned}$$

One may verify that operator $\widehat{\Lambda}_1$ possesses a strict contraction property, while operator $\widehat{\Lambda}_2$ is C.C. Indeed, for arbitrary elements $\bar{x}_1,\bar{x}_2 \in \widehat{\mathbb{B}}_\rho$, and $\forall\ \tau \in (\tau_\kappa,\tau_{\kappa+1}]$, we derive the subsequent estimate for the norm of the impulsive component $\widehat{\Lambda}_1$:

$$\begin{aligned}
\mathbb{E}\Big\|\widehat{\Lambda}_1(\bar{x}_1)(\tau)-\widehat{\Lambda}_1(\bar{x}_2)(\tau)\Big\|^2 &\le 2\mathbb{E}\Bigg\|\sum_{\kappa=1}^{\gamma}\bar{Y}^{-1}\mathfrak{R}\mathfrak{V}_{\hbar}(\tau-\tau_\kappa)\bar{Y}\Big[\mathcal{I}_\kappa(\bar{x}_1(\tau_\kappa))-\mathcal{I}_\kappa(\bar{x}_2(\tau_\kappa))\Big]\Bigg\|^2\\
&\quad + 2\mathbb{E}\Bigg\|\sum_{\kappa=1}^{\gamma}\bar{Y}^{-1}\mathfrak{R}\mathfrak{K}_{\hbar}(\tau-\tau_\kappa)\bar{Y}\Big[\widehat{\mathcal{I}}_\kappa(\bar{x}_1(\tau_\kappa))-\widehat{\mathcal{I}}_\kappa(\bar{x}_2(\tau_\kappa))\Big]\Bigg\|^2\\
&\le 2\gamma\left(\widehat{\aleph}\aleph\theta\widehat{M}\right)^2\cdot(\iota_1^*+\iota_2^*)\,\mathbb{E}\|\bar{x}_1-\bar{x}_2\|^2.
\end{aligned}$$

Consequently, if the subsequent adequate condition is met, operator $\widehat{\Lambda}_1$ exhibits a contraction attribute:

$$2\gamma\left(\widehat{\aleph}\aleph\theta\widehat{M}\right)^2\cdot(\iota_1^*+\iota_2^*) < 1.$$

Lemma 3.2. *Let $\{\mathfrak{Q}_{\hbar}(\tau)\}_{\tau\ge 0}$ represent an analytic semi-group engendered by a sectorial operator on the BS $\mathbb{X}$. Consequently, $\forall$ fixed $\tau > 0$, $\mathfrak{Q}_{\hbar}(\tau)$ constitutes a compact linear transformation on $\mathbb{X}$.*

We next establish that the mapping $\widehat{\Lambda}_2$ is $\mathbb{USC}$ and C.C. To this end, the verification is decomposed into three distinct claims.

Claim 1. Operator $\widehat{\Lambda}_2$ maps bd subsets of $\widehat{\mathbb{B}}_\rho$ into uniformly bd families. This inference is an immediate consequence of the procedure delineated in Step 2.
Claim 2. The family of operators

$$\Big\{\widehat{\Lambda}_2(\bar{x}) : \bar{x} \in \mathbb{B}_\epsilon\Big\}$$

constitutes an equi-continuous set.

$\forall\ \bar{x} \in \mathbb{B}_\epsilon$ and $\bar{\Xi} \in \widehat{\Lambda}_2(\bar{x})$, $\exists$ a selection $\mathfrak{g} \in \mathbb{N}(\bar{x})$ **s.t.** (3.1) is satisfied $\forall\ \tau \in \mathfrak{J}$. In particular, for $0 < \tau_1 < \tau_2 < \rho$, we obtain

$$\begin{aligned}
\mathbb{E}\Big\|\bar{\Xi}(\tau_2) - \bar{\Xi}(\tau_1)\Big\|^2 &\leq 7\mathbb{E}\Big\|\bar{Y}^{-1}\mathfrak{R}\Big\{\mathfrak{V}_\hbar(\tau_2) - \mathfrak{V}_\hbar(\tau_1)\Big\}\bar{Y}\Big\{-\vartheta(y+\widehat{x})\Big\} + \bar{Y}^{-1}\mathfrak{R}\Big\{\mathfrak{K}_\hbar(\tau_2) - \mathfrak{K}_\hbar(\tau_1)\Big\} \\
&\quad \times \bar{Y}\Big\{x_1 - \beta(y+\widehat{x})\Big\}\Big\|^2 + 7\mathbb{E}\Big\|\frac{\wp\mathfrak{R}(2-\hbar)}{\mathfrak{M}(\hbar-1)\Gamma(\hbar)}\Big\{\int_0^{\tau_2}\bar{Y}^{-1}(\tau_2-\bar{s})^{\hbar-1}\Big\{\mho U(\bar{s}) + \mathfrak{g}(\bar{s})\Big\}d\bar{s} \\
&\quad - \int_0^{\tau_1}\bar{Y}^{-1}(\tau_1-\bar{s})^{\hbar-1}\Big\{\mho U(\bar{s}) + \mathfrak{g}(\bar{s})\Big\}d\bar{s}\Big\}\Big\|^2 \\
&\quad + 7\mathbb{E}\Big\|\frac{\wp\mathfrak{R}(2-\hbar)}{\mathfrak{M}(\hbar-1)\Gamma(\hbar)}\Big\{\int_0^{\tau_2}\bar{Y}^{-1}(\tau_2-\bar{s})^{\hbar-1}\zeta\Big(\bar{s}, y_{\bar{s}} + \widehat{x_{\bar{s}}}, \int_0^{\bar{s}}\mathbb{H}(\bar{s},\mathfrak{U},y_{\mathfrak{U}} + \widehat{x_{\mathfrak{U}}})\,d\mathfrak{U}\Big)d\bar{W}(\bar{s}) \\
&\quad - \int_0^{\tau_1}\bar{Y}^{-1}(\tau_1-\bar{s})^{\hbar-1}\zeta\Big(\bar{s}, y_{\bar{s}} + \widehat{x_{\bar{s}}}, \int_0^{\bar{s}}\mathbb{H}(\bar{s},\mathfrak{U},y_{\mathfrak{U}} + \widehat{x_{\mathfrak{U}}})\,d\mathfrak{U}\Big)d\bar{W}(\bar{s})\Big\}\Big\|^2 \\
&\quad + 7\mathbb{E}\Big\|\frac{\wp\mathfrak{R}(2-\hbar)}{\mathfrak{M}(\hbar-1)\Gamma(\hbar)}\Big\{\int_0^{\tau_2}\bar{Y}^{-1}(\tau_2-\bar{s})^{\hbar-1}\mathfrak{F}\Big(\bar{s}, y_{\bar{s}} + \widehat{x_{\bar{s}}}, \int_0^{\bar{s}}\mathbb{G}(\bar{s},\mathfrak{U},y_{\mathfrak{U}} + \widehat{x_{\mathfrak{U}}})\,d\mathfrak{U}\Big)d\bar{s} \\
&\quad - \int_0^{\tau_1}\bar{Y}^{-1}(\tau_1-\bar{s})^{\hbar-1}\mathfrak{F}\Big(\bar{s}, y_{\bar{s}} + \widehat{x_{\bar{s}}}, \int_0^{\bar{s}}\mathbb{G}(\bar{s},\mathfrak{U},y_{\mathfrak{U}} + \widehat{x_{\mathfrak{U}}})\,d\mathfrak{U}\Big)d\bar{s}\Big\}\Big\|^2 \\
&\quad + 7\mathbb{E}\Big\|\frac{(\hbar-1)\mathfrak{R}^2}{\mathfrak{M}(\hbar-1)}\Big\{\int_0^{\tau_2}\bar{Y}^{-1}\mathfrak{Q}_\hbar(\tau_2-\bar{s})\Big\{\mho U(\bar{s}) + \mathfrak{g}(\bar{s})\Big\}d\bar{s} \\
&\quad - \int_0^{\tau_1}\bar{Y}^{-1}\mathfrak{Q}_\hbar(\tau_1-\bar{s})\Big\{\mho U(\bar{s}) + \mathfrak{g}(\bar{s})\Big\}d\bar{s}\Big\}\Big\|^2 \\
&\quad + 7\mathbb{E}\Big\|\frac{(\hbar-1)\mathfrak{R}^2}{\mathfrak{M}(\hbar-1)}\Big\{\int_0^{\tau_2}\bar{Y}^{-1}\mathfrak{Q}_\hbar(\tau_2-\bar{s})\zeta\Big(\bar{s}, y_{\bar{s}} + \widehat{x_{\bar{s}}}, \int_0^{\bar{s}}\mathbb{H}(\bar{s},\mathfrak{U},y_{\mathfrak{U}} + \widehat{x_{\mathfrak{U}}})\,d\mathfrak{U}\Big)d\bar{W}(\bar{s}) \\
&\quad - \int_0^{\tau_1}\bar{Y}^{-1}\mathfrak{Q}_\hbar(\tau_1-\bar{s})\zeta\Big(\bar{s}, y_{\bar{s}} + \widehat{x_{\bar{s}}}, \int_0^{\bar{s}}\mathbb{H}(\bar{s},\mathfrak{U},y_{\mathfrak{U}} + \widehat{x_{\mathfrak{U}}})\,d\mathfrak{U}\Big)d\bar{W}(\bar{s})\Big\}\Big\|^2 \\
&\quad + 7\mathbb{E}\Big\|\frac{(\hbar-1)\mathfrak{R}^2}{\mathfrak{M}(\hbar-1)}\Big\{\int_0^{\tau_2}\bar{Y}^{-1}\mathfrak{Q}_\hbar(\tau_2-\bar{s})\mathfrak{F}\Big(\bar{s}, y_{\bar{s}} + \widehat{x_{\bar{s}}}, \int_0^{\bar{s}}\mathbb{G}(\bar{s},\mathfrak{U},y_{\mathfrak{U}} + \widehat{x_{\mathfrak{U}}})\,d\mathfrak{U}\Big)d\bar{s} \\
&\quad - \int_0^{\tau_1}\bar{Y}^{-1}\mathfrak{Q}_\hbar(\tau_1-\bar{s})\mathfrak{F}\Big(\bar{s}, y_{\bar{s}} + \widehat{x_{\bar{s}}}, \int_0^{\bar{s}}\mathbb{G}(\bar{s},\mathfrak{U},y_{\mathfrak{U}} + \widehat{x_{\mathfrak{U}}})\,d\mathfrak{U}\Big)d\bar{s}\Big\}\Big\|^2.
\end{aligned}$$

Thus, the compactness of $\mathfrak{Q}_\hbar$ ensures continuity in the uniform operator topology. Consequently, the preceding inequality converges to zero as $\tau_2 \to \tau_1$. This, in turn, establishes that the collection $\{\widehat{\Lambda}_2(\bar{x}) : \bar{x} \in \mathbb{B}_\epsilon\}$ forms a family of equicontinuous functions.
Claim 3. The operator $\widehat{\Lambda}_2(\bar{x})$ is C.C.

$\forall\, \tau \in \mathfrak{J}$ with $\tau > 0$, we claim that the set

$$\mathfrak{O}(\tau) := \{\hat{\Xi}(\tau) : \ \hat{\Xi} \in \widehat{\Lambda}_2(\mathbb{B}_\epsilon)\}$$

is r.c. Indeed, the compactness of $\mathfrak{O}(0)$ in $\mathbb{B}_\epsilon$ is immediate. Now, fix $\tau \in (0, \rho]$ and choose $\ell \in (0, \tau)$. For any $\bar{x} \in \mathbb{B}_\epsilon$, we define

$$\begin{aligned}
\hat{\Xi}^\ell(\tau) \;=\;& \bar{Y}^{-1}\mathfrak{R}\mathfrak{V}_\hbar(\tau)\bar{Y}\Big\{-\vartheta(y+\widehat{x})\Big\} + \bar{Y}^{-1}\mathfrak{R}\mathfrak{K}_\hbar(\tau)\bar{Y}\Big\{x_1 - \beta(y+\widehat{x})\Big\} \\
&+ \frac{\wp\mathfrak{R}(2-\hbar)}{\mathfrak{M}(\hbar-1)\Gamma(\hbar)}\int_0^{\tau-\ell} \bar{Y}^{-1}(\tau-\bar{s})^{\hbar-1}\Big\{\mho U(\bar{s}) + \mathfrak{g}(\bar{s})\Big\}d\bar{s} \\
&+ \frac{\wp\mathfrak{R}(2-\hbar)}{\mathfrak{M}(\hbar-1)\Gamma(\hbar)}\int_0^{\tau-\ell} \bar{Y}^{-1}(\tau-\bar{s})^{\hbar-1}\zeta\Big(\bar{s}, y_{\bar{s}}+\widehat{x_{\bar{s}}}, \int_0^{\bar{s}} \mathbb{H}(\bar{s},\mathfrak{U}, y_{\mathfrak{U}}+\widehat{x_{\mathfrak{U}}})\, d\mathfrak{U}\Big)d\bar{W}(\bar{s}) \\
&+ \frac{\wp\mathfrak{R}(2-\hbar)}{\mathfrak{M}(\hbar-1)\Gamma(\hbar)}\int_0^{\tau-\ell} \bar{Y}^{-1}(\tau-\bar{s})^{\hbar-1}\mathfrak{F}\Big(\bar{s}, y_{\bar{s}}+\widehat{x_{\bar{s}}}, \int_0^{\bar{s}} \mathbb{G}(\bar{s},\mathfrak{U}, y_{\mathfrak{U}}+\widehat{x_{\mathfrak{U}}})\, d\mathfrak{U}\Big)d\bar{s} \\
&+ \frac{(\hbar-1)\mathfrak{R}^2}{\mathfrak{M}(\hbar-1)}\int_0^{\tau-\ell} \bar{Y}^{-1}\mathfrak{Q}_\hbar(\tau-\bar{s})\Big\{\mho U(\bar{s}) + \mathfrak{g}(\bar{s})\Big\}d\bar{s} \\
&+ \frac{(\hbar-1)\mathfrak{R}^2}{\mathfrak{M}(\hbar-1)}\int_0^{\tau-\ell} \bar{Y}^{-1}\mathfrak{Q}_\hbar(\tau-\bar{s})\zeta\Big(\bar{s}, y_{\bar{s}}+\widehat{x_{\bar{s}}}, \int_0^{\bar{s}} \mathbb{H}(\bar{s},\mathfrak{U}, y_{\mathfrak{U}}+\widehat{x_{\mathfrak{U}}})\, d\mathfrak{U}\Big)d\bar{W}(\bar{s}) \\
&+ \frac{(\hbar-1)\mathfrak{R}^2}{\mathfrak{M}(\hbar-1)}\int_0^{\tau-\ell} \bar{Y}^{-1}\mathfrak{Q}_\hbar(\tau-\bar{s})\mathfrak{F}\Big(\bar{s}, y_{\bar{s}}+\widehat{x_{\bar{s}}}, \int_0^{\bar{s}} \mathbb{G}(\bar{s},\mathfrak{U}, y_{\mathfrak{U}}+\widehat{x_{\mathfrak{U}}})\, d\mathfrak{U}\Big)d\bar{s},
\end{aligned}$$

where $\mathfrak{g} \in \mathbb{N}(\bar{x})$. Invoking the compact character of the family $\{\mathfrak{Q}_\hbar(\tau)\}_{\tau>0}$, one infers that the induced operator $\mathfrak{O}(\tau)$ exhibits relative compactness when restricted to the bd set $\widehat{\mathbb{B}}_\rho$, for all $\ell \in (0, \rho)$. In addition, one obtains

$$\begin{aligned}
\mathbb{E}\left\|\hat{\Xi}(\tau) - \hat{\Xi}^\ell(\tau)\right\|^2 \le\;& 8\mathbb{E}\left\|\frac{\wp\mathfrak{R}(2-\hbar)}{\mathfrak{M}(\hbar-1)\Gamma(\hbar)}\int_{\tau-\ell}^{\tau} \bar{Y}^{-1}(\tau-\bar{s})^{\hbar-1}\mho U(\bar{s})d\bar{s}\right\|^2 \\
&+ 8\mathbb{E}\left\|\frac{\wp\mathfrak{R}(2-\hbar)}{\mathfrak{M}(\hbar-1)\Gamma(\hbar)}\int_{\tau-\ell}^{\tau} \bar{Y}^{-1}(\tau-\bar{s})^{\hbar-1}\mathfrak{g}(\bar{s})d\bar{s}\right\|^2 \\
&+ 8\mathbb{E}\left\|\frac{\wp\mathfrak{R}(2-\hbar)}{\mathfrak{M}(\hbar-1)\Gamma(\hbar)}\int_{\tau-\ell}^{\tau} \bar{Y}^{-1}(\tau-\bar{s})^{\hbar-1}\zeta\Big(\bar{s}, y_{\bar{s}}+\widehat{x_{\bar{s}}}, \int_0^{\bar{s}} \mathbb{H}(\bar{s},\mathfrak{U}, y_{\mathfrak{U}}+\widehat{x_{\mathfrak{U}}})\, d\mathfrak{U}\Big)d\bar{W}(\bar{s})\right\|^2 \\
&+ 8\mathbb{E}\left\|\frac{\wp\mathfrak{R}(2-\hbar)}{\mathfrak{M}(\hbar-1)\Gamma(\hbar)}\int_{\tau-\ell}^{\tau} \bar{Y}^{-1}(\tau-\bar{s})^{\hbar-1}\mathfrak{F}\Big(\bar{s}, y_{\bar{s}}+\widehat{x_{\bar{s}}}, \int_0^{\bar{s}} \mathbb{G}(\bar{s},\mathfrak{U}, y_{\mathfrak{U}}+\widehat{x_{\mathfrak{U}}})\, d\mathfrak{U}\Big)d\bar{s}\right\|^2 \\
&+ 8\mathbb{E}\left\|\frac{(\hbar-1)\mathfrak{R}^2}{\mathfrak{M}(\hbar-1)}\int_{\tau-\ell}^{\tau} \bar{Y}^{-1}\mathfrak{Q}_\hbar(\tau-\bar{s})\mho U(\bar{s})d\bar{s}\right\|^2 \\
&+ 8\mathbb{E}\left\|\frac{(\hbar-1)\mathfrak{R}^2}{\mathfrak{M}(\hbar-1)}\int_{\tau-\ell}^{\tau} \bar{Y}^{-1}\mathfrak{Q}_\hbar(\tau-\bar{s})\mathfrak{g}(\bar{s})d\bar{s}\right\|^2 \\
&+ 8\mathbb{E}\left\|\frac{(\hbar-1)\mathfrak{R}^2}{\mathfrak{M}(\hbar-1)}\int_{\tau-\ell}^{\tau} \bar{Y}^{-1}\mathfrak{Q}_\hbar(\tau-\bar{s})\zeta\Big(\bar{s}, y_{\bar{s}}+\widehat{x_{\bar{s}}}, \int_0^{\bar{s}} \mathbb{H}(\bar{s},\mathfrak{U}, y_{\mathfrak{U}}+\widehat{x_{\mathfrak{U}}})\, d\mathfrak{U}\Big)d\bar{W}(\bar{s})\right\|^2 \\
&+ 8\mathbb{E}\left\|\frac{(\hbar-1)\mathfrak{R}^2}{\mathfrak{M}(\hbar-1)}\int_{\tau-\ell}^{\tau} \bar{Y}^{-1}\mathfrak{Q}_\hbar(\tau-\bar{s})\mathfrak{F}\Big(\bar{s}, y_{\bar{s}}+\widehat{x_{\bar{s}}}, \int_0^{\bar{s}} \mathbb{G}(\bar{s},\mathfrak{U}, y_{\mathfrak{U}}+\widehat{x_{\mathfrak{U}}})\, d\mathfrak{U}\Big)d\bar{s}\right\|^2.
\end{aligned}$$

Consequently, one obtains

$$\mathbb{E}\Big\|\hat{\Xi}(\tau)-\hat{\Xi}^{\ell}(\tau)\Big\|^{2}\longrightarrow 0 \quad \text{as} \quad \ell\to 0^{+}.$$

Hence, the collection $\mathfrak{O}(\tau)$ is r.c. As a result, operator $\widehat{\Lambda}_2$ is C.C.

Step 5. The $\mathbb{MV}$-mapping $\widehat{\Lambda}_2$ possesses a cl graph.

Assume that $\bar{x}_n\to\bar{x}_*$ as $n\to\infty$ in $\widehat{\mathbb{B}}_\rho$, and that $\bar{\Xi}_n\to\bar{\Xi}_*$ as $n\to\infty$, where $\bar{\Xi}_n\in\widehat{\Lambda}_2(\bar{x}_n)$ for every n. We aim to establish that $\bar{\Xi}_*\in\widehat{\Lambda}_2(\bar{x}_*)$. Indeed, the inclusion $\bar{\Xi}_n\in\widehat{\Lambda}_2(\bar{x}_n)$ entails the existence of an element $\mathrm{g}_n\in\mathbb{N}(\bar{x}_n)$ **s.t.**, $\forall\,\tau\in(0,\rho)$,

$$\begin{aligned}
\bar{\Xi}_n(\tau) \;=\;& \bar{Y}^{-1}\mathfrak{R}\mathfrak{V}_{\hbar}(\tau)\bar{Y}\Big\{-\vartheta(y+\widehat{x_n})\Big\}+\bar{Y}^{-1}\mathfrak{R}\mathfrak{K}_{\hbar}(\tau)\bar{Y}\Big\{x_1-\beta(y+\widehat{x_n})\Big\}\\
&+\frac{\wp\mathfrak{R}(2-\hbar)}{\mathfrak{M}(\hbar-1)\Gamma(\hbar)}\int_0^{\tau}\bar{Y}^{-1}(\tau-\bar{s})^{\hbar-1}\Big\{\mho U(\bar{s})+\mathrm{g}_n(\bar{s})\Big\}d\bar{s}\\
&+\frac{\wp\mathfrak{R}(2-\hbar)}{\mathfrak{M}(\hbar-1)\Gamma(\hbar)}\int_0^{\tau}\bar{Y}^{-1}(\tau-\bar{s})^{\hbar-1}\zeta\Big(\bar{s},y_{\bar{s}}+\widehat{x^n_{\bar{s}}},\int_0^{\bar{s}}\mathbb{H}(\bar{s},\mathfrak{U},y_{\mathfrak{U}}+\widehat{x^n_{\mathfrak{U}}})\,d\mathfrak{U}\Big)d\bar{W}(\bar{s})\\
&+\frac{\wp\mathfrak{R}(2-\hbar)}{\mathfrak{M}(\hbar-1)\Gamma(\hbar)}\int_0^{\tau}\bar{Y}^{-1}(\tau-\bar{s})^{\hbar-1}\mathfrak{F}\Big(\bar{s},y_{\bar{s}}+\widehat{x^n_{\bar{s}}},\int_0^{\bar{s}}\mathbb{G}(\bar{s},\mathfrak{U},y_{\mathfrak{U}}+\widehat{x^n_{\mathfrak{U}}})\,d\mathfrak{U}\Big)d\bar{s}\\
&+\frac{(\hbar-1)\mathfrak{R}^2}{\mathfrak{M}(\hbar-1)}\int_0^{\tau}\bar{Y}^{-1}\mathfrak{Q}_{\hbar}(\tau-\bar{s})\Big\{\mho U(\bar{s})+\mathrm{g}_n(\bar{s})\Big\}d\bar{s}\\
&+\frac{(\hbar-1)\mathfrak{R}^2}{\mathfrak{M}(\hbar-1)}\int_0^{\tau}\bar{Y}^{-1}\mathfrak{Q}_{\hbar}(\tau-\bar{s})\zeta\Big(\bar{s},y_{\bar{s}}+\widehat{x^n_{\bar{s}}},\int_0^{\bar{s}}\mathbb{H}(\bar{s},\mathfrak{U},y_{\mathfrak{U}}+\widehat{x^n_{\mathfrak{U}}})\,d\mathfrak{U}\Big)d\bar{W}(\bar{s})\\
&+\frac{(\hbar-1)\mathfrak{R}^2}{\mathfrak{M}(\hbar-1)}\int_0^{\tau}\bar{Y}^{-1}\mathfrak{Q}_{\hbar}(\tau-\bar{s})\mathfrak{F}\Big(\bar{s},y_{\bar{s}}+\widehat{x^n_{\bar{s}}},\int_0^{\bar{s}}\mathbb{G}(\bar{s},\mathfrak{U},y_{\mathfrak{U}}+\widehat{x^n_{\mathfrak{U}}})\,d\mathfrak{U}\Big)d\bar{s}.
\end{aligned}$$

Invoking hypotheses $(H1)$–$(H6)$, it follows immediately that the collection

$$\Big\{\bar{x}_n,\ \mathrm{g}_n,\ \zeta(\cdot,\cdot,\bar{x}_n),\ \mathfrak{F}(\cdot,\cdot,\bar{x}_n)\Big\}_{n\ge 1}$$

is uniformly bd. Furthermore, $\exists$ limit elements $\bar{x}_*$ and g_* **s.t.**, up to a subsequence,

$$\Big(\bar{x}_n,\ \mathrm{g}_n,\ \zeta(\cdot,\cdot,\bar{x}_n),\ \mathfrak{F}(\cdot,\cdot,\bar{x}_n)\Big)\rightharpoonup\Big(\bar{x}_*,\ \mathrm{g}_*,\ \zeta(\cdot,\cdot,\bar{x}_*),\ \mathfrak{F}(\cdot,\cdot,\bar{x}_*)\Big)$$

weakly in the product space

$$\widehat{\mathbb{B}}_\rho\times\mathbb{L}_2(\mathfrak{J},\mathbb{X})\times\mathbb{L}_0^2\times\mathbb{X}.$$

In light of the compactness of the operator $\mathfrak{Q}_{\hbar}$, it consequently follows that

$$\begin{aligned}
\bar{\Xi}_n(\tau) \;\to\;& \bar{Y}^{-1}\mathfrak{R}\mathfrak{V}_{\hbar}(\tau)\bar{Y}\Big\{-\vartheta(y+\widehat{x_*})\Big\}+\bar{Y}^{-1}\mathfrak{R}\mathfrak{K}_{\hbar}(\tau)\bar{Y}\Big\{x_1-\beta(y+\widehat{x_*})\Big\}\\
&+\frac{\wp\mathfrak{R}(2-\hbar)}{\mathfrak{M}(\hbar-1)\Gamma(\hbar)}\int_0^{\tau}\bar{Y}^{-1}(\tau-\bar{s})^{\hbar-1}\Big\{\mho U(\bar{s})+\mathrm{g}_*(\bar{s})\Big\}d\bar{s}\\
&+\frac{\wp\mathfrak{R}(2-\hbar)}{\mathfrak{M}(\hbar-1)\Gamma(\hbar)}\int_0^{\tau}\bar{Y}^{-1}(\tau-\bar{s})^{\hbar-1}\zeta\Big(\bar{s},y_{\bar{s}}+\widehat{x^*_{\bar{s}}},\int_0^{\bar{s}}\mathbb{H}(\bar{s},\mathfrak{U},y_{\mathfrak{U}}+\widehat{x^*_{\mathfrak{U}}})\,d\mathfrak{U}\Big)d\bar{W}(\bar{s})\\
&+\frac{\wp\mathfrak{R}(2-\hbar)}{\mathfrak{M}(\hbar-1)\Gamma(\hbar)}\int_0^{\tau}\bar{Y}^{-1}(\tau-\bar{s})^{\hbar-1}\mathfrak{F}\Big(\bar{s},y_{\bar{s}}+\widehat{x^*_{\bar{s}}},\int_0^{\bar{s}}\mathbb{G}(\bar{s},\mathfrak{U},y_{\mathfrak{U}}+\widehat{x^*_{\mathfrak{U}}})\,d\mathfrak{U}\Big)d\bar{s}
\end{aligned}$$

$$+\frac{(\hbar-1)\Re^2}{\mathfrak{M}(\hbar-1)}\int_0^\tau \bar{Y}^{-1}\mathfrak{Q}_\hbar(\tau-\bar{s})\Big\{\mho U(\bar{s})+\mathfrak{g}_*(\bar{s})\Big\}d\bar{s}$$
$$+\frac{(\hbar-1)\Re^2}{\mathfrak{M}(\hbar-1)}\int_0^\tau \bar{Y}^{-1}\mathfrak{Q}_\hbar(\tau-\bar{s})\zeta\Big(\bar{s},y_{\bar{s}}+\widehat{x}^*_{\bar{s}},\int_0^{\bar{s}}\mathbb{H}(\bar{s},\mathfrak{U},y_{\mathfrak{U}}+\widehat{x}^*_{\mathfrak{U}})\,d\mathfrak{U}\Big)d\bar{W}(\bar{s})$$
$$+\frac{(\hbar-1)\Re^2}{\mathfrak{M}(\hbar-1)}\int_0^\tau \bar{Y}^{-1}\mathfrak{Q}_\hbar(\tau-\bar{s})\mathfrak{F}\Big(\bar{s},y_{\bar{s}}+\widehat{x}^*_{\bar{s}},\int_0^{\bar{s}}\mathbb{G}(\bar{s},\mathfrak{U},y_{\mathfrak{U}}+\widehat{x}^*_{\mathfrak{U}})\,d\mathfrak{U}\Big)d\bar{s}. \qquad (3.2)$$

Focusing on $\bar{\Xi}_n \to \bar{\Xi}_*$ in $\widehat{\mathbb{B}}_\rho$ and invoking Lemma 3.1, we infer that the limit element satisfies $\mathfrak{g}_* \in \mathbb{N}(\bar{x}_*)$. Consequently, $\bar{\Xi}_* \in \widehat{\Lambda}_2(\bar{x}_*)$. This verifies that the $\mathbb{MV}$-mapping $\widehat{\Lambda}_2$ enjoys a cl graph and exhibits complete continuity with compact images. As a direct corollary, $\widehat{\Lambda}_2$ is $\mathbb{USC}$. Therefore, in view of Theorem 2.1, operator Λ admits at least one FP in $\mathbb{B}_\epsilon$, which corresponds to a $\mathbb{MS}$ of problem (1.1). □

The subsequent result addresses the $\mathbb{ACO}$ of problem (1.1). For the entirety of the subsequent analysis, we presuppose the veracity of the ensuing hypothesis.

(A1) $\exists$ a constant $\widetilde{V}_1 > 0$, satisfying

$$\mathbb{E}\Big\|\mathfrak{F}(\tau,x_1,x_2)\Big\|^2 \le \widetilde{V}_1, \quad \forall\,\tau\in\mathfrak{J},\ x_1\in\mathbb{B},\ x_2\in\mathbb{X}.$$

(A2) $\exists$ a constant $\widetilde{V}_2 > 0$ **s.t.**

$$\mathbb{E}\Big\|\zeta(\tau,x_1,x_2)\Big\|^2 \le \widetilde{V}_2, \quad \forall\,\tau\in\mathfrak{J},\ x_1\in\mathbb{B},\ x_2\in\mathbb{X}.$$

Theorem 3.2. *Assume that the premises of Theorem 3.1 are satisfied and that conditions (A1) and (A2) are verified. Moreover, suppose that the linearized variant of system* (2.5) *is approximately controllable over $\mathfrak{J}$. Then, it follows that the stochastic control system* (1.1)*, governed by a FD, inherits $\mathbb{ACO}$ on $\mathfrak{J}$.*

Proof. Let $x^\xi(\cdot)$ be a FP of Λ within the space $\mathbb{B}_\rho$. Pursuant to Theorem 3.1, any such FP inherently represents a $\mathbb{MS}$ of system (1.1). Furthermore, by virtue of the stochastic Fubini theorem, it ensues that any FP of Λ qualifies as a $\mathbb{MS}$ of (1.1), under the condition that $x^\xi(\tau)$ satisfies

$$x^\xi(\rho) = x_\rho - \xi\mathcal{R}(\xi,\mathfrak{Q}_0^\rho)\Theta(x^\xi(\cdot)),$$

where

$$\begin{aligned}\Theta(x^\xi(\cdot)) \;=\; &\mathbb{E}[x_\rho]+\int_0^\rho \widehat{\varrho}(\bar{s})\,d\bar{W}(\bar{s})-\bar{Y}^{-1}\Re\mathfrak{V}_\hbar(\rho)\bar{Y}\Big\{\phi(0)-\vartheta(x^\xi)\Big\}-\bar{Y}^{-1}\Re\mathfrak{K}_\hbar(\rho)\bar{Y}\Big\{x_1-\beta(x^\xi)\Big\}\\
&-\frac{\wp\Re(2-\hbar)}{\mathfrak{M}(\hbar-1)\Gamma(\hbar)}\int_0^\rho \bar{Y}^{-1}(\rho-\bar{s})^{\hbar-1}\mathfrak{g}^\xi(\bar{s})d\bar{s}\\
&-\frac{\wp\Re(2-\hbar)}{\mathfrak{M}(\hbar-1)\Gamma(\hbar)}\int_0^\rho \bar{Y}^{-1}(\rho-\bar{s})^{\hbar-1}\zeta\Big(\bar{s},x^\xi_{\bar{s}},\int_0^{\bar{s}}\mathbb{H}(\bar{s},\mathfrak{U},x^\xi_{\mathfrak{U}})\,d\mathfrak{U}\Big)d\bar{W}(\bar{s})\\
&-\frac{\wp\Re(2-\hbar)}{\mathfrak{M}(\hbar-1)\Gamma(\hbar)}\int_0^\rho \bar{Y}^{-1}(\rho-\bar{s})^{\hbar-1}\mathfrak{F}\Big(\bar{s},x^\xi_{\bar{s}},\int_0^{\bar{s}}\mathbb{G}(\bar{s},\mathfrak{U},x^\xi_{\mathfrak{U}})\,d\mathfrak{U}\Big)d\bar{s}\\
&-\frac{(\hbar-1)\Re^2}{\mathfrak{M}(\hbar-1)}\int_0^\rho \bar{Y}^{-1}\mathfrak{Q}_\hbar(\rho-\bar{s})\mathfrak{g}^\xi(\bar{s})d\bar{s}\end{aligned}$$

$$
\begin{aligned}
&-\frac{(\hbar-1)\mathfrak{R}^2}{\mathfrak{M}(\hbar-1)}\int_0^\rho \bar{Y}^{-1}\mathfrak{Q}_\hbar(\rho-\bar{s})\zeta\Big(\bar{s},x_{\bar{s}}^\xi,\int_0^{\bar{s}}\mathbb{H}(\bar{s},\mathfrak{U},x_{\mathfrak{U}}^\xi)\,d\mathfrak{U}\Big)d\bar{W}(\bar{s})\\
&-\frac{(\hbar-1)\mathfrak{R}^2}{\mathfrak{M}(\hbar-1)}\int_0^\rho \bar{Y}^{-1}\mathfrak{Q}_\hbar(\rho-\bar{s})\mathfrak{F}\Big(\bar{s},x_{\bar{s}}^\xi,\int_0^{\bar{s}}\mathbb{G}(\bar{s},\mathfrak{U},x_{\mathfrak{U}}^\xi)\,d\mathfrak{U}\Big)d\bar{s}\\
&-\sum_{\kappa=1}^{\gamma}\bar{Y}^{-1}\mathfrak{R}\mathfrak{B}_\hbar(\rho-\tau_\kappa)\bar{Y}\mathcal{I}_\kappa(x^\xi(\tau_\kappa))-\sum_{\kappa=1}^{\gamma}\bar{Y}^{-1}\mathfrak{R}\mathfrak{K}_\hbar(\rho-\tau_\kappa)\bar{Y}\widehat{\mathcal{I}}_\kappa(x^\xi(\tau_\kappa)).
\end{aligned}
$$

Consequently, $\exists$ a subsequence

$$
\Big\{\mathfrak{g}^\xi(\bar{s}),\ \zeta(\bar{s},x_{\bar{s}}^\xi,\int_0^{\bar{s}}\mathbb{H}(\bar{s},\mathfrak{U},x_{\mathfrak{U}}^\xi)\,d\mathfrak{U}),\ \mathfrak{F}(\bar{s},x_{\bar{s}}^\xi,\int_0^{\bar{s}}\mathbb{G}(\bar{s},\mathfrak{U},x_{\mathfrak{U}}^\xi)\,d\mathfrak{U})\Big\}
$$

which converges weakly to

$$
\Big\{\mathfrak{g}(\bar{s}),\ \zeta(\bar{s}),\ \mathfrak{F}(\bar{s})\Big\}\quad\text{in}\quad\mathbb{L}_2(\mathfrak{J},\mathbb{X})\times\mathbb{L}_0^2\times\mathbb{X},
$$

respectively. Conversely, though $\xi\mathcal{R}(\xi,\mathfrak{Q}_0^\rho)$ exhibits strong convergence as $\xi\to0^+$ $\forall$ $\bar{s}\in\mathfrak{J}$. Therefore, by invoking the $\mathbb{LDCT}$, we deduce that $\forall$ $\tau\in\mathfrak{J}$,

$$
\begin{aligned}
\mathbb{E}\left\|x^\xi(\rho)-x_\rho\right\|^2\le 9\mathbb{E}\Big\|\xi\mathcal{R}(\xi,\mathfrak{Q}_0^\rho)\Big[\mathbb{E}[x_\rho]&+\int_0^\rho\widehat{\varrho}(\bar{s})\,d\bar{W}(\bar{s})-\bar{Y}^{-1}\mathfrak{R}\mathfrak{B}_\hbar(\rho)\bar{Y}\Big\{\phi(0)-\vartheta(x^\xi)\Big\}\\
&-\bar{Y}^{-1}\mathfrak{R}\mathfrak{K}_\hbar(\rho)\bar{Y}\Big\{x_1-\beta(x^\xi)\Big\}\Big]\Big\|^2\\
&+9\mathbb{E}\left\|\frac{\wp\mathfrak{R}(2-\hbar)}{\mathfrak{M}(\hbar-1)\Gamma(\hbar)}\int_0^\rho\xi\mathcal{R}(\xi,\mathfrak{Q}_0^\rho)\bar{Y}^{-1}(\rho-\bar{s})^{\hbar-1}\Big[\mathfrak{g}^\xi(\bar{s})-\mathfrak{g}(\bar{s})\Big]d\bar{s}\right\|^2\\
&+9\mathbb{E}\left\|\frac{\wp\mathfrak{R}(2-\hbar)}{\mathfrak{M}(\hbar-1)\Gamma(\hbar)}\int_0^\rho\xi\mathcal{R}(\xi,\mathfrak{Q}_0^\rho)\bar{Y}^{-1}(\rho-\bar{s})^{\hbar-1}\mathfrak{g}(\bar{s})d\bar{s}\right\|^2\\
&+9\mathbb{E}\left\|\frac{\wp\mathfrak{R}(2-\hbar)}{\mathfrak{M}(\hbar-1)\Gamma(\hbar)}\int_0^\rho\xi\mathcal{R}(\xi,\mathfrak{Q}_0^\rho)\bar{Y}^{-1}(\rho-\bar{s})^{\hbar-1}\Big[\zeta\Big(\bar{s},x_{\bar{s}}^\xi,\int_0^{\bar{s}}\mathbb{H}(\bar{s},\mathfrak{U},x_{\mathfrak{U}}^\xi)\,d\mathfrak{U}\Big)-\zeta(\bar{s})\Big]d\bar{W}(\bar{s})\right\|^2\\
&+9\mathbb{E}\left\|\frac{\wp\mathfrak{R}(2-\hbar)}{\mathfrak{M}(\hbar-1)\Gamma(\hbar)}\int_0^\rho\xi\mathcal{R}(\xi,\mathfrak{Q}_0^\rho)\bar{Y}^{-1}(\rho-\bar{s})^{\hbar-1}\zeta(\bar{s})d\bar{W}(\bar{s})\right\|^2\\
&+9\mathbb{E}\left\|\frac{\wp\mathfrak{R}(2-\hbar)}{\mathfrak{M}(\hbar-1)\Gamma(\hbar)}\int_0^\rho\xi\mathcal{R}(\xi,\mathfrak{Q}_0^\rho)\bar{Y}^{-1}(\rho-\bar{s})^{\hbar-1}\Big[\mathfrak{F}\Big(\bar{s},x_{\bar{s}}^\xi,\int_0^{\bar{s}}\mathbb{G}(\bar{s},\mathfrak{U},x_{\mathfrak{U}}^\xi)\,d\mathfrak{U}\Big)-\mathfrak{F}(\bar{s})\Big]d\bar{s}\right\|^2\\
&+9\mathbb{E}\left\|\frac{\wp\mathfrak{R}(2-\hbar)}{\mathfrak{M}(\hbar-1)\Gamma(\hbar)}\int_0^\rho\xi\mathcal{R}(\xi,\mathfrak{Q}_0^\rho)\bar{Y}^{-1}(\rho-\bar{s})^{\hbar-1}\mathfrak{F}(\bar{s})d\bar{s}\right\|^2\\
&+9\mathbb{E}\left\|\frac{(\hbar-1)\mathfrak{R}^2}{\mathfrak{M}(\hbar-1)}\int_0^\rho\xi\mathcal{R}(\xi,\mathfrak{Q}_0^\rho)\bar{Y}^{-1}\mathfrak{Q}_\hbar(\rho-\bar{s})\Big[\mathfrak{g}^\xi(\bar{s})-\mathfrak{g}(\bar{s})\Big]d\bar{s}\right\|^2\\
&+9\mathbb{E}\left\|\frac{(\hbar-1)\mathfrak{R}^2}{\mathfrak{M}(\hbar-1)}\int_0^\rho\xi\mathcal{R}(\xi,\mathfrak{Q}_0^\rho)\bar{Y}^{-1}\mathfrak{Q}_\hbar(\rho-\bar{s})\mathfrak{g}(\bar{s})d\bar{s}\right\|^2\\
&+9\mathbb{E}\left\|\frac{(\hbar-1)\mathfrak{R}^2}{\mathfrak{M}(\hbar-1)}\int_0^\rho\xi\mathcal{R}(\xi,\mathfrak{Q}_0^\rho)\bar{Y}^{-1}\mathfrak{Q}_\hbar(\rho-\bar{s})\Big[\zeta\Big(\bar{s},x_{\bar{s}}^\xi,\int_0^{\bar{s}}\mathbb{H}(\bar{s},\mathfrak{U},x_{\mathfrak{U}}^\xi)\,d\mathfrak{U}\Big)-\zeta(\bar{s})\Big]d\bar{W}(\bar{s})\right\|^2
\end{aligned}
$$

$$+ 9\mathbb{E}\left\|\frac{(\hbar-1)\mathfrak{R}^2}{\mathfrak{M}(\hbar-1)}\int_0^{\rho}\xi\mathcal{R}(\xi,\mathfrak{Q}_0^{\rho})\bar{Y}^{-1}\mathfrak{Q}_{\hbar}(\rho-\bar{s})\zeta(\bar{s})d\bar{W}(\bar{s})\right\|^2$$
$$+ 9\mathbb{E}\left\|\frac{(\hbar-1)\mathfrak{R}^2}{\mathfrak{M}(\hbar-1)}\int_0^{\rho}\xi\mathcal{R}(\xi,\mathfrak{Q}_0^{\rho})\bar{Y}^{-1}\mathfrak{Q}_{\hbar}(\rho-\bar{s})\Big[\mathfrak{F}\Big(\bar{s},x_{\bar{s}}^{\xi},\int_0^{\bar{s}}\mathbb{G}(\bar{s},\mathfrak{U},x_{\mathfrak{U}}^{\xi})\,d\mathfrak{U}\Big)-\mathfrak{F}(\bar{s})\Big]d\bar{s}\right\|^2$$
$$+ 9\mathbb{E}\left\|\frac{(\hbar-1)\mathfrak{R}^2}{\mathfrak{M}(\hbar-1)}\int_0^{\rho}\xi\mathcal{R}(\xi,\mathfrak{Q}_0^{\rho})\bar{Y}^{-1}\mathfrak{Q}_{\hbar}(\rho-\bar{s})\mathfrak{F}(\bar{s})d\bar{s}\right\|^2$$
$$+ 9\mathbb{E}\left\|\sum_{\kappa=1}^{\gamma}\xi\mathcal{R}(\xi,\mathfrak{Q}_0^{\rho})\bar{Y}^{-1}\mathfrak{R}\mathfrak{B}_{\hbar}(\rho-\tau_\kappa)\bar{Y}\mathcal{I}_\kappa(x^{\xi}(\tau_\kappa))\right\|^2$$
$$+ 9\mathbb{E}\left\|\sum_{\kappa=1}^{\gamma}\xi\mathcal{R}(\xi,\mathfrak{Q}_0^{\rho})\bar{Y}^{-1}\mathfrak{R}\mathfrak{K}_{\hbar}(\rho-\tau_\kappa)\bar{Y}\widehat{\mathcal{I}}_\kappa(x^{\xi}(\tau_\kappa))\right\|^2.$$

Since the operator $\xi\mathcal{R}(\xi,\mathfrak{Q}_0^{\rho})$ converges strongly to 0 as $\xi\to 0^+$ and satisfies $\|\xi\mathcal{R}(\xi,\mathfrak{Q}_0^{\rho})\|\le 1$, it follows, by virtue of the $\mathbb{LDCT}$ combined with the compactness of $\mathfrak{Q}_{\hbar}(\tau)$, that

$$\mathbb{E}\left\|x^{\xi}(\rho)-x_{\rho}\right\|^2 \longrightarrow 0 \quad \text{as} \quad \xi\to 0^+.$$

□

Remark 3.2. *(Conceptual algorithm for existence and approximate controllability)*

To facilitate practical understanding of the proposed analytical framework, we summarize the major steps used to verify the existence of $\mathbb{MS}$*s and* $\mathbb{ACO}$ *of system* (1.1) *in the form of a conceptual algorithm. Conceptual steps for solving Sobolev-type stochastic impulsive differential inclusions:*

1. ***Model setup:*** *Define the stochastic impulsive system with higher-order AB-FD, specifying impulsive times, Brownian motion processes, and multi-valued nonlinearities via Clarke's sub-differential.*
2. ***Function space selection:*** *Identify a suitable HS* $\mathcal{X}$ *and define the appropriate BS of stochastic processes for* $\mathbb{MS}$*s.*
3. ***Operator formulation:*** *Rewrite the differential inclusion as an abstract integral equation using the fractional resolvent operator associated with the AB derivative.*
4. ***Verification of analytical conditions:*** *Check the key assumptions such as Lipschitz continuity, compactness, and boundedness of the multi-valued mappings* $\mathfrak{F}$, ζ *and impulsive operators* $\mathcal{I}_k$.
5. ***Existence of mild solution:*** *Apply stochastic FP theorems or* $\mathbb{MV}$*-analysis (e.g., Martelli Fixed Point Theorem) to prove the existence of* $\mathbb{MS}$*s.*
6. ***Approximate controllability analysis:***
   - *(a) Construct a control operator and define an appropriate control function space.*
   - *(b) Demonstrate that the reachable set is dense in the state space, using properties of the fractional resolvent and* $\mathbb{MV}$ *maps.*
7. ***Conclusion:*** *Verify that all conditions are satisfied, ensuring that the system admits mild solutions and is approximately controllable under the given assumptions.*

This algorithm provides a structured overview of the analytical methodology, enabling readers to follow the procedure conceptually and apply it to related fractional stochastic systems with impulsive and multi-valued dynamics.

Remark 3.3. *While the presented framework establishes rigorous existence and $\mathbb{ACO}$ results for Sobolev-type stochastic impulsive differential inclusions with higher-order AB-FDs, certain limitations warrant attention:*

- *The analysis assumes that the linear operators involved are sectorial and generate strongly continuous semigroups, which may not hold for all infinite-dimensional systems.*
- *The stochastic perturbations are modeled via $\mathcal{Q}$-Brownian motion processes, and the results may not directly extend to systems driven by more general Lévy processes or non-Gaussian noise.*
- *The multi-valued nonlinearities are treated using Clarke's generalized sub-differential, requiring the functions to be locally Lipschitz and convex-valued. Nonconvex or discontinuous mappings beyond this class would necessitate further analytical techniques.*
- *The fractional derivative is taken in the Caputo sense of AB type, and the current methodology may not immediately extend to other non-singular or singular fractional operators without modifications.*

These constraints define the current scope of applicability and indicate directions for future extensions of the framework.

4. Demonstrative case study

We examine a class of neutral impulsive $\mathbb{SSPDI}$s subject to non-local constraints, incorporating the ABC-FD, admissible control inputs, and non-smooth perturbations formalized through the Clarke sub-differential.

$$\begin{aligned}
{}^{\mathfrak{ABC}}D^{\hbar}_{0+}\Big(1-\frac{\partial^2}{\partial\mathfrak{Z}^2}\Big)\mathfrak{E}(\tau,\mathfrak{Z}) &\in \frac{\partial^2}{\partial\mathfrak{Z}^2}\mathfrak{E}(\tau,\mathfrak{Z})+\nu(\tau,\mathfrak{Z})+\int_{-\infty}^{\tau}\chi_1(\bar{s}-\tau)\mathfrak{E}(\mathfrak{U},\bar{s})\,d\bar{s}\\
&\quad+\int_0^{\tau}\int_{-\infty}^{0}\chi_2(\bar{s},\mathfrak{U},\varpi-\bar{s})\mathfrak{E}(\varpi,\mathfrak{U})\,d\varpi\,d\bar{s}\\
&\quad+\Big(\int_{-\infty}^{\tau}\widehat{\chi}_1(\bar{s}-\tau)\mathfrak{E}(\mathfrak{U},\bar{s})\,d\bar{s}\\
&\qquad+\int_0^{\tau}\int_{-\infty}^{0}\widehat{\chi}_2(\bar{s},\mathfrak{U},\varpi-\bar{s})\mathfrak{E}(\varpi,\mathfrak{U})\,d\varpi\,d\bar{s}\Big)\frac{d\bar{W}(\tau)}{d\tau}\\
&\quad+\partial S\Big(\frac{\tau^2-\mid\mathfrak{E}(\tau,\mathfrak{Z})\mid}{4}\Big),\quad \tau\in(0,1],\quad \mathfrak{Z}\in[0,\pi],\quad \tau\neq\tau_\kappa,\\
\mathfrak{E}(\tau,0) &= \mathfrak{E}(\tau,\pi)=0,\ \ \mathfrak{E}'(\tau,0)=\mathfrak{E}'(\tau,\pi)=0 \qquad \tau\in(0,1],\\
\mathfrak{E}(\tau_\kappa^+,\mathfrak{Z})-\mathfrak{E}(\tau_\kappa^-,\mathfrak{Z}) &= \frac{1}{2+\mathfrak{Z}}\arctan(\mathfrak{E}(\mathfrak{Z})),\quad \forall\mathfrak{Z}\in[0,\pi],\\
\mathfrak{E}'(\tau_\kappa^+,\mathfrak{Z})-\mathfrak{E}'(\tau_\kappa^-,\mathfrak{Z}) &= \frac{1}{1+\mathfrak{Z}^2}\sin(\mathfrak{E}(\mathfrak{Z})),\quad \forall\mathfrak{Z}\in[0,\pi],\\
\mathfrak{E}(0,\mathfrak{Z})+\int_0^{\pi}\mathfrak{H}_1(\mathfrak{Z},\mathfrak{U})\mathfrak{E}(\tau,\mathfrak{U})d\mathfrak{U} &= \varphi_{\mathfrak{Z}},\quad \tau\in(-\infty,0],\\
\mathfrak{E}'(0,\mathfrak{Z})+\int_0^{\pi}\mathfrak{H}_2(\mathfrak{Z},\mathfrak{U})\mathfrak{E}(\tau,\mathfrak{U})d\mathfrak{U} &= \widehat{\varphi}_{\mathfrak{Z}},\quad \tau\in(-\infty,0],
\end{aligned}\tag{4.1}$$

where ${}^{\mathfrak{ABC}}D^{\hbar}_{0+}$ denotes the AB-FD of order $\hbar \in (1,2)$ in the Caputo sense. The underlying functional framework is the separable HS $\mathbb{X} = \mathbb{Y} = \mathbb{L}_2([0,\pi],(0,1])$. In the foregoing exemplar, the parameter ranging over $[0,\pi]$ embodies the spatial variable upon which the associated partial differential operator exerts its action. Consequently, the system dynamics are formulated over the spatial domain $[0,\pi]$ under the imposition of suitable boundary constraints. At this juncture, let us consider the impulsive mappings delineated by

$$\mathcal{I}_\kappa(\mathfrak{E})(\mathfrak{Z}) = \frac{1}{2+\mathfrak{Z}} \arctan(\mathfrak{E}(\mathfrak{Z})), \tag{4.2}$$

$$\widehat{\mathcal{I}}_\kappa(\mathfrak{E})(\mathfrak{Z}) = \frac{1}{1+\mathfrak{Z}^2} \sin(\mathfrak{E}(\mathfrak{Z})), \quad \mathfrak{Z} \in [0,\pi]. \tag{4.3}$$

(i) Both mappings $\mathfrak{E} \mapsto \arctan(\mathfrak{E})$ and $\mathfrak{E} \mapsto \sin(\mathfrak{E})$ are Fréchet differentiable and bd. Owing to the smoothness and asymptotic attenuation of the coefficients $\frac{1}{2+\mathfrak{Z}}$ and $\frac{1}{1+\mathfrak{Z}^2}$ with respect to $\mathfrak{Z}$, the mappings $\mathcal{I}_\kappa$ and $\widehat{\mathcal{I}}_\kappa$ act as regularizing operators, carrying bd subsets of $\mathbb{X}$ into families that are simultaneously uniformly bd and equi-continuous. By the Arzelà–Ascoli theorem and compact embedding into $\mathbb{X}$, this guarantees that both operators are C.C. Moreover, the uniform moment bounds hold:

$$\mathbb{E}\|\mathcal{I}_\kappa(\mathfrak{E})\|^2 \le \int_0^\pi \frac{\pi^2}{4(2+\mathfrak{Z})^2} d\mathfrak{Z} =: d_1^\kappa,$$
$$\mathbb{E}\|\widehat{\mathcal{I}}_\kappa(\mathfrak{E})\|^2 \le \int_0^\pi \frac{1}{(1+\mathfrak{Z}^2)^2} d\mathfrak{Z} =: d_2^\kappa.$$

These bounds are independent of $\mathfrak{E}$, satisfying $(H5)(i)$.

(ii) Using the Lipschitz property of $\arctan(\mathfrak{E})$ and $\sin(\mathfrak{E})$, we obtain

$$|\mathcal{I}_\kappa(\mathfrak{E})(\mathfrak{Z}) - \mathcal{I}_\kappa(\bar{\mathfrak{E}})(\mathfrak{Z})| \le \frac{1}{2+\mathfrak{Z}} |\mathfrak{E}(\mathfrak{Z}) - \bar{\mathfrak{E}}(\mathfrak{Z})|,$$
$$|\widehat{\mathcal{I}}_\kappa(\mathfrak{E})(\mathfrak{Z}) - \widehat{\mathcal{I}}_\kappa(\bar{\mathfrak{E}})(\mathfrak{Z})| \le \frac{1}{1+\mathfrak{Z}^2} |\mathfrak{E}(\mathfrak{Z}) - \bar{\mathfrak{E}}(\mathfrak{Z})|.$$

Squaring and integrating yields

$$\mathbb{E}\|\mathcal{I}_\kappa(\mathfrak{E}) - \mathcal{I}_\kappa(\bar{\mathfrak{E}})\|^2 \le \frac{1}{4} \mathbb{E}\|\mathfrak{E} - \bar{\mathfrak{E}}\|^2,$$
$$\mathbb{E}\|\widehat{\mathcal{I}}_\kappa(\mathfrak{E}) - \widehat{\mathcal{I}}_\kappa(\bar{\mathfrak{E}})\|^2 \le \frac{1}{(1+\pi^2)^2} \mathbb{E}\|\mathfrak{E} - \bar{\mathfrak{E}}\|^2.$$

Thus, condition $(H5)(ii)$ is satisfied with $\iota_1^* = \frac{1}{4}$ and $\iota_2^* = \frac{1}{(1+\pi^2)^2}$.

Consequently, the operators $\mathcal{I}_\kappa$ and $\widehat{\mathcal{I}}_\kappa$ defined in (4.2) and (4.3) satisfy all components of assumption $(H5)$.

Contemplate the operatorial constructs Z and $\bar{Y}$, which are prescribed via

$$Z = \frac{\partial^2}{\partial \mathfrak{Z}^2}, \qquad \bar{Y} = 1 - \frac{\partial^2}{\partial \mathfrak{Z}^2},$$

with common domain

$$\mathbb{D}(Z) = \mathbb{D}(\bar{Y}) = \Big\{\mathfrak{E} \in \mathbb{X} : \mathfrak{E}, \frac{\partial \mathfrak{E}}{\partial \mathfrak{Z}} \text{ are absolutely continuous}, \frac{\partial^2 \mathfrak{E}}{\partial \mathfrak{Z}^2} \in \mathbb{X},\ \mathfrak{E}(0) = \mathfrak{E}(\pi) = 0\Big\}.$$

Then, for $\mathfrak{E} \in \mathbb{D}(Z) = \mathbb{D}(\bar{Y})$, the spectral representations are

$$Z\mathfrak{E} = \sum_{n=1}^{\infty} n^2 \langle \mathfrak{E}, \mathfrak{E}_n \rangle \mathfrak{E}_n,$$
$$\bar{Y}\mathfrak{E} = \sum_{n=1}^{\infty} (1 + n^2) \langle \mathfrak{E}, \mathfrak{E}_n \rangle \mathfrak{E}_n,$$

where

$$\mathfrak{E}_n(\mathfrak{Z}) = \sqrt{\frac{2}{\pi}} \sin(n\mathfrak{Z}), \quad n = 1, 2, 3, \ldots$$

forms an orthonormal eigenbasis of Z.

Consequently, for $\mathfrak{E} \in \mathbb{X}$, one has

$$\bar{Y}^{-1}\mathfrak{E} = \sum_{n=1}^{\infty} \frac{1}{1 + n^2} \langle \mathfrak{E}, \mathfrak{E}_n \rangle \mathfrak{E}_n,$$
$$Z\bar{Y}^{-1}\mathfrak{E} = \sum_{n=1}^{\infty} \frac{n^2}{1 + n^2} \langle \mathfrak{E}, \mathfrak{E}_n \rangle \mathfrak{E}_n.$$

It ensues that $Z\bar{Y}^{-1}$ is self-adjoint and induces an analytic semi-group $\{\mathfrak{Q}_\alpha(\tau)\}_{\tau \geq 0}$ on $\mathbb{X}$.

Now, consider the weight function

$$\mathfrak{G}(\tau) = e^{4\tau}, \quad \tau < 0.$$

Then,

$$k = \int_{-\infty}^{0} \mathfrak{G}(\bar{s})\, d\bar{s} = \frac{1}{4},$$

and we define the associated history-dependent norm

$$\|\mathfrak{E}\|_{\mathbb{B}} = \int_{-\infty}^{0} \mathfrak{G}(\bar{s}) \sup_{\theta \in [\bar{s}, 0]} \left(\mathbb{E}\|\mathfrak{E}(\theta)\|^2\right)^{1/2} d\bar{s}.$$

Thus, $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$ is a BS.

For $(\tau, \mathfrak{E}) \in \mathfrak{J} \times \mathbb{B}$, with

$$\mathfrak{E}(\theta)(\mathfrak{Z}) = \mathfrak{E}(\theta, \mathfrak{Z}), \quad (\theta, \mathfrak{Z}) \in (-\infty, 0] \times [0, \pi],$$

we write

$$\mathfrak{E}(\tau)(\mathfrak{Z}) = \mathfrak{E}(\tau, \mathfrak{Z}).$$

Finally, we introduce the bd linear control operator $\mho : \mathcal{U} \to \mathbb{X}$ as

$$(\mho U)(\tau)(\mathfrak{Z}) = \nu(\tau, \mathfrak{Z}), \quad \mathfrak{Z} \in [0, \pi], \quad U \in \mathcal{U}.$$

Define the nonlinear and history–dependent terms as follows:

$$\partial\mathcal{S}(\tau,\mathfrak{E}(\tau)) = \partial\mathcal{S}\left(\frac{\tau^2 - |\mathfrak{E}(\tau,\mathfrak{Z})|}{4}\right),$$

$$\begin{aligned}\mathfrak{F}\left(\tau,\mathfrak{E}_\tau,\int_0^\tau \mathbb{G}(\tau,\bar{s},\mathfrak{E}_{\bar{s}})\,d\bar{s}\right) &= \int_{-\infty}^{\tau}\chi_1(\bar{s}-\tau)\,\mathfrak{E}(\mathfrak{U},\bar{s})\,d\bar{s}\\ &+\int_0^\tau\int_{-\infty}^{0}\chi_2(\bar{s},\mathfrak{U},\varpi-\bar{s})\,\mathfrak{E}(\varpi,\mathfrak{U})\,d\varpi\,d\bar{s},\\ \zeta\left(\tau,x_\tau,\int_0^\tau \mathbb{H}(\tau,\bar{s},x_{\bar{s}})\,d\bar{s}\right) &= \int_{-\infty}^{\tau}\widehat{\chi}_1(\bar{s}-\tau)\,\mathfrak{E}(\mathfrak{U},\bar{s})\,d\bar{s}\\ &+\int_0^\tau\int_{-\infty}^{0}\widehat{\chi}_2(\bar{s},\mathfrak{U},\varpi-\bar{s})\,\mathfrak{E}(\varpi,\mathfrak{U})\,d\varpi\,d\bar{s}.\end{aligned}$$

With these constructions, system (4.1) can be reinterpreted as a particular realization of the abstract framework (1.1). Under the stipulated hypotheses, all structural requirements of Theorems 3.1 and 3.2 are fulfilled. As a consequence, system (4.1) admits $\mathbb{ACO}$ on the interval $(0, 1]$.

5. Conclusions

In this study, we have articulated a unified and rigorous analytical framework for the existence of solutions and the $\mathbb{ACO}$ of a novel class of $\mathbb{SSIDI}$s driven by a higher-order AB-FD in the Caputo sense. The suggested scheme accommodates hereditary effects through fractional memory, stochastic excitations represented by $\mathbb{BM}$ processes, and abrupt state variations induced by impulsive actions. The presence of non-smooth and potentially non-convex $\mathbb{MV}$ nonlinearities was systematically addressed via Clarke's generalized sub-differential, thereby broadening the scope of classical controllability theory to encompass nonlocal, stochastic, and impulsive dynamical environments. The core contributions of this investigation were obtained through a synergistic integration of fractional evolution theory, infinite-dimensional stochastic analysis, $\mathbb{MEAS}$ selection methodologies, and $\mathbb{MV}$ FP-theory within the BS framework. Within this rigorous analytical edifice, verifiable sufficient criteria guaranteeing the existence of $\mathbb{MS}$s and $\mathbb{ACO}$ were rigorously established. Additionally, a meticulously constructed illustrative example was provided to substantiate the abstract hypotheses and to accentuate the efficacy and practical relevance of the proposed methodology.

Although we primarily focused on the theoretical analysis of existence and $\mathbb{ACO}$ for Sobolev-type stochastic impulsive differential inclusions, we acknowledge the importance of graphical and numerical illustrations. This theoretical investigation establishes a foundation for the analysis of Sobolev-type stochastic impulsive systems with higher-order fractional memory. In future studies, researchers could focus on several directions, including:

- Developing numerically robust schemes and simulation frameworks to visualize solution trajectories, controllability effects, and the influence of impulsive and stochastic terms.
- Investigating optimal control strategies under the combined effects of higher-order fractional memory, stochastic perturbations, and impulsive operators.

- Exploring potential applications of the proposed framework to real-world stochastic systems in engineering, physics, and finance, where memory-driven dynamics and abrupt state changes are prevalent.

Such extensions will not only complement the theoretical findings but also provide practical insights and broaden the applicability of the proposed model.

The conceptual and methodological framework developed herein engenders multiple promising trajectories for future research. A natural extension involves the incorporation of more general FOs, such as Prabhakar-type or distributed-order derivatives, thereby facilitating the representation of more sophisticated and intricate memory effects. An additional salient research trajectory involves the rigorous examination of exact controllability and the formulation of optimal control strategies in the presence of state-dependent impulsive operators or under more intricate stochastic perturbations. The incorporation of laterals, non-instantaneous impulses, and temporally fluctuating stochastic coefficients would further enhance the verisimilitude and applicability of the model to intricate real-world dynamical systems. Moreover, the construction of numerically robust and computationally efficient schemes that rigorously preserve the inherent structural characteristics of higher-order fractional, stochastic, and impulsive dynamics constitutes a formidable and open problem, meriting extensive and systematic future investigation.

Author contributions

A. M. Sayed Ahmed: Investigation, formal analysis, writing-review and editing; Taha Radwan: Resources, investigation, writing-review and editing; Yakup Yildirim: Visualization, investigation, writing-original draft; Hamdy M. Ahmed: Methodology, validation, formal analysis. All authors have read and agreed to publish the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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