Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(3): 8407–8416.
DOI: 10.3934/math.2026345
Received: 05 February 2026
Revised: 17 March 2026
Accepted: 24 March 2026
Published: 30 March 2026

Research article

Extended form of (ζ, η)-derivation of von Neumann algebra

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Abstract: Let $\mathcal{U}$ be regarded as a CSL subalgebra of a von Neumann algebra operating on a Hilbert space $\mathcal{H}$. Let $\mathcal{G}_{\mathbf{h}}, \mathbf{h} : \mathcal{U} \to \mathcal{U}$ denote two significant linear mappings that intricately fulfill certain essential algebraic identities. As a result, $\mathcal{G}_{\mathbf{h}}$ emerges as a generalized (ζ, η)-derivation, preserving a fundamental relationship with the associated (ζ, η)-derivation $\mathbf{h}$ functioning within the framework of $\mathcal{U}$.

Keywords: semiprime rings; algebra; automorphisms; generalized (ζ, η)-derivation
Mathematics Subject Classification: 16N60, 16W25, 46L10, 47L40

1. Introduction

Consider $\mathcal{H}$ as a complex Hilbert space, weaving together the threads of mathematical theory throughout this article. Denote the identity operator within the confines of $\mathcal{H}$ by K, and let the algebra, the set of linear and bounded operators on $\mathcal{H}$, be symbolized by $\mathcal{B}(\mathcal{H})$. Herein, the two terms subspace and projection ascend beyond mere abstracts, enveloping the meanings of "orthogonal projection" and "norm closed linear manifold," respectively. For simplicity of exposition, designating the projection on a subspace is a matter of calculated convenience.

$\mathcal{S}_\alpha$ emerges as a collection of subspaces within $\mathcal{H}$, and the notation $\cup\mathcal{S}_\alpha$ clearly represents the smallest subspace that encompasses each $\mathcal{S}_\alpha$, while $\cap\mathcal{S}_\alpha$ denotes the largest subspace that holds within every $\mathcal{S}_\alpha$. The revered subspace lattice $\mathcal{L}$, which precisely balances between the void of $\{0\}$ and the entirety of K, epitomizes a strongly closed subspace lattice, standing resolute, closed under the canonical operations of lattice theory, here denoted as $\cap$ and $\cup$. This ensures that every subspace within $\mathcal{L}$ remains unperturbed and invariant, which is determined by $Alg\ \mathcal{L}$, provided $\mathcal{L}$ takes the form of a subspace lattice.

The text defines key concepts in operator algebra on a Hilbert space $\mathcal{H}$. For a subspace lattice $\mathcal{L}$, the algebra $Alg\ \mathcal{L}$ consists of all bounded operators A on $\mathcal{H}$ satisfying $AR = RAR$ for every projection

$R \in \mathcal{L}$. For an operator algebra $\mathcal{U}$, the lattice $Lat\,\mathcal{U}$ consists of all subspaces R of $\mathcal{H}$ such that $AR = RAR$ for every $A \in \mathcal{U}$. An algebra $\mathcal{U}$ is called reflexive if $\mathcal{U} = Alg\,Lat\,\mathcal{U}$, and a subspace lattice $\mathcal{L}$ is reflexive if $\mathcal{L} = Lat\,Alg\,\mathcal{L}$. A commutative subspace lattice (CSL) is a subspace lattice whose projections commute pairwise, and $Alg\,\mathcal{L}$ in this case is called a CSL algebra. If such a CSL $\mathcal{L}$ has its projections contained in a von Neumann algebra N, then $\mathcal{U} = N \cap Alg\,\mathcal{L}$ is termed a CSL subalgebra of N.

Taking into account the complexities of its framework, an additive mapping designated as $\mathbf{h}$, which associates $\mathcal{U}$ with itself, is identified as a derivation, depending on the condition $\mathbf{h}(UW) = \mathbf{h}(U)W + U\mathbf{h}(W)$ for all $U, W \in \mathcal{U}$. This mapping becomes the domain of Jordan derivations upon fulfillment of the essential criterion $\mathbf{h}(U^2) = \mathbf{h}(U)U + U\mathbf{h}(U)$ for all $U \in \mathcal{U}$. Subsequently, we explore the intriguing extensions of derivations.

Introduce an additive mapping labeled $\mathcal{G}_{\mathbf{h}}$ from $\mathcal{U}$ to itself and observe its conversion into a generalized derivation, provided that for a given derivation $\mathbf{h}$ on $\mathcal{U}$ such that $\mathcal{G}_{\mathbf{h}}(UW) = \mathcal{G}_{\mathbf{h}}(U)W + U\mathbf{h}(W)$ is satisfied for every pair $U, W \in \mathcal{U}$. Meanwhile, an additive mapping $\mathcal{G}_{\mathbf{h}} : \mathcal{U} \to \mathcal{U}$ assumes its place as a generalized Jordan derivation associated with a Jordan derivation $\mathbf{h} : \mathcal{U} \to \mathcal{U}$, fulfilling $\mathcal{G}_{\mathbf{h}}(U^2) = \mathcal{G}_{\mathbf{h}}(U)U + U\mathbf{h}(U)$ for all $U \in \mathcal{U}$. It is immediately clear that any generalized derivation automatically qualifies as a generalized Jordan derivation; however, the converse implication generally does not hold.

The text defines several types of derivations on the algebra $\mathcal{U}$, given two automorphisms η and ζ. An additive map $\mathbf{h} : \mathcal{U} \to \mathcal{U}$ is a (ζ, η)-derivation if $\mathbf{h}(UW) = \mathbf{h}(U)\zeta(W) + \eta(U)\mathbf{h}(W)$ for all $U, W \in \mathcal{U}$. It is a Jordan (ζ, η)-derivation if $\mathbf{h}(U^2) = \mathbf{h}(U)\zeta(U) + \eta(U)\mathbf{h}(U)$ for all $U \in \mathcal{U}$. Every (ζ, η)-derivation is a Jordan (ζ, η)-derivation, but not conversely.

A generalized (ζ, η)-derivation is an additive map $\mathcal{G}_{\mathbf{h}} : \mathcal{U} \to \mathcal{U}$ associated with a (ζ, η)-derivation $\mathbf{h}$ such that $\mathcal{G}_{\mathbf{h}}(UW) = \mathcal{G}_{\mathbf{h}}(U)\zeta(W) + \eta(U)\mathbf{h}(W)$. Similarly, a generalized Jordan (ζ, η)-derivation is defined using a Jordan (ζ, η)-derivation $\mathbf{h}$ and the identity $\mathcal{G}_{\mathbf{h}}(U^2) = \mathcal{G}_{\mathbf{h}}(U)\zeta(U) + \eta(U)\mathbf{h}(U)$. Every generalized (ζ, η)-derivation is a generalized Jordan (ζ, η)-derivation, but the converse generally fails.

Fascinatingly, a fundamental result by Bresar and Vukman [4] reveals that on a prime ring having 2-torsion freeness, all Jordan (ζ, η)-derivation qualifies as a (ζ, η)-derivation. Subsequently, a significant generalization was advanced in [12], demonstrating that a Jordan (ζ, η)-derivation on a 2-torsion-free semiprime ring also qualifies as a (ζ, η)-derivation. Later, a pivotal proof emerged [10] confirming that a Jordan derivation on a CSL algebra transforms seamlessly into a derivation, and this milestone was further expanded in [9], establishing that a Jordan derivation on a CSL subalgebra of the von Neumann algebra indeed acts as a derivation.

Recently, more general non-associative frameworks have been used in addition to associative algebras to characterize certain classes of additive and multiplicative mappings. For example, the additivity of n-multiplicative maps and Jordan-kind and Lie-kind maps on alternative and Jordan $*$-algebras were studied in several articles by Ferreira and associates in [5, 8]. These advancements give researchers a more comprehensive viewpoint and inspiration.

The development of non-associative structures is another creative strategy in this direction. Highlighting the foundational contributions for a precise scholarly contextualization of the research direction as seen in [6, 7]. In [6], the authors obtained that if $\mathfrak{J}$ contains a nontrivial idempotent, then n-multiplicative maps and n-multiplicative derivations from $\mathfrak{J}$ to $\mathfrak{J}'$ are additive maps, where $\mathfrak{J}$ and $\mathfrak{J}'$ are Jordan rings. Further, in [7], the authors characterized additive mappings on an alternative division ring and presented some new identities concerning Lie and Jordan products, and provided a complete description of commuting maps on octonion algebras. Further, in [1, 2], the authors have extended the above mentioned results in semiprime rings, semi-simple Banach algebras, and standard operator algebras.

Given the connection to the authors, with more extensive work in [13–15], the authors of these studies relax the torsion constraint in $\mathcal{U}$ by employing tools based on the Vandermonde determinant. Then, using appropriately modified logic and substantial methodological alterations, Theorem 2.1 is established.

If we consider $\mathcal{G}_{\mathbf{h}}$ as a generalized Jordan (ζ, η)-derivation, associated with a Jordan (ζ, η)-derivation $\mathbf{h}$ on $\mathcal{U}$, then, intriguingly, the identity

$$\mathcal{G}_{\mathbf{h}}(U^{q_1+q_2}) = \mathcal{G}_{\mathbf{h}}(U^{q_1})\zeta(U^{q_2}) + \eta(U^{q_1})\mathbf{h}(U^{q_2}),$$

and

$$\mathcal{G}_{\mathbf{h}}(U^{q_1+q_2+q_3}) = \mathcal{G}_{\mathbf{h}}(U^{q_1})\zeta(U^{q_2+q_3}) + \eta(U^{q_1})\mathbf{h}(U^{q_2})\zeta(U^{q_3}) + \eta(U^{q_1+q_2})\mathbf{h}(U^{q_3})$$

universally hold for all $U \in \mathcal{U}$. Nevertheless, rather remarkably, the converse proposition typically does not hold true. This intriguing relationship implies that these identities present less stringent conditions compared to the robust identity within a generalized (ζ, η)-derivation (which is, naturally, also a generalized Jordan (ζ, η)-derivation). In this article, we thoroughly examine the requisite conditions on $\mathcal{U}$, such that $\mathcal{G}_{\mathbf{h}}$ emerges as a generalized (ζ, η)-derivation connected to a (ζ, η)-derivation $\mathbf{h}$, provided that it satisfies the intricate algebraic identities described above.

Our article aims to resolutely answer the pivotal question concerning whether $\mathcal{U}$ stands as a CSL subalgebra of the von Neumann algebra when it acts dynamically on a Hilbert space. Exploring the classifications of Jordan (ζ, η)-derivations within algebras and rings spans a diverse and far-reaching field. Unquestionably, any (ζ, η)–derivation naturally qualifies as a Jordan (ζ, η)-derivation. Ashraf et al. [3] and Lanski [12] have beautifully provided illuminating counterexamples that compellingly demonstrate the general truth that the converses are not universally valid.

In order to establish the principal theorems in this paper, we require the following lemma:

Lemma 1.1. *[11] Consider N as a von Neumann algebra operating on a Hilbert space $\mathcal{H}$, and let $\mathcal{L}$ be a CSL with its projections residing within N. Let $\mathcal{U} = N \cap Alg\ \mathcal{L}$ represent the CSL subalgebra of von Neumann algebra N. If η and ζ denote any automorphisms on $\mathcal{U}$, then it follows that a Jordan (ζ, η)-derivation on N is inherently a (ζ, η)-derivation.*

2. Main theorems

To lay the foundation of our exploration, we initially unveil the pivotal theorem.

Theorem 2.1. *Let $q_1, q_2 > 1$ be any fixed integers and $\mathcal{U} = N \cap Alg\ \mathcal{L}$ be a CSL subalgebra of von Neumann algebra N. Suppose that $\mathcal{G}_{\mathbf{h}}, \mathbf{h} : \mathcal{U} \to \mathcal{U}$ are two linear mappings that meet the conditions of the algebraic identity*

$$\begin{aligned} 2\mathcal{G}_{\mathbf{h}}(U^{q_1+q_2}) &= \mathcal{G}_{\mathbf{h}}(U^{q_1})\zeta(U^{q_2}) + \eta(U^{q_1})\mathbf{h}(U^{q_2}) \\ &\quad + \mathcal{G}_{\mathbf{h}}(U^{q_2})\zeta(U^{q_1}) + \eta(U^{q_2})\mathbf{h}(U^{q_1}), \end{aligned} \tag{2.1}$$

for all $U \in \mathcal{U}$, where $U^0 = K$, and η and ζ are automorphisms on $\mathcal{U}$. Then, $\mathcal{G}_{\mathbf{h}}$ is a generalized (ζ, η)-derivation with associated (ζ, η)-derivation $\mathbf{h}$ on $\mathcal{U}$.

Proof. It is provided that

$$\begin{aligned} 2\mathcal{G}_{\mathbf{h}}(U^{q_1+q_2}) &= \mathcal{G}_{\mathbf{h}}(U^{q_1})\zeta(U^{q_2}) + \eta(U^{q_1})\mathbf{h}(U^{q_2}) \\ &\quad + \mathcal{G}_{\mathbf{h}}(U^{q_2})\zeta(U^{q_1}) + \eta(U^{q_2})\mathbf{h}(U^{q_1}), \end{aligned} \tag{2.2}$$

for all $U \in \mathcal{U}$. Substituting K for U in the previous equation and using the facts $\zeta(K) = K$ and $\eta(K) = K$, we obtain

$$2\mathcal{G}_{\mathbf{h}}(K) = \mathcal{G}_{\mathbf{h}}(K)K + K\mathbf{h}(K) + \mathcal{G}_{\mathbf{h}}(K)K + K\mathbf{h}(K),$$

which implies that $\mathbf{h}(K) = 0$. When $U + mK$ is substituted for U in Eq (2.2), where m represents any positive integer, the equation becomes as follows:

$$\begin{aligned} &2\mathcal{G}_{\mathbf{h}}\Big(U^{q_1+q_2} + \tbinom{q_1+q_2}{1}U^{q_1+q_2-1}mK + \cdots + \tbinom{q_1+q_2}{q_1+q_2-2}U^2m^{q_1+q_2-2}K + \tbinom{q_1+q_2}{q_1+q_2-1}Um^{q_1+q_2-1}K + m^{q_1+q_2}K\Big) = \\ &\mathcal{G}_{\mathbf{h}}\Big(U^{q_1} + \tbinom{q_1}{1}U^{q_1-1}mK + \cdots + \tbinom{q_1}{q_1-2}U^2m^{q_1-2}K + \tbinom{q_1}{q_1-1}Um^{q_1-1}K + m^{q_1}K\Big)\zeta\Big(U^{q_2} + \tbinom{q_2}{1}U^{q_2-1}mK + \cdots + \\ &\tbinom{q_2}{q_2-2}U^2m^{q_2-2}K + \tbinom{q_2}{q_2-1}Um^{q_2-1}K + m^{q_2}K\Big) + \eta\Big(U^{q_1} + \tbinom{q_1}{1}U^{q_1-1}mK + \cdots + \tbinom{q_1}{q_1-2}U^2m^{q_1-2}K + \\ &\tbinom{q_1}{q_1-1}Um^{q_1-1}K + m^{q_1}K\Big)\mathbf{h}\Big(U^{q_2} + \tbinom{q_2}{1}U^{q_2-1}mK + \cdots + \tbinom{q_2}{q_2-2}U^2m^{q_2-2}K + \tbinom{q_2}{q_2-1}Um^{q_2-1}K + m^{q_2}K\Big) + \\ &\mathcal{G}_{\mathbf{h}}\Big(U^{q_2} + \tbinom{q_2}{1}U^{q_2-1}mK + \cdots + \tbinom{q_2}{q_2-2}U^2m^{q_2-2}K + \tbinom{q_2}{q_2-1}Um^{q_2-1}K + m^{q_2}K\Big)\zeta\Big(U^{q_1} + \tbinom{q_1}{1}U^{q_1-1}mK + \cdots + \\ &\tbinom{q_1}{q_1-2}U^2m^{q_1-2}K + \tbinom{q_1}{q_1-1}Um^{q_1-1}K + m^{q_1}K\Big) + \eta\Big(U^{q_2} + \tbinom{q_2}{1}U^{q_2-1}mK + \cdots + \tbinom{q_2}{q_2-2}U^2m^{q_2-2}K + \\ &\tbinom{q_2}{q_2-1}Um^{q_2-1}K + m^{q_2}K\Big)\mathbf{h}\Big(U^{q_1} + \tbinom{q_1}{1}U^{q_1-1}mK + \cdots + \tbinom{q_1}{q_1-2}U^2m^{q_1-2}K + \tbinom{q_1}{q_1-1}Um^{q_1-1}K + m^{q_1}K\Big), \end{aligned}$$

for all $U \in \mathcal{U}$. Utilizing Eq (2.2), transform the aforementioned expression into

$$mf_1(U,K) + m^2f_2(U,K) + \cdots + m^{q_1+q_2-1}f_{q_1+q_2-1}(U,K) = 0,$$

where the coefficients of m^i are $f_i(U,K)$ for each instance of $i = (q_1+q_2-1), (q_1+q_2-2), \cdots, 2, 1$. When methodically switching m with $(q_1+q_2-1), (q_1+q_2-2), \cdots, 2, 1$, we obtain a (q_1+q_2-1) system of linear equations, and this system results in a homogeneous system. This leads to the formation of a Vandermonde matrix given as

$$\mathcal{V} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 2 & 2^2 & \cdots & 2^{q_1+q_2-1} \\ . & . & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ (q_1+q_2-2) & (q_1+q_2-2)^2 & \cdots & (q_1+q_2-2)^{q_1+q_2-1} \\ (q_1+q_2-1) & (q_1+q_2-1)^2 & \cdots & (q_1+q_2-1)^{q_1+q_2-1} \end{bmatrix}.$$

This indicates the same for every case of U within $\mathcal{U}$, and for each instance of $i = (q_1+q_2-1), (q_1+q_2-2), \cdots, 2, 1$, the trivial equation dynamically unfolds as $f_i(U,K) = 0$. Notably, especially considering $i = q_1+q_2-1$, we derive the subsequent results consistently across all $U \in \mathcal{U}$:

$$\begin{aligned} 2(\tbinom{q_1+q_2}{q_1+q_2-1})\mathcal{G}_{\mathbf{h}}(U) &= \tbinom{q_2}{q_2-1}\mathcal{G}_{\mathbf{h}}(K)\zeta(U) + \tbinom{q_1}{q_1-1}\mathcal{G}_{\mathbf{h}}(U) \\ &\quad + \tbinom{q_2}{q_2-1}\mathbf{h}(U) + \tbinom{q_1}{q_1-1}\mathcal{G}_{\mathbf{h}}(K)\zeta(U) \\ &\quad + \tbinom{q_2}{q_2-1}\mathcal{G}_{\mathbf{h}}(U) + \tbinom{q_1}{q_1-1}\mathbf{h}(U), \end{aligned}$$

for all $U \in \mathcal{U}$. This yields that

$$2(q_1+q_2)\mathcal{G}_{\mathbf{h}}(U) = (q_1+q_2)\mathcal{G}_{\mathbf{h}}(K)\zeta(U) + (q_1+q_2)\mathcal{G}_{\mathbf{h}}(U) + (q_1+q_2)\mathbf{h}(U).$$

Then, this leads to the following:

$$\mathcal{G}_{\mathbf{h}}(U) = \mathcal{G}_{\mathbf{h}}(K)\zeta(U) + \mathbf{h}(U) \tag{2.3}$$

for all $U \in \mathcal{U}$. Further, as $f_{q_1+q_2-2}(U,K) = 0$ with $\mathbf{h}(K) = 0$, we have

$$\begin{aligned} 2(\tbinom{q_1+q_2}{q_1+q_2-2})\mathcal{G}_{\mathbf{h}}(U^2) \;=\;& \tbinom{q_2}{q_2-2}\mathcal{G}_{\mathbf{h}}(K)\zeta(U^2) + \tbinom{q_1}{q_1-1}\tbinom{q_2}{q_2-1}\mathcal{G}_{\mathbf{h}}(U)\zeta(U) \\ &+\tbinom{q_1}{q_1-2}\mathcal{G}_{\mathbf{h}}(U^2) + \tbinom{q_2}{q_2-2}\mathbf{h}(U^2) \\ &+\tbinom{q_1}{q_1-1}\tbinom{q_2}{q_2-1}\eta(a)\mathbf{h}(U) + \tbinom{q_1}{q_1-2}\mathcal{G}_{\mathbf{h}}(K)\zeta(U^2) \\ &+\tbinom{q_2}{q_2-1}\tbinom{q_1}{q_1-1}\mathcal{G}_{\mathbf{h}}(U)\zeta(U) + \tbinom{q_2}{q_2-2}\mathcal{G}_{\mathbf{h}}(U^2) \\ &+\tbinom{q_1}{q_1-2}\mathbf{h}(U^2) + \tbinom{q_2}{q_2-1}\tbinom{q_1}{q_1-1}\eta(U)\mathbf{h}(U), \end{aligned}$$

for all $U \in \mathcal{U}$. Consequently,

$$\begin{aligned} (q_1+q_2-1)(q_1+q_2)\mathcal{G}_{\mathbf{h}}(U^2) \;=\;& [\tfrac{q_2(q_2-1)}{2} + \tfrac{q_1(q_1-1)}{2}][\mathcal{G}_{\mathbf{h}}(K)\zeta(U^2) + \mathbf{h}(U^2)] \\ &+q_1q_2\mathcal{G}_{\mathbf{h}}(U)\zeta(U) + \tfrac{q_1(q_1-1)}{2}\mathcal{G}_{\mathbf{h}}(U^2) \\ &+q_1q_2\eta(U)\mathbf{h}(U) + q_1q_2\mathcal{G}_{\mathbf{h}}(U)\zeta(U) \\ &+\tfrac{q_2(q_2-1)}{2}\mathcal{G}_{\mathbf{h}}(U^2) + q_2q_1\eta(U)\mathbf{h}(U). \end{aligned} \tag{2.4}$$

Replacing U by U^2 in (2.3), we arrive at

$$\mathcal{G}_{\mathbf{h}}(U^2) = \mathcal{G}_{\mathbf{h}}(K)\zeta(U^2) + \mathbf{h}(U^2),$$

for all U in $\mathcal{U}$. Combining this relation with Eq (2.4), we obtain

$$\begin{aligned} (q_1+q_2-1)(q_1+q_2)\mathcal{G}_{\mathbf{h}}(U^2) \;&= [q_2(q_2-1) + q_1(q_1-1)]\mathcal{G}_{\mathbf{h}}(U^2) \\ &+2q_1q_2\mathcal{G}_{\mathbf{h}}(U)\zeta(U) + 2q_1q_2\eta(U)\mathbf{h}(U). \end{aligned}$$

With careful and exact computations, we clearly demonstrate that

$$2q_2q_1\mathcal{G}_{\mathbf{h}}(U^2) \;=\; 2q_2q_1[\mathcal{G}_{\mathbf{h}}(U)\zeta(U) + \eta(U)\mathbf{h}(U)].$$

This implies that

$$\mathcal{G}_{\mathbf{h}}(U^2) = \mathcal{G}_{\mathbf{h}}(U)\zeta(U) + \eta(U)\mathbf{h}(U), \text{ for all } U \in \mathcal{U}. \tag{2.5}$$

Next, putting U^2 in place of U in (2.3), we obtain

$$\begin{aligned} \mathbf{h}(U^2) \;&=\; \mathcal{G}_{\mathbf{h}}(U^2) - \mathcal{G}_{\mathbf{h}}(K)\zeta(U^2) \\ &=\; \mathcal{G}_{\mathbf{h}}(U)\zeta(U) + \eta(U)\mathbf{h}(U) - \mathcal{G}_{\mathbf{h}}(K)\zeta(U^2) \\ &=\; [\mathcal{G}_{\mathbf{h}}(U) - \mathcal{G}_{\mathbf{h}}(K)\zeta(U)]\zeta(U) + \eta(U)\mathbf{h}(U) \\ &=\; \mathbf{h}(U)\zeta(U) + \eta(U)\mathbf{h}(U). \end{aligned}$$

From the above observation, we conclude that $\mathbf{h}$ is a Jordan (ζ,η)-derivation. Thus, we get that $\mathbf{h}$ is an associated (ζ,η)-derivation using Lemma 1.1. Again,

$$\begin{aligned}
\mathcal{G}_{\mathbf{h}}(UW) &= \mathcal{G}_{\mathbf{h}}(K)\zeta(U)\zeta(W) + \mathbf{h}(UW)\\
&= \mathcal{G}_{\mathbf{h}}(K)\zeta(U)\zeta(W) + \mathbf{h}(U)\zeta(W) + \eta(U)\mathbf{h}(W)\\
&= [\mathcal{G}_{\mathbf{h}}(K)\zeta(U) + \mathbf{h}(U)]\zeta(W) + \eta(U)\mathbf{h}(W)\\
&= \mathcal{G}_{\mathbf{h}}(U)\zeta(W) + \eta(U)\mathbf{h}(W).
\end{aligned}$$

Hence, we deduce that $\mathcal{G}_{\mathbf{h}}$ is a generalized (ζ, η)-derivation on $\mathcal{U}$ having associated (ζ, η)-derivation $\mathbf{h}$. □

Theorem 2.2. *Let $q_1, q_2 \geq 1$ and $q_3 \geq 0$ be any fixed integer, ζ, η be automorphisms, and $\mathcal{U} = N \cap Alg\, \mathcal{L}$ be a CSL subalgebra of von Neumann algebra N. Suppose that $\mathcal{G}_{\mathbf{h}}, \mathbf{h} : \mathcal{U} \to \mathcal{U}$ are two additive mappings which satisfy the algebraic identity*

$$\mathcal{G}_{\mathbf{h}}(U^{q_1+q_2+q_3}) = \mathcal{G}_{\mathbf{h}}(U^{q_1})\zeta(U^{q_2+q_3}) + \eta(U^{q_1})\mathbf{h}(U^{q_2})\zeta(U^{q_3}) + \eta(U^{q_1+q_2})\mathbf{h}(U^{q_3}) \tag{2.6}$$

for all $U \in \mathcal{U}$, where $U^0 = K$. Then, $\mathcal{G}_{\mathbf{h}}$ is a generalized (ζ, η)-derivation on $\mathcal{U}$ with corresponding (ζ, η)-derivation $\mathbf{h}$.

Proof. Given that

$$\begin{aligned}
\mathcal{G}_{\mathbf{h}}(U^{q_1+q_2+q_3}) &= \mathcal{G}_{\mathbf{h}}(U^{q_1})\zeta(U^{q_2+q_3}) + \eta(U^{q_1})\mathbf{h}(U^{q_2})\zeta(U^{q_3})\\
&\quad + \eta(U^{q_1+q_2})\mathbf{h}(U^{q_3}) \text{ for all } U \in \mathcal{U},
\end{aligned} \tag{2.7}$$

substituting U for K and using the fact $\zeta(K) = K$ and $\eta(K) = K$, we obtain

$$\mathcal{G}_{\mathbf{h}}(K) = \mathcal{G}_{\mathbf{h}}(K)K + K\mathbf{h}K + K\mathbf{h}(K) \text{ for all } U \in \mathcal{U}.$$

This implies that $\mathbf{h}(K) = 0$.

Next, we continue by applying condition (2.7), where U is substituted with $U + nK$, resulting in

$$\begin{aligned}
&\mathcal{G}_{\mathbf{h}}\Big(U^{q_1+q_2+q_3} + n\tbinom{q_1+q_2+q_3}{1}U^{q_1+q_2+q_3-1}K + \cdots + n^{q_1+q_2+q_3-2}\tbinom{q_1+q_2+q_3}{q_1+q_2+q_3-2}U^2K +\\
&n^{q_1+q_2+q_3-1}\tbinom{q_1+q_2+q_3}{q_1+q_2+q_3-1}UK + n^{q_1+q_2+q_3}K\Big) = \mathcal{G}_{\mathbf{h}}\Big(U^{q_1} + n\tbinom{q_1}{1}U^{q_1-1}K + \cdots +\\
&n^{q_1-2}\tbinom{q_1}{q_1-2}U^2K + n^{q_1-1}\tbinom{q_1}{q_1-1}UK + n^{q_1}K\Big)\zeta\Big(U^{q_2+q_3} + n\tbinom{q_2+q_3}{1}U^{q_2+q_3-1}K + \cdots\\
&+ n^{q_2+q_3-2}\tbinom{q_2+q_3}{q_2+q_3-2}U^2K + n^{q_2+q_3-1}\tbinom{q_2+q_3}{q_2+q_3-1}UK + n^{q_2+q_3}K\Big) + \eta\Big(U^{q_1} + n\tbinom{q_1}{1}U^{q_1-1}K\\
&+ \cdots + n^{q_1-2}\tbinom{q_1}{q_1-2}U^2K + n^{q_1-1}\tbinom{q_1}{q_1-1}UK + n^{q_1}K\Big)\mathbf{h}\Big(U^{q_2} + n\tbinom{q_2}{1}U^{q_2-1}K + \cdots +\\
&n^{q_2-2}\tbinom{q_2}{q_2-2}U^2K + n^{q_2-1}\tbinom{q_2}{q_2-1}UK + n^{q_2}K\Big)\zeta\Big(U^{q_3} + n\tbinom{q_3}{1}U^{q_3-1}K + \cdots +\\
&n^{q_3-2}\tbinom{q_3}{q_3-2}U^2K + n^{q_3-1}\tbinom{q_3}{q_3-1}UK + n^{q_3}K\Big) + \eta\Big(U^{q_1+q_2} + n\tbinom{q_1+q_2}{1}(U^{q_1+q_2-1})K + \cdots +\\
&n^{q_1+q_2-2}\tbinom{q_1+q_2}{q_1+q_2-2}U^2K + n^{q_1+q_2-1}\tbinom{q_1+q_2}{q_1+q_2-1}UK + n^{q_1+q_2}K\Big)\mathbf{h}\Big(U^{q_3} + n\tbinom{q_3}{1}U^{q_3-1}K + \cdots +\\
&n^{q_3-2}\tbinom{q_3}{q_3-2}U^2K + n^{q_3-1}\tbinom{q_3}{q_3-1}UK + n^{q_3}K\Big)
\end{aligned}$$

for all $U \in \mathcal{U}$ and $n \geq 1$. Transform the given expression by employing the impressive (2.7) as

$$n\mathcal{P}_1(U, K) + n^2\mathcal{P}_2(U, K) + \cdots + (n^{q_1+q_2+q_3-1})\mathcal{P}_{q_1+q_2+q_3-1}(U, K) = 0,$$

where $\mathcal{P}_i(U, K)$ represents the coefficients of n^i's across every instance of $i = 1, 2, \cdots, (q_1+q_2+q_3-1)$. If we successively replace n by $1, 2, \ldots, (q_1 + q_2 + q_3 - 1)$, we obtain a system of $(q_1 + q_2 + q_3 -$

1) homogeneous equations. This insightful manipulation results in the construction of a powerful Vandermonde matrix

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 2 & 2^2 & \cdots & 2^{q_1+q_2+q_3-1} \\ . & . & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ (q_1+q_2+q_3-2) & (q_1+q_2+q_3-2)^2 & \cdots & (q_1+q_2+q_3-2)^{q_1+q_2+q_3-1} \\ (q_1+q_2+q_3-1) & (q_1+q_2+q_3-1)^2 & \cdots & (q_1+q_2+q_3-1)^{q_1+q_2+q_3-1} \end{bmatrix}.$$

This results in that $\mathcal{P}_i(U,K) = 0$ for each $U \in \mathcal{U}$ and for $i = 1, 2, .., (q_1 + q_2 + q_3 - 1)$. Specifically, $\mathcal{P}_{q_1+q_2+q_3-1}(U,K) = 0$, and we arrive at

$$\begin{aligned} \tbinom{q_1+q_2+q_3}{q_1+q_2+q_3-1}\mathcal{G}_{\mathbf{h}}(K)U &= \tbinom{q_2+q_3}{q_2+q_3-1}\mathcal{G}_{\mathbf{h}}(K)\zeta(U) + \tbinom{q_1}{q_1-1}\mathcal{G}_{\mathbf{h}}(U) + \tbinom{q_3}{q_3-1}\mathbf{h}(K)\zeta(U) \\ &+\tbinom{q_2}{q_2-1}\mathbf{h}(U) + \tbinom{q_3}{q_3-1}\mathbf{h}(K)\zeta(U) + \tbinom{q_3}{q_3-1}\mathbf{h}(U) + \tbinom{q_1+q_2}{q_1+q_2-1}\eta(U)\mathbf{h}(K), \end{aligned}$$

for all $U \in \mathcal{U}$. By simplifying the last relation, we obtain

$$(q_2 + q_3)\mathcal{G}_{\mathbf{h}}(U) = (q_2 + q_3)\mathcal{G}_{\mathbf{h}}(K)\zeta(U) + (q_2 + q_3)\mathbf{h}(U),$$

for all $U \in \mathcal{U}$. It follows that

$$\mathcal{G}_{\mathbf{h}}(U) = \mathcal{G}_{\mathbf{h}}(K)\zeta(U) + \mathbf{h}(U), \quad \text{for all } U \in \mathcal{U}. \tag{2.8}$$

Let us now turn our attention to $\mathcal{P}_2(U,K) = 0$, implementing the fact $\mathbf{h}(K) = 0$; through this consideration, we derive

$$\begin{aligned} \tbinom{q_1+q_2+q_3}{q_1+q_2+q_3-2}\mathcal{G}_{\mathbf{h}}(U^2) &= \tbinom{q_2+q_3}{q_2+q_3-2}\mathcal{G}_{\mathbf{h}}(K)\zeta(U^2) + \tbinom{q_1}{q_1-1}\tbinom{q_2+q_3}{q_2+q_3-1}\mathcal{G}_{\mathbf{h}}(U)\zeta(U) \\ &+\tbinom{q_1}{q_1-2}\mathcal{G}_{\mathbf{h}}(U^2) + \tbinom{q_2}{q_2-1}\tbinom{q_3}{q_3-1}\mathbf{h}(U)\zeta(U) + \tbinom{q_2}{q_2-2}\mathbf{h}(U^2) \\ &+\tbinom{q_1}{q_1-1}\tbinom{q_2}{q_2-1}\eta(U)\mathbf{h}(U) + \tbinom{q_3}{q_3-2}\mathbf{h}(U^2) + \tbinom{q_1+q_2}{q_1+q_2-1}\tbinom{q_3}{q_3-1}\eta(U)\mathbf{h}(U). \end{aligned}$$

A subtle manipulation results in

$$\begin{aligned} &(q_1 + q_2 + q_3)(q_1 + q_2 + q_3 - 1)\mathcal{G}_{\mathbf{h}}(U^2) = (q_2 + q_3)(q_2 + q_3 - 1)\mathcal{G}_{\mathbf{h}}(K)\zeta(U^2) \\ &+2q_1(q_2 + q_3)\mathcal{G}_{\mathbf{h}}(U)\zeta(U) + q_1(q_1 - 1)\mathcal{G}_{\mathbf{h}}(U^2)2q_2q_3\mathbf{h}(U)\eta(U) \\ &+q_2(q_2 - 1)\mathbf{h}(U^2) + 2q_1q_2\eta(U)\mathbf{h}(U) + q_3(q_3 - 1)\mathbf{h}(U^2) + 2q_3(q_1 + q_2)\eta(U)\mathbf{h}(U), \end{aligned}$$

for all $U \in \mathcal{U}$. Next, evaluate the term

$$\begin{aligned} &[(q_1 + q_2 + q_3)(q_1 + q_2 + q_3 - 1) - q_1(q_1 - 1)]\mathcal{G}_{\mathbf{h}}(U^2) = (q_2 + q_3)(q_2 + q_3 - 1)[\mathcal{G}_{\mathbf{h}}(U^2) \\ &-\mathbf{h}(U^2)] + 2q_1(q_2 + q_3)\mathcal{G}_{\mathbf{h}}(U)\zeta(U) + (2q_1q_2 + 2q_2q_3 + 2q_3q_1)\eta(U)\mathbf{h}(U) \\ &+({q_2}^2 - q_2 + {q_3}^2 - q_3)\mathbf{h}(U^2) + 2q_2q_3\mathbf{h}(U)\zeta(U). \end{aligned}$$

By the above observation, we obtain

$$\begin{aligned} &[(q_1 + q_2 + q_3)(q_1 + q_2 + q_3 - 1) - q_1(q_1 - 1) - (q_2 + q_3)(q_2 + q_3 - 1)]\mathcal{G}_{\mathbf{h}}(U^2) \\ &= -2q_2q_3\mathbf{h}(U^2) + 2q_1(q_2 + q_3)\mathcal{G}_{\mathbf{h}}(U)\zeta(U) + (2q_1q_2 + 2q_2q_3 + 2q_3q_1)\eta(U)\mathbf{h}(U) \\ &+ 2q_2q_3\mathbf{h}(U)\zeta(U), \end{aligned}$$

which implies that

$$\begin{aligned}(2q_1q_2 + 2q_1q_3)\mathcal{G}_{\mathbf{h}}(U^2) &= -2q_2q_3\mathbf{h}(U^2) + 2q_1(q_2 + q_3)\mathcal{G}_{\mathbf{h}}(U)\zeta(U)\\ &\quad +(2q_1q_2 + 2q_2q_3 + 2q_3q_1)\eta(U)\mathbf{h}(U) + 2q_2q_3\mathbf{h}(U)\zeta(U).\end{aligned}$$

Substituting U^2 for U in (2.8) and applying the above relation, we obtain

$$\begin{aligned}(2q_1q_2 + 2q_1q_3)[\mathcal{G}_{\mathbf{h}}(K)U^2 + \mathbf{h}(U^2)] &= -2q_2q_3\mathbf{h}(U^2) + 2q_1(q_2 + q_3)[\mathcal{G}_{\mathbf{h}}(K)\zeta(U^2)\\ &\quad +\mathbf{h}(U)\zeta(U)] + (2q_1q_2 + 2q_2q_3 + 2q_3q_1)\eta(U)\mathbf{h}(U)\\ &\quad +2q_2q_3\mathbf{h}(U)\zeta(U).\end{aligned}$$

Subsequently, with some calculations, we arrive at

$$\begin{aligned}&(2q_1q_2 + 2q_1q_3 + 2q_2q_3)\mathbf{h}(U^2) = 2q_1(q_2 + q_3)\mathbf{h}(U)\zeta(U)\\ &+(2q_1q_2 + 2q_2q_3 + 2q_3q_1)\eta(U)\mathbf{h}(U) + 2q_2q_3\mathbf{h}(U)\zeta(U),\end{aligned}$$

for every $U \in \mathcal{U}$. Thus, we obtain $\mathbf{h}(U^2) = \mathbf{h}(U)\zeta(U) + \eta(U)\mathbf{h}(U)$, for each $U \in \mathcal{U}$. Hence, $\mathbf{h}$ is a Jordan (ζ, η)-derivation on $\mathcal{U}$. The conclusion follows from Lemma 1.1, and $\mathbf{h}$ is a (ζ, η)-derivation on $\mathcal{U}$. Moreover, by adeptly implementing analogous techniques as demonstrated in the preceding theorem, $\mathcal{G}_{\mathbf{h}}$ emerges as a generalized (ζ, η)-derivation on $\mathcal{U}$, intrinsically linked with the (ζ, η)-derivation $\mathbf{h}$.

Example 2.1. Consider

$$\mathsf{A}_3 = \left\{ \begin{bmatrix} a & b & c \\ 0 & 0 & 0 \\ 0 & 0 & \kappa \end{bmatrix} \mid a, b, c, \kappa \in \mathbb{C} \right\}.$$

Then, A_3 is a CSL subalgebra of von Neumann algebra $B(\mathbb{C}^3)$, where $B(\mathbb{C}^3)$ is the von Neumann algebra of all bounded operators on a Hilbert space $\mathbb{C}^3$. Define linear mappings $\mathcal{G}_{\mathbf{h}}, \mathbf{h}, \zeta, \eta : \mathsf{A}_3 \to \mathsf{A}_3$ by

$$\mathcal{G}_{\mathbf{h}}\left(\begin{bmatrix} a & b & c \\ 0 & 0 & 0 \\ 0 & 0 & \kappa \end{bmatrix}\right) = \begin{bmatrix} a & b & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \mathbf{h}\left(\begin{bmatrix} a & b & c \\ 0 & 0 & 0 \\ 0 & 0 & \kappa \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \kappa \end{bmatrix},$$

$$\zeta\left(\begin{bmatrix} a & b & c \\ 0 & 0 & 0 \\ 0 & 0 & \kappa \end{bmatrix}\right) = \begin{bmatrix} a & b & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \kappa \end{bmatrix} \eta\left(\begin{bmatrix} a & b & c \\ 0 & 0 & 0 \\ 0 & 0 & \kappa \end{bmatrix}\right) = \begin{bmatrix} -a & -b & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

It is clear that $\mathcal{G}_{\mathbf{h}}$ is a generalized (ζ, η)-derivation associated with (ζ, η)-derivation $\mathbf{h}$. Further, $\mathcal{G}_{\mathbf{h}}$ satisfies the identity of the second theorem for $q_1 = 1, q_2 = 2$, and $q_3 = 0$. This shows the existence and validity of our hypothesis.

3. Conclusions

The engaging study of generalized (ζ, η)-derivations over rings, together with the deeply complex (ζ, η)-derivation frameworks on rings and the exuberantly complex CSL subalgebra of von Neumann algebra $\mathcal{U}$, presents an extraordinarily compelling and fertile realm of academic inquiry. We conclude that under what conditions the given mapping acts as a generalized (ζ, η)-derivation linked with (ζ, η)-derivation while satisfying some identities. As a leading edge of mathematical research, the ambitious endeavor of unveiling continuity theorems across a vast and varied array of algebraic structures, ranging from the mathematical elegance of Banach algebra, through the robust simplicity of semisimple Banach algebra, to the structural intricacies of Lie algebra, and the mesmerizing domain of C^* algebra in a promising expedition within the expansively intriguing and intellectually stimulating framework of this study.

Furthermore, it is crucial to emphasize that the reader is invited to explore a diverse spectrum of functional identities intricately tied to specific derivation types, including the sophisticated realm of generalized (ζ, η)-derivations within semiprime rings possessing involution, and the cutting-edge domain of generalized-(ζ, η)-higher derivations. This revelation of diverse forms of additive maps relevant to both rings and their interconnected subsets has been brilliantly articulated through the pure artistry of algebraic methodologies, uncovering profound layers of mathematical sophistication and deep theoretical understanding.

4. Open problems

The following problem would logically integrate the current findings with a more comprehensive algebraic program and greatly elevate the conceptual contribution.

Let $\mathfrak{A}$ be a suitable non-associative algebra (for instance, an alternative algebra, a Jordan algebra, or a non-associative operator algebra analogue). Suppose that mappings $\mathcal{G}_{\mathbf{h}}, h : \mathfrak{A} \longrightarrow \mathfrak{A}$ satisfy identities analogous to (2.1) and (2.6). Under which structural conditions on $\mathfrak{A}$ does $\mathcal{G}_{\mathbf{h}}$ become a generalized (ζ, η)-derivation associated with a (ζ, η)-derivation h?

For future research, it would be fascinating to stay focused on the following facts: (i) Can the Vandermonde-based technique be adapted to alternative or Jordan-type operator structures?; and (ii) Do the main theorems remain valid in alternative division rings or Jordan operator algebras?

Author contributions

Abu Zaid Ansari: Writing–original draft, Methodology, Conceptualization; Faiza Shujat: Writing–review & editing, Validation; Ahlam Fallatah: Supervision, Data curation, Formal analysis. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors sincerely thank the reviewers for their insightful suggestions and recommendations, which enriched and strengthened our manuscript. The authors extend their appreciation to the Deanship of Scientific Research at the Islamic University of Madinah, Saudi Arabia, for funding this research.

Conflict of interest

The contributors declare that they don't have competing interests. The current paper has been read and approved for publishing by all authors.

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