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Research article

Lie symmetry and conservation law of the conformable fractional Burgers' equation

Zhihan Liu and Qi Wang*

College of Sciences, Northeastern University, No. 3-11, Wenhua Road, Shenyang, 110819, China

* **Correspondence:** Email: wangqimath@mail.neu.edu.cn.

Abstract: In this work, we studied Lie symmetry and the conservation law of the conformable fractional Burgers' equation. We discussed the physical significance of the conformable Burgers' equation, as well as the behavior of the fluid that satisfies this equation. As an application, we calculated the group invariant solutions corresponding to an optimal system of one-dimensional subalgebras of the Lie algebra of infinitesimal point symmetries of this equation under the condition that the three spatial derivative orders of the equation are consistent.

Keywords: conformable Burgers' equation; Lie symmetry; conservation law
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1. Introduction

Fractional calculus has played a significant role in many applied fields and has aroused considerable interest of scientists, and fractional partial differential equations, as generalizations of classical partial differential equations, have been proposed and investigated in many research fields. In 2014, a new definition of a fractional derivative, the conformable fractional derivative, was introduced by Khalil et al. in [1].

The most common way to construct fractional differential equations is to replace some or all of the derivatives in integer-order differential equations with fractional derivatives. In this work, we replace the integer-order derivatives of the Burgers' equation,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u\,\frac{\partial u}{\partial x}, \tag{1.1}$$

by the conformable fractional derivatives. That is, we consider the conformable fractional Burgers' equation

$$\mathrm{D}_t^{\alpha} u = \mathrm{D}_x^{\beta}\,\mathrm{D}_x^{\gamma} u + u\,\mathrm{D}_x^{\delta} u, \tag{1.2}$$

where $u = u(x,t)$ denotes an unknown function on independent variables $x > 0$ and $t > 0$; the fixed real numbers α, β, γ, and δ are the orders of conformable derivatives with $0 < \alpha, \beta, \gamma, \delta \leq 1$; and D_t^α and D_x^β stand for the conformable fractional derivatives of order α and β of u with respect to t and x, respectively. Here, α is called the order of time derivative, while β, γ, and δ are called the orders of spatial derivatives of Eq (1.2).

First of all, we review the definition of a conformable derivative:

Definition 1.1 ([1]). Let $f : (0, +\infty) \to \mathbb{R}$ be a function, $\alpha \in (0,1]$, and $t \in (0, +\infty)$. If the following limit exists, then it is called the conformable fractional derivative of order α of f at t, and is denoted by $\mathrm{D}^\alpha f(t)$:

$$\mathrm{D}^\alpha f(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}.$$

In this case, f is said to be conformably differentiable of order α at t.

Note that $\mathrm{D}^1 f(t)$ coincides with the classical definition of the first derivative $f'(t)$. But a conformable derivative does not satisfy $\mathrm{D}^\alpha \mathrm{D}^\beta = \mathrm{D}^{\alpha+\beta}$ in general, and neither does the Riemann-Liouville derivative or Caputo derivative. On the other hand, a conformable fractional derivative satisfies the constant rule, sum rule, product rule, and quotient rule as a classical derivative:

Lemma 1.2 ([1]). *Let $\alpha \in (0,1]$, $t \in (0,+\infty)$, $c, d \in \mathbb{R}$, and f and g be functions conformably differentiable of order α at t. Then we have*

(1) $\mathrm{D}_t^\alpha c = 0$;

(2) $\mathrm{D}_t^\alpha t^c = ct^{c-\alpha}$;

(3) $\mathrm{D}_t^\alpha (cf + dg)(t) = c\,\mathrm{D}_t^\alpha f(t) + d\,\mathrm{D}_t^\alpha g(t)$;

(4) $\mathrm{D}_t^\alpha (fg)(t) = g\,\mathrm{D}_t^\alpha f(t) + f\,\mathrm{D}_t^\alpha g(t)$;

(5) $\mathrm{D}_t^\alpha (f/g)(t) = (g\,\mathrm{D}_t^\alpha f(t) - f\,\mathrm{D}_t^\alpha g(t))/g(t)^2$;

(6) *If, in addition, f is differentiable at t, then* $\mathrm{D}_t^\alpha = t^{1-\alpha} f'(t)$.

Further properties of the conformable fractional derivative, such as the intermediate value theorem and extensions to fractional transforms, are explored in [2–4], building on the foundational definition in [1]. Based on this notion of a fractional derivative, a large number of conformable fractional differential equations have been proposed and investigated in recent years. Besides the study of mathematical properties, many researchers focus on the physical significance of conformable differential equations and applications of conformable models in physical, chemical, and biological processes and engineering; see, for instance, [5–7] and the references therein. Some of these works considered conformable Burgers-type equations or their variants, which suggests that equations of the form (1.2) may possess meaningful physical interpretations.

In this work, we study Eq (1.2) from the perspective of Lie symmetry and the conservation law, in order to help reveal further structural and physical implications of the model. Lie symmetry analysis and conservation laws provide fundamental tools for understanding the structure of differential equations and the physical phenomena described by them. Classical theories of Lie symmetries and conservation laws can be found in [8,9]. Recently, the extension of Lie group methods to conformable fractional differential equations has also attracted attention; see, for example, [10,11].

The exact and numerical solutions of some special or modified cases of Eq (1.2) or related Burgers-type equations have been considered in [12–14]. For instance, [12] used the first integral method to establish the exact solutions for the time-conformable Burgers equation; [13] employed the generalized Kudryashov method to obtain soliton-type solutions of the time-conformable Burgers equation; and [14] proposed a fully discrete computational technique involving the implicit finite difference technique and cubic Hermite splines to numerically solve the conformable damped Burgers equation with variable coefficients. On the other hand, Lie symmetries and group-invariant solutions for a more general form of (1.2) have been provided in [15]. These studies further motivate us to analyze Lie symmetries on (1.2) and construct its group-invariant solutions.

In this work, we study the Lie symmetry structure and conservation laws of the conformable fractional Burgers equation (1.2), with the aim of clarifying its structural properties and exploring the possible physical interpretation of this conformable model. The main content is as follows:

(1) First, Theorem 2.1 provides the Lie algebra of infinitesimal point symmetries of (1.2). We see that if $\beta = \gamma = \delta$, then the fractional-order equation (1.2) has an isomorphic Lie algebra of infinitesimal point symmetries as the integer-order equation (1.1), which is of dimension 5; otherwise, if at least two of β, γ, δ are distinct, then the Lie algebra is of dimension 2, i.e., the "quantity" of symmetries is greatly reduced. Perhaps surprisingly, although Eq (1.2) has four parameters α, β, γ, and δ, its Lie symmetry only falls into the two cases mentioned above. This somewhat aligns with the intuition that "α is the order of the time derivative, while β, γ, and δ are orders of the spatial derivatives". We discuss the one-parameter transformation groups generated by these infinitesimal point symmetries one by one in Remark 2.2, which may help to understand the physical sense of the conformable Burgers equation or even the conformable derivative.

(2) Second, Theorem 2.5 provides the conservation laws of (1.2). We see that if $\delta \neq \beta$, then (1.2) has only a trivial conservation law, and if $\delta = \beta$, then (1.2) has the same "quantity" of conservation laws as integer-order equation (1.1). Note that the fractional-order equation (1.2) always has non-constant characteristics of the conservation laws unless $\alpha = \beta = \delta = 1$ (γ is arbitrary), which is different from the integer-order equation (1.1). Also note that the conservation law of (1.2) only falls into these two cases. Based on this theorem, we define the conformable momentum for the fluid satisfying the conformable Burgers equation and discuss the potential physical significance of such hypothetical fluid in Remark 2.6.

(3) Finally, we calculate the group invariant solutions corresponding to an optimal system of one-dimensional subalgebras of the Lie algebra of infinitesimal point symmetries of (1.2) in §3 under the condition that the three spatial derivative orders are consistent, i.e., $\beta = \gamma = \delta$.

2. Point symmetry and conservation law

A point symmetry (i.e., geometric symmetry) group of a given differential equation is defined to be a local transformation group whose elements transform a smooth solution of this equation to another, and a conservation law of a differential equation is defined to be a divergence expression which vanishes for all smooth solutions of this equation. Thus, we can focus on the smooth solutions of (1.2) when we study its point symmetries and conservation laws. We conclude from Lemma 1.2 that a smooth function $u(x,t)$ is a solution of (1.2) if and only if it is a solution of the following equation:

$$t^{1-\alpha}\frac{\partial u}{\partial t}-\Big((1-\gamma)x^{1-\beta-\gamma}+x^{1-\delta}u\Big)\frac{\partial u}{\partial x}-x^{2-\beta-\gamma}\frac{\partial^2 u}{\partial x^2}=0. \tag{2.1}$$

Hence, a local transformation group is a point symmetry group of (1.2) if and only if it is that of (2.1).

The following theorem provides the Lie algebra of infinitesimal point symmetries (i.e., geometric symmetries) of the conformable fractional Burgers equation (1.2). The proof of the theorem is quite standard, so we put it in Appendix A.

Theorem 2.1. (1) *When at least two of β,γ,δ are distinct (i.e., $\beta\neq\gamma$, or $\gamma\neq\delta$, or $\beta\neq\delta$), the Lie algebra $\mathfrak{g}$ of infinitesimal symmetries of Eq* (1.2) *is spanned by the following vector fields:*

$$\begin{aligned}
v_1&=t^{1-\alpha}\frac{\partial}{\partial t},\\
v_2&=\alpha x\frac{\partial}{\partial x}+(\beta+\gamma)t\frac{\partial}{\partial t}+\alpha(\delta-\beta-\gamma)u\frac{\partial}{\partial u}.
\end{aligned}$$

(2) *When $\beta=\gamma=\delta$, the Lie algebra $\mathfrak{g}$ of infinitesimal symmetries of Eq* (1.2) *is spanned by the following vector fields:*

$$\begin{aligned}
v_1&=t^{1-\alpha}\frac{\partial}{\partial t},\\
v_2&=\alpha x\frac{\partial}{\partial x}+2\beta t\frac{\partial}{\partial t}-\alpha\beta u\frac{\partial}{\partial u},\\
v_3&=x^{1-\beta}\frac{\partial}{\partial x},\\
v_4&=x^{1-\beta}t^{\alpha}\frac{\partial}{\partial x}-\alpha\frac{\partial}{\partial u},\\
v_5&=\alpha x t^{\alpha}\frac{\partial}{\partial x}+\beta t^{\alpha+1}\frac{\partial}{\partial t}-\alpha\left(\alpha x^{\beta}+\beta t^{\alpha}u\right)\frac{\partial}{\partial u}.
\end{aligned}$$

We see that when $\alpha=\beta=\gamma=\delta=1$, the theorem coincides with the classical result of the integral order Burgers equation. Moreover, Eq (1.2) has more symmetries when $\beta=\gamma=\delta$, which accords with the intuition because β, γ, and δ are all the orders of conformable derivatives on the spatial variable x. On the other hand, the order α of the conformable derivative on the time variable t does not affect the amount of symmetries.

Remark 2.2. We consider the point symmetry groups generated by the v_i's one by one, which may help us to understand the physical sense of the conformable Burgers equation or even the conformable derivative:

(1) The group G_1 generated by v_1 transforms (x,t,u) to $\left(x,(t^{\alpha}+\alpha\varepsilon)^{1/\alpha},u\right)$ for $\varepsilon\in\mathbb{R}$ and $|\varepsilon|$ sufficiently small. If we write $\tau=t^{\alpha}$, then the action of G_1 on the new coordinate system (x,τ,u) is given by $(x,\tau,u)\mapsto(x,\tau+\alpha\varepsilon,u)$. Hence, the symmetry generated by v_1 may be interpreted as an α-order conformable time-translation invariance, reflecting the invariance of the equation under translations in the transformed time variable t^{α}. Similarly, when $\beta=\gamma=\delta$, the group G_3 generated by v_3 transforms (x,t,u) to $\left((x^{\beta}+\beta\varepsilon)^{1/\beta},t,u\right)$ for $\varepsilon\in\mathbb{R}$ and $|\varepsilon|$ sufficiently small.

If we write $\xi = x^\beta$, then the action of G_3 is given by $(\xi, t, u) \mapsto (\xi + \beta\varepsilon, t, u)$. Hence, the symmetry generated by v_3 may be interpreted as a β-order conformable space-translation invariance, reflecting the invariance of the equation under translations in the transformed spatial variable x^β.

(2) The group G_2 generated by v_2 transforms (x, t, u) to $(xe^{\alpha\varepsilon}, te^{(\beta+\gamma)\varepsilon}, ue^{-\alpha(\beta+\gamma-\delta)\varepsilon})$ for $\varepsilon \in \mathbb{R}$. We write $\lambda = e^\varepsilon$. Then the action of G_2 is given by

$$(x, t, u) \mapsto (\lambda^\alpha x, \lambda^{\beta+\gamma} t, \lambda^{-\alpha(\beta+\gamma-\delta)} u).$$

Hence, G_2 is a scaling transformation group, and it reflects a scale invariance of the conformable Burgers equation as it does that of the integer-order Burgers equation, but the scales are influenced by the parameters α, β, γ, and δ. The scaling symmetry indicates that the relative scaling behavior of u depends on the relation between $\beta + \gamma$ and δ.

(3) When $\alpha = \beta = \gamma = \delta = 1$, the group G_4 generated by v_4 reflects the Galilean invariance of the integer-order Burgers equation. That is, the map $(x, t, u) \mapsto (x + vt, t, u + v)$ transforms solutions of the integer-order Burgers equation to other solutions, where v denotes the velocity of the system. For general values of α and $\beta = \gamma = \delta$, the group G_4 generated by v_4 transforms (x, t, u) to $\left((x^\beta + \beta\varepsilon t^\alpha)^{1/\beta}, t, u - \alpha\varepsilon\right)$ for $\varepsilon \in \mathbb{R}$ and $|\varepsilon|$ sufficiently small. We write $v = \varepsilon$. Then under the new coordinate system (ξ, τ, u) defined above, the action of G_5 is given by $(\xi, \tau, u) \mapsto (\xi + \beta v\tau, \tau, u - \alpha v)$. This is almost entirely consistent with the form of the Galilean invariance, and the only difference is that the velocities of the response in the ξ component and the u component have a ratio of $\beta : \alpha$. In particular, when $\alpha = \beta = \gamma = \delta$ (not equal to 1 necessarily), viewed from a reference (ξ, τ, u)-frame moving at a constant velocity, the conformable Burgers equation remains unchanged. We may consider G_4 as reflecting a conformable Galilean-type invariance of the conformable Burgers equation.

(4) When $\alpha = \beta = \gamma = \delta = 1$, the group G_5 generated by v_5 reflects a special projective symmetry of the integer-order Burgers equation, which transforms (x, t, u) to

$$\left(\frac{x}{1 - st}, \frac{t}{1 - st}, u - s(tu + x)\right).$$

It can be interpreted as a transformation to a uniformly accelerated reference frame. While Galilean invariance deals with constant velocity, the invariance generalizes this to constant acceleration. For general values of α and $\beta = \gamma = \delta$, the group G_5 generated by v_5 transforms (x, t, u) to

$$\left(x\,(1 - \alpha\beta\varepsilon t^\alpha)^{-1/\beta}, t\,(1 - \alpha\beta\varepsilon t^\alpha)^{-1/\alpha}, u - \alpha\varepsilon(\beta t^\alpha u + \alpha x^\beta)\right),$$

for $\varepsilon \in \mathbb{R}$ and $|\varepsilon|$ sufficiently small. We write $s = \alpha\varepsilon$. Then under the new coordinate system (ξ, τ, u) defined above, the action of G_5 is given by

$$(\xi, \tau, u) \mapsto \left(\frac{\xi}{1 - \beta s\tau}, \frac{\tau}{1 - \beta s\tau}, u - s(\beta\tau u + \alpha\xi)\right).$$

This is almost entirely consistent with the form of the special projective invariance of the integer-order Burgers equation, and the only difference is that the parameters occur in u component. In particular, when $\alpha = \beta = \gamma = \delta$ (not equal to 1 necessarily), viewed from a reference (ξ, τ, u)-frame moving at a constant acceleration, the conformable Burgers equation satisfies a similar invariance as the integer-order Burgers equation. We may consider G_5 as reflecting a conformable special projective symmetry of the conformable Burgers equation.

The commutator table of v_1, v_2 in the case where at least two of β, γ, δ are distinct is given in Table 1, and that of $v_1, \ldots, v_5$ in the case where $\beta = \gamma = \delta$ is given in Table 2. That is, the entries in row i and column j of these tables represent the Lie bracket $[v_i, v_j] = v_i v_j - v_j v_i$. The adjoint tables of v_1, v_2 and $v_1, \ldots, v_5$ are given in Tables 3 and 4, respectively. That is, the entries in row i and column j of these tables represent the adjoint representation $\mathrm{Ad}_{\varepsilon v_i}(v_j)$.

Table 1. Commutator table of Lie algebra of point symmetries when at least two of β, γ, δ are distinct.

	v_1	v_2
v_1	0	$\alpha(\beta+\gamma)v_1$
v_2	$-\alpha(\beta+\gamma)v_1$	0

Table 2. Commutator table of Lie algebra of point symmetries when $\beta = \gamma = \delta$.

	v_1	v_2	v_3	v_4	v_5
v_1	0	$2\ \alpha\beta v_1$	0	αv_3	αv_2
v_2	$-2\alpha\beta v_1$	0	$-\alpha\beta v_3$	$\alpha\beta v_4$	$2\alpha\beta v_5$
v_3	0	$\alpha\beta v_3$	0	0	$\alpha\beta v_4$
v_4	$-\alpha v_3$	$-\alpha\beta v_4$	0	0	0
v_5	$-\alpha v_2$	$-2\alpha\beta v_5$	$-\alpha\beta v_4$	0	0

Table 3. Adjoint table of Lie algebra of point symmetries when at least two of β, γ, δ are distinct.

	v_1	v_2
v_1	v_1	$-\alpha(\beta+\gamma)\varepsilon v_1 + v_2$
v_2	$e^{\alpha(\beta+\gamma)\varepsilon} v_1$	v_2

Table 4. Adjoint table of Lie algebra of point symmetries when $\beta = \gamma = \delta$.

	v_1	v_2	v_3	v_4	v_5
v_1	v_1	$-2\alpha\beta\varepsilon v_1 + v_2$	v_3	$-\alpha\varepsilon v_3 + v_4$	$\alpha^2\beta\varepsilon^2 v_1 - \alpha\varepsilon v_2 + v_5$
v_2	$e^{2\alpha\beta\varepsilon} v_1$	v_2	$e^{\alpha\beta\varepsilon} v_3$	$e^{-\alpha\beta\varepsilon} v_4$	$e^{-2\alpha\beta\varepsilon} v_5$
v_3	v_1	$v_2 - \alpha\beta\varepsilon v_3$	v_3	v_4	$-\alpha\beta\varepsilon v_4 + v_5$
v_4	$v_1 + \alpha\varepsilon v_3$	$v_2 + \alpha\beta\varepsilon v_4$	v_3	v_4	v_5
v_5	$v_1 + \alpha\varepsilon v_2 + \alpha^2\beta\varepsilon^2 v_5$	$v_2 + 2\alpha\beta\varepsilon v_5$	$v_3 + \alpha\beta\varepsilon v_4$	v_4	v_5

Remark 2.3. In this remark, we assume that $\beta = \gamma = \delta$, and let $\mathfrak{g}$ be the Lie algebra spanned by $v_1, \ldots, v_5$ defined in Theorem 2.1. We also write $\tilde{\mathfrak{g}}$ for the Lie algebra of point symmetries of the

integer-order Burgers equation (1.1) and write $u_1, \ldots, u_5$ for the classical basis of $\tilde{\mathfrak{g}}$. That is,

$$u_1 = \frac{\partial}{\partial t}, \qquad u_2 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}, \qquad u_3 = \frac{\partial}{\partial x},$$
$$u_4 = t\frac{\partial}{\partial x} - \frac{\partial}{\partial u}, \qquad u_5 = xt\frac{\partial}{\partial x} + t^2\frac{\partial}{\partial t} - (x + tu)\frac{\partial}{\partial u}.$$

We define a linear map $\varphi : \tilde{\mathfrak{g}} \to \mathfrak{g}$ by letting

$$\varphi(u_1) = \frac{1}{\alpha^2\beta}v_1, \quad \varphi(u_2) = \frac{1}{\alpha\beta}v_2, \quad \varphi(u_3) = v_3, \quad \varphi(u_4) = \alpha\beta v_4, \quad \varphi(u_5) = v_5.$$

Then we check that φ is an isomorphism of Lie algebras.

The following corollary provides an optimal system of one-dimensional subalgebras of the Lie algebra spanned by the vector fields v_1 and v_2 (when at least two of β, γ, δ are distinct) or $v_1, \ldots, v_5$ (when $\beta = \gamma = \delta$) in Theorem 2.1. The proof of the theorem is quite standard, so we put the proof in Appendix B.

Theorem 2.4. *Let $\mathfrak{g}$ be the Lie algebra spanned by the vector fields v_1 and v_2 (when at least two of β, γ, δ are distinct) or $v_1, \ldots, v_5$ (when $\beta = \gamma = \delta$) in Theorem 2.1.*

(1) *When at least two of β, γ, δ are distinct, an optimal system of one-dimensional subalgebras of $\mathfrak{g}$ is given by*

$$v_1, \quad v_2.$$

(2) *When $\beta = \gamma = \delta$, an optimal system of one-dimensional subalgebras of $\mathfrak{g}$ is given by*

$$v_1 \pm v_5, \quad v_2, \quad v_3, \quad v_3 \pm v_5, \quad v_4.$$

Now we consider the conservation law of (1.2). By the conservation law of (1.2), we mean a pair (p, q) of functions on the variables $x, t, u, u_x, u_t, u_{xx}, \ldots$ of the jet space such that the restriction of $\mathrm{D}_t p + \mathrm{D}_x q$ on any smooth solution of (1.2) vanishes, where D_t and D_x denote the total derivatives on t and x, respectively. In this case, p and q are called the conserved density and flux of the conservation law. The conservation law plays an important role in the study of differential equations. We take two subintervals $[t_1, t_2]$ and $[x_1, x_2]$ of $(0, +\infty)$. For a fixed conservation law (p, q) of (1.2), we have

$$\int_{x_1}^{x_2} p|_{t=t_2}\,\mathrm{d}x - \int_{x_1}^{x_2} p|_{t=t_1}\,\mathrm{d}x = \int_{t_1}^{t_2} q|_{x=x_2}\,\mathrm{d}t - \int_{t_1}^{t_2} q|_{x=x_1}\,\mathrm{d}t. \tag{2.2}$$

In particular, if $q|_{x=x_i} \equiv 0$ for $i = 1$ and 2, then the value $\int_{x_1}^{x_2} p\,\mathrm{d}x$ is a constant independent of t.

A conservation law (p, q) is called trivial, if p and q both vanish on any smooth solution of (1.2), or if $\mathrm{D}_t p + \mathrm{D}_x q$ itself (not the restriction) always vanishes, or if (p, q) is a linear combination of the two cases above. Two conservation laws (p, q) and (p', q') are called equivalent if $(p - p', q - q')$ is trivial. In most applications, we are only interested in studying conservation laws up to equivalence. The following theorem provides all the equivalence classes of conservation laws of (1.2). We put the proof in Appendix C.

Theorem 2.5. (1) *When $\delta \neq \beta$, Eq* (1.2) *has only a trivial conservation law. That is, an arbitrary conservation law of* (1.2) *has an equivalent conservation law whose conserved density p and flux q are both* 0.

(2) *When $\delta = \beta$, we write*

$$p_0 = x^{\beta-1}u, \quad q_0 = -t^{\alpha-1}\Big(x^{1-\gamma}u_x + \frac{1}{2}u^2\Big).$$

Then the pair (p_0, q_0) *forms the conserved density and flux of a conservation law of* (1.2), *and every conservation law of* (1.2) *has an equivalent conservation law whose conserved density is* cp_0 *and flux is* c_0q *for some constant* c_0.

Remark 2.6. To consider the physical meaning of Theorem 2.5, we assume that $\delta = \beta$ in this remark, and let $x_1 > x_0 > 0$ and $t_1 > t_0 > 0$. For a smooth function $u(x,t)$ defined on a neighborhood W of $[x_0, x_1] \times [t_0, t_1]$, we regard $u(x,t)$ as the velocity function of some one-dimensional fluid. We call

$$M_\beta(t) = \int_{x_0}^{x_1} x^{\beta-1}u(x,t)\,\mathrm{d}x$$

the β-order conformable momentum of the fluid on $[x_0, x_1]$ at time t, which generalizes the classical momentum in the framework of conformable calculus. Then for any smooth solution $u(x,t)$ of (1.2), we deduce from (2.2) and Theorem 2.5 that

$$\begin{aligned} M_\beta(t) - M_\beta(t_0) &= -\int_{t_0}^{t} s^{\alpha-1}\Big[x_2^{1-\gamma}u_x(x_2,s) + \frac{1}{2}u(x_2,s)^2\Big]\mathrm{d}s + \int_{t_0}^{t} s^{\alpha-1}\Big[x_1^{1-\gamma}u_x(x_1,s) + \frac{1}{2}u(x_1,s)^2\Big]\mathrm{d}s \\ &= \int_{t_0}^{t} s^{\alpha-1}\Big[\,\mathrm{D}_x^\gamma u(x_1,s) - \mathrm{D}_x^\gamma u(x_2,s)\Big]\mathrm{d}s + \int_{t_0}^{t} s^{\alpha-1}\Big[\frac{1}{2}u(x_1,s)^2 - \frac{1}{2}u(x_2,s)^2\Big]\mathrm{d}s, \end{aligned}$$

for any $t \in [t_0, t_1]$. In particular, if

$$\mathrm{D}_x^\gamma u(x_1,t) = \mathrm{D}_x^\gamma u(x_2,t) = u(x_1,t) = u(x_2,t) = 0, \quad \forall t \in [t_0, t_1],$$

then the β-order conformable momentum M_β is a constant, independent of t. When $\alpha = \beta = \gamma = \delta = 1$, this is just the conservation of momentum of the integer-order Burgers equation. That is, the rate of change of the total momentum on the interval $[x_0, x_1]$ is determined solely by the difference in the total flux at the boundaries, and the total flux is the sum of the convective (inertial) momentum flux $u^2/2$ and the diffusive (viscous) momentum flux u_x across the boundaries. For general values of α, β, and $\gamma = \delta$, the original convective and diffusive fluxes both need to be multiplied by an additional factor $t^{\alpha-1}$; moreover, the diffusive momentum flux u_x should be replaced with $\mathrm{D}_x^\gamma u$. In particular, assume that the velocity field u of a certain hypothetical one-dimensional fluid satisfies the following spatial conformable Burgers equation:

$$\frac{\partial u}{\partial t} = \mathrm{D}_x^\beta\,\mathrm{D}_x^\gamma u + u\,\mathrm{D}_x^\delta u,$$

and then the conformable momentum density of this hypothetical fluid involves both the velocity field u and a spatial weight $x^{\beta-1}$, which reflects the influence of the conformable structure on the momentum distribution; and the diffusive momentum flux of this hypothetical fluid is determined by the conformable acceleration $\mathrm{D}_x u$ (not only the normal acceleration u_x like ordinary fluid). On the other hand, the convective flux of this hypothetical fluid is exactly the same as ordinary fluid.

3. Group invariant solutions

Most interesting group-invariant solutions of (1.2) occur in the case where $\beta = \gamma = \delta$. Hence, in this section we always assume that $\beta = \gamma = \delta$ and hence all of $v_1, \dots, v_5$ in Theorem 2.1 are well-defined. We calculate the group-invariant solutions corresponding to the optimal system of one-dimensional subalgebras provided in Theorem 2.4. Any other group-invariant solutions can be obtained from these solutions by point transformations. Since a group-invariant solution corresponding to v_3 must be constant, it remains to consider those corresponding to $v_1 \pm v_5$, v_2, $v_3 \pm v_5$, and v_4. Using the global invariants of these infinitesimal Lie symmetries, the conformable fractional Burgers equation can be reduced to several ordinary differential equations. Since these reduced equations are generally nonlinear and complicated, their analytical solutions are obtained with the assistance of symbolic computation using the computer algebra system Mathematica.

3.1. Group-invariant solution of $v_1 + v_5$

In this subsection, we let G be the one-parameter symmetry group of (1.2) generated by the infinitesimal generator $v_1 + v_5$. From the characteristic equation

$$\frac{\mathrm{d}x}{\alpha x t^{\alpha}} = \frac{\mathrm{d}t}{\beta t^{1+\alpha} + t^{1-\alpha}} = \frac{\mathrm{d}u}{-\alpha\left(\alpha x^{\beta} + \beta t^{\alpha} u\right)}$$

of $v_1 + v_5$, we see that G has two functionally independent invariants

$$y = x\left(\alpha + \alpha\beta t^{2\alpha}\right)^{-\frac{1}{2\beta}} \quad \text{and} \quad v = u\sqrt{\alpha\left(\beta t^{2\alpha} + 1\right)} + \alpha^2 t^{\alpha} y^{\beta}.$$

Regarding y, v, and t as independent, dependent, and parametric variables, respectively, we deduce from (2.1) that

$$y^2\frac{\mathrm{d}^2 v}{\mathrm{d}y^2} + y\left(v y^{\beta} + 1 - \beta\right)\frac{\mathrm{d}v}{\mathrm{d}y} + \alpha^4 y^{3\beta} = 0.$$

By using the computer algebra system Mathematica, we see that its general solution is

$$\begin{aligned} v(y) = {} & -\frac{\mathrm{i}\alpha^2 y^{\beta}}{\sqrt{\beta}\left[L\left(-C_1, -\frac{1}{2}, \frac{\mathrm{i}\alpha^2 y^{2\beta}}{2\beta^{3/2}}\right) + C_2 U\left(C_1, \frac{1}{2}, \frac{\mathrm{i}\alpha^2 y^{2\beta}}{2\beta^{3/2}}\right)\right]} \\ & \times\left\{C_2\left[U\left(C_1, \frac{1}{2}, \frac{\mathrm{i}\alpha^2 y^{2\beta}}{2\beta^{3/2}}\right) + 2C_1 U\left(C_1 + 1, \frac{3}{2}, \frac{\mathrm{i}\alpha^2 y^{2\beta}}{2\beta^{3/2}}\right)\right]\right. \\ & \left. + 2L\left(-C_1 - 1, \frac{1}{2}, \frac{\mathrm{i}\alpha^2 y^{2\beta}}{2\beta^{3/2}}\right) + L\left(-C_1, -\frac{1}{2}, \frac{\mathrm{i}\alpha^2 y^{2\beta}}{2\beta^{3/2}}\right)\right\}, \end{aligned} \tag{3.1}$$

where $U(a, b, x)$ denotes the Tricomi confluent hypergeometric function, $L(n, \alpha, x)$ denotes the generalized Laguerre polynomial, and C_1 and C_2 are arbitrary constants. Therefore, the G-invariant solution of (1.2) is given by

$$\begin{aligned} u(x,t) = {} & \frac{-\alpha x^{\beta}}{\sqrt{\beta}(\beta t^{2\alpha}+1)}\left[L\left(-C_1,-\frac{1}{2},w\right)+C_2U\left(C_1,\frac{1}{2},w\right)\right]^{-1} \\ & \times\left\{2\,\mathrm{i}C_1C_2U\left(C_1+1,\frac{3}{2},w\right)+C_2(\sqrt{\beta}t^{\alpha}+\mathrm{i})U\left(C_1,\frac{1}{2},w\right)\right. \\ & \left.+\,2\,\mathrm{i}L\left(-C_1-1,\frac{1}{2},w\right)+(\sqrt{\beta}t^{\alpha}+\mathrm{i})L\left(-C_1,-\frac{1}{2},w\right)\right\}, \end{aligned} \tag{3.2}$$

where

$$w=\frac{\mathrm{i}\alpha x^{2\beta}}{\beta^{3/2}\,(2\beta t^{2\alpha}+2)}.$$

In general, $v(y)$ defined in (3.1) and $u(x,t)$ defined in (3.2) are complex-valued functions. When

$$C_1=\frac{1}{4}+\mathrm{i}C_0,\quad C_2=0,\quad C_0\in\mathbb{R}, \tag{3.3}$$

u and v are real-valued for any $x>0$, $t>0$, and $0<\alpha,\ \beta\le 1$. We cannot find any other conditions for C_1 and C_2 to make u and v real-valued for all possible variables and parameters. In the case of (3.3), we have

$$v(y)=-\frac{\mathrm{i}\alpha^2y^{\beta}}{\sqrt{\beta}}\left(1+\frac{2L\left(-\mathrm{i}C_0-\frac{5}{4},\frac{1}{2},\frac{\mathrm{i}\alpha^2y^{2\beta}}{2\beta^{3/2}}\right)}{L\left(-\mathrm{i}C_0-\frac{1}{4},-\frac{1}{2},\frac{\mathrm{i}\alpha^2y^{2\beta}}{2\beta^{3/2}}\right)}\right), \tag{3.4}$$

and

$$\begin{aligned} u_1(x,t) &= -\frac{\alpha^2x^{\beta}}{\alpha+\alpha\beta t^{2\alpha}}\left\{\frac{\mathrm{i}}{\sqrt{\beta}}+t^{\alpha}+\frac{2\,\mathrm{i}L\left(-\mathrm{i}C_0-\frac{5}{4},\frac{1}{2},w\right)}{\sqrt{\beta}L\left(-\mathrm{i}C_0-\frac{1}{4},-\frac{1}{2},w\right)}\right\} \\ &= -\frac{\alpha^2x^{\beta}}{\alpha+\alpha\beta t^{2\alpha}}\left\{t^{\alpha}+\Re\left(\frac{2\,\mathrm{i}L\left(-\mathrm{i}C_0-\frac{5}{4},\frac{1}{2},w\right)}{\sqrt{\beta}L\left(-\mathrm{i}C_0-\frac{1}{4},-\frac{1}{2},w\right)}\right)\right\}, \end{aligned} \tag{3.5}$$

where $\Re(z)$ denotes the real part of z.

3.2. Group-invariant solution of $v_1 - v_5$

In this subsection, we let G be the one-parameter symmetry group of (1.2) generated by the infinitesimal generator $v_1 - v_5$. From the characteristic equation

$$\frac{\mathrm{d}x}{-\alpha x t^{\alpha}}=\frac{\mathrm{d}t}{t^{1-\alpha}-\beta t^{\alpha+1}}=\frac{\mathrm{d}u}{\alpha\,(\alpha x^{\beta}+\beta t^{\alpha}u)}$$

of $v_1 - v_5$, we see that G has two functionally independent invariants

$$y=x\left(\alpha\left(\beta t^{2\alpha}-1\right)\right)^{-\frac{1}{2\beta}}\quad\text{and}\quad v=\sqrt{\alpha}\left(u\sqrt{\beta t^{2\alpha}-1}+\alpha^{3/2}t^{\alpha}y^{\beta}\right).$$

Regarding y, v, and t as independent, dependent, and parametric variables, respectively, we deduce from (2.1) that

$$y^2\frac{\mathrm{d}^2v}{\mathrm{d}y^2}+y\left(y^{\beta}v-\beta+1\right)\frac{\mathrm{d}v}{\mathrm{d}y}-\alpha^4y^{3\beta}=0.$$

By using the computer algebra system Mathematica, we see that its general solution is

$$
\begin{aligned}
v(y) = &-\frac{\alpha^2 y^\beta}{\sqrt{\beta}\left[L\left(-C_1, -\frac{1}{2}, \frac{\alpha^2 y^{2\beta}}{2\beta^{3/2}}\right) + C_2 U\left(C_1, \frac{1}{2}, \frac{\alpha^2 y^{2\beta}}{2\beta^{3/2}}\right)\right]} \\
&\times \left\{C_2\left[U\left(C_1, \frac{1}{2}, \frac{\alpha^2 y^{2\beta}}{2\beta^{3/2}}\right) + 2C_1 U\left(C_1 + 1, \frac{3}{2}, \frac{\alpha^2 y^{2\beta}}{2\beta^{3/2}}\right)\right]\right. \\
&\left. + 2L\left(-C_1 - 1, \frac{1}{2}, \frac{\alpha^2 y^{2\beta}}{2\beta^{3/2}}\right) + L\left(-C_1, -\frac{1}{2}, \frac{\alpha^2 y^{2\beta}}{2\beta^{3/2}}\right)\right\},
\end{aligned}
\tag{3.6}
$$

where $U(a, b, x)$ denotes the Tricomi confluent hypergeometric function, $L(n, \alpha, x)$ denotes the generalized Laguerre polynomial, and C_1 and C_2 are arbitrary constants. Therefore, the G-invariant solution of (1.2) is given by

$$
\begin{aligned}
u_2(x, t) = &-\frac{\alpha x^\beta}{\sqrt{\beta}\,(\beta t^{2\alpha} - 1)}\left[L\left(-C_1, -\frac{1}{2}, w\right) + C_2 U\left(C_1, \frac{1}{2}, w\right)\right]^{-1} \\
&\times \left\{2C_1 C_2 U\left(C_1 + 1, \frac{3}{2}, w\right) + C_2(\sqrt{\beta} t^\alpha + 1) U\left(C_1, \frac{1}{2}, w\right)\right. \\
&\left. + 2L\left(-C_1 - 1, \frac{1}{2}, w\right) + (\sqrt{\beta} t^\alpha + 1) L\left(-C_1, -\frac{1}{2}, w\right)\right\},
\end{aligned}
\tag{3.7}
$$

where

$$
w = \frac{\alpha x^{2\beta}}{2\beta^{3/2}\,(\beta t^{2\alpha} - 1)}.
$$

In general, $v(y)$ defined in (3.6) and $u_2(x, t)$ defined in (3.7) are complex-valued functions. When

$$
C_1 = -2, -3, -4, \ldots, \qquad \text{and} \qquad C_2 \in \mathbb{R}, \tag{3.8}
$$

u and v are real-valued for any $x > 0$, $t > 0$, and $0 < \alpha, \beta \leq 1$.

3.3. Group-invariant solution of v_2

In this subsection, we let G be the one-parameter symmetry group of (1.2) generated by the infinitesimal generator v_2. From the characteristic equation

$$
\frac{\mathrm{d}x}{\alpha x} = \frac{\mathrm{d}t}{2\beta t} = \frac{\mathrm{d}u}{-\alpha\beta u}
$$

of v_2, we see that G has two functionally independent invariants

$$
y = x t^{-\frac{\alpha}{2\beta}} \quad \text{and} \quad v = t^{\alpha/2} u.
$$

Regarding y, v, and t as independent, dependent, and parametric variables, respectively, we deduce from (2.1) that

$$
2\beta y^2 \frac{\mathrm{d}^2 v}{\mathrm{d}y^2} + y\left(\alpha y^{2\beta} + 2\beta y^\beta v - 2(\beta^2 - \beta)\right)\frac{\mathrm{d}v}{\mathrm{d}y} + \alpha\beta y^{2\beta} v = 0.
$$

By using the computer algebra system Mathematica, we see that its general solution is

$$
\begin{aligned}
v(y) = {} & \frac{1}{\beta\left[C_2 M\left(C_1 - \frac{1}{2}, \frac{1}{2}, -\frac{y^{2\beta}\alpha}{4\beta^2}\right) + 3\sqrt{\alpha}y^{\beta} M\left(C_1, \frac{3}{2}, -\frac{y^{2\beta}\alpha}{4\beta^2}\right)\right]} \\
& \times \left\{\alpha y^{\beta}\left[(1-2C_1)C_2 M\left(C_1 + \frac{1}{2}, \frac{3}{2}, -\frac{\alpha y^{2\beta}}{4\beta^2}\right) - 2\sqrt{\alpha}C_1 y^{\beta} M\left(C_1 + 1, \frac{5}{2}, -\frac{\alpha y^{2\beta}}{4\beta^2}\right)\right]\right. \\
& \left. + 6\sqrt{\alpha}\beta^2 M\left(C_1, \frac{3}{2}, -\frac{\alpha y^{2\beta}}{4\beta^2}\right)\right\},
\end{aligned}
\tag{3.9}
$$

where $M(a, b, x)$ denotes the Kummer confluent hypergeometric function, and C_1 and C_2 are arbitrary constants. Therefore, the G-invariant solution of (1.2) is given by

$$
\begin{aligned}
u_3(x, t) = {} & \frac{t^{-\alpha}}{\beta}\left[C_2 t^{\alpha/2} M\left(C_1 - \frac{1}{2}, \frac{1}{2}, w\right) + 3\sqrt{\alpha}x^{\beta} M\left(C_1, \frac{3}{2}, w\right)\right]^{-1} \\
& \times \left\{\alpha x^{\beta}\left[(1-2C_1)C_2 t^{\alpha/2} M\left(C_1 + \frac{1}{2}, \frac{3}{2}, w\right) - 2C_1\sqrt{\alpha}x^{\beta} M\left(C_1 + 1, \frac{5}{2}, w\right)\right]\right. \\
& \left. + 6\sqrt{\alpha}\beta^2 t^{\alpha} M\left(C_1, \frac{3}{2}, w\right)\right\},
\end{aligned}
\tag{3.10}
$$

where

$$
w = -\frac{\alpha x^{2\beta}}{4\beta^2 t^{\alpha}}.
$$

When $C_1, C_2 \in \mathbb{R}$, $v(y)$ defined in (3.9) and $u_3(x, t)$ defined in (3.10) are real-valued functions.

3.4. Group-invariant solution of $v_3 + v_5$

In this subsection, we let G be the one-parameter symmetry group of (1.2) generated by the infinitesimal generator $v_3 + v_5$. From the characteristic equation of $v_3 + v_5$, we see that G has two functionally independent invariants

$$
y = \frac{2\alpha x^{\beta} t^{\alpha} + 1}{2\alpha t^{2\alpha}} \quad \text{and} \quad v = \frac{2\beta u t^{2\alpha} + 2\alpha y t^{2\alpha} + 1}{2\beta t^{\alpha}}.
$$

Regarding y, v, and t as independent, dependent, and parametric variables, respectively, we deduce from (2.1) that

$$
\beta^3 \frac{\mathrm{d}^2 v}{\mathrm{d}y^2} + \beta^2 v \frac{\mathrm{d}v}{\mathrm{d}y} - \alpha = 0.
$$

By using the computer algebra system Mathematica, we see that its general solution is

$$
v(y) = \frac{\sqrt[3]{\frac{4\alpha}{\beta}}\left\{C_2\,\mathrm{Ai}'\left(\frac{y\alpha - C_1/2}{\sqrt[3]{2}\alpha^{2/3}\beta^{4/3}}\right) + \mathrm{Bi}'\left(\frac{y\alpha - C_1/2}{\sqrt[3]{2}\alpha^{2/3}\beta^{4/3}}\right)\right\}}{C_2\,\mathrm{Ai}\left(\frac{y\alpha - C_1/2}{\sqrt[3]{2}\alpha^{2/3}\beta^{4/3}}\right) + \mathrm{Bi}\left(\frac{y\alpha - C_1/2}{\sqrt[3]{2}\alpha^{2/3}\beta^{4/3}}\right)},
\tag{3.11}
$$

where $\mathrm{Ai}(z)$ and $\mathrm{Bi}(z)$ denote the Airy functions, $\mathrm{Ai}'(z)$ and $\mathrm{Bi}'(z)$ denote their derivatives, and C_1 and C_2 are arbitrary constants. Therefore, the G-invariant solution of (1.2) is given by

$$\begin{aligned} u_4(x,t) = \frac{t^{-2\alpha}}{\beta\,(C_2\,\mathrm{Ai}(w)+\mathrm{Bi}(w))} \times \Big\{ & C_2\left[\sqrt[3]{4\alpha\beta^2}t^{\alpha}\,\mathrm{Ai}'(w) - \left(\alpha t^{\alpha}x^{\beta}+1\right)\mathrm{Ai}(w)\right] \\ & - (\alpha t^{\alpha}x^{\beta}+1)\,\mathrm{Bi}(w) + \sqrt[3]{4\alpha\beta^2}t^{\alpha}\,\mathrm{Bi}'(w)\Big\}, \end{aligned} \tag{3.12}$$

where

$$w = \frac{2\alpha x^{\beta}t^{\alpha} - C_1 t^{2\alpha} + 1}{2\sqrt[3]{2\alpha^2\beta^4}t^{2\alpha}}.$$

When $C_1, C_2 \in \mathbb{R}$, $v(y)$ defined in (3.9) and $u_3(x,t)$ defined in (3.10) are real-valued functions.

3.5. *Group-invariant solution of* $v_3 - v_5$

In this subsection, we let G be the one-parameter symmetry group of (1.2) generated by the infinitesimal generator $v_3 - v_5$. From the characteristic equation of $v_3 - v_5$, we see that G has two functionally independent invariants

$$y = \frac{2\alpha x^{\beta}t^{\alpha} - 1}{2\alpha t^{2\alpha}} \quad \text{and} \quad v = \frac{2\beta u t^{2\alpha} + 2\alpha y t^{2\alpha} - 1}{2\beta t^{\alpha}}.$$

Regarding y, v, and t as independent, dependent, and parametric variables, respectively, we deduce from (2.1) that

$$\beta^3\frac{\mathrm{d}^2 v}{\mathrm{d}y^2} + \beta^2 v\frac{\mathrm{d}v}{\mathrm{d}y} + \alpha = 0.$$

By using the computer algebra system Mathematica, we see that its general solution is

$$v(y) = \frac{\theta\sqrt[3]{\frac{4\alpha}{\beta}}\left\{C_2\,\mathrm{Ai}'\left(\frac{\theta(y\alpha+C_1/2)}{\sqrt[3]{2}\alpha^{2/3}\beta^{4/3}}\right) + \mathrm{Bi}'\left(\frac{\theta(y\alpha+C_1/2)}{\sqrt[3]{2}\alpha^{2/3}\beta^{4/3}}\right)\right\}}{C_2\,\mathrm{Ai}\left(\frac{\theta(y\alpha+C_1/2)}{\sqrt[3]{2}\alpha^{2/3}\beta^{4/3}}\right) + \mathrm{Bi}\left(\frac{\theta(y\alpha+C_1/2)}{\sqrt[3]{2}\alpha^{2/3}\beta^{4/3}}\right)}, \qquad \theta = \exp(\mathrm{i}\pi/3), \tag{3.13}$$

where $\mathrm{Ai}(z)$ and $\mathrm{Bi}(z)$ denote the Airy functions, $\mathrm{Ai}'(z)$ and $\mathrm{Bi}'(z)$ denote their derivatives, and C_1 and C_2 are arbitrary constants. Therefore, the G-invariant solution of (1.2) is given by

$$\begin{aligned} u(x,t) = \frac{t^{-2\alpha}}{\beta\,(C_2\,\mathrm{Ai}(w)+\mathrm{Bi}(w))} \times \Big\{ & C_2\left[\theta\sqrt[3]{4\alpha\beta^2}t^{\alpha}\,\mathrm{Ai}'(w) - \left(\alpha t^{\alpha}x^{\beta}-1\right)\mathrm{Ai}(w)\right] \\ & - (\alpha t^{\alpha}x^{\beta}-1)\,\mathrm{Bi}(w) + \theta\sqrt[3]{4\alpha\beta^2}t^{\alpha}\,\mathrm{Bi}'(w)\Big\}, \end{aligned} \tag{3.14}$$

where

$$w = \frac{\theta\left(2\alpha x^{\beta}t^{\alpha} + C_1 t^{2\alpha} - 1\right)}{2\sqrt[3]{2\alpha^2\beta^4}t^{2\alpha}}, \qquad \theta = \exp(\mathrm{i}\pi/3).$$

In general, $v(y)$ defined in (3.13) and $u(x,t)$ defined in (3.14) are complex-valued functions. When

$$C_1 \in \mathbb{R}, \quad C_2 \in \{\pm\sqrt{3}, \mathrm{i}, -3\,\mathrm{i}\}, \tag{3.15}$$

u and v are real-valued for any $x > 0$, $t > 0$, and $0 < \alpha,\beta \leq 1$. In the case of (3.15), we have

$$\begin{aligned}
v(y) &= -\frac{(2\alpha y + C_1)^2 F\left(\frac{5}{3}, -\frac{(2y\alpha+C_1)^3}{144\alpha^2\beta^4}\right)}{8\alpha\beta^3 F\left(\frac{2}{3}, -\frac{(2y\alpha+C_1)^3}{144\alpha^2\beta^4}\right)}, && \text{when } C_2 = \sqrt{3},\\
v(y) &= \frac{4\alpha\beta F\left(\frac{1}{3}, -\frac{(2y\alpha+C_1)^3}{144\alpha^2\beta^4}\right)}{(2\alpha y + C_1) F\left(\frac{4}{3}, -\frac{(2y\alpha+C_1)^3}{144\alpha^2\beta^4}\right)}, && \text{when } C_2 = -\sqrt{3},\\
v(y) &= -\frac{2^{2/3}\sqrt[3]{\frac{\alpha}{\beta}}\,\mathrm{Ai}'\left(-\frac{2y\alpha+C_1}{2\sqrt[3]{2}\alpha^{2/3}\beta^{4/3}}\right)}{\mathrm{Ai}\left(-\frac{2y\alpha+C_1}{2\sqrt[3]{2}\alpha^{2/3}\beta^{4/3}}\right)}, && \text{when } C_2 = \mathrm{i},\\
v(y) &= -\frac{2^{2/3}\sqrt[3]{\frac{\alpha}{\beta}}\,\mathrm{Bi}'\left(-\frac{2y\alpha+C_1}{2\sqrt[3]{2}\alpha^{2/3}\beta^{4/3}}\right)}{\mathrm{Bi}\left(-\frac{2y\alpha+C_1}{2\sqrt[3]{2}\alpha^{2/3}\beta^{4/3}}\right)}, && \text{when } C_2 = -3\,\mathrm{i},
\end{aligned} \tag{3.16}$$

and

$$\begin{aligned}
u_5(x,t) &= \frac{t^{-2\alpha}\left(1 - \alpha t^{\alpha} x^{\beta}\right)}{\beta} - \frac{t^{-5\alpha}\left(C_1 t^{2\alpha} + 2\alpha t^{\alpha} x^{\beta} - 1\right)^2 F\left(\frac{5}{3}, w_1\right)}{8\alpha\beta^3 F\left(\frac{2}{3}, w_1\right)}, && \text{when } C_2 = \sqrt{3},\\
u_6(x,t) &= \beta^{-1} t^{-2\alpha}\left[\frac{4\alpha\beta^2 t^{3\alpha} F\left(\frac{1}{3}, w_1\right)}{\left(C_1 t^{2\alpha} + 2\alpha x^{\beta} t^{\alpha} - 1\right) F\left(\frac{4}{3}, w_1\right)} - \alpha x^{\beta} t^{\alpha} + 1\right], && \text{when } C_2 = -\sqrt{3},\\
u_7(x,t) &= \beta^{-1} t^{-2\alpha}\left[1 - \alpha x^{\beta} t^{\alpha} - \frac{\sqrt[3]{4\alpha\beta^2} t^{\alpha}\,\mathrm{Ai}'(w_2)}{\mathrm{Ai}(w_2)}\right], && \text{when } C_2 = \mathrm{i},\\
u_8(x,t) &= \beta^{-1} t^{-2\alpha}\left[1 - \alpha x^{\beta} t^{\alpha} - \frac{\sqrt[3]{4\alpha\beta^2} t^{\alpha}\,\mathrm{Bi}'(w_2)}{\mathrm{Bi}(w_2)}\right], && \text{when } C_2 = -3\,\mathrm{i},
\end{aligned} \tag{3.17}$$

where $F(a,z) = {}_0F_1(a,z)$ denotes the confluent hypergeometric limit function, and

$$w_1 = -\frac{\left(\left(2t^{\alpha} x^{\beta}\alpha - 1\right) t^{-2\alpha} + C_1\right)^3}{144\alpha^2\beta^4}, \qquad w_2 = -\frac{\left(2t^{\alpha} x^{\beta}\alpha - 1\right) t^{-2\alpha} + C_1}{2\sqrt[3]{2}\alpha^{2/3}\beta^{4/3}}.$$

4. Conclusions

In this work, we investigated the Lie symmetry and conservation laws of the conformable fractional Burgers equation. By applying the classical Lie symmetry method to the conformable fractional framework, the infinitesimal generators of the symmetry group were derived and the corresponding Lie algebra was obtained. Furthermore, an optimal system of one-dimensional subalgebras was constructed, which enabled us to perform systematic symmetry reductions of the equation. Based on these reductions, several group invariant solutions were obtained under the condition that the three spatial derivative orders are consistent.

In addition, conservation laws associated with the conformable fractional Burgers equation were derived. These conservation laws provide further insight into the intrinsic structure of the equation and help reveal the physical properties of the fluid system described by the model. This allows us to define a conformable momentum for hypothetical fluids satisfying the equation, highlighting potential applications in modeling anomalous diffusion or viscous flows.

The present study demonstrates that Lie symmetry analysis remains an effective tool for investigating conformable fractional differential equations. The approach used here can also be applied to other nonlinear conformable fractional models arising in applied mathematics and mathematical physics. In future work, it would be interesting to explore more general fractional Burgers-type equations and investigate their symmetry structures, exact solutions, and potential physical applications.

Author contributions

Zhihan Liu: Formal analysis, writing – original draft; Qi Wang: Conceptualization, methodology, project administration, supervision, writing – review & editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflicts of interest

All authors declare no conflicts of interest in this paper.

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A. Proof of Theorem 2.1

In this section, we calculate the Lie algebra of infinitesimal symmetries for Eq (1.2). We refer the reader to [8] or [9] for the principle of the calculation. Let X and U denote $\mathbb{R}^2$ and $\mathbb{R}^1$ with coordinate system (x,t) and (u), respectively. For an open subset $M \subseteq X \times U$, we write $M^{(n)}$ for the jet space of M with order n. By Lemma 1.2, the corresponding algebraic equation of (1.2) in the jet space $M^{(2)}$ is given by

$$\Delta := t^{1-\alpha}u_t - (1-\gamma)x^{1-\beta-\gamma}u_x - x^{2-\beta-\gamma}u_{xx} - x^{1-\delta}uu_x = 0, \tag{A.1}$$

namely,

$$u_t = t^{\alpha-1}\Big[(1-\gamma)x^{1-\beta-\gamma}u_x + x^{2-\beta-\gamma}u_{xx} + x^{1-\delta}uu_x\Big]. \tag{A.2}$$

Let v be a smooth vector field on M with the following form:

$$v = X(x,t,u)\frac{\partial}{\partial x} + T(x,t,u)\frac{\partial}{\partial t} + U(x,t,u)\frac{\partial}{\partial u}. \tag{A.3}$$

Then its prolongation of order 2 has the following form:

$$\mathrm{pr}^{(2)}\, v = v + U^x\frac{\partial}{\partial u_x} + U^t\frac{\partial}{\partial u_t} + U^{xx}\frac{\partial}{\partial u_{xx}} + U^{xt}\frac{\partial}{\partial u_{xt}} + U^{tt}\frac{\partial}{\partial u_{tt}}. \tag{A.4}$$

If the one-parameter transformation group G_v generated by v is a symmetry group of (1.2), then

$$\mathrm{pr}^{(2)}\, v(\Delta) = X\frac{\partial\Delta}{\partial x} + T\frac{\partial\Delta}{\partial t} + U\frac{\partial\Delta}{\partial u} + U^x\frac{\partial\Delta}{\partial u_x} + U^t\frac{\partial\Delta}{\partial u_t} + U^{xx}\frac{\partial\Delta}{\partial u_{xx}} \tag{A.5}$$

is identically zero on the hypersurface defined by $\Delta = 0$ in $M^{(2)}$. It is well-known that the functions U^x, U^t, and U^{xx} in (A.4) and (A.5) are given by

$$\begin{aligned}
U^x &= U_x + (-X_x + U_u)u_x - T_x u_t - X_u u_x^2 - T_u u_x u_t,\\
U^t &= U_t - X_t u_x + (-T_t + U_u)u_t - X_u u_x u_t - T_u u_t^2,\\
U^{xx} &= U_{xx} + (-X_{xx} + 2U_{xu})u_x - T_{xx}u_t\\
&\quad + (-2X_{xu} + U_{uu})u_x^2 - 2T_{xu}u_x u_t - X_{uu}u_x^3 - T_{uu}u_x^2 u_t\\
&\quad + (-2X_x + U_u)u_{xx} - 2T_x u_{xt} - 3X_u u_x u_{xx} - 2T_u u_x u_{xt} - T_u u_t u_{xx}.
\end{aligned} \tag{A.6}$$

Step 1. Substituting (A.6) into (A.5) and replacing u_{xx} by (A.2), the restriction of $\mathrm{pr}^{(2)}\, v(\Delta)$ on the hypersurface (A.1) of $M^{(2)}$ can be regarded as a polynomial on u_x, u_{xx}, and u_{xxx} with coefficients of functions on x, t, and u. The coefficients of $u_x u_{xxx}$ and u_{xxx} are

$$2t^{\alpha-1}x^{2(2-\beta-\gamma)}\frac{\partial T}{\partial x} \quad \text{and} \quad 2t^{\alpha-1}x^{2(2-\beta-\gamma)}\frac{\partial T}{\partial u},$$

respectively. It follows that T is a one-variable function of t, i.e.,

$$T = T(t). \tag{A.7}$$

Then the coefficient of $u_x u_{xx}$ is

$$2x^{-\beta-\gamma+2}\frac{\partial X}{\partial u}.$$

It follows that X is a two-variable function of x and t, i.e.,

$$X = X(x,t). \tag{A.8}$$

Then the coefficient of u_x^2 in $\mathrm{pr}^{(2)}\, v(\Delta)$ becomes

$$-x^{-\beta-\gamma+2}\frac{\partial^2 U}{\partial u^2}.$$

Thus, U has the following form:

$$U = A(x,t)u + B(x,t) \tag{A.9}$$

for some undetermined functions $A(x,t)$ and $B(x,t)$.

Step 2. Now the restriction of $\mathrm{pr}^{(2)}\, v(\Delta)$ on $\Delta = 0$ can be regarded as a polynomial on u, u_x, and u_{xx} with coefficients of functions on x and t. Then the coefficient of u^2 is

$$-x^{1-\delta}\frac{\partial A}{\partial x}.$$

Thus, A has the following form:

$$A = A(t). \tag{A.10}$$

Then the coefficient of u is

$$t^{1-\alpha}\frac{\mathrm{d}A}{\mathrm{d}t} - x^{1-\delta}\frac{\partial B}{\partial x}.$$

Thus, B has the following form:

$$B = \delta^{-1} x^{\delta} t^{1-\alpha} \frac{\mathrm{d}A}{\mathrm{d}t} + F(t), \tag{A.11}$$

for some undetermined function $F(t)$. Then the coefficient of u_{xx} is

$$\frac{x^{-\beta-\gamma+1}}{t}\left[x\left(2t\frac{\partial X}{\partial x} - t\frac{\mathrm{d}T}{\mathrm{d}t} + (1-\alpha)T\right) + t(\beta+\gamma-2)X\right].$$

Thus, X has the following form:

$$X = \frac{x}{t(\beta+\gamma)}\left[(\alpha-1)T(t) + t\left(\frac{\mathrm{d}T}{\mathrm{d}t} + (\beta+\gamma)x^{\frac{1}{2}(-\beta-\gamma)}G(t)\right)\right] \tag{A.12}$$

for some undetermined function $G(t)$. Then the coefficient of uu_x is

$$-\frac{x^{\frac{1}{2}(-\beta-\gamma-2\delta+2)}}{2t(\beta+\gamma)}\left\{(\beta+\gamma)(\beta+\gamma-2\delta)tG + 2(\beta+\gamma)x^{\frac{\beta+\gamma}{2}}tA + 2(\beta+\gamma-\delta)x^{\frac{\beta+\gamma}{2}}\left(t\frac{\mathrm{d}T}{\mathrm{d}t} + (\alpha-1)T\right)\right\}.$$

Case 1: $\beta + \gamma \neq 2\delta$.

In this case, we see from the coefficient of uu_x that

$$G = 0, \quad A = -\frac{(\beta+\gamma-\delta)}{(\beta+\gamma)t}\left(t\frac{\mathrm{d}T}{\mathrm{d}t} + (\alpha-1)T\right). \tag{A.13}$$

Then the coefficient of u_x is

$$x\left\{\frac{t^{-\alpha-1}(\beta+\gamma-2\delta)}{\delta(\beta+\gamma)}\left[t^2\frac{\mathrm{d}^2T}{\mathrm{d}t^2} - (\alpha-1)\left(T - t\frac{\mathrm{d}T}{\mathrm{d}t}\right)\right] - F(t)x^{-\delta}\right\}.$$

Then we obtain

$$F = 0, \qquad t^2\frac{\mathrm{d}^2T}{\mathrm{d}t^2} - (\alpha-1)\left(T(t) - t\frac{\mathrm{d}T}{\mathrm{d}t}\right) = 0 \quad \Rightarrow \quad T = c_1 t + c_2 t^{1-\alpha}, \tag{A.14}$$

and hence

$$X = \frac{c_1\alpha x}{\beta+\gamma}, \quad U = \frac{c_1\alpha\,(\delta-\beta-\gamma)\,u}{\beta+\gamma}. \tag{A.15}$$

Case 2: $\beta + \gamma = 2\delta$.

In this case, we see from the coefficient of uu_x that

$$A = -\frac{1}{2t}\left(t\frac{\mathrm{d}T}{\mathrm{d}t} + (\alpha-1)T\right). \tag{A.16}$$

Then the coefficient of u is

$$\frac{1}{4}t^{-\alpha}x^{-\frac{3\beta}{2}-\frac{3\gamma}{2}+1}\left[(\beta^2-\gamma^2)t^{\alpha}G - 4x^{\beta+\gamma}\left(t^{\alpha}F + t\frac{\mathrm{d}G}{\mathrm{d}t}\right)\right].$$

Thus, we obtain

$$(\beta-\gamma)G = 0, \quad F = -t^{1-\alpha}\frac{\mathrm{d}G}{\mathrm{d}t}. \tag{A.17}$$

Case 2.1: $\beta + \gamma = 2\delta$ and $\beta \neq \gamma$.

In this case, we see from the coefficient of u that

$$F = G = 0. \tag{A.18}$$

Then the restriction of $\mathrm{pr}^{(2)}\, v(\Delta)$ on $\Delta = 0$ becomes

$$\frac{t^{-2\alpha-1} x^{\frac{1}{2}(-\beta-\gamma)}}{4(\beta+\gamma)} \left\{ (\gamma^2 - \beta^2) t^\alpha \left[(\alpha - 1)\left(T - t\frac{\mathrm{d}T}{\mathrm{d}t}\right) - t^2 \frac{\mathrm{d}^2 T}{\mathrm{d}t^2} \right] - 4x^{\beta+\gamma} \left[t^3 \frac{\mathrm{d}^3 T}{\mathrm{d}t^3} + \left(\alpha^2 - 1\right)\left(T - t\frac{\mathrm{d}T}{\mathrm{d}t}\right) \right] \right\}.$$

Thus, we obtain

$$(\alpha - 1)\left(T - t\frac{\mathrm{d}T}{\mathrm{d}t}\right) - t^2 \frac{\mathrm{d}^2 T}{\mathrm{d}t^2} = 0 \quad \Rightarrow \quad T = c_1 t + c_2 t^{1-\alpha}, \tag{A.19}$$

and hence

$$X = \frac{c_1 \alpha x}{\beta + \gamma}, \quad U = -\frac{c_1 \alpha u}{2}. \tag{A.20}$$

Case 2.2: $\beta = \gamma = \delta$.

In this case, the restriction of $\mathrm{pr}^{(2)}\, v(\Delta)$ on $\Delta = 0$ becomes

$$\frac{t^{-2\alpha-1}}{2\beta} \left\{ x^\beta \left[-t^3 \frac{\mathrm{d}^3 T}{\mathrm{d}t^3} - \left(\alpha^2 - 1\right)\left(T(t) - t\frac{\mathrm{d}T}{\mathrm{d}t}\right) \right] - 2\beta t^2 \left(t\frac{\mathrm{d}^2 G}{\mathrm{d}t^2} - (\alpha - 1)\frac{\mathrm{d}G}{\mathrm{d}t} \right) \right\}.$$

Thus, we obtain

$$T = c_1 t + c_2 t^{1-\alpha} + c_3 t^{1+\alpha}, \quad G = c_4 t^\alpha + c_5, \tag{A.21}$$

and hence

$$\begin{aligned} X &= \frac{\frac{\alpha c_1 x}{2} + \alpha c_3 t^\alpha x + \beta x^{1-\beta}\left(c_4 t^\alpha + c_5\right)}{\beta}, \\ U &= -\frac{\alpha^2 c_3 x^\beta}{\beta} - \frac{\alpha c_1 u}{2} - \alpha c_3 t^\alpha u - \alpha c_4. \end{aligned} \tag{A.22}$$

Then we complete the proof of Theorem 2.1.

B. Proof of Theorem 2.4

Case 1: At least two of β, γ, δ are distinct.

We write $u = a_1 v_1 + a_2 v_2$ for a nonzero vector of $\mathfrak{g}$. In particular, at least one of a_1 and a_2 is nonzero. When $a_2 \neq 0$, by taking $\varepsilon_1 = a_1/(\alpha(\beta+\gamma)a_2)$, we have

$$\mathrm{Ad}_{\varepsilon_1 v_1} u = a_2 v_2.$$

That is, by adjoint maps and scale multiplications, we can transform u with $a_2 \neq 0$ to v_2. When $a_2 = 0$, we must have $a_1 \neq 0$, so by a scale multiplication, we can transform u with $a_2 = 0$ to v_1.

Case 2: $\beta = \gamma = \delta$.

We write $u = a_1 v_1 + \cdots + a_5 v_5$ for a nonzero vector of $\mathfrak{g}$. In particular, at least one of $a_1, \ldots, a_5$ is nonzero. We see that $\Delta = a_1 a_5 - \beta a_2^2$ is invariant under the adjoint maps of $\mathfrak{g}$.

Case 2.1: $\Delta < 0$.

We see that

$$\mathrm{Ad}_{\varepsilon v_1} u = (\alpha^2 a_5 \beta \varepsilon^2 - 2\alpha a_2 \beta \varepsilon + a_1)v_1 + (a_2 - \alpha a_5 \varepsilon)v_2 + w_{3,4,5},$$

where $w_{3,4,5}$ denotes a vector spanned by $\{v_3, v_4, v_5\}$, and similar notations will be used in the rest of this section. Since $\Delta < 0$, there are two distinct ε's vanishing the coefficient of v_1 and keeping the coefficient of v_2 nonzero. Thus, by an adjoint map and a scale multiplication, we may assume that $a_1 = 0$ and $a_2 = 1$ in u. Then we calculate that

$$\mathrm{Ad}_{\varepsilon_3 v_3} \mathrm{Ad}_{\varepsilon_4 v_4} \mathrm{Ad}_{\varepsilon_5 v_5} u = v_2, \qquad \varepsilon_3 = \frac{a_3}{\alpha\beta}, \quad \varepsilon_4 = -\frac{a_4}{\alpha\beta}, \quad \varepsilon_5 = -\frac{a_5}{2\alpha\beta}.$$

Therefore, by adjoint maps and scale multiplications, we can transform u with $\Delta < 0$ to v_2.

Case 2.2: $\Delta = 0$ **and** $a_1 = a_2 = a_5 = 0$**.**

Then we have

$$\mathrm{Ad}_{\varepsilon_1 v_1} u = a_4 v_4, \quad \varepsilon_1 = \frac{a_3}{\alpha a_4}, \quad \text{when } a_4 \neq 0,$$
$$\mathrm{Ad}_{\varepsilon_5 v_5} u = a_3 v_3, \quad \varepsilon_5 = -\frac{a_4}{\alpha\beta a_3}, \quad \text{when } a_3 \neq 0.$$

Therefore, by adjoint maps and scale multiplications, we can transform u with $\Delta = 0$ and $a_1 = a_2 = a_5 = 0$ to v_3 or v_4.

Case 2.3: $\Delta = 0$ **and** $(a_1, a_2, a_5) \neq (0, 0, 0)$**.**

Since

$$\mathrm{Ad}_{\varepsilon v_1} u = (\alpha^2 a_1 \beta \varepsilon^2 + 2\alpha a_2 \beta \varepsilon + a_5)v_5 + w_{1,2,3,4},$$

we may always assume that $a_5 \neq 0$ in this case. Then by a similar discussion as in Case 2.1, we may assume that $a_1 = a_2 = 0$ in u. Then we have

$$\mathrm{Ad}_{\varepsilon v_3} u = a_3 v_3 + a_5 v_5, \quad \varepsilon = \frac{a_4}{\alpha\beta a_5}.$$

Thus, by an adjoint map and a scale multiplication, we may assume that $a_1 = a_2 = a_4 = 0$ and $a_5 = 1$ in u. When $a_3 = 0$, we have $u = v_5$; and when $a_3 \neq 0$, we have

$$\mathrm{Ad}_{\varepsilon v_2} u = \sqrt[3]{a_3^2}(\pm v_3 + v_5), \quad \varepsilon = -\frac{\ln |a_3|}{3\alpha\beta}.$$

Therefore, by adjoint maps and scale multiplications, we can transform u with $\Delta = 0$ and $(a_1, a_2, a_5) \neq (0, 0, 0)$ to v_5 or $v_3 \pm v_5$.

Case 2.4: $\Delta > 0$**.**

In this case, we must have $a_1 a_5 > 0$. Then we calculate that

$$\mathrm{Ad}_{\varepsilon_3 v_3} \mathrm{Ad}_{\varepsilon_4 v_4} \mathrm{Ad}_{\varepsilon_1 v_1} u = \frac{\Delta}{a_5} v_1 + a_5 v_5, \qquad \varepsilon_1 = \frac{a_2}{\alpha a_5}, \quad \varepsilon_3 = \frac{a_4}{\alpha a_5 \beta}, \quad \varepsilon_4 = \frac{a_2 a_4 - a_3 a_5}{\alpha\Delta}.$$

Thus, by adjoint maps and scale multiplication, we may assume that $a_1 = 1$, $a_5 \neq 0$, and $a_2 = a_3 = a_4 = 0$ in u. Then we have

$$\mathrm{Ad}_{\varepsilon v_2} u = \sqrt{|a_5|}(v_1 \pm v_5), \quad \varepsilon = \frac{1}{4\alpha\beta} \ln |a_5|.$$

Therefore, by adjoint maps and scale multiplications, we can transform u with $\Delta > 0$ to $v_1 \pm v_5$.

C. Proof of Theorem 2.5

In this section, we write u_k and $u_{k,t}$ for the variables in the jet space corresponding to the partial derivatives $\frac{\partial^k u}{\partial x^k}$ and $\frac{\partial^{k+1} u}{\partial x^k \partial t}$, respectively. As mentioned at the beginning of §2, a divergence expression is a conservation law of (1.2) if and only if it is that of (2.1).

First, we see that the Fréchet derivative of Δ is given by

$$\mathbb{D}_\Delta = t^{1-\alpha}\,\mathrm{D}_t - (x^{1-\delta}u + (1-\gamma)x^{1-\beta-\gamma})\,\mathrm{D}_x - x^{2-\beta-\gamma}\,\mathrm{D}_x^2 - x^{1-\delta}u_x,$$

where D_x and D_t denote the total derivatives on x and t, respectively, as we mentioned above. We calculate that the adjoint operator $\mathbb{D}_\Delta^*$ of $\mathbb{D}_\Delta$ is given by

$$\begin{aligned}\mathbb{D}_\Delta^* =& -(1-\alpha)t^{-\alpha} - t^{1-\alpha}\,\mathrm{D}_t + ((1-\delta)x^{-\delta}u + x^{1-\delta}u_x + (1-\gamma)(1-\beta-\gamma)x^{-\beta-\gamma})\\ &+ (x^{1-\delta}u + (1-\gamma)x^{1-\beta-\gamma})\,\mathrm{D}_x - (2-\beta-\gamma)(1-\beta-\gamma)x^{-\beta-\gamma}\\ &- 2(2-\beta-\gamma)x^{1-\beta-\gamma}\,\mathrm{D}_x - x^{2-\beta-\gamma}\,\mathrm{D}_x^2 - x^{1-\delta}u_x\\ =& -t^{1-\alpha}\,\mathrm{D}_t + (x^{1-\delta}u - (3-2\beta-\gamma)x^{1-\beta-\gamma})\,\mathrm{D}_x - x^{2-\beta-\gamma}\,\mathrm{D}_x^2\\ &- (1-\alpha)t^{-\alpha} + (1-\delta)x^{-\delta}u - (1-\beta)(1-\beta-\gamma)x^{-\beta-\gamma}.\end{aligned}$$

Since we always have $\mathbb{D}_\Delta \neq \mathbb{D}_\Delta^*$ no matter what values the parameters α, β, γ, and δ take, the conformable fractional Burgers equation (1.2) is never the Euler-Lagrange equation for any variational problem.

Lemma C.1. *The characteristic r of a conservation law of Eq* (1.2) *has an equivalent characteristic of the following form:*

$$r = cx^{\delta-1}t^{\alpha-1}\exp\Big(-\frac{(\delta-\beta)(\delta-\beta-\gamma)t^\alpha}{\alpha x^{\beta+\gamma}}\Big),$$

where c is an arbitrary constant.

Proof. Let r be the characteristic of a conservation law of Eq (1.2). Then r has an equivalent characteristic of the form $r(x, t, u, u_1, \ldots, u_n)$ for some non-negative integer n, and the restriction of $\mathbb{D}_\Delta^* r$ on any solution of (1.2) is zero.

One checks that $\mathbb{D}_\Delta^* r$ itself (not the restriction) is a polynomial on the variables $u_{n,t}, u_{n+1}, u_{n+2}$ of degree 1 with coefficients of functions on $x, t, u, u_1, \ldots, u_n$. The coefficients of u_{n+2} and $u_{n,t}$ of $\mathbb{D}_\Delta^* r$ are

$$-x^{2-\beta-\gamma}\frac{\partial r}{\partial u_n}, \quad -t^{1-\alpha}\frac{\partial r}{\partial u_n},$$

respectively. Hence, the restriction of $\mathbb{D}_\Delta^* r$ on the solution of (1.2) can be regarded as a polynomial on the variables u_{n+1}, u_{n+2} of degree 1 with coefficients of functions on $x, t, u, u_1, \ldots, u_n$, where the coefficient of u_{n+2} is

$$-2x^{2-\beta-\gamma}\frac{\partial r}{\partial u_n}.$$

Then we obtain $\partial r/\partial u_n = 0$, i.e., r is independent on u_n. By induction on n, we conclude that a characteristic of the conservation law of (2.1) must be a smooth function only on x and t.

For such a characteristic $r = r(x,t)$, the restriction of $\mathbb{D}_{\Delta}^{*}r$ on the solution of (1.2) has no other expression other than $\mathbb{D}_{\Delta}^{*}r$ itself, so we must have $\mathbb{D}_{\Delta}^{*}r = 0$. One checks that $\mathbb{D}_{\Delta}^{*}r$ is a polynomial of degree 1 on u with coefficients of functions on x and t, and the coefficient of u is

$$x^{-\delta}\left(x\frac{\partial r}{\partial x} - (\delta - 1)r\right).$$

We conclude from the coefficient of u being 0 that

$$r = x^{\delta-1}s(t),$$

for some smooth $s(t)$. Then

$$\mathbb{D}_{\Delta}^{*}r = \frac{x^{\delta-\beta-\gamma-1}}{t^{\alpha}}\Big\{\big[(\alpha-1)x^{\beta+\gamma} - (\delta-\beta)(\delta-\beta-\gamma)t^{\alpha}\big]s(t) - x^{\beta+\gamma}t\frac{\mathrm{d}s}{\mathrm{d}t}\Big\}.$$

Then we deduce that

$$s = ct^{\alpha-1}\exp\Big(-\frac{(\delta-\beta)(\delta-\beta-\gamma)t^{\alpha}}{\alpha x^{\beta+\gamma}}\Big)$$

and hence

$$r = cx^{\delta-1}t^{\alpha-1}\exp\Big(-\frac{(\delta-\beta)(\delta-\beta-\gamma)t^{\alpha}}{\alpha x^{\beta+\gamma}}\Big),$$

for an arbitrary constant c, which completes the proof. □

Now we begin to prove Theorem 2.5, and suppose that (1.2) has a non-trivial conservation law (p', q'). Then there is a conservation law (p, q) equivalent to (p', q') whose characteristic is given by Lemma C.1. We may assume that the characteristic of (p, q) is

$$r = x^{\delta-1}t^{\alpha-1}\exp\Big(-\frac{(\delta-\beta)(\delta-\beta-\gamma)t^{\alpha}}{\alpha x^{\beta+\gamma}}\Big). \tag{C.1}$$

That is,

$$\mathrm{D}_t p + \mathrm{D}_x q = r\Delta, \tag{C.2}$$

where Δ is defined in (A.1). On the right-hand side of (C.2), the only term containing a jet-space variable involving the derivative on t (i.e., the variables u_t, u_{xt}, u_{tt}, and so on) is

$$\exp\Big(-\frac{(\delta-\beta)(\delta-\beta-\gamma)t^{\alpha}}{\alpha x^{\beta+\gamma}}\Big)x^{\delta-1}u_t,$$

so we must have

$$p = \exp\Big(-\frac{(\delta-\beta)(\delta-\beta-\gamma)t^{\alpha}}{\alpha x^{\beta+\gamma}}\Big)x^{\delta-1}u. \tag{C.3}$$

It follows that

$$\begin{aligned}\mathrm{D}_x q = &-\exp\Big(-\frac{(\delta-\beta)(\delta-\beta-\gamma)t^{\alpha}}{\alpha x^{\beta+\gamma}}\Big)\\ &\times\Big[t^{\alpha-1}u_x u + x^{\delta-\beta-\gamma+1}t^{\alpha-1}u_{xx} + (1-\gamma)x^{\alpha-\beta-\gamma}t^{\alpha-1}u_x\Big].\end{aligned} \tag{C.4}$$

Then q must be a linear combination of u_x and u^2 with coefficients of functions on x and t. The coefficients of u_x and u^2 in q are determined by the coefficients of u_{xx} and $u_x u$ in the right-hand side of (C.4), respectively, so we obtain

$$q = -\exp\Big(-\frac{(\delta-\beta)(\delta-\beta-\gamma)t^{\alpha}}{\alpha x^{\beta+\gamma}}\Big)\Big[x^{1+\delta-\beta-\gamma}t^{\alpha-1}u_x + \frac{1}{2}t^{\alpha-1}u^2\Big]. \tag{C.5}$$

Substituting (C.3) and (C.5) into (C.2), we obtain

$$\begin{aligned}&\frac{(\beta-\delta)t^{\alpha-1}}{2\alpha x^{2\beta+2\gamma+1}}\Big\{(\beta+\gamma-\delta)x^{\beta+\gamma}u[(\beta+\gamma)t^{\alpha}u + 2\alpha x^{\delta}]\\ &-2x^{\delta+1}u_x[\alpha x^{\beta+\gamma} - (\beta+\gamma)(\beta+\gamma-\delta)t^{\alpha}]\Big\} = 0,\end{aligned}$$

for arbitrary x, t, u, and u_x. Therefore, we must have $\delta = \beta$, which completes the proof.

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