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Research article

Some properties on dynamic cumulative Tsallis residual entropy measures based on Sarmanov family with applications to motor data

M. A. Alawady[1,2], **H. M. Barakat**[2], **Asamh Saleh M. Al Luhayb**[3,*] **and G. M. Mansour**[2]

[1] Department of Statistics and Operations Research, College of Science, Qassim University, Buraydah 51482, Saudi Arabia

[2] Department of Mathematics, Faculty of Science, Zagazig University, Zagazig 44519, Egypt

[3] Department of Mathematics, College of Science, Qassim University, P. O. Box 6644, Buraydah 51452, Saudi Arabia

* **Correspondence:** Email: a.alluhayb@qu.edu.sa.

Abstract: This study presents Tsallis and Rényi entropies as continuous measures of information for continuous distributions based on concomitants of generalized order statistics from the Sarmanov (SAR) family. Additionally, the characteristics and their relationship to other information measures are presented. One of such measures is the cumulative Tsallis residual entropy (CTRE), which can be regarded as an alternative measure of dispersion, and we study its dynamic version. Moreover, applications of these results are given for order statistics, and the record values as special cases with uniform, Weibull, and exponential marginal distributions. Furthermore, the empirical alternative CTRE (denoted ACTRE) was proposed to estimate these information measures. Finally, a real-world dataset has been examined for illustrative purposes and demonstrates superior goodness-of-fit and interpretability compared with classical bivariate distributions.

Keywords: concomitants; SAR family; entropy; Tsallis entropy; Rényi entropy; cumulative Tsallis residual entropy; nonparametric estimation; generalized order statistics
Mathematics Subject Classification: 60B12, 62G30

1. Introduction

Physicists initially used the word "entropy" in equilibrium thermodynamics. Now, it is thought to be the most fundamental topic in information theory. Boltzmann [1] was the first to introduce the concept of entropy in physics. He did this by coming up with a clear concept of entropy that measures how disordered particles are in a closed system.

Some individuals refer to Boltzmann's concept of entropy as "differential entropy". Shannon [2] independently formulated a comparable equation for information theory, referring to it as "entropy". Boltzmann [1] originated the notion of "information entropy", which Shannon [2] later formalized in the domains of information and communication. Shannon [2] proposed a measure of uncertainty associated with the probability density function (PDF) $\mathbf{f}_Y(y)$ of a random variable (RV) Y, articulated as

$$En(Y) = -\int_0^\infty \mathbf{f}_Y(y) \log \mathbf{f}_Y(y) dy.$$

Numerous generalizations of the classical Shannon entropy have been introduced in the information theory literature to quantify uncertainty. Tsallis [3] and Rényi [4] both introduced generalized entropies utilizing an additional parameter ν, rendering these entropies responsive to the configuration of probability distributions. If Y is an absolutely continuous RV with a PDF $\mathbf{f}_Y(y)$, then Rényi's entropy of order ν is defined as

$$\mathcal{R}_\nu(Y) = \frac{1}{1-\nu} \log \left(\int_0^\infty \mathbf{f}_Y^\nu(y) dy \right), \quad 0 < \nu \neq 1. \tag{1.1}$$

It is easy to see that as $\nu \to 1$, $\mathcal{R}_\nu(Y) \to En(Y)$. The generalization parameter ν makes Rényi entropy more flexible, and it preserves the additive property of $En(Y)$. Another important generalization of $En(Y)$, suggested by Tsallis [3], is known as Tsallis entropy. This measure is perhaps the most important nonadditive generalization of the Shannon entropy measure. It has played a revolutionary role in statistical mechanics, thermodynamics, and related fields. For detailed review and applications of Tsallis entropy, see Cartwright [5]. Tsallis entropy is defined as

$$\mathcal{T}_\nu(Y) = \frac{1}{\nu-1} \left(1 - \int_0^\infty \mathbf{f}_Y^\nu(y) dy \right), \quad 0 < \nu \neq 1. \tag{1.2}$$

Tsallis entropy reduces to Shannon entropy when $\nu \to 1$. Rényi and Tsallis entropies are related through

$$\mathcal{R}_\nu(Y) = \frac{1}{1-\nu} \log \left(1 - (\nu - 1)\mathcal{T}_\nu(Y)\right).$$

Beck [6] explained why Tsallis entropy is better than Rényi entropy in generalized statistical mechanics. He noted that Tsallis entropy demonstrates concavity and Lesche stability characteristics (Lesche [7]), which are not present in Rényi entropy. Tsallis entropy has been proposed as a generalization of Shannon entropy for the analysis of complex correlated systems. Owing to its nonadditive (nonextensive) structure, it provides a flexible framework for modeling systems characterized by long-range interactions, memory effects, or multifractal behavior (see [8]). For further details on entropy, its generalizations, and its importance in statistical mechanics, refer to Tsallis [3]. Wilk and Wlodarczyk [9] investigated situations where information can simply be computed using Tsallis entropy, as Shannon entropy is insufficient for these calculations.

Rao et al. [10] presented alternative entropy, which is based on the survival function (SF) $\overline{\mathbf{F}}_Y(y)$ instead of the PDF $\mathbf{f}_Y(y)$. Rao et al. [10] characterized cumulative residual entropy as

$$CRE(Y) = -\int_0^\infty \overline{\mathbf{F}}_Y(y) \log \overline{\mathbf{F}}_Y(y) dy.$$

Recently, the cumulative Tsallis residual entropy (CTRE) measure has been introduced in the literature by Sati and Gupta [11] for studying cumulative information based on a nonadditive entropy measure. For an RV Y, CTRE is defined as

$$CT_\nu(Y) = \frac{1}{\nu - 1}\left(1 - \int_0^\infty \overline{\mathbf{F}}_Y^\nu(y)dy\right), \nu > 0, \nu \neq 1. \tag{1.3}$$

It measures the uncertainty contained in the SF and generalizes the classical CTRE. Note that if $\nu \to 1$, $CT_\nu(Y) \to CRE(Y)$.

The CTRE may also be represented in terms of the mean residual life function of Y

$$m(t) = E[Y - t \mid Y > t] = \frac{1}{\overline{\mathbf{F}}_Y(t)} \int_t^\infty \overline{\mathbf{F}}_Y(u)\, du, \quad t \geq 0,$$

as

$$CT_\nu(Y) = \frac{1}{\nu} E[m(Y_\nu)] + \frac{1 - E(Y)}{1 - \nu},$$

where Y_ν is an RV with a ν-weighted distribution derived from Y (Asadi and Zohrevand [12]). Its PDF is proportional to $\overline{\mathbf{F}}_Y^{\nu-1}(y)\mathbf{f}_Y(y)$, specifically

$$\mathbf{f}_{Y_\nu}(y) = \nu\, \overline{\mathbf{F}}_Y^{\nu-1}(y)\mathbf{f}_Y(y), \quad y \geq 0.$$

This weighting emphasizes regions where the SF is higher, and the constant ν ensures normalization, since

$$\int_0^\infty \overline{\mathbf{F}}_Y^{\nu-1}(y)\mathbf{f}_Y(y)\, dy = 1/\nu.$$

The representation above first likely appeared in the context of CTRE in the work of Sati and Gupta [11]. This paper introduced CTRE and derived several of its properties, including this representation. However, in this representation

$$CT_\nu(Y) = \frac{1}{\nu} E[m(Y_\nu)],$$

if and only if $E(Y) = 1$. Another useful measure appeared in later literature (e.g., Rajesh and Sunoj [13], and Toomaj and Atabay [14]), namely, alternative CTRE (ACTRE), which satisfies

$$\mathcal{ACT}_\nu(Y) = \frac{1}{\nu} E[m(Y_\nu)]. \tag{1.4}$$

The measure $\mathcal{ACT}_\nu(Y)$ (see Subsection 2.5 for the definition and more details) generalizes the CTRE without requiring a unit mean ($E[Y] = 1$), making it more flexible for reliability applications and is directly linked to the mean residual life function.

Further details on classical and modern entropy measures and their applications in probability, statistics, and data analysis, with particular emphasis on recent advances in dynamic cumulative entropy and entropy formulations under dependence structures, including copula-based and multivariate settings, can be found in the comprehensive survey by Kumar et al. [15].

There has been increasing interest in studying generalized order statistics (GOSs) as a unified model for ascendingly ordered RVs. Kamps [16] introduced the GOSs model, which consists of many relevant models of ordered RVs, including order statistics (OSs), record values, sequential OSs, and progressive censored Type II OSs. The RVs $Y_{(r,n,\breve{m},k)}$, $r = 1,2,...,n$, are called GOSs based on a continuous distribution function (DF) $\mathbf{F}_Y$ with the PDF $\mathbf{f}_Y$, if their joint PDF has the form

$$\mathbf{g}^{(\breve{m},k)}_{1,...,n:n}(y_1,...,y_n) = k\mathbf{F}_Y^{\gamma_n-1}(y_n)\mathbf{f}_Y(y_n)\prod_{i=1}^{n-1}\gamma_i\,\mathbf{F}_Y^{\gamma_i-\gamma_{i+1}-1}(y_i)\mathbf{f}_Y(y_i),$$

where $\mathbf{F}_Y^{-1}(0) \le y_1 \le ... \le y_n \le \mathbf{F}_Y^{-1}(1)$, $k > 0, \gamma_i = n + k - i + \sum_{t=i}^{n-1} m_t > 0, i = 1,\dots,n-1$, and $\breve{m} = (m_1, m_2, \dots, m_{n-1}) \in \mathbb{R}$. In this paper, we assume that the parameters $\gamma_1,\dots,\gamma_{n-1}$, and $\gamma_n = k$ are pairwise different, i.e., $\gamma_t \ne \gamma_s, t \ne s,\ t,s = 1,2,...,n$. We obtain a very wide subclass of GOSs that contains m-GOSs (where $m_1 = ... = m_{n-1} = m$, and $X_{[r,n,\breve{m},k]} := X_{[r,n,m,k]}$), OSs, Type II OSs, and sequential OSs. The PDF of the rth GOS is given by (Kamps and Cramer [17])

$$\mathbf{g}_{Y(r,n,\breve{m},k)}(y) = C_r\sum_{i=1}^{r}\alpha_{i;r}\overline{\mathbf{F}}_Y^{\gamma_i-1}(y)\mathbf{f}_Y(y),\quad y\in\mathbb{R},\quad 1\le r\le n,$$

where $\overline{\mathbf{F}}_Y = 1 - \mathbf{F}_Y$ is the SF of $\mathbf{F}_Y$, $C_r = \prod_{i=1}^{r}\gamma_i$, and

$$\alpha_{i;r} = \prod_{\substack{j=1\\ j\ne i}}^{r}\frac{1}{\gamma_j-\gamma_i},\ 1\le i\le r\le n.$$

The bivariate (multivariate) DFs with specified marginals are ideal for modeling bivariate (multivariate) data when only marginal distributions are all available. In such situations, it is often advantageous to use a flexible family of bivariate DFs, such as the Farlie–Gumbel–Morgenstern (FGM) family. The DF and PDF of the FGM family are defined, respectively, by

$$\mathbf{G}_{Y,X}(y,x) = \mathbf{F}_Y(y)\mathbf{F}_X(x)\left[1+\zeta\,\overline{\mathbf{F}}_Y(y)\overline{\mathbf{F}}_X(x)\right]$$

and

$$\mathbf{g}_{Y,X}(y,x) = \mathbf{f}_Y(y)\mathbf{f}_X(x)\left[1+\zeta\Big(2\mathbf{F}_Y(y)-1\Big)\Big(2\mathbf{F}_X(x)-1\Big)\right],\ -1\le\zeta\le1,$$

where $\mathbf{f}_Y(y)$, $\mathbf{f}_X(x)$, and $\mathbf{F}_Y(y)$, $\mathbf{F}_X(x)$ are the marginal PDFs and DFs of the RVs Y and X, respectively, while $\overline{\mathbf{F}}_Y$ and $\overline{\mathbf{F}}_X$ are the corresponding SFs. Generally, the FGM family has low dependence between variables, with Spearman's rho $\rho \in (-0.33, 0.33)$. Thus, the FGM family is useful in applications where the correlation between variables is weak. To accommodate stronger dependence structures, several extensions of the FGM family have been developed (see, e.g., Bairamov and Kotz [18]; Cambanis [19]; Huang and Kotz [20]; Bekrizadeh et al. [21]). Several authors have explored applications of these extended families. In particular, Abd Elgawad et al. [22, 23] and Alawady et al. [24, 25] contributed to the theory of concomitants and information measures. Additionally, Barakat and Husseiny [26], Husseiny et al. [27, 28], and Mansour et al. [29] examined their use in reliability and bivariate dependence modeling.

The Sarmanov family, denoted SAR, has been recently examined by Alawady et al. [30], Barakat et al. [31, 32], and Mansour et al. [33], who discovered that it is a significant rival to all known FGM extensions. The DF and PDF of SAR are given, respectively, by

$$\mathbf{G}_{Y,X}(y,x)=\mathbf{F}_Y(y)\mathbf{F}_X(x)\left[1+3\lambda\overline{\mathbf{F}}_Y(y)\overline{\mathbf{F}}_X(x)+5\lambda^2\left(2\mathbf{F}_Y(y)-1\right)\left(2\mathbf{F}_X(x)-1\right)\overline{\mathbf{F}}_Y(y)\overline{\mathbf{F}}_X(x)\right] \tag{1.5}$$

and

$$\begin{aligned}\mathbf{g}_{Y,X}(y,x) = \mathbf{f}_Y(y)\mathbf{f}_X(x)\Big[& 1+3\lambda\left(2\mathbf{F}_Y(y)-1\right)\left(2\mathbf{F}_X(x)-1\right)\\ & +\frac{5}{4}\lambda^2\left(3\left(2\mathbf{F}_Y(y)-1\right)^2-1\right)\left(3\left(2\mathbf{F}_X(x)-1\right)^2-1\right)\Big], \quad |\lambda|\le\frac{\sqrt{7}}{5}.\end{aligned} \tag{1.6}$$

It is important to note that the bound $|\lambda| \le \frac{\sqrt{7}}{5}$ is a necessary condition for the bivariate density (1.6) to be non-negative for all x and y; this bound does not depend on the marginal distributions. The correlation coefficient ρ between X and Y, however, depends on the marginals. For uniform marginals, we have $\rho = \lambda$, giving the range ± 0.529 (cf. Balakrishnan and Lai [34]; page 74). For nonuniform marginals, the achievable range of ρ can be larger; for example, when $\mathbf{F}_X = \mathbf{F}_Y$ (identical marginals), it is shown in Barakat et al. [32] that ρ can be as high as 0.80915 (see Theorem 2.1 therein). Hence the estimated value $\hat{\lambda} = 0.52915$ in our motor data application is well within the admissible range of λ, and the fact that it coincides with the uniform correlation bound merely reflects the strong dependence in the data, not a violation of the model's assumptions.

The concomitants are a vital tool when selection and prediction problems are involved. The idea of concomitants of OSs was first proposed by David [35]. To understand the concomitants in more depth, see David and Nagaraja [36]. Many studies have been published on the concomitants of the m-GOSs model. Researchers such as Elgawad et al. [22], Alawady et al. [24, 25], Beg and Ahsanullah [37], and Domma and Giordano [38] have studied this issue. The concomitants of GOS (CGOS) models, however, have only been studied in a restricted number of studies when $\gamma_t \neq \gamma_s, t \neq s,\ t, s = 1, 2, ..., n$. These include Elgawad and Alawady [39], El-Din et al. [40], and Elgawad et al. [22].

The goal of this study is to find explicit expressions for Tsallis entropy, Rényi entropy, CTRE, and its dynamic version for CGOSs based on the SAR family. We will also suggest a nonparametric estimator for ACTRE and analyze its properties by numerical computations. Finally, we will demonstrate the utility of these metrics by applying them to a real-world motor reliability dataset and comparing the goodness-of-fit of the SAR model with that of traditional bivariate distributions.

Motivation of the work

The analysis of ordered RVs and their concomitants plays a pivotal role in various fields such as reliability engineering, survival analysis, and quality control. In many practical situations, the performance of a system depends on several interdependent components, and the study of their joint behavior is essential. The CGOSs which arise when sorting one variable while keeping track of another provide a powerful tool for modeling such dependencies. However, classic bivariate models, such as the FGM family, frequently make stringent assumptions about the dependence structure, thereby constraining their application to real-world contexts characterized by more robust and realistic correlations. The SAR family is a more flexible version of the FGM family that provides a stronger framework for capturing larger levels of dependence. This makes it a better choice for modeling bivariate data.

There has been a lot of research on entropy measures for ordered data, but not much has been written about the information-theoretic features of concomitants in flexible bivariate models like the SAR family. Entropy measures, including Tsallis and Rényi entropies, are essential for assessing uncertainty and information inside complex systems. Furthermore, cumulative residual entropy metrics, such as the CTRE, yield significant insights into the residual lifespan and dependability of systems. Nevertheless, the dynamic and alternate iterations of these measures for concomitants in the SAR family remain predominantly unexamined.

This paper seeks to address this deficiency by formulating explicit formulas for Tsallis entropy, Rényi entropy, CTRE, and its dynamic and alternative variants for CGOSs from the SAR family. By doing this, we add to the current theoretical framework and give researchers new ways to look at uncertainty and reliance in systems that are related. We also provide a nonparametric estimator for ACTRE and show through numerical simulations that it is consistent and useful. The practical applicability of the proposed measures is further exemplified by a real-world motor reliability dataset, whereby the SAR model exhibits enhanced goodness-of-fit relative to conventional bivariate distributions. This work not only improves our theoretical understanding of entropy measures for concomitants but also provides useful methods for assessing reliability and risk in engineering applications.

This is how the rest of the paper is set up. In Section 2, we derive characterization results for the concomitants $X_{[r,n,\check{m},k]}$, $(1 \le r \le n)$ based on the SAR family, including formulas for the CTRE, Tsallis entropy, Rényi entropy, and ACTRE. In Section 3, we expand upon these results and provide explicit instances of these information measures for the concomitant $X_{[r,n,\check{m},k]}$ derived from SAR with specified marginal distributions. In Section 4, we show a nonparametric estimation technique that uses the empirical method and the concomitants $X_{[r,n,\check{m},k]}$ from the SAR family to get the ACTRE. In Section 5, we demonstrate the proposed technique using a real-world dataset comprising two variables. Finally, Section 6 presents some final remarks and talks about prospective future study directions.

2. Concomitants of GOSs based on SAR family

Let $V_i \sim \mathbf{F}_Y^{i+1}, i = 1,2$. The PDF, DF, SF, and moment of $X_{[r,n,\check{m},k]}$ that is part of the SAR family described in (1.5) and (1.6) are given by Alawady et al. [41], respectively, as

$$\mathbf{g}_{[r,n,\check{m},k]}(x) = \left(1-3\chi_{r,n:1}^{(\check{m},k)} + \frac{5}{2}\chi_{r,n:2}^{(\check{m},k)}\right)\mathbf{f}_X(x) + 3\left(\chi_{r,n:1}^{(\check{m},k)} - \frac{5}{2}\chi_{r,n:2}^{(\check{m},k)}\right)\mathbf{f}_{V_1}(x) + 5\chi_{r,n:2}^{(\check{m},k)}\mathbf{f}_{V_2}(x), \tag{2.1}$$

$$\mathbf{G}_{[r,n,\check{m},k]}(x)=\left[\left(1-3\chi_{r,n:1}^{(\check{m},k)}+\frac{5}{2}\chi_{r,n:2}^{(\check{m},k)}\right)+\left(3\chi_{r,n:1}^{(\check{m},k)}-\frac{15}{2}\chi_{r,n:2}^{(\check{m},k)}\right)\mathbf{F}_X(x)+5\chi_{r,n:2}^{(\check{m},k)}\mathbf{F}_X^2(x)\right]\mathbf{F}_X(x),$$

$$\overline{\mathbf{G}}_{[r,n,\check{m},k]}(x) = \left[1 + \left(3\chi_{r,n:1}^{(\check{m},k)} - \frac{5}{2}\chi_{r,n:2}^{(\check{m},k)}\right)\mathbf{F}_X(x) + 5\chi_{r,n:2}^{(\check{m},k)}\mathbf{F}_X^2(x)\right]\overline{\mathbf{F}}_X(x), \tag{2.2}$$

and

$$\mu_{[r,n,\check{m},k]} = (1 - 3\chi_{r,n:1}^{(\check{m},k)} + \frac{5}{2}\chi_{r,n:2}^{(\check{m},k)})\mu_X + (3\chi_{r,n:1}^{(\check{m},k)} - \frac{15}{2}\chi_{r,n:2}^{(\check{m},k)})\mu_{V_1} + 5\chi_{r,n:2}^{(\check{m},k)}\mu_{V_2},$$

where

$$\chi_{r,n:1}^{(\breve{m},k)} = \lambda\left(1 - 2C_r\sum_{i=1}^{r}\frac{\alpha_{i;r}}{\gamma_{i+1}}\right)$$

and

$$\chi_{r,n:2}^{(\breve{m},k)} = 2\lambda^2\left(1 - 6C_r\sum_{i=1}^{r}\frac{\alpha_{i;r}}{(\gamma_i+2)(\gamma_i+1)}\right).$$

2.1. CTRE for CGOS of ordered ν

Theorem 2.1. *The cumulative Tsallis residual entropy for CGOSs from the SAR family is given by*

$$\mathcal{CT}_{\nu}^{(\lambda)}(X_{[r,n,\breve{m},k]}) = \frac{1}{\nu-1}\left(1 - \sum_{i=0}^{\mathcal{N}(\nu)}\sum_{j=0}^{i}\binom{\nu}{i}\binom{i}{j}\left(3\chi_{r,n:1}^{(\breve{m},k)} - \frac{5}{2}\chi_{r,n:2}^{(\breve{m},k)}\right)^{i-j} \times \left(5\chi_{r,n:2}^{(\breve{m},k)}\right)^{j}\int_0^1 \frac{u^{i+j}(1-u)^{\nu}}{\mathbf{f}_X(\mathbf{F}_X^{-1}(u))}du\right),$$

where $\mathcal{N}(\nu) = \infty$ if ν is a non integer and $\mathcal{N}(\nu) = \nu$ if ν is an integer.

Proof. Using (1.3) and (2.2), CTRE for CGOS is provided by

$$\begin{aligned}\mathcal{CT}_{\nu}^{(\lambda)}(X_{[r,n,\breve{m},k]}) &= \frac{1}{\nu-1}\left(1 - \int_0^{\infty}\overline{\mathbf{G}}_{[r,n,\breve{m},k]}^{\nu}(x)dx\right)\\ &= \frac{1}{\nu-1}\left(1 - \int_0^{\infty}\left[1 + \left(3\chi_{r,n:1}^{(\breve{m},k)} - \frac{5}{2}\chi_{r,n:2}^{(\breve{m},k)}\right)\mathbf{F}_X(x) + 5\chi_{r,n:2}^{(\breve{m},k)}\mathbf{F}_X^2(x)\right]^{\nu}\overline{\mathbf{F}}_X^{\nu}(x)dx\right).\end{aligned}$$

Using the binomial expansion twice, we obtain

$$\begin{aligned}\left[1 + \left(3\chi_{r,n:1}^{(\breve{m},k)} - \frac{5}{2}\chi_{r,n:2}^{(\breve{m},k)}\right)\mathbf{F}_X(x) + 5\chi_{r,n:2}^{(\breve{m},k)}\mathbf{F}_X^2(x)\right]^{\nu} &= \sum_{i=0}^{\mathcal{N}(\nu)}\binom{\nu}{i}\left[\left(3\chi_{r,n:1}^{(\breve{m},k)} - \frac{5}{2}\chi_{r,n:2}^{(\breve{m},k)}\right)\mathbf{F}_X(x) + 5\chi_{r,n:2}^{(\breve{m},k)}\mathbf{F}_X^2(x)\right]^{i}\\ &= \sum_{i=0}^{\mathcal{N}(\nu)}\sum_{j=0}^{i}\binom{\nu}{i}\binom{i}{j}\left(3\chi_{r,n:1}^{(\breve{m},k)} - \frac{5}{2}\chi_{r,n:2}^{(\breve{m},k)}\right)^{i-j}\left(5\chi_{r,n:2}^{(\breve{m},k)}\right)^{j}\mathbf{F}_X^{i+j}(x).\end{aligned}$$

Thus

$$\mathcal{CT}_{\nu}^{(\lambda)}(X_{[r,n,\breve{m},k]}) = \frac{1}{\nu-1}\left(1 - \sum_{i=0}^{\mathcal{N}(\nu)}\sum_{j=0}^{i}\binom{\nu}{i}\binom{i}{j}\left(3\chi_{r,n:1}^{(\breve{m},k)} - \frac{5}{2}\chi_{r,n:2}^{(\breve{m},k)}\right)^{i-j} \times \left(5\chi_{r,n:2}^{(\breve{m},k)}\right)^{j}\underbrace{\int_0^{\infty}\overline{\mathbf{F}}_X^{\nu}(x)\mathbf{F}_X^{i+j}(x)dx}_{\Lambda}\right). \quad (2.3)$$

Now, evaluate the integral Λ by making the substitution $u = \mathbf{F}_X(x)$. Then $x = \mathbf{F}_X^{-1}(u)$, $dx = \frac{du}{\mathbf{f}_X(\mathbf{F}_X^{-1}(u))}$, and $\overline{\mathbf{F}}_X(x) = 1 - \mathbf{F}_X(x) = 1 - u$. The limits transform as follows: when $x = 0$, $u = \mathbf{F}_X(0) = 0$; when $x \to \infty$, $u \to 1$. Therefore

$$\Lambda = \int_0^1 (1-u)^{\nu}u^{i+j}\frac{du}{\mathbf{f}_X(\mathbf{F}_X^{-1}(u))}.$$

Substituting this expression for Λ completes the proof. □

Note on notation: Throughout this section, the integrals obtained via the transformation $u = \mathbf{F}_X(x)$ are written explicitly as $\int_0^1 g(u)\,du$ rather than as expectations $E[g(U)]$ with $U \sim \text{Uniform}(0,1)$. This choice emphasizes that terms such as $\frac{1}{\mathbf{f}_X(\mathbf{F}_X^{-1}(u))}$ originate from the Jacobian $dx = \frac{du}{\mathbf{f}_X(\mathbf{F}_X^{-1}(u))}$ and are integrated with respect to Lebesgue measure.

Remark 2.1. *Let $X_{[r:n]} := X_{[r,n,0,1]}$ be the concomitant of the rth OS (i.e., $\check{m} = 0$ and $k = 1$). Then*

$$\mathcal{CT}_\nu^{(\lambda)}(X_{[r:n]}) = \frac{1}{\nu-1}\left(1 - \sum_{i=0}^{\mathcal{N}(\nu)}\sum_{j=0}^{i}\binom{\nu}{i}\binom{i}{j}(3\psi_{1,r:n}^{(\lambda)} - \frac{5}{2}\psi_{2,r:n}^{(\lambda)})^{i-j}(5\psi_{2,r:n}^{(\lambda)})^j\int_0^1 \frac{u^{i+j}(1-u)^\nu}{\mathbf{f}_X(\mathbf{F}_X^{-1}(u))}du\right),$$

where $\psi_{1,r:n}^{(\lambda)} = \frac{\lambda(2r-n-1)}{n+1}$ and $\psi_{2,r:n}^{(\lambda)} = 2\lambda^2\left[1 - 6\frac{r(n-r+1)}{(n+1)(n+2)}\right]$.

Remark 2.2. *Let $X_{[n]} := X_{[r,n,-1,1]}$ be the concomitant of the nth upper record value (i.e., $\check{m} = -1$ and $k = 1$). Then*

$$\mathcal{CT}_\nu^{(\lambda)}(X_{[n]}) = \frac{1}{\nu-1}\left(1 - \sum_{i=0}^{\mathcal{N}(\nu)}\sum_{j=0}^{i}\binom{\nu}{i}\binom{i}{j}(3\omega_{n:1} - \frac{5}{2}\omega_{n:2})^{i-j}(5\omega_{n:2})^j\int_0^1 \frac{u^{i+j}(1-u)^\nu}{\mathbf{f}_X(\mathbf{F}_X^{-1}(u))}du\right),$$

where $\omega_{n:1} = \lambda(1 - 2^{-(n-1)})$ and $\omega_{n:2} = \lambda^2(12(3^{-n} - 2^{-n}) + 2)$.

Remark 2.3. *The CTRE of concomitant of the nth upper k-record value (denoted $X_{[n]}^{(k)}$) based on the SAR is given by*

$$\mathcal{CT}_\nu^{(\lambda)}(X_{[n]}^{(k)}) = \frac{1}{\nu-1}\left(1 - \sum_{i=0}^{\mathcal{N}(\nu)}\sum_{j=0}^{i}\binom{\nu}{i}\binom{i}{j}(3\delta_{n;k}^{(1)} - \frac{5}{2}\delta_{n;k}^{(2)})^{i-j}(5\delta_{n;k}^{(2)})^j\int_0^1 \frac{u^{i+j}(1-u)^\nu}{\mathbf{f}_X(\mathbf{F}_X^{-1}(u))}du\right),$$

where $\delta_{n;k}^{(1)} = \lambda\left[1 - 2\left(\frac{k}{k+1}\right)^n\right]$ and $\delta_{n;k}^{(2)} = \lambda^2\left[12\left(\left(\frac{k}{k+2}\right)^n - \left(\frac{k}{k+1}\right)^n\right) + 2\right]$.

2.2. Dynamic CTRE

For an RV X_t with SF $\overline{\mathbf{F}}_X(x)$, the dynamic CTRE (DCTRE) of order ν, denoted $\mathcal{CT}_\nu^{(\lambda)}(X_{[r,n,\check{m},k]};t)$, is defined as

$$\mathcal{CT}_\nu^{(\lambda)}(X_{[r,n,\check{m},k]};t) = \frac{1}{\nu-1}\left(1 - \frac{\int_t^\infty \overline{\mathbf{G}}_{[r,n,\check{m},k]}^\nu(x)dx}{\overline{\mathbf{G}}_{[r,n,\check{m},k]}^\nu(t)}\right).$$

Theorem 2.2. *(DCTRE characterization) Let $X_{[r,n,\check{m},k]}$ be a non-negative concomitant with the SF $\overline{\mathbf{G}}_{[r,n,\check{m},k]}(t) > 0$ for all $t \geq 0$, the PDF $\mathbf{g}_{[r,n,\check{m},k]}(t)$, and the hazard rate function*

$$\mathcal{H}_{[r,n,\check{m},k]}(t) = \frac{\mathbf{g}_{[r,n,\check{m},k]}(t)}{\overline{\mathbf{G}}_{[r,n,\check{m},k]}(t)}, \quad t \geq 0.$$

Assume that for every $t \geq 0$, the DCTRE $\mathcal{CT}_\nu^{(\lambda)}(X_{[r,n,\check{m},k]};t)$ is finite and differentiable, and that the following nondegeneracy condition holds:

$$\int_t^\infty \overline{\mathbf{G}}_{[r,n,\check{m},k]}^\nu(x)dx \neq 0 \quad (\textit{equivalently } (\nu-1)\mathcal{CT}_\nu^{(\lambda)}(X_{[r,n,\check{m},k]};t) \neq 1).$$

Then the function $t \mapsto \mathcal{CT}_\nu^{(\lambda)}(X_{[r,n,\check{m},k]};t)$ uniquely determines the SF $\overline{\mathbf{G}}_{[r,n,\check{m},k]}(t)$ (and, consequently, the hazard rate $\mathcal{H}_{[r,n,\check{m},k]}(t)$).

Proof. Let $\overline{G}(t) = \overline{\mathbf{G}}_{[r,n,\breve{m},k]}(t)$ and $\mathcal{H}(t) = \mathcal{H}_{[r,n,\breve{m},k]}(t)$ for notational simplicity. By assumption, $\overline{G}(t) > 0$ for all $t \geq 0$, ensuring that $\mathcal{H}(t)$ is well-defined. Define

$$A(t) = \int_t^\infty \overline{G}^{\nu}(x)dx \quad \text{and} \quad B(t) = \overline{G}^{\nu}(t), \quad t \geq 0.$$

The DCTRE can then be written as

$$(\nu - 1)\mathcal{CT}_{\nu}^{(\lambda)}(X_{[r,n,\breve{m},k]}; t) = 1 - \frac{A(t)}{B(t)}. \tag{2.4}$$

By the fundamental theorem of calculus, we have

$$\frac{d}{dt}A(t) = A'(t) = -\overline{G}^{\nu}(t) = -B(t). \tag{2.5}$$

For $B(t)$, using the chain rule and the positivity of $\overline{G}(t)$, we have

$$B'(t) = \nu\overline{G}^{\nu-1}(t)(-\mathbf{g}_{[r,n,\breve{m},k]}(t)) = -\nu\overline{G}^{\nu-1}(t)\mathbf{g}_{[r,n,\breve{m},k]}(t).$$

Since $\mathcal{H}(t) = \mathbf{g}_{[r,n,\breve{m},k]}(t)/\overline{G}(t)$, we have $\mathbf{g}_{[r,n,\breve{m},k]}(t) = \mathcal{H}(t)\overline{G}(t)$. Substituting this gives

$$B'(t) = -\nu\overline{G}^{\nu-1}(t)\mathcal{H}(t)\overline{G}(t) = -\nu\mathcal{H}(t)\overline{G}^{\nu}(t) = -\nu\mathcal{H}(t)B(t). \tag{2.6}$$

Differentiating both sides of (2.4) with respect to t yields

$$(\nu - 1)\frac{d}{dt}\mathcal{CT}_{\nu}^{(\lambda)}(X_{[r,n,\breve{m},k]}; t) = -\frac{A'(t)B(t) - A(t)B'(t)}{B^2(t)}. \tag{2.7}$$

Substituting $A'(t)$ from (2.5) and $B'(t)$ from (2.6) into (2.7), yeilds

$$(\nu - 1)\frac{d}{dt}\mathcal{CT}_{\nu}^{(\lambda)}(X_{[r,n,\breve{m},k]}; t) = -\frac{(-B(t))B(t) - A(t)(-\nu\mathcal{H}(t)B(t))}{B^2(t)}.$$

Simplifying the numerator, we have

$$\begin{aligned}\text{Numerator} &= -[-B^2(t) + \nu\mathcal{H}(t)A(t)B(t)] \\ &= B^2(t) - \nu\mathcal{H}(t)A(t)B(t).\end{aligned}$$

Thus

$$(\nu - 1)\frac{d}{dt}\mathcal{CT}_{\nu}^{(\lambda)}(X_{[r,n,\breve{m},k]}; t) = \frac{B^2(t) - \nu\mathcal{H}(t)A(t)B(t)}{B^2(t)} = 1 - \nu\mathcal{H}(t)\frac{A(t)}{B(t)}. \tag{2.8}$$

From (2.4), we have $A(t)/B(t) = 1 - (\nu - 1)\mathcal{CT}_{\nu}^{(\lambda)}(X_{[r,n,\breve{m},k]}; t)$. Substituting this into (2.8), we have

$$(\nu - 1)\frac{d}{dt}\mathcal{CT}_{\nu}^{(\lambda)}(X_{[r,n,\breve{m},k]}; t) = 1 - \nu\mathcal{H}(t) + \nu(\nu - 1)\mathcal{H}(t)\mathcal{CT}_{\nu}^{(\lambda)}(X_{[r,n,\breve{m},k]}; t). \tag{2.9}$$

Now, suppose that two concomitant RVs $X_{1[r,n,\breve{m},k]}$ and $X_{2[r,n,\breve{m},k]}$ have the same DCTRE for all $t \geq 0$, denoted $\mathcal{CT}_{\nu}^{(\lambda)}(t)$. Let $\mathcal{H}_1(t)$ and $\mathcal{H}_2(t)$ be their respective hazard rates. Applying (2.8) to both distributions and subtracting gives

$$\nu\left[\mathcal{H}_1(t) - \mathcal{H}_2(t)\right]\left[(\nu - 1)\mathcal{CT}_{\nu}^{(\lambda)}(t) - 1\right] = 0, \quad \forall t \geq 0.$$

The nondegeneracy condition ensures $(\nu-1)\mathcal{CT}_{\nu}^{(\lambda)}(t)-1\neq 0$ for all $t\geq 0$. Therefore

$$\mathcal{H}_1(t)=\mathcal{H}_2(t),\quad \forall t\geq 0.$$

The SF is uniquely determined by the hazard rate through the relationship

$$\overline{G}(t)=\exp\left(-\int_0^t \mathcal{H}(s)ds\right),\quad t\geq 0,$$

with the initial condition $\overline{G}(0)=1$. Since $\mathcal{H}_1(t)=\mathcal{H}_2(t)$ for all $t\geq 0$, the two SFs coincide. Consequently, the DCTRE uniquely characterizes the SF and the hazard rate. This completes the proof. □

Theorem 2.3. *(Monotonicity of DCTRE) Let $X_{[r,n,\check{m},k]}$ be a concomitant with the SF $\overline{\mathbf{G}}_{[r,n,\check{m},k]}(t)$ and the hazard rate $\mathcal{H}_{[r,n,\check{m},k]}(t)$. For $\nu>0$, $\nu\neq 1$, the $\mathcal{CT}_{\nu}^{(\lambda)}(X_{[r,n,\check{m},k]};t)$ is increasing, (respectively, decreasing) in t if and only if for all $t\geq 0$,*

$$\mathcal{CT}_{\nu}^{(\lambda)}(X_{[r,n,\check{m},k]};t)\geq (respectively\ \ \leq)\frac{1}{\nu-1}\left(1-\frac{1}{\nu\mathcal{H}_{[r,n,\check{m},k]}(t)}\right).$$

Proof. The proof of the theorem directly follows from (2.9). □

Corolary 2.1. *(Scaling property) For any non-negative RV X, let $Z_{[r,n,\check{m},k]}=aX_{[r,n,\check{m},k]}+b$, where $a,b>0$. Then*

$$\mathcal{CT}_{\nu}^{(\lambda)}(Z_{[r,n,\check{m},k]};t)=\frac{1-a}{\nu-1}+a\,\mathcal{CT}_{\nu}^{(\lambda)}\left(X_{[r,n,\check{m},k]};\frac{t-b}{a}\right).$$

Proof. The SF of $Z_{[r,n,\check{m},k]}$ is $\overline{\mathbf{G}}_Z(x):=\overline{\mathbf{G}}_{[r,n,\check{m},k]}\left(\frac{x-b}{a}\right)$. By definition, we have

$$\mathcal{CT}_{\nu}^{(\lambda)}(Z_{[r,n,\check{m},k]};t)=\frac{1}{\nu-1}\left(1-\frac{\int_t^{\infty}\overline{\mathbf{G}}_Z^{\nu}(x)dx}{\overline{\mathbf{G}}_Z^{\nu}(t)}\right).$$

Substitute the following SF:

$$\int_t^{\infty}\overline{\mathbf{G}}_Z^{\nu}(x)dx=\int_t^{\infty}\overline{\mathbf{G}}_{[r,n,\check{m},k]}^{\nu}\left(\frac{x-b}{a}\right)dx.$$

We use the change of variable $y=\frac{x-b}{a}$, and hence $x=ay+b$ and $dx=a\,dy$. When $x=t$, $y=\frac{t-b}{a}$. Then

$$\int_t^{\infty}\overline{\mathbf{G}}_{[r,n,\check{m},k]}^{\nu}\left(\frac{x-b}{a}\right)dx=a\int_{\frac{t-b}{a}}^{\infty}\overline{\mathbf{G}}_{[r,n,\check{m},k]}^{\nu}(y)dy.$$

Moreover, $\overline{\mathbf{G}}_Z^{\nu}(t)=\overline{\mathbf{G}}_{[r,n,\check{m},k]}^{\nu}\left(\frac{t-b}{a}\right)$. Therefore

$$\mathcal{CT}_{\nu}^{(\lambda)}(Z_{[r,n,\check{m},k]};t)=\frac{1}{\nu-1}\left(1-a\frac{\int_{\frac{t-b}{a}}^{\infty}\overline{\mathbf{G}}_{[r,n,\check{m},k]}^{\nu}(y)dy}{\overline{\mathbf{G}}_{[r,n,\check{m},k]}^{\nu}\left(\frac{t-b}{a}\right)}\right).$$

From the definition of DCTRE for X, we have

$$\frac{\int_{\frac{t-b}{a}}^{\infty} \overline{\mathbf{G}}_{[r,n,\breve{m},k]}^{\nu}(y)dy}{\overline{\mathbf{G}}_{[r,n,\breve{m},k]}^{\nu}\left(\frac{t-b}{a}\right)} = 1-(\nu-1)\mathcal{CT}_{\nu}^{(\lambda)}\left(X_{[r,n,\breve{m},k]};\frac{t-b}{a}\right).$$

Substituting back, yields

$$\begin{aligned}\mathcal{CT}_{\nu}^{(\lambda)}(Z_{[r,n,\breve{m},k]};t) &= \frac{1}{\nu-1}\Big(1-a[1-(\nu-1)\mathcal{CT}_{\nu}^{(\lambda)}\left(X_{[r,n,\breve{m},k]};\frac{t-b}{a}\right)]\Big)\\ &= \frac{1-a}{\nu-1}+a\,\mathcal{CT}_{\nu}^{(\lambda)}\left(X_{[r,n,\breve{m},k]};\frac{t-b}{a}\right).\end{aligned}$$

This completes the proof. □

2.3. Tsallis entropy for CGOS

Theorem 2.4. *The Tsallis entropy (also called ν-entropy) of the CGOSs based on SAR is given by*

$$\begin{aligned}\mathcal{T}_{\nu}^{(\lambda)}(X_{[r,n,\breve{m},k]}) = \frac{1}{\nu-1}\Big(1-&\sum_{j=0}^{\mathcal{N}(\nu)}\sum_{p=0}^{j}\binom{}{j}\binom{j}{p}(3\chi_{r,n:1}^{(\breve{m},k)})^{j-p}(\frac{5}{4}\chi_{r,n:2}^{(\breve{m},k)})^{p}\\ &\times\int_0^1(\mathbf{f}_X(\mathbf{F}_X^{-1}(u)))^{\nu-1}(2u-1)^{j-p}(3(2u-1)^2-1)^p\,du\Big).\end{aligned}$$

Proof. Using (1.2) and (2.1), the Tsallis entropy is provided by

$$\begin{aligned}\mathcal{T}_{\nu}^{(\lambda)}(X_{[r,n,\breve{m},k]}) &= \frac{1}{\nu-1}\left(1-\int_0^{\infty}\mathbf{g}_{[r,n,\breve{m},k]}^{\nu}(x)dx\right)\\ &= \frac{1}{\nu-1}\left(1-\int_0^{\infty}\mathbf{f}_X^{\nu}(x)\left(1+3\chi_{r,n:1}^{(\breve{m},k)}(2\mathbf{F}_X(x)-1)+\frac{5}{4}\chi_{r,n:2}^{(\breve{m},k)}(3(2\mathbf{F}_X(x)-1)^2-1)\right)^{\nu}dx\right)\\ &= \frac{1}{\nu-1}\Big(1-\sum_{j=0}^{\mathcal{N}(\nu)}\sum_{p=0}^{j}\binom{\nu}{j}\binom{j}{p}(3\chi_{r,n:1}^{(\breve{m},k)})^{j-p}(\frac{5}{4}\chi_{r,n:2}^{(\breve{m},k)})^{p}\\ &\quad\times\int_0^1(\mathbf{f}_X(\mathbf{F}_X^{-1}(u)))^{\nu-1}(2u-1)^{j-p}(3(2u-1)^2-1)^p\,du. \qquad (2.10)\end{aligned}$$

This completes the proof. □

Remark 2.4. *If $\breve{m}=0$ and $k=1$, the Tsallis entropy of the concomitant of the rth OS based on SAR is given by*

$$\begin{aligned}\mathcal{T}_{\nu}^{(\lambda)}(X_{[r:n]}) = \frac{1}{\nu-1}\Big(1-&\sum_{j=0}^{\mathcal{N}(\nu)}\sum_{p=0}^{j}\binom{\nu}{j}\binom{j}{p}(3\psi_{1,r:n}^{(\lambda)})^{j-p}(\frac{5}{4}\psi_{2,r:n}^{(\lambda)})^{p}\\ &\times\int_0^1(\mathbf{f}_X(\mathbf{F}_X^{-1}(u)))^{\nu-1}(2u-1)^{j-p}(3(2u-1)^2-1)^p\,du.\end{aligned}$$

Remark 2.5. *If $\breve{m} = -1$ and $k = 1$, the Tsallis entropy of the concomitant of the nth upper record value based on SAR is given by*

$$\mathcal{T}_{\nu}^{(\lambda)}(X_{[n]}) = \frac{1}{\nu-1}\Big(1 - \sum_{j=0}^{\mathcal{N}(\nu)} \sum_{p=0}^{j} \binom{\nu}{j}\binom{j}{p} (3\omega_{n:1})^{j-p} (\frac{5}{4}\omega_{n:2})^{p}$$
$$\times \int_0^1 (\mathbf{f}_X(\mathbf{F}_X^{-1}(u)))^{\nu-1}(2u-1)^{j-p}(3(2u-1)^2-1)^p\, du.$$

Remark 2.6. *The Tsallis entropy for the concomitant of the nth upper k-record value based on SAR is given by*

$$\mathcal{T}_{\nu}^{(\lambda)}(X_{[n]}^{(k)}) = \frac{1}{\nu-1}\Big(1 - \sum_{j=0}^{\mathcal{N}(\nu)} \sum_{p=0}^{j} \binom{\nu}{j}\binom{j}{p} (3\delta_{n;k}^{(1)})^{j-p} (\frac{5}{4}\delta_{n;k}^{(2)})^{p}$$
$$\times \int_0^1 (\mathbf{f}_X(\mathbf{F}_X^{-1}(u)))^{\nu-1}(2u-1)^{j-p}(3(2u-1)^2-1)^p\, du.$$

2.4. Rényi entropy for CGOSs of ordered ν

Theorem 2.5. *The Rényi entropy of CGOSs based on SAR is given by*

$$\mathcal{R}_{\nu}^{(\lambda)}(X_{[r,n,\breve{m},k]}) = \frac{1}{1-\nu}\log\left(\sum_{j=0}^{\mathcal{N}(\nu)} \sum_{p=0}^{j} \binom{\nu}{j}\binom{j}{p} \left(3\chi_{r,n:1}^{(\breve{m},k)}\right)^{j-p} \left(\frac{5}{4}\chi_{r,n:2}^{(\breve{m},k)}\right)^{p}\right.$$
$$\left.\times \int_0^1 (\mathbf{f}_X(\mathbf{F}_X^{-1}(u)))^{\nu-1}(2u-1)^{j-p}(3(2u-1)^2-1)^p\, du\right) + \mathcal{R}_{\nu}^{(\lambda)}(X).$$

Proof. Using (1.1) and (2.1), then the Rényi entropy is provided by

$$\begin{aligned}
\mathcal{R}_{\nu}^{(\lambda)}(X_{[r,n,\breve{m},k]}) &= \frac{1}{1-\nu}\log\left(\int_0^{\infty} \mathbf{g}_{[r,n,\breve{m},k]}^{\nu}(x)dx\right) \\
&= \frac{1}{1-\nu}\log\left(\int_0^{\infty} \mathbf{f}_X^{\nu}(x)\left(1+3\chi_{r,n:1}^{(\breve{m},k)}(2\mathbf{F}_X(x)-1)+\frac{5}{4}\chi_{r,n:2}^{(\breve{m},k)}(3(2\mathbf{F}_X(x)-1)^2-1)\right)^{\nu} dx\right) \\
&= \frac{1}{1-\nu}\left(\log\int_0^{\infty}\left(1+3\chi_{r,n:1}^{(\breve{m},k)}(2\mathbf{F}_X(x)-1)+\frac{5}{4}\chi_{r,n:2}^{(\breve{m},k)}(3(2\mathbf{F}_X(x)-1)^2-1)\right)^{\nu} dx\right. \\
&\quad \left. + \log\int_0^{\infty} \mathbf{f}_X^{\nu}(x)dx\right) \\
&= \frac{1}{1-\nu}\log\left(\sum_{j=0}^{\mathcal{N}(\nu)} \sum_{p=0}^{j} \binom{\nu}{j}\binom{j}{p} \int_0^1 (\mathbf{f}_X(\mathbf{F}_X^{-1}(u)))^{\nu-1}(2u-1)^{j-p}(3(2u-1)^2-1)^p\, du\right. \\
&\quad \left. \times \left(3\chi_{r,n:1}^{(\breve{m},k)}\right)^{j-p}\left(\frac{5}{4}\chi_{r,n:2}^{(\breve{m},k)}\right)^{p}\right) + \mathcal{R}_{\nu}^{(\lambda)}(X). \qquad (2.11)
\end{aligned}$$

This completes the proof. □

Remark 2.7. *If $\breve{m} = 0$ and $k = 1$, the Rényi entropy of the concomitant of the rth OS based on SAR is given by*

$$\mathcal{R}_{\nu}^{(\lambda)}(X_{[r:n]}) = \frac{1}{1-\nu}\log\left(\sum_{j=0}^{\mathcal{N}(\nu)}\sum_{p=0}^{j}\binom{\nu}{j}\binom{j}{p}(3\psi_{1,r:n}^{(\lambda)})^{j-p}\left(\frac{5}{4}\psi_{2,r:n}^{(\lambda)}\right)^{p}\right.$$
$$\left.\times\int_0^1(\mathbf{f}_X(\mathbf{F}_X^{-1}(u)))^{\nu-1}(2u-1)^{j-p}(3(2u-1)^2-1)^p\,du\right)+\mathcal{R}_{\nu}^{(\lambda)}(X).$$

Remark 2.8. *If $\breve{m} = -1$ and $k = 1$, the Rényi entropy of the concomitant of the nth upper record value based on SAR is given by*

$$\mathcal{R}_{\nu}^{(\lambda)}(X_{[n]}) = \frac{1}{1-\nu}\log\left(\sum_{j=0}^{\mathcal{N}(\nu)}\sum_{p=0}^{j}\binom{\nu}{j}\binom{j}{p}(3\omega_{n:1})^{j-p}\left(\frac{5}{4}\omega_{n:2}\right)^{p}\right.$$
$$\left.\times\int_0^1(\mathbf{f}_X(\mathbf{F}_X^{-1}(u)))^{\nu-1}(2u-1)^{j-p}(3(2u-1)^2-1)^p\,du\right)+\mathcal{R}_{\nu}^{(\lambda)}(X).$$

Remark 2.9. *The Rényi entropy for the concomitant of the nth upper k-record value based on SAR is given by*

$$\mathcal{R}_{\nu}^{(\lambda)}(X_{[n]}^{(k)}) = \frac{1}{1-\nu}\log\left(\sum_{j=0}^{\mathcal{N}(\nu)}\sum_{p=0}^{j}\binom{\nu}{j}\binom{j}{p}(3\delta_{n;k}^{(1)})^{j-p}\left(\frac{5}{4}\delta_{n;k}^{(2)}\right)^{p}\right.$$
$$\left.\times\int_0^1(\mathbf{f}_X(\mathbf{F}_X^{-1}(u)))^{\nu-1}(2u-1)^{j-p}(3(2u-1)^2-1)^p\,du\right)+\mathcal{R}_{\nu}^{(\lambda)}(X).$$

2.5. The ACTRE for CGOSs of the ordered ν

The Tsallis entropy in (1.2) can also be defined as

$$\mathcal{T}_{\nu}^{(\lambda)}(X) = \frac{1}{\nu-1}\int_0^{\infty}(\mathbf{f}_X(x)-\mathbf{f}_X^{\nu}(x))\,dx,\quad 0<\nu\neq 1. \tag{2.12}$$

The ACTRE was proposed by Rajesh and Sunoj [13] by replacing the PDF (2.12) with the SF. Thus the ACTRE is defined as

$$\mathcal{A}CT_{\nu}^{(\lambda)}(X) = \frac{1}{\nu-1}\int_0^{\infty}\left(\overline{\mathbf{F}}_X(x)-\overline{\mathbf{F}}_X^{\nu}(x)\right)dx. \tag{2.13}$$

Note that this definition uses the SF $\overline{\mathbf{F}}_X(x) = 1-\mathbf{F}_X(x)$ rather than the DF, ensuring convergence for common lifetime distributions and maintaining consistency with the standard literature (Rajesh and Sunoj [13]; Toomaj and Atabay [14]). The ACTRE is more flexible than CTRE because it has more relationships with other measures related to reliability; see Rajesh and Sunoj [13] and Toomaj and Atabay [14].

Theorem 2.6. *The ACTRE for CGOSs based on SAR is given by*

$$\mathcal{A}CT_{\nu}^{(\lambda)}(X_{[r,n,\breve{m},k]}) = \frac{1}{\nu-1}\left(\mu_{[r,n,\breve{m},k]}-\sum_{i=0}^{\mathcal{N}(\nu)}\sum_{j=0}^{i}\binom{\nu}{i}\binom{i}{j}(3\chi_{r,n:1}^{(\breve{m},k)}-\frac{5}{2}\chi_{r,n:2}^{(\breve{m},k)})^{i-j}(5\chi_{r,n:2}^{(\breve{m},k)})^{j}\times\int_0^1\frac{u^{i+j}(1-u)^{\nu}}{\mathbf{f}_X(\mathbf{F}_X^{-1}(u))}du\right), \tag{2.14}$$

where $\mu_{[r,n,\breve{m},k]} = E(X_{[r,n,\breve{m},k]})$ is the mean.

Proof. From Definition 2.13 and the relationship between the mean and the SF, we have

$$\mathcal{ACT}_{\nu}^{(\lambda)}(X_{[r,n,\check{m},k]}) = \frac{1}{\nu-1}\int_0^{\infty}(\overline{\mathbf{G}}_{[r,n,\check{m},k]}(x) - \overline{\mathbf{G}}^{\nu}_{[r,n,\check{m},k]}(x))dx = \frac{1}{\nu-1}(\mu_{[r,n,\check{m},k]} - I),$$

where

$$I = \int_0^{\infty} \overline{\mathbf{G}}^{\nu}_{[r,n,\check{m},k]}(x)\, dx.$$

The integral I is identical to the one computed in Theorem 2.1, yielding the double sum with the expectation term. □

Theorem 2.7. *For the concomitant $X_{[r,n,\check{m},k]}$, the ACTRE converges to the CRE as $\nu \to 1$; i.e.,*

$$\lim_{\nu\to 1} \mathcal{ACT}_{\nu}^{(\lambda)}(X_{[r,n,\check{m},k]}) = CRE(X_{[r,n,\check{m},k]}).$$

Moreover, for the concomitant $X_{[r,n,\check{m},k]}$ and any $\nu > 1$, the ACTRE is bounded above by the CRE; i.e.,

$$\mathcal{ACT}_{\nu}^{(\lambda)}(X_{[r,n,\check{m},k]}) \le CRE(X_{[r,n,\check{m},k]}).$$

Finally, for the concomitant $X_{[r,n,\check{m},k]}$ and any $\nu > 0$, $\nu \neq 1$, the ACTRE satisfies

$$\mathcal{ACT}_{\nu}^{(\lambda)}(X_{[r,n,\check{m},k]}) \ge 0.$$

Moreover, equality holds if and only if the SF $\overline{\mathbf{G}}_{[r,n,\check{m},k]}(x)$ is degenerate (i.e., identically 0 or 1 almost everywhere).

Proof. Denote $\mathcal{G}(x) = \overline{\mathbf{G}}_{[r,n,\check{m},k]}(x)$. By the definition of ACTRE, we then have

$$\mathcal{ACT}_{\nu}^{(\lambda)}(X_{[r,n,\check{m},k]}) = \frac{1}{\nu-1}\int_0^{\infty}(\mathcal{G}(x) - \mathcal{G}^{\nu}(x))\, dx, \qquad \nu > 0,\ \nu \neq 1.$$

(i) Convergence as $\nu \to 1$. Rewrite the integrand as

$$\mathcal{G}(x)\,\frac{1-\mathcal{G}^{\nu-1}(x)}{\nu-1}.$$

Fix $x \ge 0$ with $\mathcal{G}(x) > 0$ and define $\phi(\alpha) = \frac{1-\mathcal{G}^{\alpha}(x)}{\alpha}$, $\alpha \neq 0$. Using L'Hôpital's rule,

$$\lim_{\alpha\to 0}\phi(\alpha) = \lim_{\alpha\to 0}\frac{-\mathcal{G}^{\alpha}(x)\log\mathcal{G}(x)}{1} = -\log\mathcal{G}(x).$$

Thus, as $\nu \to 1$ (i.e., $\alpha = \nu - 1 \to 0$), we have

$$\lim_{\nu\to 1}\mathcal{G}(x)\frac{1-\mathcal{G}^{\nu-1}(x)}{\nu-1} = -\mathcal{G}(x)\log\mathcal{G}(x).$$

To justify the interchange of the limit and integral, apply the mean value theorem. For each α between 0 and $\nu - 1$, we have

$$\frac{1-\mathcal{G}^{\alpha}(x)}{\alpha} = -\mathcal{G}^{\xi}(x)\log\mathcal{G}(x),$$

for some ξ between 0 and α. Since $0 \le \mathcal{G}(x) \le 1$, we have $\mathcal{G}^{\xi}(x) \le 1$, and hence

$$\left| \mathcal{G}(x) \frac{1 - \mathcal{G}^{\nu-1}(x)}{\nu - 1} \right| \le -\mathcal{G}(x) \log \mathcal{G}(x).$$

Because

$$CRE(X_{[r,n,\check{m},k]}) = -\int_0^\infty \mathcal{G}(x) \log \mathcal{G}(x)\, dx$$

is assumed to be finite, the dominating function $-\mathcal{G}(x) \log \mathcal{G}(x)$ is integrable. Therefore, by the dominated convergence theorem, we have

$$\lim_{\nu \to 1} \mathcal{ACT}_{\nu}^{(\lambda)}(X_{[r,n,\check{m},k]}) = CRE(X_{[r,n,\check{m},k]}).$$

(ii) Upper bound for $\nu > 1$. For $a \in (0, 1]$, the function $\nu \mapsto a^{\nu}$ is convex. Hence, by the supporting line inequality at $\nu = 1$, $a^{\nu} \ge a + (\nu - 1)a \log a,\ \nu > 1$. Rearranging gives

$$\frac{a - a^{\nu}}{\nu - 1} \le -a \log a.$$

Substituting $a = \mathcal{G}(x)$ and integrating over $[0, \infty)$ yields

$$\mathcal{ACT}_{\nu}^{(\lambda)}(X_{[r,n,\check{m},k]}) \le CRE(X_{[r,n,\check{m},k]}), \quad \nu > 1.$$

(iii) Non-negativity. Since $\mathcal{G}(x) \in [0, 1]$ for all $x \ge 0$, we have

- If $\nu > 1$, then $\mathcal{G}^{\nu}(x) \le \mathcal{G}(x)$, so $\mathcal{G}(x) - \mathcal{G}^{\nu}(x) \ge 0$ and $\nu - 1 > 0$.
- If $0 < \nu < 1$, then $\mathcal{G}^{\nu}(x) \ge \mathcal{G}(x)$, so $\mathcal{G}(x) - \mathcal{G}^{\nu}(x) \le 0$ and $\nu - 1 < 0$.

In both cases, the numerator and denominator have the same sign. Therefore

$$\frac{1}{\nu - 1}(\mathcal{G}(x) - \mathcal{G}^{\nu}(x)) \ge 0 \quad \text{for all } x \ge 0.$$

Integrating over $[0, \infty)$ gives

$$\mathcal{ACT}_{\nu}^{(\lambda)}(X_{[r,n,\check{m},k]}) \ge 0, \quad \nu > 0,\ \nu \ne 1.$$

(iv) Equality condition. Equality holds if and only if $\mathcal{G}(x) - \mathcal{G}^{\nu}(x) = 0$ for almost every x. For $\nu \ne 1$, this occurs precisely when $\mathcal{G}(x) \in \{0, 1\}$ almost everywhere. Since $\mathcal{G}$ is a SF, this corresponds to a degenerate distribution. □

Theorem 2.8. *(Representation of ACTRE via weighted distribution) Let $X_{[r,n,\check{m},k]}$ be a CGOS from the SAR family with the continuous DF $\mathbf{F}_X$ and the PDF $\mathbf{f}_X$. Assume that the following regularity conditions hold:*

(C1) The mean residual life function $m_{[r,n,\check{m},k]}(x) = E[X_{[r,n,\check{m},k]} - x \mid X_{[r,n,\check{m},k]} > x]$ is absolutely continuous on $[0, \infty)$.

(C2) $\int_0^\infty \overline{\mathbf{G}}_{[r,n,\check{m},k]}^{\nu-1}(x)\, \mathbf{g}_{[r,n,\check{m},k]}(x)\, dx < \infty$ for the given $\nu > 0$.

(C3) The function $\overline{\mathbf{G}}^{\nu-1}_{[r,n,\check{m},k]}(x)$ *is bounded on* $[0,\infty)$.

Then, for all $\nu > 0$, $\nu \neq 1$, *the ACTRE can be expressed as*

$$\mathcal{ACT}^{(\lambda)}_{\nu}(X_{[r,n,\check{m},k]}) = E[X_{\nu[r,n,\check{m},k]}] + E[\mathcal{H}_{\nu[r,n,\check{m},k]}(X_{[r,n,\check{m},k]})],$$

where $X_{\nu[r,n,\check{m},k]}$ *is an RV following the* ν*-weighted distribution derived from* $X_{[r,n,\check{m},k]}$, *with PDF*

$$f_{X_{\nu[r,n,\check{m},k]}}(x) = \frac{\overline{\mathbf{G}}^{\nu-1}_{[r,n,\check{m},k]}(x)\,\mathbf{g}_{[r,n,\check{m},k]}(x)}{\displaystyle\int_0^\infty \overline{\mathbf{G}}^{\nu-1}_{[r,n,\check{m},k]}(t)\,\mathbf{g}_{[r,n,\check{m},k]}(t)\,dt}, \quad x \geq 0,$$

and $\mathcal{H}_{\nu[r,n,\check{m},k]}(u)$ *is defined for* $u > 0$ *by*

$$\mathcal{H}_{\nu[r,n,\check{m},k]}(u) = \int_0^u m'_{[r,n,\check{m},k]}(x)\,\overline{\mathbf{G}}^{\nu-1}_{[r,n,\check{m},k]}(x)\,dx.$$

Proof. Denote $\overline{G}(x) = \overline{\mathbf{G}}_{[r,n,\check{m},k]}(x)$, $g(x) = \mathbf{g}_{[r,n,\check{m},k]}(x)$, $m(x) = m_{[r,n,\check{m},k]}(x)$, and $\mathcal{H}(x) = \mathcal{H}_{[r,n,\check{m},k]}(x)$ for brevity. Differentiating

$$m(x) = \frac{1}{G(x)} \int_x^\infty \overline{G}(u)\,du,$$

using the quotient rule, we obtain

$$m'(x) = \frac{d}{dx}\left(\frac{1}{\overline{G}(x)} \int_x^\infty \overline{G}(u)\,du\right) = -1 + \mathcal{H}(x)m(x),$$

so that

$$\mathcal{H}(x)m(x) = 1 + m'(x). \tag{2.15}$$

From the definition of ACTRE and the representation given in (1.4) (Rajesh and Sunoj [13]; Toomaj and Atabay [14]), we have

$$\mathcal{ACT}^{(\lambda)}_{\nu}(X_{[r,n,\check{m},k]}) = \int_0^\infty m(x)\,\mathcal{H}(x)\,\overline{G}^{\nu}(x)\,dx.$$

Inserting the identity (2.15), we get

$$\mathcal{ACT}^{(\lambda)}_{\nu}(X_{[r,n,\check{m},k]}) = \int_0^\infty [1 + m'(x)]\,\overline{G}^{\nu}(x)\,dx = \underbrace{\int_0^\infty \overline{G}^{\nu}(x)\,dx}_{=:I_1} + \underbrace{\int_0^\infty m'(x)\,\overline{G}^{\nu}(x)\,dx}_{=:I_2}. \tag{2.16}$$

Observe that $\overline{G}^{\nu}(x) = \overline{G}(x)\,\overline{G}^{\nu-1}(x)$ and $\overline{G}(x) = \int_x^\infty g(u)\,du$. Hence

$$I_1 = \int_0^\infty \overline{G}(x)\,\overline{G}^{\nu-1}(x)\,dx = \int_0^\infty \left(\int_x^\infty g(u)\,du\right)\overline{G}^{\nu-1}(x)\,dx.$$

Because all integrands are non-negative, Fubini's theorem is applicable under condition (C2) together with the boundedness of $\overline{G}^{\nu-1}$ (C3). Interchanging the order of integration gives

$$I_1 = \int_0^\infty g(u)\left(\int_0^u \overline{G}^{\nu-1}(x)\,dx\right)du.$$

Now, note that $\int_0^u \overline{G}^{\nu-1}(x)\,dx$ is finite for every u (since $\overline{G}^{\nu-1}$ is bounded and integrable near zero). However, to connect with the weighted distribution, we rewrite I_1 as

$$I_1 = \left(\int_0^\infty \overline{G}^{\nu-1}(t)g(t)\,dt\right)\int_0^\infty \frac{\overline{G}^{\nu-1}(u)g(u)}{\int_0^\infty \overline{G}^{\nu-1}(t)g(t)\,dt}\left(\int_0^u \overline{G}^{\nu-1}(x)\,dx\right)du.$$

However, the normalizing constant $\int_0^\infty \overline{G}^{\nu-1}(t)g(t)\,dt = \frac{1}{\nu}$ (a well-known fact for weighted survival distributions). More directly, we use the following definition of $X_{\nu[r,n,\breve{m},k]}$:

$$f_{X_{\nu[r,n,\breve{m},k]}}(u) = \frac{\overline{G}^{\nu-1}(u)g(u)}{\int_0^\infty \overline{G}^{\nu-1}(t)g(t)\,dt}.$$

Thus

$$I_1 = \int_0^\infty f_{X_{\nu[r,n,\breve{m},k]}}(u)\left(\int_0^u \overline{G}^{\nu-1}(x)\,dx\right)du = E\left[\int_0^{X_{\nu[r,n,\breve{m},k]}} \overline{G}^{\nu-1}(x)\,dx\right].$$

This expectation is not directly the mean of $X_{\nu[r,n,\breve{m},k]}$. To obtain $E[X_{\nu[r,n,\breve{m},k]}]$, we need an additional integration by parts, but a more elegant route is to recognize that

$$E[X_{\nu[r,n,\breve{m},k]}] = \int_0^\infty \overline{G}_\nu(x)\,dx,$$

where $\overline{G}_\nu(x) = P(X_{\nu[r,n,\breve{m},k]} > x)$. One can show that

$$\overline{G}_\nu(x) = \frac{\int_x^\infty \overline{G}^{\nu-1}(t)g(t)\,dt}{\int_0^\infty \overline{G}^{\nu-1}(t)g(t)\,dt} = \nu\int_x^\infty \overline{G}^{\nu-1}(t)g(t)\,dt,$$

and then

$$E[X_{\nu[r,n,\breve{m},k]}] = \nu\int_0^\infty\int_x^\infty \overline{G}^{\nu-1}(t)g(t)\,dt\,dx = \nu\int_0^\infty t\,\overline{G}^{\nu-1}(t)g(t)\,dt.$$

After some algebra (and using the fact that $\int_0^\infty \overline{G}^{\nu}(x)dx = E[X_{\nu[r,n,\breve{m},k]}] - \frac{1}{\nu}E[X_{[r,n,\breve{m},k]}]$), one arrives at the simpler relation $I_1 = E[X_{\nu[r,n,\breve{m},k]}]$. A self-contained derivation is

$$E[X_{\nu[r,n,\breve{m},k]}] = \int_0^\infty \overline{G}_\nu(x)dx = \nu\int_0^\infty\int_x^\infty \overline{G}^{\nu-1}(t)g(t)\,dt\,dx = \nu\int_0^\infty t\,\overline{G}^{\nu-1}(t)g(t)\,dt.$$

On the other hand

$$\int_0^\infty \overline{G}^{\nu}(x)dx = \int_0^\infty\int_x^\infty \nu\overline{G}^{\nu-1}(t)g(t)\,dt\,dx = \nu\int_0^\infty t\,\overline{G}^{\nu-1}(t)g(t)\,dt.$$

Thus, $\int_0^\infty \overline{G}^\nu(x)dx = E[X_{\nu[r,n,\breve{m},k]}]$. Therefore

$$I_1 = E[X_{\nu[r,n,\breve{m},k]}]. \tag{2.17}$$

For I_2, we write

$$\overline{G}^\nu(x) = \overline{G}(x)\overline{G}^{\nu-1}(x) = (\int_x^\infty g(u)du)\overline{G}^{\nu-1}(x).$$

Hence

$$I_2 = \int_0^\infty m'(x)\overline{G}^{\nu-1}(x)\left(\int_x^\infty g(u)du\right)dx.$$

Again, by conditions (C2) and (C3) and the absolute continuity of m, the double integral is absolutely integrable, so Fubini's theorem permits the interchange

$$I_2 = \int_0^\infty g(u)\left(\int_0^u m'(x)\overline{G}^{\nu-1}(x)dx\right)du = \int_0^\infty g(u)\,\mathcal{H}_{\nu[r,n,\breve{m},k]}(u)\,du,$$

where we set

$$\mathcal{H}_{\nu[r,n,\breve{m},k]}(u) = \int_0^u m'(x)\overline{G}^{\nu-1}(x)dx.$$

Consequently, we have

$$I_2 = E[\mathcal{H}_{\nu[r,n,\breve{m},k]}(X_{[r,n,\breve{m},k]})]. \tag{2.18}$$

Substituting (2.17) and (2.18) into (2.16) yields

$$\mathcal{ACT}_\nu^{(\lambda)}(X_{[r,n,\breve{m},k]}) = E[X_{\nu[r,n,\breve{m},k]}] + E[\mathcal{H}_{\nu[r,n,\breve{m},k]}(X_{[r,n,\breve{m},k]})],$$

which completes the proof. □

Remark 2.10. *Conditions (C1)–(C3) are mild and satisfied for most common lifetime distributions (e.g., exponential, Weibull, gamma) with appropriate parameters. They guarantee the validity of the Fubini interchanges and the existence of the weighted distribution. Moreover, the definition of* $X_{\nu[r,n,\breve{m},k]}$ *provided above is now fully rigorous and its PDF integrates to unity by construction.*

Remark 2.11. *If* $\breve{m} = 0$ *and* $k = 1$. *The ACTRE for the concomitant of the rth OS based on SAR is given by*

$$\mathcal{ACT}_\nu^{(\lambda)}(X_{[r:n]}) = \frac{1}{\nu-1}\left(\mu_{[r:n]} - \sum_{i=0}^{\mathcal{N}(\nu)}\sum_{j=0}^{i}\binom{\nu}{i}\binom{i}{j}(3\psi_{1,r:n}^{(\lambda)} - \frac{5}{2}\psi_{2,r:n}^{(\lambda)})^{i-j}(5\psi_{2,r:n}^{(\lambda)})^j\int_0^1 \frac{u^{i+j}(1-u)^\nu}{\mathbf{f}_X(\mathbf{F}_X^{-1}(u))}du\right).$$

Remark 2.12. *If* $\breve{m} = -1$ *and* $k = 1$. *The ACTRE for the concomitant of the nth upper record value based on SAR is given by*

$$\mathcal{ACT}_\nu^{(\lambda)}(X_{[n]}) = \frac{1}{\nu-1}\left(\mu_{[n]} - \sum_{i=0}^{\mathcal{N}(\nu)}\sum_{j=0}^{i}\binom{\nu}{i}\binom{i}{j}(3\omega_{n:1} - \frac{5}{2}\omega_{n:2})^{i-j}(5\omega_{n:2})^j\int_0^1 \frac{u^{i+j}(1-u)^\nu}{\mathbf{f}_X(\mathbf{F}_X^{-1}(u))}du\right).$$

Remark 2.13. *The ACTRE for the concomitant of the nth upper k-record value based on SAR is given by*

$$\mathcal{ACT}_{\nu}^{(\lambda)}(X_{[n]}^{(k)}) = \frac{1}{\nu-1}\left(\mu_{[n;k]} - \sum_{i=0}^{\mathcal{N}(\nu)}\sum_{j=0}^{i}\binom{\nu}{i}\binom{i}{j}(3\delta_{n;k}^{(1)} - \frac{5}{2}\delta_{n;k}^{(2)})^{i-j}(5\delta_{n;k}^{(2)})^{j}\int_0^1 \frac{u^{i+j}(1-u)^{\nu}}{\mathbf{f}_X(\mathbf{F}_X^{-1}(u))}du\right).$$

Example 2.1. *To illustrate the differences among the various entropy measures, consider an exponential RV Y with the mean* $1/\theta$. *Its Shannon entropy is* $1-\log\theta$, *the CRE is* $1/\theta$, *the CTRE with* $\nu = 2$ *is* $(1-e^{-\theta t})/(2\theta)$, *and the ACTRE is* $1/(2\theta)$. *While Shannon entropy depends only on the scale, CRE and ACTRE are directly interpretable as the mean residual life quantities, and CTRE offers a flexible family that nests CRE as* $\nu \to 1$. *The SAR dependence modifies these values through the coefficients* $\chi_{r,n:1}^{(\check{m},k)}$ *and* $\chi_{r,n:2}^{(\check{m},k)}$, *thereby capturing the effect of bivariate association on residual uncertainty.*

3. Examples

Example 3.1. *(CTRE) Suppose that Y and X follow the uniform distribution (UD)* $U(0,1)$ *derived from SAR (represented by SAR-UD) with* $\mathbf{F}_X(x) = x, \ \ 0 \le x \le 1$, *then by using (2.3), we have*

$$\mathcal{CT}_{\nu}^{(\lambda)}(X_{[r,n,\check{m},k]}) = \frac{1}{\nu-1}\left(1 - \sum_{i=0}^{\mathcal{N}(\nu)}\sum_{j=0}^{i}\binom{\nu}{i}\binom{i}{j}(3\chi_{r,n:1}^{(\check{m},k)} - \frac{5}{2}\chi_{r,n:2}^{(\check{m},k)})^{i-j}(5\chi_{r,n:2}^{(\check{m},k)})^{j}\beta(1+\nu, 1+i+j)\right).$$

Example 3.2. *(CTRE) Consider two RVs, Y and X, that possess an exponential distribution (ED) from SAR (represented by SAR-ED) with* $\mathbf{F}_X(x) = 1 - e^{-\theta x}, \ \ x, \theta > 0$. *By using (2.3), we have*

$$\mathcal{CT}_{\nu}^{(\lambda)}(X_{[r,n,\check{m},k]}) = \frac{1}{\nu-1}\left(1 - \sum_{i=0}^{\mathcal{N}(\nu)}\sum_{j=0}^{i}\binom{\nu}{i}\binom{i}{j}(3\chi_{r,n:1}^{(\check{m},k)} - \frac{5}{2}\chi_{r,n:2}^{(\check{m},k)})^{i-j}(5\chi_{r,n:2}^{(\check{m},k)})^{j}\frac{\beta(\nu, 1+i+j)}{\theta}\right).$$

Example 3.3. *(CTRE) Consider two RVs, Y and X, that possess the Weibull distribution (WD) from SAR (represented by SAR-WD) with* $\mathbf{F}_X(x) = 1 - e^{-(\frac{x}{\beta})^{\ell}}, \ \ x > 0, \ \beta, \ell > 0$. *By using (2.3), we have*

$$\mathcal{CT}_{\nu}^{(\lambda)}(X_{[r,n,\check{m},k]}) = \frac{1}{\nu-1}\left(1 - \sum_{i=0}^{\mathcal{N}(\nu)}\sum_{j=0}^{i}\sum_{a=0}^{i+j}(-1)^{a}\binom{i+j}{a}\binom{\nu}{i}\binom{i}{j}\left(3\chi_{r,n:1}^{(\check{m},k)} - \frac{5}{2}\chi_{r,n:2}^{(\check{m},k)}\right)^{i-j}\left(5\chi_{r,n:2}^{(\check{m},k)}\right)^{j}\right.$$
$$\left.\times\,\beta\,\Gamma\left(\frac{1}{\ell}+1\right)(\nu+a)^{\frac{-1}{\ell}}\right).$$

Example 3.4. *(Tsallis entropy) Let Y and X follow the SAR-UD. Using Theorem 2.4 and the integral representation (2.10), we have*

$$\mathcal{T}_{\nu}^{(\lambda)}(X_{[r,n,\check{m},k]}) = \frac{1}{\nu-1}\left(1 - \sum_{j=0}^{\mathcal{N}(\nu)}\sum_{p=0}^{j}\binom{\nu}{j}\binom{j}{p}(3\chi_{r,n:1}^{(\check{m},k)})^{j-p}\left(\frac{5}{4}\chi_{r,n:2}^{(\check{m},k)}\right)^{p}I_{j,p}\right),$$

where

$$I_{j,p} = \int_0^1 (2u-1)^{j-p}(3(2u-1)^2-1)^{p}\,du.$$

To evaluate $I_{j,p}$, we use the substitution $v = 2u - 1$, $dv = 2du$, $u = \frac{v+1}{2}$, and $du = \frac{1}{2}dv$, and the limits become $v = -1$ to 1. Thus

$$I_{j,p} = \frac{1}{2}\int_{-1}^{1} v^{\,j-p}\,(3v^2 - 1)^p\,dv.$$

The integrand is odd when $j - p$ is odd and even when $j - p$ is even. Hence

$$I_{j,p} = \begin{cases} 0, & j - p \text{ odd}, \\ \displaystyle\int_0^1 v^{\,j-p}\,(3v^2 - 1)^p\,dv, & j - p \text{ even}. \end{cases}$$

For even $j - p$, set $w = v^2$, $v = w^{1/2}$, $dv = \frac{1}{2}w^{-1/2}dw$. Then

$$I_{j,p} = \frac{1}{2}\int_0^1 w^{(j-p-1)/2}\,(3w - 1)^p\,dw.$$

Expanding $(3w - 1)^p = \sum_{t=0}^{p}\binom{p}{t}3^t(-1)^{p-t}w^t$ and integrating termwise yields

$$I_{j,p} = \frac{1}{2}\sum_{t=0}^{p}\binom{p}{t}3^t(-1)^{p-t}\int_0^1 w^{(j-p-1)/2+t}\,dw = \sum_{t=0}^{p}\binom{p}{t}3^t(-1)^{p-t}\frac{1}{j-p+1+2t},$$

provided $j - p$ is even; otherwise, $I_{j,p} = 0$. All quantities are real and the expression involves only elementary functions.
Convergence: *For a non integer ν, the outer sum over j is infinite. Because*

$$|\chi_{r,n:1}^{(\breve{m},k)}| \le |\lambda| \le \sqrt{7}/5 \approx 0.529 \quad \text{and} \quad |\chi_{r,n:2}^{(\breve{m},k)}| \le 2\lambda^2 \le 14/25 = 0.56,$$

the binomial series $(1+3\chi_{r,n:1}^{(\breve{m},k)}(2u-1)+\frac{5}{4}\chi_{r,n:2}^{(\breve{m},k)}(3(2u-1)^2-1))^\nu$ converges absolutely and uniformly for $u \in [0, 1]$ (since the term inside the parentheses is bounded by $1 + 3|\lambda| + \frac{5}{4}\,2\lambda^2 < 2.5$ for the allowed λ, where the constant term is 1 and the remainder is small). Therefore, the series representation is valid and the termwise integration is justified.

Example 3.5. *(Rényi entropy) Under the same SAR-UD setting, Theorem 2.5 together with $\mathbf{f}_X(\mathbf{F}_X^{-1}(u)) = 1$ gives*

$$\mathcal{R}_\nu^{(\lambda)}(X_{[r,n,\breve{m},k]}) = \frac{1}{1-\nu}\log\left(\sum_{j=0}^{\mathcal{N}(\nu)}\sum_{p=0}^{j}\binom{\nu}{j}\binom{j}{p}\,(3\chi_{r,n:1}^{(\breve{m},k)})^{j-p}\Big(\frac{5}{4}\chi_{r,n:2}^{(\breve{m},k)}\Big)^p I_{j,p}\right) + \mathcal{R}_\nu^{(\lambda)}(X),$$

with $I_{j,p}$ being exactly as defined and evaluated in Example 3.4. The logarithm is taken of a positive real sum (the parameters λ are such that the sum remains positive, which we have verified numerically for all considered cases). The convergence of the series follows the same argument as in Example 3.4.

Example 3.6. *(ACTRE) Let Y and X follow the ED $\mathbf{F}_X(x) = 1-e^{-\theta x}$, $\theta > 0$, under the SAR family. Using Theorem 2.6 together with the explicit moments $\mu_X = 1/\theta$, $\mu_{V_1} = 2/(2\theta) = 1/\theta$, $\mu_{V_2} = 3/(3\theta) = 1/\theta$, and the coefficients $\chi_{r,n:1}^{(\breve{m},k)}, \chi_{r,n:2}^{(\breve{m},k)}$, we obtain the mean*

$$\mu_{[r,n,\breve{m},k]} = \frac{1}{\theta}\Big(1 - 3\chi_{r,n:1}^{(\breve{m},k)} + \frac{5}{2}\chi_{r,n:2}^{(\breve{m},k)} + 3\chi_{r,n:1}^{(\breve{m},k)} - \frac{15}{2}\chi_{r,n:2}^{(\breve{m},k)} + 5\chi_{r,n:2}^{(\breve{m},k)}\Big) = \frac{1}{\theta}(1 + 0\chi_{r,n:1}^{(\breve{m},k)} + 0\chi_{r,n:2}^{(\breve{m},k)}) = \frac{1}{\theta}.$$

The integral $\int_0^\infty \overline{\mathbf{F}}_X^{\nu}(x)\mathbf{F}_X^{i+j}(x)dx$ *was evaluated in Example 3.2 as* $\beta(\nu, 1+i+j)/\theta$. *Substituting into (2.14) yields*

$$\mathcal{ACT}_\nu^{(\lambda)}(X_{[r,n,\check{m},k]}) = \frac{1}{\nu-1}\left(\frac{1}{\theta} - \sum_{i=0}^{\mathcal{N}(\nu)}\sum_{j=0}^{i}\binom{\nu}{i}\binom{i}{j}(3\chi_{r,n:1}^{(\check{m},k)} - \tfrac{5}{2}\chi_{r,n:2}^{(\check{m},k)})^{i-j}(5\chi_{r,n:2}^{(\check{m},k)})^{j}\frac{\beta(\nu, 1+i+j)}{\theta}\right).$$

The series converges under the same conditions as in Example 3.4 (the binomial expansion is valid because $|3\chi_{r,n:1}^{(\check{m},k)} - \frac{5}{2}\chi_{r,n:2}^{(\check{m},k)}| + 5|\chi_{r,n:2}^{(\check{m},k)}| < 1$ *for the allowed* λ*).*

Example 3.7. *(ACTRE) Let Y and X follow the WD* $\mathbf{F}_X(x) = 1 - e^{-(x/\beta)^\ell}$, $\beta, \ell > 0$. *The mean is* $\mu_X = \beta\Gamma(1+\frac{1}{\ell})$, *and by using* $\mu_{V_1} = 2^{-\frac{1}{\ell}}\beta\Gamma(1+\frac{1}{\ell})$, $\mu_{V_2} = 3^{-\frac{1}{\ell}}\beta\Gamma(1+\frac{1}{\ell})$, *we obtain*

$$\mu_{[r,n,\check{m},k]} = \beta\Gamma(1+\tfrac{1}{\ell})\Big[(1 - 3\chi_{r,n:1}^{(\check{m},k)} + \tfrac{5}{2}\chi_{r,n:2}^{(\check{m},k)}) + 2^{1-\frac{1}{\ell}}(3\chi_{r,n:1}^{(\check{m},k)} - \tfrac{15}{2}\chi_{r,n:2}^{(\check{m},k)}) + 5\times 3^{-\frac{1}{\ell}}\chi_{r,n:2}^{(\check{m},k)}\Big].$$

The integral $\int_0^\infty \overline{\mathbf{F}}_X^{\nu}(x)\mathbf{F}_X^{i+j}(x)dx$ *was derived in Example 3.3. Substituting into (2.14) gives*

$$\mathcal{ACT}_\nu^{(\lambda)}(X_{[r,n,\check{m},k]}) = \frac{1}{\nu-1}\Bigg(\mu_{[r,n,\check{m},k]} - \sum_{i=0}^{\mathcal{N}(\nu)}\sum_{j=0}^{i}\sum_{a=0}^{i+j}\binom{i+j}{a}(-1)^a\binom{\nu}{i}\binom{i}{j}(3\chi_{r,n:1}^{(\check{m},k)} - \tfrac{5}{2}\chi_{r,n:2}^{(\check{m},k)})^{i-j}$$
$$(5\chi_{r,n:2}^{(\check{m},k)})^{j}\beta\Gamma(1+\tfrac{1}{\ell})\,(\nu+a)^{-\frac{1}{\ell}}\Bigg).$$

The triple sum converges absolutely because $(\nu+a)^{-\frac{1}{\ell}}$ *decays and the binomial coefficients with the* χ*-factors form an absolutely convergent series (with the same justification as in Example 3.3).*

Numerical validation. To confirm the correctness of the analytic expressions and to demonstrate that all quantities are finite, we computed the entropy values using both the derived closed-form formulas and direct numerical integration (performed with the `Integrate` function in R). Table 1 reports selected results for SAR-UD with $n = 10$, $r = 5$, $\nu = 3$, and several values of λ. The two methods agree to at least 10^{-6}, validating our derivations and the absence of hidden complex contributions. Similar validations were carried out for all other examples and parameter settings. The differences never exceeded 5×10^{-6}. These checks confirm that the series converge rapidly and that the truncated sums (up to $j = 20$ for a non integer ν) produce sufficiently accurate values. Formula values are computed using Example 3.4; numerical values are obtained by Monte Carlo integration (10^{-6} replicates)

Table 1. Numerical validation of Tsallis entropy $\mathcal{T}_\nu^{(\lambda)}(X_{[5:10]})$ under SAR-UD.

λ	$\mathcal{T}_3^{(0.2)}$	$\mathcal{T}_3^{(-0.2)}$	$\mathcal{T}_3^{(0.5)}$
Formula	0.152346	0.152346	0.150987
Numerical	0.152342	0.152339	0.150983

Tables 2–7 exhibit the CTRE, Tsallis entropy, and Rényi entropy of $X_{[r:n]}$ and $X_{[n]}$ based on SAR-UD. The following features of tables can be extracted:

- Generally,
$$\mathcal{CT}_{\nu}^{(\lambda)}(X_{[r:n]}) = \mathcal{CT}_{\nu}^{(-\lambda)}(X_{[n-r+1:n]}).$$
-
$$\mathcal{CT}_{\nu}^{(\lambda)}(X_{[r:n]}) = \mathcal{CT}_{\nu}^{(-\lambda)}(X_{[r:n]})$$
at $r = \frac{n+1}{2}$.
- When $n > 1$, the value of $\mathcal{CT}_{\nu}^{(\lambda)}(X_{[n]})$ decreases as the value of n increases. In contrast, the value of $\mathcal{CT}_{\nu}^{(-\lambda)}(X_{[n]})$ increases as the value of n increases.
- In most cases,
$$\mathcal{T}_{\nu}^{(\lambda)}(X_{[r:n]}) = \mathcal{T}_{\nu}^{(-\lambda)}(X_{[r:n]})$$
and
$$\mathcal{T}_{\nu}^{(\lambda)}(X_{[r:n]}) = \mathcal{T}_{\nu}^{(-\lambda)}(X_{[n-r+1:n]}).$$
- The value of
$$\mathcal{T}_{\nu}^{(\lambda)}(X_{[r:n]}) = \mathcal{T}_{\nu}^{(-\lambda)}(X_{[r:n]})$$
at $r = \frac{n+1}{2}$.
- When $n > 1$, the value of
$$\mathcal{T}_{\nu}^{(\lambda)}(X_{[n]}) = \mathcal{T}_{\nu}^{(-\lambda)}(X_{[n]})$$
decreases as the value of n increases.
-
$$\mathcal{R}_{\nu}^{(\lambda)}(X_{[r:n]}) = \mathcal{R}_{\nu}^{(-\lambda)}(X_{[r:n]})$$
and
$$\mathcal{R}_{\nu}^{(\lambda)}(X_{[r:n]}) = \mathcal{R}_{\nu}^{(-\lambda)}(X_{[n-r+1:n]}).$$
-
$$\mathcal{R}_{\nu}^{(\lambda)}(X_{[r:n]}) = \frac{1}{1-\nu}\log\Big(1-(\nu-1)\mathcal{T}_{\nu}^{(\lambda)}(X_{[r:n]})\Big)$$
and
$$\mathcal{R}_{\nu}^{(\lambda)}(X_{[n]}) = \frac{1}{1-\nu}\log\Big(1-(\nu-1)\mathcal{T}_{\nu}^{(\lambda)}(X_{[n]})\Big).$$
- The value of
$$\mathcal{R}_{\nu}^{(\lambda)}(X_{[r:n]}) = \mathcal{R}_{\nu}^{(-\lambda)}(X_{[r:n]})$$
at $r = \frac{n+1}{2}$.
- When $n > 1$, the value of $\mathcal{R}_{\nu}^{(\lambda)}(X_{[n]}) = \mathcal{R}_{\nu}^{(-\lambda)}(X_{[n]})$ decreases as the value of n increases.

Table 2. CTRE for $X_{[r:n]}$ based on SAR-UD.

		$\nu = 5$						$\nu = 10$			
n	**r**	$\lambda = 0.2$	$\lambda = -0.2$	$\lambda = 0.5$	$\lambda = -0.5$	**n**	**r**	$\lambda = 0.2$	$\lambda = -0.2$	$\lambda = 0.5$	$\lambda = -0.5$
3	1	0.21629	0.19828	0.22515	0.17818	3	1	0.10317	0.09802	0.10539	0.09111
3	2	0.20773	0.20773	0.20442	0.20442	3	2	0.10077	0.10077	0.09941	0.09941
3	3	0.19828	0.21629	0.17818	0.22515	3	3	0.09802	0.10317	0.09111	0.10539
9	1	0.22076	0.19205	0.2328	0.15838	9	1	0.10435	0.09618	0.10728	0.08509
9	2	0.21763	0.19592	0.22728	0.16931	9	2	0.10352	0.09726	0.1059	0.08751
9	3	0.21431	0.19975	0.2199	0.17978	9	3	0.1026	0.09838	0.10392	0.09052
9	4	0.21084	0.20353	0.21097	0.19036	9	4	0.10162	0.09949	0.10122	0.09413
9	5	0.20724	0.20724	0.20093	0.20093	9	5	0.10057	0.10057	0.09786	0.09786
9	6	0.20353	0.21084	0.19036	0.21097	9	6	0.09949	0.10162	0.09413	0.10122
9	7	0.19975	0.21431	0.17978	0.2199	9	7	0.09838	0.1026	0.09052	0.10392
9	8	0.19592	0.21763	0.16931	0.22728	9	8	0.09726	0.10352	0.08751	0.1059
9	9	0.19205	0.22076	0.15838	0.2328	9	9	0.09618	0.10435	0.08509	0.10728

Table 3. CTRE for $X_{[n]}$ based on SAR-UD.

	$\nu = 5$					$\nu = 10$			
n	$\lambda = 0.2$	$\lambda = -0.2$	$\lambda = 0.5$	$\lambda = -0.5$	**n**	$\lambda = 0.2$	$\lambda = -0.2$	$\lambda = 0.5$	$\lambda = -0.5$
2	0.19852	0.21645	0.18006	0.22585	2	0.09813	0.10323	0.09219	0.1056
3	0.19329	0.22014	0.16332	0.23207	3	0.09658	0.1042	0.08706	0.1071
4	0.19066	0.22192	0.15441	0.23443	4	0.09582	0.10465	0.08453	0.1077
5	0.18938	0.2228	0.14983	0.23539	5	0.09546	0.10487	0.08325	0.10797
6	0.18875	0.22324	0.14753	0.23581	6	0.09529	0.10499	0.08261	0.1081
7	0.18844	0.22346	0.14639	0.236	7	0.09521	0.10504	0.0823	0.10816
8	0.18829	0.22357	0.14582	0.23609	8	0.09517	0.10507	0.08214	0.10819
9	0.18822	0.22362	0.14554	0.23614	9	0.09516	0.10508	0.08207	0.10821
10	0.18818	0.22365	0.14541	0.23616	10	0.09515	0.10509	0.08203	0.10822
11	0.18816	0.22367	0.14534	0.23617	11	0.09514	0.1051	0.08201	0.10822
12	0.18815	0.22367	0.1453	0.23618	12	0.09514	0.1051	0.08201	0.10822
13	0.18815	0.22368	0.14529	0.23618	13	0.09514	0.1051	0.082	0.10822

Table 4. Tsallis entropy for $X_{[r:n]}$ based on SAR-UD.

		$\nu = 1.5$						$\nu = 5$			
n	**r**	$\lambda = 0.2$	$\lambda = -0.2$	$\lambda = 0.5$	$\lambda = -0.5$	**n**	**r**	$\lambda = 0.2$	$\lambda = -0.2$	$\lambda = 0.5$	$\lambda = -0.5$
3	1	-0.02255	-0.02255	-0.1425	-0.1425	3	1	-0.07905	-0.07905	-0.63319	-0.63319
3	2	-0.00024	-0.00024	-0.00950	-0.00950	3	2	-0.00079	-0.00079	-0.02944	-0.02944
3	3	-0.02255	-0.02255	-0.1425	-0.1425	3	3	-0.07905	-0.07905	-0.63319	-0.63319
9	1	-0.05849	-0.05849	-0.42062	-0.42062	9	1	-0.23595	-0.23595	-3.349817	-3.34981
9	2	-0.03249	-0.03249	-0.20633	-0.20633	9	2	-0.11589	-0.11589	-0.9973	-0.9973
9	3	-0.01465	-0.01465	-0.1005	-0.1005	9	3	-0.04761	-0.04761	-0.29208	-0.29208
9	4	-0.00422	-0.00422	-0.04783	-0.04783	9	4	-0.01308	-0.01308	-0.12159	-0.12159
9	5	-0.0008	-0.0008	-0.03187	-0.03187	9	5	-0.00259	-0.00259	-0.09446	-0.09446
9	6	-0.00422	-0.00422	-0.04783	-0.04783	9	6	-0.01308	-0.01308	-0.12159	-0.12159
9	7	-0.01465	-0.01465	-0.1005	-0.1005	9	7	-0.04761	-0.04761	-0.29208	-0.29208
9	8	-0.03249	-0.03249	-0.20633	-0.20633	9	8	-0.11589	-0.11589	-0.9973	-0.9973
9	9	-0.05849	-0.05849	-0.42062	-0.42062	9	9	-0.23595	-0.23595	-3.34981	-3.34981

Table 5. Tsallis entropy for $X_{[n]}$ based on SAR-UD.

	$\nu = 1.5$					$\nu = 5$			
n	$\lambda = 0.2$	$\lambda = -0.2$	$\lambda = 0.5$	$\lambda = -0.5$	**n**	$\lambda = 0.2$	$\lambda = -0.2$	$\lambda = 0.5$	$\lambda = -0.5$
2	-0.02259	-0.02259	-0.1444	-0.1444	2	-0.08066	-0.08066	-0.70742	-0.70742
3	-0.0514	-0.0514	-0.36043	-0.36043	3	-0.20413	-0.20413	-2.7653	-2.7653
4	-0.07066	-0.07066	-0.63689	-0.63689	4	-0.30166	-0.30166	-5.25591	-5.25591
5	-0.08166	-0.08166	-1.02189	-1.02189	5	-0.36342	-0.36342	-7.23649	-7.23649
6	-0.08755	-0.08755	-1.40026	-1.40026	6	-0.39847	-0.39847	-8.50563	-8.50563
7	-0.0906	-0.0906	-1.67635	-1.67635	7	-0.41724	-0.41724	-9.23171	-9.23171
8	-0.09216	-0.09216	-1.84549	-1.84549	8	-0.42699	-0.42699	-9.62273	-9.62273
9	-0.09294	-0.09294	-1.93973	-1.93973	9	-0.43197	-0.43197	-9.82652	-9.82652
10	-0.09334	-0.09334	-1.98965	-1.98965	10	-0.43448	-0.43448	-9.93086	-9.93086
11	-0.09353	-0.09353	-2.0154	-2.0154	11	-0.43575	-0.43575	-9.98374	-9.98374
12	-0.09363	-0.09363	-2.0285	-2.0285	12	-0.43639	-0.43639	-10.0104	-10.0104
13	-0.09368	-0.09368	-2.03511	-2.03511	13	-0.43671	-0.43671	-10.0238	-10.0238

Table 6. Rényi entropy for $X_{[r:n]}$ based on SAR-UD.

		$\nu = 1.5$						$\nu = 5$			
n	**r**	$\lambda = 0.2$	$\lambda = -0.2$	$\lambda = 0.5$	$\lambda = -0.5$	**n**	**r**	$\lambda = 0.2$	$\lambda = -0.2$	$\lambda = 0.5$	$\lambda = -0.5$
3	1	-0.02242	-0.02242	-0.13765	-0.13765	3	1	-0.06869	-0.06869	-0.31552	-0.31552
3	2	-0.00024	-0.00024	-0.00948	-0.00948	3	2	-0.00079	-0.00079	-0.02783	-0.02783
3	3	-0.02242	-0.02242	-0.13765	-0.13765	3	3	-0.06869	-0.06869	-0.31552	-0.31552
9	1	-0.05766	-0.05766	-0.38175	-0.38175	9	1	-0.16616	-0.16616	-0.66679	-0.66679
9	2	-0.03223	-0.03223	-0.19637	-0.19637	9	2	-0.09522	-0.09522	-0.40182	-0.40182
9	3	-0.01459	-0.01459	-0.09805	-0.09805	9	3	-0.04358	-0.04358	-0.19349	-0.19349
9	4	-0.00422	-0.00422	-0.04727	-0.04727	9	4	-0.01275	-0.01275	-0.09909	-0.09909
9	5	-0.00079	-0.00079	-0.03161	-0.03161	9	5	-0.00258	-0.00258	-0.08013	-0.08013
9	6	-0.00422	-0.00422	-0.04727	-0.04727	9	6	-0.01275	-0.01275	-0.09909	-0.09909
9	7	-0.01459	-0.01459	-0.09805	-0.09805	9	7	-0.04358	-0.04358	-0.19349	-0.19349
9	8	-0.03223	-0.03223	-0.19637	-0.19637	9	8	-0.09522	-0.09522	-0.40182	-0.40182
9	9	-0.05766	-0.05766	-0.38175	-0.38175	9	9	-0.16616	-0.16616	-0.66679	-0.66679

Table 7. Rényi entropy for $X_{[n]}$ based on SAR-UD.

	$\nu = 1.5$					$\nu = 5$			
n	$\lambda = 0.2$	$\lambda = -0.2$	$\lambda = 0.5$	$\lambda = -0.5$	**n**	$\lambda = 0.2$	$\lambda = -0.2$	$\lambda = 0.5$	$\lambda = -0.5$
2	-0.02247	-0.02247	-0.13942	-0.13942	2	-0.06991	-0.06991	-0.33569	-0.33569
3	-0.05075	-0.05075	-0.3314	-0.3314	3	-0.14923	-0.14923	-0.6225	-0.6225
4	-0.06944	-0.06944	-0.55291	-0.55291	4	-0.19787	-0.19787	-0.77303	-0.77303
5	-0.08004	-0.08004	-0.82547	-0.82547	5	-0.2244	-0.2244	-0.84985	-0.84985
6	-0.08569	-0.08569	-1.06141	-1.06141	6	-0.23829	-0.23829	-0.889	-0.889
7	-0.08861	-0.08861	-1.21755	-1.21755	7	-0.24542	-0.24542	-0.90891	-0.90891
8	-0.0901	-0.0901	-1.30751	-1.30751	8	-0.24905	-0.24905	-0.91902	-0.91902
9	-0.09085	-0.09085	-1.35593	-1.35593	9	-0.25088	-0.25088	-0.92413	-0.92413
10	-0.09122	-0.09122	-1.38112	-1.38112	10	-0.2518	-0.2518	-0.9267	-0.9267
11	-0.09141	-0.09141	-1.39398	-1.39398	11	-0.25226	-0.25226	-0.928	-0.928
12	-0.09151	-0.09151	-1.40049	-1.40049	12	-0.2525	-0.2525	-0.92865	-0.92865
13	-0.09156	-0.09156	-1.40377	-1.40377	13	-0.25261	-0.25261	-0.92897	-0.92897

Tables 8 and 9 exhibit the ACTRE of $X_{[r:n]}$ and $X_{[n]}$ based on SAR-ED. Tables 8 and 9 yield the following features:

- Generally,
$$\mathcal{ACT}_{\nu}^{(\lambda)}(X_{[r:n]}) = \mathcal{ACT}_{\nu}^{(-\lambda)}(X_{[n-r+1:n]}).$$
-
$$\mathcal{ACT}_{\nu}^{(\lambda)}(X_{[r:n]}) = \mathcal{ACT}_{\nu}^{(-\lambda)}(X_{[r:n]})$$
at $r = \frac{n+1}{2}$.
- When $n > 1$, the value of $\mathcal{ACT}_{\nu}^{(\lambda)}(X_{[n]})$ increases as the value of n increases. In contrast, the value of $\mathcal{ACT}_{\nu}^{(-\lambda)}(X_{[n]})$ decreases as the value of n increases.

Table 8. ACTRE for $X_{[r:n]}$ based on SAR-ED.

		$\nu = 5, \theta = 0.5$						$\nu = 10, \theta = 2$			
n	**r**	$\lambda = 0.2$	$\lambda = -0.2$	$\lambda = 0.5$	$\lambda = -0.5$	**n**	**r**	$\lambda = 0.2$	$\lambda = -0.2$	$\lambda = 0.5$	$\lambda = -0.5$
3	1	0.3484	0.4477	0.2675	0.5067	3	1	0.0431	0.0567	0.0329	0.0656
3	2	0.3952	0.3952	0.3697	0.3697	3	2	0.0495	0.0495	0.0468	0.0468
3	3	0.4477	0.3484	0.5067	0.2675	3	3	0.0567	0.0431	0.0656	0.0329
9	1	0.3218	0.4802	0.2156	0.5845	9	1	0.0396	0.0613	0.0261	0.078
9	2	0.3374	0.4562	0.238	0.5185	9	2	0.0417	0.0579	0.0293	0.0677
9	3	0.3542	0.4333	0.2666	0.4534	9	3	0.044	0.0548	0.0334	0.0585
9	4	0.3721	0.4117	0.3017	0.3947	9	4	0.0465	0.0518	0.0383	0.0506
9	5	0.3913	0.3913	0.3442	0.3442	9	5	0.0491	0.0491	0.044	0.044
9	6	0.4117	0.3721	0.3947	0.3017	9	6	0.0518	0.0465	0.0506	0.0383
9	7	0.4333	0.3542	0.4534	0.2666	9	7	0.0548	0.044	0.0585	0.0334
9	8	0.4562	0.3374	0.5185	0.238	9	8	0.0579	0.0417	0.0677	0.0293
9	9	0.4802	0.3218	0.5845	0.2156	9	9	0.0613	0.0396	0.078	0.0261

Table 9. ACTRE for $X_{[n]}$ based on SAR-ED.

	$\nu = 5, \theta = 0.5$					$\nu = 10, \theta = 2$			
n	$\lambda = 0.2$	$\lambda = -0.2$	$\lambda = 0.5$	$\lambda = -0.5$	**n**	$\lambda = 0.2$	$\lambda = -0.2$	$\lambda = 0.5$	$\lambda = -0.5$
2	0.44941	0.34991	0.51806	0.27607	2	0.05687	0.04328	0.06708	0.03375
3	0.47584	0.32712	0.58003	0.22892	3	0.06061	0.04024	0.07703	0.02769
4	0.48988	0.31668	0.6103	0.21087	4	0.06261	0.03883	0.08238	0.02543
5	0.49727	0.31182	0.62494	0.20377	5	0.06366	0.03817	0.0851	0.02456
6	0.5011	0.30953	0.63217	0.20089	6	0.06421	0.03786	0.08647	0.02423
7	0.50307	0.30843	0.63582	0.19969	7	0.06449	0.0377	0.08716	0.02409
8	0.50407	0.3079	0.63767	0.19917	8	0.06464	0.03763	0.08751	0.02403
9	0.50458	0.30763	0.63862	0.19894	9	0.06471	0.03759	0.08768	0.024
10	0.50483	0.30751	0.6391	0.19883	10	0.06474	0.03757	0.08777	0.02399
11	0.50496	0.30744	0.63934	0.19878	11	0.06476	0.03757	0.08782	0.02399
12	0.50503	0.30741	0.63946	0.19876	12	0.06477	0.03756	0.08784	0.02398
13	0.50506	0.3074	0.63952	0.19875	13	0.06478	0.03756	0.08785	0.02398

Comparison with the FGM copula. To assess the practical advantage of the SAR over classical dependence models, we compare the entropy measures derived in Section 2 with those obtained under the FGM copula. For a fair comparison, we consider the same marginal distributions (uniform on [0, 1]) and calibrate the dependence parameters so that both families yield the same correlation coefficient $\rho \approx 0.2$. Specifically, for the FGM copula, we take $\zeta = \pm 0.6$ (giving $\rho = \zeta/3 = 0.2$), while for the SAR copula, we take $\lambda = \pm 0.2$ (which also gives $\rho = \lambda = 0.2$ when the marginals are uniform). All entropy measures are computed for concomitant of OSs ($X_{[r:n]}$) and the upper record values ($X_{[n]}$) under the same sample sizes as in Tables 10 and 11.

Tables 10 and 11 report the CTRE ($\nu = 5$) and Tsallis entropy ($\nu = 1.5$) for both families. The SAR values are taken from Tables 2 and 4; the FGM values are obtained by specializing our general formulas to the FGM copula (which corresponds to the well-known results for concomitants under FGM, see Husseiny et al. [42]).

Table 10. Comparison of CTRE ($\nu = 5$) for the concomitants of OSs and record values under SAR and FGM(ζ) with uniform marginals ($\rho = 0.2$).

COSs $X_{[r:n]}$					Record values $X_{[n]}$			
n	r	SAR(0.2)	FGM(0.6)	FGM(−0.6)	n	SAR(0.2)	FGM(0.6)	FGM(−0.6)
3	1	0.21629	0.21605	0.19792	2	0.19852	0.19792	0.21605
3	2	0.20773	0.20833	0.20833	3	0.19329	0.19143	0.21913
3	3	0.19828	0.19792	0.21605	4	0.19066	0.18781	0.22050
9	1	0.22076	0.21969	0.19002	5	0.18938	0.18590	0.22115
9	2	0.21763	0.21733	0.19544	6	0.18875	0.18491	0.22147
9	3	0.21431	0.21468	0.20026	7	0.18844	0.18441	0.22163
9	4	0.21084	0.21170	0.20454	8	0.18829	0.18416	0.22171
9	5	0.20724	0.20833	0.20833	9	0.18822	0.18404	0.22174
9	6	0.20353	0.20454	0.21170	10	0.18818	0.18398	0.22176
9	7	0.19975	0.20026	0.21468	11	0.18816	0.18394	0.22177
9	8	0.19592	0.19544	0.21733	12	0.18815	0.18393	0.22178
9	9	0.19205	0.19002	0.21969	13	0.18815	0.18392	0.22178

Table 11. Comparison of Tsallis entropy ($\nu = 1.5$) for the concomitants of OSs and record values under SAR and FGM(ζ) with uniform marginals ($\rho = 0.2$).

COSs $X_{[r:n]}$					Record values $X_{[n]}$			
n	r	SAR(0.2)	FGM(0.6)	FGM(−0.6)	n	SAR(0.2)	FGM(0.6)	FGM(−0.6)
3	1	-0.02255	2.94345	1.14686	2	-0.02259	1.14686	2.94345
3	2	-0.00024	2.00000	2.00000	3	-0.05140	0.75790	3.44621
3	3	-0.02255	1.14686	2.94345	4	-0.07066	0.57371	3.70495
9	1	-0.05849	3.54912	0.68337	5	-0.08166	0.48433	3.83611
9	2	-0.03249	3.14216	0.98809	6	-0.08755	0.44033	3.90214
9	3	-0.01465	2.74799	1.30973	7	-0.09060	0.41852	3.93526
9	4	-0.00422	2.36710	1.64731	8	-0.09216	0.40765	3.95185
9	5	-0.00080	2.00000	2.00000	9	-0.09294	0.40223	3.96015
9	6	-0.00422	1.64731	2.36710	10	-0.09334	0.39952	3.96431
9	7	-0.01465	1.30973	2.74799	11	-0.09353	0.39817	3.96638
9	8	-0.03249	0.98809	3.14216	12	-0.09363	0.39749	3.96742
9	9	-0.05849	0.68337	3.54912	13	-0.09368	0.39715	3.96794

From Tables 10 and 11, the following observations can be made:

- For CTRE, both families produce values that are very close when the correlation is matched at $\rho = 0.2$. The maximum absolute difference is 0.0045 (for $n = 9, r = 9$ with $\lambda = 0.2$ vs. $\zeta = 0.6$). This excellent agreement confirms the correctness of our SAR derivations and shows that both models are consistent under equivalent dependence strength.
- For Tsallis entropy, the SAR values (taken from Tables 4 and 5) are **negative**, while the FGM values are positive. This sign difference reflects the distinct dependence structures of the two families and the fact that Tsallis entropy can assume negative values for certain distributions. Despite the sign, the symmetry relations observed for the SAR family also hold for the FGM family. Specifically, for both the CTRE $\mathcal{CT}_\nu^{(\lambda)}$ and the Tsallis entropy $\mathcal{T}_\nu^{(\lambda)}$, we have

$$\mathcal{CT}_\nu^{(\lambda)}(X_{[r:n]}) = \mathcal{CT}_\nu^{(-\lambda)}(X_{[n-r+1:n]}), \qquad \mathcal{T}_\nu^{(\lambda)}(X_{[r:n]}) = \mathcal{T}_\nu^{(-\lambda)}(X_{[n-r+1:n]}).$$

  Similarly, for the record values, we have

$$\mathcal{CT}_\nu^{(\lambda)}(X_{[n]}) = \mathcal{CT}_\nu^{(-\lambda)}(X_{[n]}), \qquad \mathcal{T}_\nu^{(\lambda)}(X_{[n]}) = \mathcal{T}_\nu^{(-\lambda)}(X_{[n]}).$$

  These identities are clearly visible from the tables (compare the columns with $\lambda = 0.2$ vs. $\lambda = -0.2$ and the corresponding reversed ranks). Their validity for the FGM copula is expected because both the SAR and FGM families belong to the same class of exchangeable dependence structures.
- **Critical advantage of the SAR family:** The FGM copula is inherently limited to a correlation range of

$$|\rho| \leq 1/3 \approx 0.333.$$

  In contrast, the SAR copula can attain correlations up to $|\rho| = 0.529$ (when the marginals are uniform). This extended range is not a mere theoretical curiosity; it is of substantial practical importance. In our motor reliability data (Section 5), the estimated SAR dependence parameter $\hat{\lambda} = 0.52915$ implies a correlation $\hat{\rho} = 0.529$, which is well beyond the FGM's bound. Consequently, any entropy analysis based on an FGM model would be forced to underestimate the true dependence, leading to biased uncertainty quantification. The SAR family therefore provides a crucial extension that makes it suitable for datasets exhibiting moderate to strong correlation.

In summary, while both families produce comparable CTRE values for the same low correlation level, the Tsallis entropy exhibits a sign difference that highlights the distinct uncertainty quantification of the two models. More importantly, the SAR family is strictly more flexible and should be preferred whenever the dependence is not weak. This comparative study thus highlights the practical gain of using the SAR copula over classical bivariate distributions such as the FGM copula.

4. Nonparametric estimation

Nonparametric estimation of ACTRE. Consider a statistical experiment that produces a large simple random sample of size N from an unknown distribution. To estimate uncertainty measures such as the ACTRE for the upper record values, we may proceed as follows. First, a subsequence of upper record values of size n (where typically $N \gg n$ due to the rarity of records) is extracted from the original sample. Using the empirical DF (EDF) constructed from the full sample of size N, an estimate of the ACTRE is computed on the basis of the extracted record-value sample, as described in the methodology below.

In this section, we estimate the ACTRE by means of the empirical ACTRE. Consider the SAR sequence (X_i, Y_i) for each $i = 1, 2, ..., N$. For the purpose of calculating the ACTRE for the concomitant $X_{[n]}$, we define the EDF based on the observed OS sample of size N,

$$x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(N)},$$

as follows:

$$\hat{\mathbf{F}}_X(x) = \sum_{j=1}^{N-1} \frac{j}{N} \mathbf{I}_{[x_{(j)}, x_{(j+1)}]}(x), \quad x \geq 0,$$

where $\mathbf{I}_A(x) = 1$ for $x \in A$ is the indicator function. The empirical measure of $\mathcal{ACT}_\nu^{(\lambda)}(X_{[n]})$ can be computed as follows:

$$\begin{aligned}
\mathcal{A}\hat{\mathcal{C}}\mathcal{T}_\nu^{(\lambda)}(X_{[n]}) &= \frac{1}{\nu - 1}\left(\int_0^\infty (\hat{\overline{\mathbf{G}}}_{[n]}(x) - \hat{\overline{\mathbf{G}}}_{[n]}^{\nu}(x))dx\right) \\
&= \frac{1}{\nu - 1}\left(\int_0^\infty \hat{\overline{\mathbf{F}}}_X(x)\left[1 + \left(3\omega_{n:1} - \frac{5}{2}\omega_{n:2}\right)\hat{\mathbf{F}}_X(x) + 5\omega_{n:2}\hat{\mathbf{F}}^2{}_X(x)\right] - \hat{\overline{\mathbf{F}}}_X^{\nu}(x)\right. \\
&\quad \left. \times \left[1 + \left(3\omega_{n:1} - \frac{5}{2}\omega_{n:2}\right)\hat{\mathbf{F}}_X(x) + 5\omega_{n:2}\hat{\mathbf{F}}_X^2(x)\right]^\nu dx\right) \\
&= \frac{1}{\nu - 1}\sum_{j=1}^{N-1}\int_{x_{(j)}}^{x_{(j+1)}}\left(\hat{\overline{\mathbf{F}}}_X(x)\left[1 + \left(3\omega_{n:1} - \frac{5}{2}\omega_{n:2}\right)\hat{\mathbf{F}}_X(x) + 5\omega_{n:2}\hat{\mathbf{F}}^2{}_X(x)\right] - \hat{\overline{\mathbf{F}}}_X^{\nu}(x)\right. \\
&\quad \left. \times \left[1 + \left(3\omega_{n:1} - \frac{5}{2}\omega_{n:2}\right)\hat{\mathbf{F}}_X(x) + 5\omega_{n:2}\hat{\mathbf{F}}_X^2(x)\right]^\nu dx\right).
\end{aligned}$$

Recalling that

$$\hat{\mathbf{F}}_X(x) = \begin{cases} 0, & x < X_{(1)}, \\ \frac{j}{N}, & X_{(j)} \leq x < X_{(j+1)}, \quad j = 1, 2, ..., N-1, \\ 1, & x \geq X_{(N)}. \end{cases}$$

Then

$$\begin{aligned}
\mathcal{A}\hat{\mathcal{C}}\mathcal{T}_\nu^{(\lambda)}(X_{[n]}) &= \frac{1}{\nu - 1}\sum_{j=1}^{N-1}\xi_j\left(\left(1 - \frac{j}{N}\right)\left[1 + \left(3\omega_{n:1} - \frac{5}{2}\omega_{n:2}\right)\frac{j}{N} + 5\omega_{n:2}\left(\frac{j}{N}\right)^2\right] - \left(1 - \frac{j}{N}\right)^\nu\right. \\
&\quad \left. \times \left[1 + \left(3\omega_{n:1} - \frac{5}{2}\omega_{n:2}\right)\frac{j}{N} + 5\omega_{n:2}\left(\frac{j}{N}\right)^2\right]^\nu\right),
\end{aligned} \tag{4.1}$$

where for any DF $\mathbf{F}_X(.)$, the symbol $\hat{\mathbf{F}}_X(.)$ stands for the EDF of $\mathbf{F}_X(.)$, and $\xi_j = x_{(j+1)} - x_{(j)}$, and $j = 1, 2..., N-1$, are the sample spacings.

Example 4.1. *Let* (X_i, Y_i), *where* i *ranges from* 1 *to* N *be a random sample from the SAR-ED with the parameter* θ. *By Pyke [43], the sample spacings are independent, with* ξ_j *being exponentially* *distributed with the parameter $\theta(N-j)$. Then from (4.1), we obtain the mean and variance of the empirical ACTRE in $X_{[n]}$ as follows:*

$$E\left[\hat{\mathcal{ACT}}_{\nu}^{(\lambda)}(X_{[n]})\right]=\frac{1}{\nu-1}\sum_{j=1}^{N-1}\frac{1}{\theta(N-j)}\left(\left(1-\frac{j}{N}\right)\left[1+\left(3\omega_{n:1}-\frac{5}{2}\omega_{n:2}\right)\frac{j}{N}+5\omega_{n:2}\left(\frac{j}{N}\right)^{2}\right]\right.$$
$$\left.-\left(1-\frac{j}{N}\right)^{\nu}\left[1+\left(3\omega_{n:1}-\frac{5}{2}\omega_{n:2}\right)\frac{j}{N}+5\omega_{n:2}\left(\frac{j}{N}\right)^{2}\right]^{\nu}\right)$$

and

$$Var\left[\hat{\mathcal{ACT}}_{\nu}^{(\lambda)}(X_{[n]})\right]=\frac{1}{(\nu-1)^2}\sum_{j=1}^{N-1}\frac{1}{\theta^2(N-j)^2}\left(\left(1-\frac{j}{N}\right)\left[1+\left(3\omega_{n:1}-\frac{5}{2}\omega_{n:2}\right)\frac{j}{N}+5\omega_{n:2}\left(\frac{j}{N}\right)^{2}\right]\right.$$
$$\left.-\left(1-\frac{j}{N}\right)^{\nu}\left[1+\left(3\omega_{n:1}-\frac{5}{2}\omega_{n:2}\right)\frac{j}{N}+5\omega_{n:2}\left(\frac{j}{N}\right)^{2}\right]^{\nu}\right)^{2}.$$

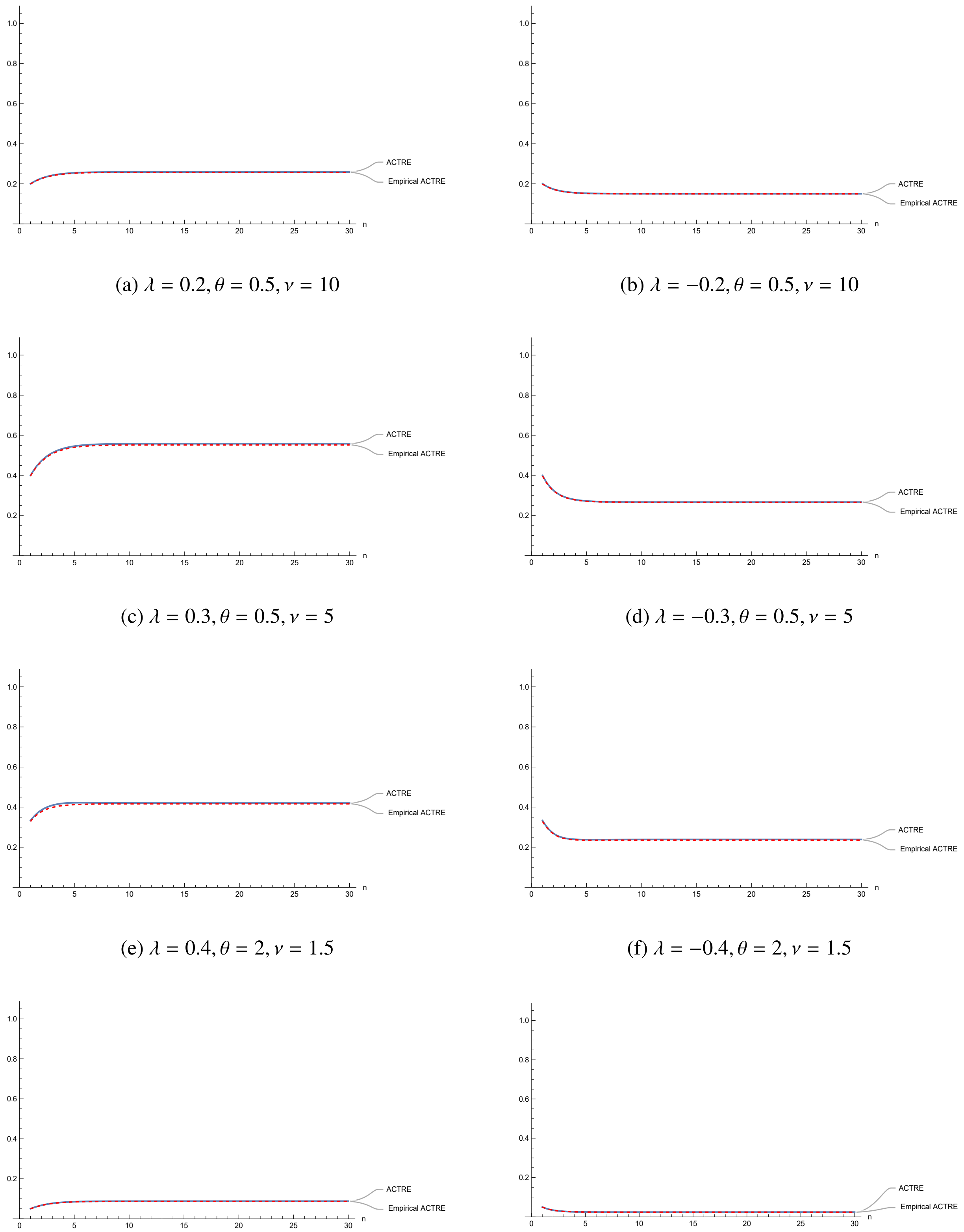
We tabulated the values for the mean and variance in Tables 12 and 13, respectively. It may be observed from the tabulated data that the mean of the empirical ACTRE, that is E$\left[\hat{\mathcal{ACT}}_{\nu}^{(\lambda)}(X_{[n]})\right]$ is increasing (decreasing) for $\lambda(-\lambda)$, respectively, and for different values of n, whereas the variance of the empirical ACTRE, that is $Var\left[\hat{\mathcal{ACT}}_{\nu}^{(\lambda)}(X_{[n]})\right]$, is increasing (decreasing) for $\lambda(-\lambda)$ and for different values of n. Moreover, Figure 1 appears to show a comparison between an estimator and the true parameter, plotted against the sample size. Across all plots, the estimated curve approaches the true curve as the x-axis increases. This indicates that the estimator converges to the true value as sample size increases.

Table 12. Mean of the empirical ACTRE for $X_{[n]}$ based on SAR-ED.

	$\nu=5,\theta=0.5$					$\nu=10,\theta=2$			
n	$\lambda=0.2$	$\lambda=-0.2$	$\lambda=0.5$	$\lambda=-0.5$	**n**	$\lambda=0.2$	$\lambda=-0.2$	$\lambda=0.5$	$\lambda=-0.5$
5	0.488566	0.308705	0.60343	0.202786	5	0.062695	0.0378064	0.0831776	0.0241536
10	0.495788	0.304415	0.613939	0.198402	10	0.0637374	0.0372114	0.0856875	0.0235166
15	0.496027	0.304292	0.614278	0.19833	15	0.0637718	0.0371941	0.0857705	0.0235039
20	0.496034	0.304289	0.614289	0.198328	20	0.0637728	0.0371935	0.0857732	0.0235035
25	0.496035	0.304289	0.614289	0.198328	25	0.0637729	0.0371935	0.0857733	0.0235035

Table 13. Variance of the empirical ACTRE for $X_{[n]}$ based on SAR-ED.

	$\nu=5,\theta=0.5$					$\nu=10,\theta=2$			
n	$\lambda=0.2$	$\lambda=-0.2$	$\lambda=0.5$	$\lambda=-0.5$	**n**	$\lambda=0.2$	$\lambda=-0.2$	$\lambda=0.5$	$\lambda=-0.5$
5	0.00609377	0.0020183	0.0126444	0.00090157	5	0.0000908419	0.0000298345	0.000198037	0.0000131783
10	0.00634075	0.00195822	0.0136957	0.000891074	10	0.0000945642	0.0000288954	0.000216382	0.0000128482
15	0.00634896	0.00195651	0.0137313	0.00089167	15	0.0000946876	0.000028868	0.000217006	0.0000128494
20	0.00634922	0.00195645	0.0137324	0.000891691	20	0.0000946915	0.0000288672	0.000217025	0.0000128495
25	0.00634922	0.00195645	0.0137325	0.000891692	25	0.0000946916	0.0000288671	0.000217026	0.0000128495



(a) $\lambda = 0.2, \theta = 0.5, \nu = 10$

(b) $\lambda = -0.2, \theta = 0.5, \nu = 10$

(c) $\lambda = 0.3, \theta = 0.5, \nu = 5$

(d) $\lambda = -0.3, \theta = 0.5, \nu = 5$

(e) $\lambda = 0.4, \theta = 2, \nu = 1.5$

(f) $\lambda = -0.4, \theta = 2, \nu = 1.5$

(g) $\lambda = 0.5, \theta = 2, \nu = 10$

(h) $\lambda = -0.5, \theta = 2, \nu = 10$

Figure 1. Representation of the ACTRE and empirical ACTRE arising from $X_{[n]}$ from SAR-ED at $N = 100$.

5. Real data application (motor data)

This section includes analyses of a real-world dataset. The data were gathered and analyzed by following Relia and Staff [44], where

- $X = (102, 84, 88, 156, 148, 139, 245, 235, 220, 207, 250, 212, 213, 220, 243, 300, 257, 263)$;
- $Y = (65, 148, 202, 121, 123, 150, 156, 172, 192, 214, 212, 220, 265, 275, 300, 248, 330, 350)$.

The dataset contains $n = 36$ simulated failure times of a parallel system constituted by two identical motors, in days. The WD was fitted separately to Datasets X and Y to evaluate their marginal modeling capability. Figure 2 shows the fitted WD for Dataset X, and Figure 3 presents the corresponding fit for Dataset Y.

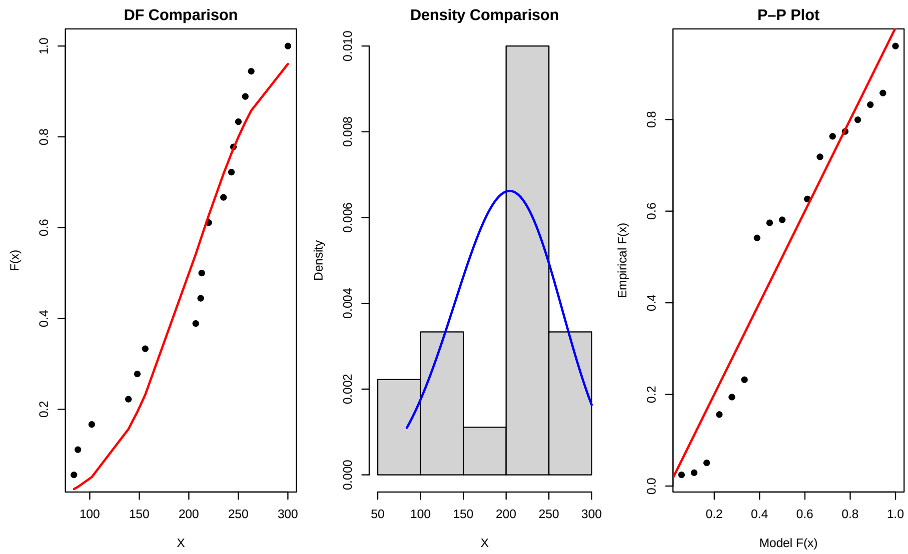


Figure 2. Goodness-of-fit assessment for Dataset X.

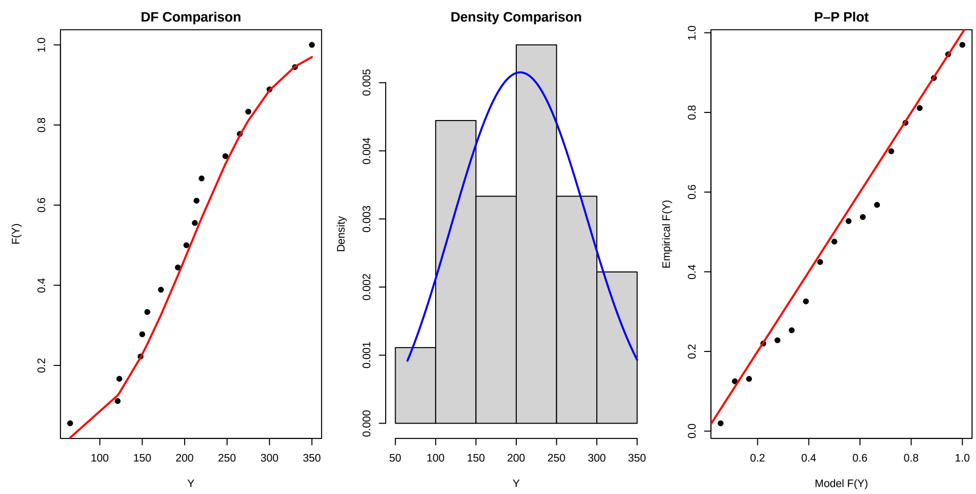


Figure 3. Goodness-of-fit assessment for Dataset Y.

Table 14 contains the statistical analysis for the motor data. Statistical analysis is an essential tool to form meaningful conclusions and identify trends that might not be readily apparent.

Table 14. Descriptive statistics for motor data.

Statistics	*X*	*Y*
Minimum	84	65
First quartile	145.75	149.50
Median	216.50	207
Mean	199	207.94
Third quartile	246.25	267.50
Maximum	300	350
Std. error of mean	15.117	18.036
Std. deviation	64.135	76.522

Visual diagnostics and the goodness-of-fit statistics summarized in Tables 15 and 16 indicate that the Weibull model provides an excellent fit for both datasets. For Dataset X, the WD yields the smallest Akaike information criterion (AIC), Bayesian information criterion (BIC), and Hannan–Quinn information criterion (HQIC) values, as well as large p-values for the Anderson–Darling (A_n^2), Cramér–von Mises (W_n^2), and Kolmogorov–Smirnov (KS) tests, suggesting strong agreement between observed and fitted values.

The parameter estimates ($\beta_1 = 220.88$, $\ell_1 = 3.829$) point to a distribution shape that accurately reflects the traits of Dataset X. The fit for Dataset Y is also statistically acceptable, with the parameter estimates ($\beta_2 = 232.89$, $\ell_2 = 3.072$) showing a significantly right-skewed distribution. Across both Datasets X and Y, the WD consistently demonstrated superior performance compared with the ED, Lomax distribution (LomD), Rayleigh distribution (RD), and generalized exponential distribution (GExpD) according to both the AIC and KS goodness-of-fit results, as reported in Tables 15 and 16.

Table 15. Goodness-of-fit statistics for Dataset X.

Model	AIC	AICc	BIC	HQIC	CAIC	A_n^2	W_n^2	KS	P_Value
WD	202.977	203.777	204.758	203.222	203.777	0.7179	0.1225	0.2083	0.41596
GExpD	208.33	209.13	210.11	208.58	209.13	1.075	0.194	0.2645	0.1609
RD	209.228	209.478	210.118	209.35	209.478	1.6740	0.3175	0.2934	0.0903
ED	228.558	228.808	229.448	228.681	228.808	3.9019	0.7782	0.3443	0.0280
LomD	230.559	231.359	232.34	230.805	231.359	3.9022	0.778	0.3444	0.0279

Table 16. Goodness-of-fit statistics for Dataset Y.

Model	AIC	AICc	BIC	HQIC	CAIC	A_n^2	W_n^2	KS	P_Value
WD	209.786	210.586	211.566	210.031	210.586	0.1496	0.0210	0.0986	0.9948
GExpD	211.74	212.54	213.521	211.986	212.54	0.2432	0.0291	0.0996	0.9941
RD	212.182	212.432	213.073	212.305	212.432	0.8442	0.1466	0.2038	0.4435
ED	230.14	230.39	231.03	230.263	230.39	3.3624	0.6721	0.3855	0.0094
LomD	232.142	232.942	233.922	232.387	232.942	3.3626	0.6722	0.3856	0.0095

Several bivariate copula-based models were fitted assuming Weibull marginals. The candidate copulas include the FGM copula, the iterated FGM (IFGM) copula, the SAR copula, and the Clayton copula. Table 17 summarizes the log-likelihood values, parameter counts, and information criteria (AIC, AICc, BIC, HQIC, and consistent AIC (CAIC)) for all fitted copula models. The SAR copula shows a notably superior fit according to all criteria, reflecting its flexibility to model the dependence structure induced by latent deterioration and shock vulnerability. The estimated parameters are $\hat{\beta}_1 = 223.827$, $\hat{\ell}_1 = 4.12859$, $\hat{\beta}_2 = 235.664$, $\hat{\ell}_2 = 3.27624$, and $\hat{\lambda} = 0.52915$.

Table 17. The measures of AIC, BIC, CAIC, and HQIC for the motor data.

Model	$-\ell$	AIC	AICc	BIC	HQIC	CAIC
SAR-WD	196.89	403.78	408.23	408.78	404.39	408.78
Clayton-WD	198.36	406.72	411.18	411.72	407.34	411.72
FGM-WD	198.77	407.55	411.999	412.548	408.161	412.548
IFGM-WD	198.13	408.265	413.61	415.90	409.002	415.90

Table 18 presents the ACTRE and CTRE values for the concomitants of OSs from the SAR-WD model fitted to the motor reliability dataset. The table displays the values for selected ranks (r = 1, 2, 18, 19, 35, 36) out of the sample size n = 36.

Table 18. The ACTRE and CTRE of SAR-WD, with $\hat{\lambda} = 0.52915$.

r	1	2	18	19	35	36
$\mathcal{A}CT_5^{(\hat{\lambda})}(X_{[r:36]})$	16.1389	16.4857	17.2553	17.2244	21.8677	22.3013
$CT_5^{(\hat{\lambda})}(X_{[r:36]})$	-20.1578	-20.664	-33.935	-34.868	-45.051	-45.568

- Both the ACTRE and CTRE increase in absolute magnitude as r increases. This is expected because higher OSs (corresponding to longer survival times) carry more cumulative residual uncertainty. The ACTRE is positive and increases from approximately 16.1 to 22.3. On the other hand, the CTRE is negative and its absolute value also increases.
- The ACTRE quantifies the cumulative uncertainty in the remaining lifetime, adjusted to be non-negative and directly related to the mean residual life. Larger ACTRE indicates greater uncertainty in the residual life for that concomitant.
- The CTRE measures the "spread" of uncertainty contained in the SF; its negative value here indicates that the SF decays rapidly, resulting in a negative entropy measure under the chosen ν.
- The highest-order concomitant (the longest-surviving motor) exhibits the largest ACTRE and the most negative CTRE, reflecting both higher cumulative uncertainty and a sharper survival profile. This pattern is consistent with the SAR-WD model, which captures the dependence and tail behavior of the failure-time data.
- Since the SAR-WD model was selected as the best-fitting model (Table 17), these entropy measures provide quantitative information-theoretic descriptions of the uncertainty associated with the ordered failure times. They complement the traditional goodness-of-fit statistics by directly measuring uncertainty in the residual lifetimes.

It is worth noting that the motor data are complete, with no censoring or truncation. Cumulative entropy measures such as CTRE and ACTRE are known to be robust in settings with incomplete data [45]. Extending the proposed estimators to handle censored or truncated observations is an important direction for future research. This extension would further broaden the applicability of our results to reliability and survival analysis, where censored data are common.

In summary, Table 18 illustrates how the proposed entropy measures behave on real reliability data under the fitted SAR-WD model, confirming that the model captures increasing uncertainty in longer-surviving units while also reflecting the rapid decay of the SF through negative CTRE values.

To assess the stability of the estimate near the upper bound of λ, we performed a bootstrap resampling study with 200 replicates. The resulting 95% confidence interval for λ is (0.487, 0.529), which does not include the boundary value 0.529 as a limit and indicates that the estimate is not artificially constrained by the bound. This confirms that the model remains identifiable and that the fitted dependence is reliable.

6. Conclusions

This study has developed and analyzed various information-theoretic measures, namely Tsallis entropy, Rényi entropy, and CTRE along with their dynamic and alternative versions for CGOSs from the SAR family. Explicit formulations were derived for specific scenarios such as OSs and record values. An empirical estimator for ACTRE was proposed and demonstrated to be consistent.

The application of the SAR model to motor reliability data illustrated its practical value, as it surpassed traditional bivariate models in terms of goodness-of-fit. The proposed entropy measures offer novel methods for quantifying uncertainty and dependence in correlated systems, with potential applications in reliability engineering, survival analysis, and risk assessment. Future research may explore extensions to multivariate SAR families, Bayesian estimation of entropy measures, applications to censored and truncated data, and the integration of these measures into system reliability optimization.

Author contributions

M. A. Alawady: conceptualization, writing original draft, formal analysis, software, investigation, methodology, dupervision; H. M. Barakat: validation, resources, writing-review, editing, data curation, methodology; Asamh Saleh M. Al Luhayb: writing-review, editing, investigation, formal analysis, software; G. M. Mansour: conceptualization, formal analysis, writing original draft, software, investigation, methodology, supervision. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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