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Research article

Cyclotomic points on algebraic varieties and all rational a^3b-monotiles

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Abstract: By computing all cyclotomic points on specific algebraic varieties, we establish an independent and highly efficient method to determine the complete set of spherical a^3b-monotiles with rational angles, thereby rigorously completing the classification of edge-to-edge monohedral quadrilateral tilings. Previous classification attempts relied heavily on fragmented older literature, within which several errors and logical gaps have been identified.

Keywords: root of unity; trigonometric Diophantine equation; cyclotomic point; spherical tiling; quadrilateral
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1. Introduction

Tilings have numerous applications in both practical contexts and pure science. Two great books introduced their modern mathematical theory [1, 2]. Open questions like "which tetrahedra fill space ?" [3] go back to 2300+ years ago and led to the first modern study of spherical tilings in 1924 [4]. After 100 years, the classification of edge-to-edge monohedral tilings of the sphere has been carried out recently in a series of works such as [5–7], mainly by two groups in the Hong Kong University of Science & Technology and Zhejiang Normal University. Our group has also made progress on non-side-to-side triangular tilings recently. Figure 1 shows six examples of almost equilateral quadrilateral, or simply a^3b-tilings, which, together with a^4b-pentagonal tilings [8], are the hardest cases in the classification program.

An a^3b-quadrilateral is given in Figure 2, with three normal edges a, a thick edge b, and angles α, β, γ and δ as indicated. Throughout this paper, an a^3b-tiling is always an edge-to-edge tiling of the sphere by the congruent simple quadrilaterals in Figure 2, such that all vertices have degree ≥ 3.

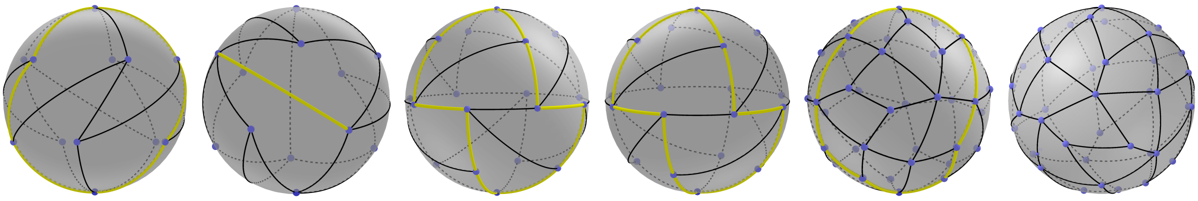

Figure 1. Some a^3b-tilings with 10, 10, 16, 16, 36 and 36 tiles.

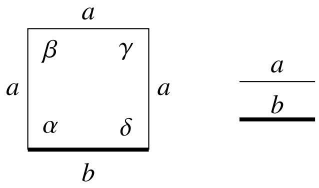

Figure 2. Notations for an a^3b-quadrilateral.

Instead of very long brute-force case studies of a^3b- or a^4b-tilings, some powerful algebraic number theory approach have helped [6, 8] to classify such tilings with all angles being rational in degree (simply called *rational* hereafter) very efficiently. Accordingly, the tilings with any irrational angle must have very limited vertex types and were classified easily in [7].

The cost is to find all rational angle solutions to a key equation ([9, Lemma 3.14]) satisfied by any a^3b-quadrilateral

$$\sin\left(\alpha - \frac{\gamma}{2}\right)\ \sin\frac{\beta}{2} = \sin\frac{\gamma}{2}\ \sin\left(\delta - \frac{\beta}{2}\right). \tag{1.1}$$

Note that a similar equation for a^4b-pentagons is much more complicated to solve [8]. Using $x = e^{i\theta}$ for each angle variable, Eq (1.1) changes to polynomial form. Solving for rational angles is then equivalent to finding root of unity solutions (or cyclotomic points on some algebraic variety). According to [10], there are two algorithms to solve such trigonometric Diophantine equations; both have super-exponential complexity-limiting practicality to either ≤ 12 monomials or ≤ 3 variables.

Both previous classifications [6, 9] applied the first algorithm to (1.1) which induces eight monomials, using many old works of different authors [11–13]. Unfortunately, quite a few errors and gaps were found in them (see [6, Remark 17], for example), and it is known that there were gaps in [12]. Moreover, it took a lot of work to translate acute angle solutions to general angles in $(0, 2\pi)$, as shown in [6].

In this paper, we will apply the other algorithm from [14–16] to get an independent and more efficient way to find all rational a^3b-monotiles for the sphere, thereby supporting our full classification results more rigorously. Although the equation has four variables, linear angle constraints from degree 3 or 4 vertices, which must appear in the tilings, can help reduce (1.1) to two or three variables.

Theorem. *There are* 15 *sporadic and* 3 *infinite sequences of rational a^3b-monotiles for the sphere, as listed in Tables 1 and 2, together with all of their different tilings.*

In Tables 1 and 2, the quantity f denotes the total number of tiles in the spherical tiling, the angles and edge lengths are expressed in units of π, and the last column counts all vertices and also all tilings when they are not uniquely determined by the vertices. A rational fraction, such as $\alpha = \frac{1}{6}$, means the precise value $\frac{\pi}{6}$. A decimal expression, such as $a \approx 0.391$, means an approximate value

$$0.391\pi \le a < 0.392\pi.$$

Table 1. Fifteen sporadic rational a^3b-quadrilaterals and their tilings.

f	$(\alpha, \beta, \gamma, \delta)$, a, b	All vertices and tilings
6	$(6, 3, 4, 3)/6$, $1/2$, $1/6$	$6\alpha\beta\delta$, $2\gamma^3$
6	$(1, 8, 4, 3)/6$, 0.391, 1	$6\alpha\beta\delta$, $2\gamma^3$
6	$(12, 4, 6, 2)/9$, 0.567, 0.174	$6\alpha\beta\delta$, $2\gamma^3$
12	$(2, 10, 3, 6)/9$, 0.339, 0.532	$12\alpha\beta\delta$, $2\gamma^6$
12	$(1, 21, 5, 8)/15$, 0.424, 0.741	$12\alpha\beta\delta$, $2\gamma^6$
12	$(4, 9, 5, 17)/15$, 0.424, 0.165	$12\alpha\beta\delta$, $2\gamma^6$
12	$(9, 28, 10, 23)/30$, 0.335, 0.415	$12\alpha\beta\delta$, $2\gamma^6$
12	$(3, 16, 10, 41)/30$, 0.469, 0.146	$12\alpha\beta\delta$, $2\gamma^6$
20	$(5, 32, 6, 23)/30$, 0.335, 0.415	$20\alpha\beta\delta$, $2\gamma^{10}$
20	$(1, 16, 6, 43)/30$, 0.469, 0.273	$20\alpha\beta\delta$, $2\gamma^{10}$
30	$(1, 42, 4, 17)/30$, 0.424, 0.549	$30\alpha\beta\delta$, $2\gamma^{15}$
18	$(3, 20, 4, 13)/18$, 0.339, 0.452	$18\alpha\beta\delta$, $2\gamma^9$
		$16\alpha\beta\delta$, $2\beta\gamma^4$, $2\alpha\gamma^5\delta$
		$14\alpha\beta\delta$, $2\alpha^2\gamma\delta^2$, $4\beta\gamma^4$
16	$(1, 4, 2, 2)/4$, $1/4$, $1/2$	$8\beta\delta^2$, $8\alpha^2\beta\gamma$, $2\gamma^4$: 2 tilings
36	$(5, 4, 7, 3)/9$, 0.174, 0.258	$18\beta\gamma^2$, $6\alpha^3\delta$, $6\alpha^2\beta^2$, $6\alpha\beta\delta^3$, $2\delta^6$
36	$(15, 6, 10, 7)/18$, 0.225, 0.118	$14\alpha^2\beta$, $8\alpha\delta^3$, $10\beta\gamma^3$, $6\beta^2\gamma\delta^2$

Table 2. Three infinite sequences of rational a^3b-quadrilaterals and their tilings.

Condition on f	All vertices and tilings
Sequence 1: $(\alpha, \beta, \gamma, \delta) = \left(\frac{4}{f}, 1 - \frac{4}{f}, \frac{4}{f}, 1\right)$	
$\forall$ even $f \ge 10$	$f\alpha\beta\delta$, $2\gamma^{\frac{f}{2}}$
$f = 4k$ $(k \ge 3)$	$(f-2)\alpha\beta\delta$, $2\alpha\gamma^{\frac{f}{4}-1}\delta$, $2\beta\gamma^{\frac{f}{4}+1}$
	$(f-4)\alpha\beta\delta$, $2\beta^2\gamma^2$, $4\alpha\gamma^{\frac{f}{4}-1}\delta$: 2 tilings
$f = 12$	$6\alpha\beta\delta$, $2\beta^3$, $6\alpha\gamma^2\delta$
Sequence 2: $(\alpha, \beta, \gamma, \delta) = \left(\frac{2}{f}, \frac{4f-4}{3f}, \frac{4}{f}, \frac{2f-2}{3f}\right)$	
$\forall$ even $f \ge 6$	$f\alpha\beta\delta$, $2\gamma^{\frac{f}{2}}$
$f = 6k + 4$ $(k \ge 1)$	$(f-2)\alpha\beta\delta$, $2\beta\gamma^{\frac{f+2}{6}}$, $2\alpha\gamma^{\frac{f-1}{3}}\delta$
	$(f-4)\alpha\beta\delta$, $2\alpha^2\gamma^{\frac{f-4}{6}}\delta^2$, $4\beta\gamma^{\frac{f+2}{6}}$: $\lfloor\frac{k+2}{2}\rfloor$ tilings
	$(f-6)\alpha\beta\delta$, $2\alpha\delta^3$, $2\alpha^2\beta\gamma^{\frac{f-4}{6}}$, $4\beta\gamma^{\frac{f+2}{6}}$: 3 tilings
Sequence 3: $(\alpha, \beta, \gamma, \delta) = \left(\frac{2}{f}, \frac{2f-4}{3f}, \frac{4}{f}, \frac{4f-2}{3f}\right)$	
$\forall$ even $f \ge 10$	$f\alpha\beta\delta$, $2\gamma^{\frac{f}{2}}$
$f = 6k + 2$ $(k \ge 2)$	$(f-2)\alpha\beta\delta$, $2\alpha\gamma^{\frac{f-2}{6}}\delta$, $2\beta\gamma^{\frac{f+1}{3}}$
	$(f-4)\alpha\beta\delta$, $4\alpha\gamma^{\frac{f-2}{6}}\delta$, $2\beta^2\gamma^{\frac{f+4}{6}}$: $\lfloor\frac{k+3}{2}\rfloor$ tilings
	$(f-6)\alpha\beta\delta$, $2\beta^3\gamma$, $6\alpha\gamma^{\frac{f-2}{6}}\delta$

In conclusion, this paper verifies that the list of rational a^3b-monotiles in our previous work [6] was complete, while it was incomplete in [9]. Specifically, the rational solutions for the case of $f = 6$ were not identified in their classification.

This paper is organized as follows. Section 2 includes basic facts from [6, 7] and some technical results specific to a^3b. Section 3 introduces the method of solving algebraic equations in roots of unity with some examples. Section 4 applies this method to all possible degree 3 vertex types, completing the proof.

2. Basic facts

We will always express angles in π radians for simplicity. So the sum of all angles at a vertex is 2. We present some basic facts and techniques in this chapter.

Let v, e and f be the numbers of vertices, edges, and tiles in a spherical quadrilateral tiling, respectively, and let v_k be the number of vertices of degree k. By applying Euler's formula $v - e + f = 2$, one can derive the following fundamental topological bounds (see [6] for detailed derivations).

Lemma 1. *In any spherical quadrilateral tiling, the total number of tiles f satisfies $f \geq 6$, and the number of degree 3 vertices v_3 satisfies $v_3 \geq 8$.*

Lemma 2. *[7, Lemma 2.1] If all tiles in a tiling of the sphere by f quadrilaterals have the same four angles $\alpha, \beta, \gamma, \delta$, then*

$$\alpha + \beta + \gamma + \delta = 2 + \frac{4}{f},$$

ranging in $(2, \frac{8}{3}]$. In particular, no vertex contains all four angles.

Lemma 3. *[5, Lemma 3] If the quadrilateral in Figure 2 is simple, then $\beta < \gamma$ is equivalent to $\alpha > \delta$.*

Lemma 4. *[6, Lemma 3] If the quadrilateral in Figure 2 is simple, then $\beta = \delta$ if and only if $\alpha = 1$. Furthermore, if it is convex with all angles < 1, then $\beta > \delta$ is equivalent to $\alpha < \gamma$ and $\beta < \delta$ is equivalent to $\alpha > \gamma$.*

Lemma 5. *[6, Lemma 4] If the quadrilateral in Figure 2 is simple and $\delta \leq 1$, then $2\alpha + \beta > 1$ and $\beta + 2\gamma > 1$.*

Lemma 6. *[6, Lemma 8] There is no tiling of the sphere by congruent quadrilaterals with two angles ≥ 1.*

Lemma 7. *[7, Lemma 2.9] Any convex spherical a^3b-quadrangle with all angles < 1 satisfies*

$$\alpha + \delta < 1 + \beta, \quad \alpha + \delta < 1 + \gamma, \quad \alpha + \beta < 1 + \delta, \quad \gamma + \delta < 1 + \alpha.$$

Lemma 8. *(parity lemma, [5, Lemma 10]) In an a^3b-tiling, the total number of ab-angles (defined as the angles α and δ adjacent to the edge b) at any vertex is even.*

Lemma 9. *(balance lemma, [5, Lemma 11]) In an a^3b-tiling by f congruent tiles, each angle of the tile appears f times in total. If no vertex contains two α or two δ, then any vertex containing α or δ must be of the form $\alpha\delta\cdots$ with exactly one α and one δ.*

Lemma 10. *[6, Proposition 10] There is no tiling of the sphere by the congruent symmetric a^3b-quadrilaterals ($\alpha = \delta$ and $\beta = \gamma$).*

In an a^3b-tiling, there are only ten possible degree 3 vertices: four a^3-vertices (β^3, $\beta^2\gamma$, $\beta\gamma^2$ and γ^3) and six a^2b-vertices ($\alpha^2\beta$, $\alpha^2\gamma$, $\beta\delta^2$, $\gamma\delta^2$, $\alpha\beta\delta$, and $\alpha\gamma\delta$).

Lemma 11. *[7, Lemma 2.11] In an a^3b-tiling, we cannot have three different a^2b-vertices. Moreover, the following are the only possible combinations of two a^2b-vertices:*

(1) $\alpha\beta\delta, \gamma\delta^2$;
(2) $\alpha\gamma\delta, \alpha^2\beta$;
(3) $\alpha^2\beta, \gamma\delta^2$.

When $\alpha\beta\delta$ (or $\alpha\gamma\delta$) appears as a vertex, we get $\gamma = \frac{4}{f}$ (or $\beta = \frac{4}{f}$). Such a quadrilateral always admits the simplest 2-layer earth map tiling. When $\alpha\beta\delta$ (or $\alpha\gamma\delta$) never appears as a vertex, we have the following two lemmas.

Lemma 12. *[7, Table 5] Suppose that $\alpha\beta\delta$ and $\alpha\gamma\delta$ are not vertices in an a^3b tiling, and there are, at most, two different degree 3 vertices. Then these two vertices are one of the following: $\{\alpha^2\beta, \beta^3\}$, $\{\alpha^2\beta, \beta^2\gamma\}$, $\{\alpha^2\beta, \gamma^3\}$, $\{\alpha^2\beta, \gamma\delta^2\}$, $\{\beta^2\gamma, \beta\delta^2\}$, $\{\beta\gamma^2, \beta\delta^2\}$, or $\{\beta\delta^2, \gamma^3\}$, up to the symmetry of interchanging $\alpha \leftrightarrow \delta$ and $\beta \leftrightarrow \gamma$.*

If there is a unique degree 3 vertex type without $\alpha\beta\delta$ and $\alpha\gamma\delta$, then it must be $\alpha^2\beta, \beta\delta^2, \beta^3$, or $\beta\gamma^2$, up to the symmetry of interchanging $\alpha \leftrightarrow \delta$ and $\beta \leftrightarrow \gamma$.

Lemma 13. *[7, Lemmas 5.1 and 5.2] In an a^3b-tiling, we have the following:*

(1) If all degree 3 vertices are $\beta\gamma^2$ or β^3, then some degree 4 vertex exists with two b-edges: One of $\{\alpha^4, \alpha^3\delta, \alpha^2\delta^2, \alpha\delta^3, \delta^4\}$ must appear.
(2) If all degree 3 vertices are $\alpha^2\beta$, then a degree 4 vertex exists without α: One of $\{\beta^4, \beta^3\gamma, \beta^2\gamma^2, \beta\gamma^3, \beta^2\delta^2, \beta\gamma\delta^2, \gamma^4, \gamma^2\delta^2, \delta^4\}$ must appear.
(3) If all degree 3 vertices are $\beta\delta^2$, then a degree 4 vertex exists without δ: One of $\{\alpha^4, \alpha^2\beta^2, \alpha^2\gamma^2, \alpha^2\beta\gamma, \beta^4, \beta^3\gamma, \beta^2\gamma^2, \beta\gamma^3, \gamma^4\}$ must appear.

The angle sums from Lemmas 12 and 13 are linear constraints helping to express four angles in terms of two angles.

3. An explicit example applying the number theory techniques

To find all rational a^3b-monotiles, we will find all reasonable rational angle solutions to (1.1) under 36 different linear angle constraints in the next chapter. The computations can be done by SageMath, Maple, or some other symbolic computation software.

As a typical example, we consider Case $\{\beta^3, \alpha^4\}$—a case from Section 4.3 (tilings with a unique degree 3 vertex type excluding $\alpha\beta\delta$ and $\alpha\gamma\delta$)—and demonstrate explicitly how to resolve it using the number theory techniques developed by Bradford and Davenport [14] and Beukers and Smyth [15].

First, by solving the linear system of angle sums, we get

$$\alpha = \frac{\pi}{2}, \quad \beta = \frac{2\pi}{3}, \quad \gamma = \frac{5\pi}{6} - \delta + \frac{4\pi}{f}.$$

Then the trigonometric Eq (1.1) reduces to

$$Q(\delta, f) := 2\sin(\frac{\pi}{4} + \frac{\delta}{2} - \frac{2\pi}{f}) - 2\cos(\frac{5\pi}{12} + \frac{\delta}{2} - \frac{2\pi}{f}) + 2\cos(\frac{\pi}{12} + \frac{\delta}{2} + \frac{2\pi}{f}) + 2\cos(\frac{\pi}{4} + \frac{3\delta}{2} - \frac{2\pi}{f}) = 0.$$

By setting $x = \mathrm{e}^{\mathrm{i}\delta}, y = \mathrm{e}^{\frac{4\mathrm{i}\pi}{f}}$, the equation above is transformed to a polynomial one

$$P(x, y) := \zeta_{12}^4 x^3 + \zeta_{12}^3 x^2 y - (\zeta_{12}^5 - \zeta_{12})x^2 + (\zeta_{12}^4 - 1)xy + \zeta_{12}^2 x + \zeta_{12} y = 0.$$

Note that the primitive 12-th root of unity ζ_{12} originates from expanding the $\cos(\frac{\pi}{12})$ term, with the full derivation of these cyclotomic coefficients demonstrated in Section 3. If we replace ζ_{12} by the other primitive 12-th roots of unity ζ_{12}^k for $k = 5, 7, 11$ in $P(x, y)$, then the product of these four polynomials will have rational coefficients by Galois theory

$$\begin{aligned}\tilde{P}(x, y) = x^{12} &- x^{10}y^2 + x^8y^4 + 12x^{10}y + 5x^{10} + 43x^8y^2 + 5x^6y^4 + 12x^8y \\ &+ 48x^6y^3 + 24x^8 + 24x^6y^2 + 24x^4y^4 + 48x^6y + 12x^4y^3 + 5x^6 \\ &+ 43x^4y^2 + 5x^2y^4 + 12x^2y^3 + x^4 - x^2y^2 + y^4.\end{aligned}$$

Note that setting $x = \mathrm{e}^{2\mathrm{i}\delta}, y = \mathrm{e}^{\frac{4\mathrm{i}\pi}{f}}$ gives a simpler polynomial

$$\begin{aligned}L(x, y) := x^4y^4 &- x^5y^2 + 5x^3y^4 + x^6 + 12x^5y + 43x^4y^2 + 48x^3y^3 + 24x^2y^4 \\ &+ 5x^5 + 12x^4y + 24x^3y^2 + 12x^2y^3 + 5xy^4 + 24x^4 + 48x^3y \\ &+ 43x^2y^2 + 12xy^3 + y^4 + 5x^3 - xy^2 + x^2,\end{aligned}$$

which is full (i.e., its monomials' exponent vectors in $\mathbb{Z}^2$ generate the full integer lattice $\mathbb{Z}^2$).

To find the roots of unity solutions (x, y) or to find all cyclotomic points on this algebraic curve, we follow [15] to compute the resultants of $L(x, y)$ with $L(-x, y)$, $L(x, -y)$, $L(-x, -y)$, $L(x^2, y^2)$, $L(-x^2, y^2)$, $L(x^2, -y^2)$, and $L(-x^2, -y^2)$. Let $\mathcal{R}$ denote the resultant. For example, we have

$$\begin{aligned}f(x) := \mathcal{R}(L(x, y), L(-x, y), x) &= 256y^8(25y^8 + 998y^6 + 1923y^4 + 998y^2 + 25)^2 \cdot \\ &(11y^8 + 6y^7 - 1217y^6 - 4182y^5 - 5112y^4 - 4182y^3 - 1217y^2 + 6y + 11)^2.\end{aligned}$$

This is a single-variable polynomial, whose root of unity solutions (or cyclotomic part) can be found by a recursive algorithm [14] computing $\gcd(f(x), f(-x))$, $\gcd(f(x), f(x^2))$, and $\gcd(f(x), f(-x^2))$. Both [14, 15] are based on a key but simple property: For ω a root of unity of order $k \equiv r \pmod 4$ ($r = 0, 1, 2, 3$), ω is conjugate to $-\omega$, ω^2, $-\omega^2$, and ω^2, respectively. It turns out that all such gcd for $f(x)$ are 1, and thus $f(x)$ has no root of unity solution.

Similarly, the following resultant produces two root of unity solutions $y = \zeta_4^i$ for $i = 1, 3$ as follows

$$\begin{aligned}\mathcal{R}(L(x, y), L(x, -y), x) = 247669456896y^{14}&(4y^8 + 67y^6 + 114y^4 + 67y^2 + 4) \\ &(y^8 + 72y^6 + 110y^4 + 72y^2 + 1)(y^2 + 1)^2.\end{aligned}$$

After computation of all seven resultants, we get six root of unity solutions $y = \zeta_{12}^i$ for $i = 2, 3, 4, 8, 9, 10$. By $y = e^{\frac{4i\pi}{f}}$, we get $f = 12$, 8, 6, 3, $\frac{8}{3}$, and $\frac{12}{5}$. Since f must be an even integer ≥ 6, we only need to consider $i = 2, 3, 4$.

For $f = 12$, substitute $y = e^{\frac{i\pi}{3}}$ in $P(x, y) = 0$, and we get $x = \zeta_{12}^i$ for $i = 2, 3, 4, 8, 9, 10$ in a similar way. This implies that $(\alpha, \beta, \gamma, \delta) = (3, 4, 2, 5)/6$, $(3, 4, 3, 4)/6$, $(3, 4, 4, 3)/6$, $(3, 4, -4, 11)/6$, $(3, 4, -3, 10)/6$ or $(3, 4, -2, 9)/6$. The last three solutions are dismissed, since they have negative angles. The first solution is dismissed, since it does not satisfy (1.1) and is a fake solution. The second and third solutions are dismissed since they violate Lemmas 4 and 10, respectively.

Similarly, for $f = 6$, we get $(\alpha, \beta, \gamma, \delta) = (3, 4, 8, 1)/6$, and for $f = 8$, we get $(\alpha, \beta, \gamma, \delta) = (3, 4, 6, 2)/6$. Only these two good solutions satisfying (1.1) and Lemmas 2–7 are shown in Table 3.

Table 3. Tilings with a unique degree 3 vertex type (excluding $\alpha\beta\delta$ and $\alpha\gamma\delta$).

Vertex 1	Vertex 2	$(\alpha, \beta, \gamma, \delta)$, f	Contradiction
$\beta\gamma^2$	α^4, $\alpha^2\delta^2$		No solution
	$\alpha^3\delta$	$(5, 4, 7, 3)/9$, 36	
	$\alpha\delta^3$	$(1, 6, 2, 3)/5$, 10	In Table 4
	δ^4	$(1, 4, 2, 2)/4$, 16	In Table 5
β^3	α^4	$(3, 4, 8, 1)/6$, 6	In Table 4 ($\alpha \leftrightarrow \delta, \beta \leftrightarrow \gamma$)
		$(3, 4, 6, 2)/6$, 8	No tiling by the balance lemma
	$\alpha^3\delta$, $\alpha^2\delta^2$, $\alpha\delta^3$, δ^4		No solution
$\alpha^2\beta$	β^4	$(9, 6, 12, 5)/12$, 6	No tiling by the balance lemma
	$\beta\gamma^3$	$(15, 6, 10, 7)/18$, 36	
	$\beta^3\gamma$, $\beta^2\gamma^2$, $\beta^2\delta^2$, $\beta\gamma\delta^2$, γ^4, $\gamma^2\delta^2$, δ^4		No solution
$\beta\delta^2$	α^4, $\alpha^2\beta^2$, β^4, $\beta^3\gamma$, $\beta^2\gamma^2$, $\beta\gamma^3$		No solution
	$\alpha^2\beta\gamma$, γ^4	$(1, 4, 2, 2)/4$, 16	In Table 5
	$\alpha^2\gamma^2$	$(2, 6, 4, 3)/6$, 8	No tiling by the balance lemma

It turns out that $(\alpha, \beta, \gamma, \delta) = (3, 4, 8, 1)/6$ is a monotile admitting the standard 2-layer earth map tiling, and it also appears in Table 4 given the symmetry $\alpha \leftrightarrow \delta, \beta \leftrightarrow \gamma$.

Table 4. Tilings with $\alpha\beta\delta$.

Vertex	$(\alpha, \beta, \gamma, \delta)$, f	$(\alpha, \beta, \gamma, \delta)$, f
$\alpha\beta\delta$	$(6, 3, 4, 3)/6$, 6	$(1, 8, 4, 3)/6$, 6
	$(12, 4, 6, 2)/9$, 6	$(2, 10, 3, 6)/9$, 12
	$(1, 21, 5, 8)/15$, 12	$(4, 9, 5, 17)/15$, 12
	$(9, 28, 10, 23)/30$, 12	$(3, 16, 10, 41)/30$, 12
	$(3, 20, 4, 13)/18$, 18	$(5, 32, 6, 23)/30$, 20
	$(1, 16, 6, 43)/30$, 20	$(1, 42, 4, 17)/30$, 30
	$(4, f-4, 4, f)/f$	$(6, 4f-4, 12, 2f-2)/3f$
	$(6, 2f-4, 12, 4f-2)/3f$	

However, $(\alpha, \beta, \gamma, \delta) = (3, 4, 6, 2)/6$ does not admit any tiling of the sphere. Otherwise, the parity lemma implies that $\gamma \cdots = \alpha^2\gamma$. Consequently, we have $\#\alpha = 2\#\gamma = 2f$, a contradiction to the balance lemma. Similarly, the following two cases in Table 3 produce no monotiles either:

(1) Case $\{\alpha^2\beta, \beta^4, (9, 6, 12, 5)/12, 6\}$.

(2) Case $\{\beta\delta^2, \alpha^2\gamma^2, (2,6,4,3)/6, 8\}$.

The discussion above resolves Case $\{\beta^3, \alpha^4\}$ completely. For more details or more examples, we refer to our work [8] on rational a^4b-tilings.

4. All rational a^3b-monotiles

By Lemma 10, we only need to consider non-symmetric a^3b-quadrilaterals ($\alpha \neq \delta$, $\beta \neq \gamma$) with even $f \geq 6$ and all rational angles in $(0, 2\pi)$ satisfying (1.1) and Lemmas 2–7 (simply called "good" hereafter). By Lemmas 11–13, there are 36 vertex combinations to help simplify (1.1), which induce all rational a^3b-monotiles summarized in Tables 3–5 in the following subsections.

4.1. *Tilings with $\alpha\beta\delta$ or $\alpha\gamma\delta$ as a vertex*

By symmetry, it is enough to consider $\alpha\beta\delta$. By Lemma 2, we get

$$\alpha = 2\pi - \beta - \delta, \quad \gamma = \tfrac{4\pi}{f}.$$

The trigonometric Eq (1.1) reduces to

$$Q(\beta, \delta, f) := 2\cos\Big(\tfrac{3\beta}{2} + \delta + \tfrac{2\pi}{f}\Big) - 2\cos\Big(\tfrac{\beta}{2} + \delta + \tfrac{2\pi}{f}\Big) + 2\cos\Big(\tfrac{\beta}{2} - \tfrac{2\pi}{f} - \delta\Big) - 2\cos\Big(\tfrac{\beta}{2} - \delta + \tfrac{2\pi}{f}\Big) = 0.$$

By setting $x = e^{i\beta}, y = e^{2i\delta}, z = e^{\frac{4i\pi}{f}}$, $Q(\beta, \delta, f)$ is transformed to a polynomial

$$P(x, y, z) := x^3yz - x^2yz - x^2z + xyz + x^2 - xy - x + 1 = 0.$$

This is the only case inducing a 3-variable polynomial equation. Following the algorithm in [16], we compute 15 resultants to get all good rational angle solutions shown below:

Case 1. $\mathcal{R}(P(x, y, z), P(-x, y, z), x)$

$(z - 1)^2 (y + 1)^2 (yz - 1)^2 = 0.$

f	$(\alpha,\beta,\gamma,\delta)$	f	$(\alpha,\beta,\gamma,\delta)$
6	$(6,3,4,3)/6$	6	$(1,8,4,3)/6$

Case 2. $\mathcal{R}(P(x, y, z), P(x, -y, z), x)$

$\left(z^2 + 1\right)(z - 1)^2 = 0.$

f	$(\alpha,\beta,\gamma,\delta)$
8	$(3,14,6,7)/12$

Case 3. $\mathcal{R}(P(x, y, z), P(x, y, -z), x)$

$(y - 1)^2 (y + 1)^2 = 0.$

f	$(\alpha,\beta,\gamma,\delta)$	f	$(\alpha,\beta,\gamma,\delta)$	f	$(\alpha,\beta,\gamma,\delta)$
6	$(6,3,4,3)/6$	6	$(1,8,4,3)/6$		$(4,f-4,4,f)/f$

We explain how f appears in the last solution $\beta = \frac{f-4}{f}\pi$. When $y = 1$, $P(x,y,z) = 0$ reduces to $(x-1)^2(xz+1) = 0$. Recall that $x = e^{i\beta}$ and $z = e^{4i\pi/f}$, and thus $(xz+1) = 0$ gives

$$\beta + \tfrac{4}{f}\pi = \pi \implies \beta = \tfrac{f-4}{f}\pi.$$

Case 4. $\mathcal{R}(P(\tilde{x},\tilde{y},z), P(-\tilde{x},-\tilde{y},z), \tilde{x}), (x = \tilde{x}\tilde{y}, y = \frac{\tilde{x}}{\tilde{y}})$

$(y-1)(y+1) = 0.$

f	$(\alpha,\beta,\gamma,\delta)$	f	$(\alpha,\beta,\gamma,\delta)$	f	$(\alpha,\beta,\gamma,\delta)$
6	$(6,3,4,3)/6$	6	$(1,8,4,3)/6$		$(4,f-4,4,f)/f$

Case 5. $\mathcal{R}(P(\tilde{x},y,z), P(-\tilde{x},y,-z), \tilde{x}), (x = \tilde{x}^2)$

$(y+1)^3\left(z^2+1\right)(y-1) = 0.$

f	$(\alpha,\beta,\gamma,\delta)$	f	$(\alpha,\beta,\gamma,\delta)$	f	$(\alpha,\beta,\gamma,\delta)$	f	$(\alpha,\beta,\gamma,\delta)$
6	$(6,3,4,3)/6$	6	$(1,8,4,3)/6$	8	$(3,14,6,7)/12$		$(4,f-4,4,f)/f$

Case 6. $\mathcal{R}(P(x,y,z), P(x,-y,-z), x)$

$(y-1)(y+1)\left(y^2+yz+z^2\right) = 0.$

f	$(\alpha,\beta,\gamma,\delta)$	f	$(\alpha,\beta,\gamma,\delta)$	f	$(\alpha,\beta,\gamma,\delta)$	f	$(\alpha,\beta,\gamma,\delta)$
6	$(6,3,4,3)/6$	6	$(1,8,4,3)/6$	8	$(3,14,6,7)/12$		$(4,f-4,4,f)/f$

Case 7. $\mathcal{R}(P(\tilde{x},\tilde{y},z), P(-\tilde{x},-\tilde{y},-z), \tilde{x}), (x = \tilde{x}\tilde{y}, y = \frac{\tilde{x}}{\tilde{y}})$

$\left(z^2+1\right)\left(y^3z-1\right)(yz-1) = 0.$

f	$(\alpha,\beta,\gamma,\delta)$	f	$(\alpha,\beta,\gamma,\delta)$	f	$(\alpha,\beta,\gamma,\delta)$	f	$(\alpha,\beta,\gamma,\delta)$
6	$(12,4,6,2)/9$	8	$(3,14,6,7)/12$		$(6,4f-4,12,2f-2)/f$		$(6,2f-4,12,4f-2)/f$

Case 8. $\mathcal{R}(P(x,y,z), P(x^2,y^2,z^2), x)$

$(z-1)^4(yz-1)^3(y-1)^2\left(y^3z-1\right) = 0.$

f	$(\alpha,\beta,\gamma,\delta)$	f	$(\alpha,\beta,\gamma,\delta)$	f	$(\alpha,\beta,\gamma,\delta)$	f	$(\alpha,\beta,\gamma,\delta)$
6	$(12,4,6,2)/9$		$(4,f-4,4,f)/f$		$(6,4f-4,12,2f-2)/f$		$(6,2f-4,12,4f-2)/f$

Case 9. $\mathcal{R}(P(x,y,z),P(-x^2,y^2,z^2),x)$

$$(y+1)^3(z-1)^2(y-1)\left(y^2+yz+z^2\right)(yz-1)^2=0.$$

f	$(\alpha,\beta,\gamma,\delta)$	f	$(\alpha,\beta,\gamma,\delta)$	f	$(\alpha,\beta,\gamma,\delta)$	f	$(\alpha,\beta,\gamma,\delta)$
6	$(6,3,4,3)/6$	6	$(1,8,4,3)/6$	8	$(3,14,6,7)/12$		$(4,f-4,4,f)/f$

Case 10. $\mathcal{R}(P(x,y,z),P(x^2,-y^2,z^2),x)$

$$(z-1)^2(y-1)(y+1)(y^6z^4-2y^6z^3-2y^5z^4+y^6z^2-y^4z^4+4y^5z^2+7y^4z^3+2y^3z^4-2y^5z-y^4z^2+y^3z^3+y^2z^4-5y^4z-10y^3z^2-5y^2z^3+y^4+y^3z-y^2z^2-2yz^3+2y^3+7y^2z+4yz^2-y^2+z^2-2y-2z+1)=0.$$

f	$(\alpha,\beta,\gamma,\delta)$	f	$(\alpha,\beta,\gamma,\delta)$	f	$(\alpha,\beta,\gamma,\delta)$	f	$(\alpha,\beta,\gamma,\delta)$
6	$(6,3,4,3)/6$	6	$(1,8,4,3)/6$	8	$(3,14,6,7)/12$	16	$(1,3,1,4)/4$
16	$(1,10,2,5)/8$	16	$(3,14,6,31)/24$		$(4,f-4,4,f)/f$		

Case 11. $\mathcal{R}(P(x,y,z),P(x^2,y^2,-z^2),x)$

$$(y-1)^3(y+1)(y^4z^6-3y^4z^5+4y^4z^4-4y^3z^5-2y^4z^3+10y^3z^4-y^2z^5+y^4z^2-9y^3z^3+8y^2z^4-yz^5+4y^3z^2-12y^2z^3+4yz^4-y^3z+8y^2z^2-9yz^3+z^4-y^2z+10yz^2-2z^3-4yz+4z^2-3z+1)=0.$$

f	$(\alpha,\beta,\gamma,\delta)$	f	$(\alpha,\beta,\gamma,\delta)$	f	$(\alpha,\beta,\gamma,\delta)$	f	$(\alpha,\beta,\gamma,\delta)$
6	$(6,3,4,3)/6$	6	$(1,8,4,3)/6$	8	$(3,14,6,7)/12$	12	$(2,10,3,6)/9$
12	$(1,21,5,8)/15$	12	$(4,9,5,17)/15$	12	$(9,28,10,23)/30$	12	$(3,16,10,41)/30$
12	$(1,2,1,3)/3$	12	$(3,22,6,11)/18$	12	$(3,10,6,23)/18$	24	$(1,5,1,6)/6$
24	$(3,46,6,23)/36$	24	$(3,22,6,47)/36$	36	$(1,8,1,9)/9$	36	$(3,70,6,35)/54$
36	$(3,37,6,71)/54$		$(4,f-4,4,f)/f$				

Case 12. $\mathcal{R}(P(x,y,z),P(-x^2,-y^2,z^2),x)$

$$y^8z^6+y^7z^7+y^6z^8-2y^8z^5-y^6z^7+y^8z^4-4y^7z^5-y^6z^6+6y^7z^4+3y^6z^5+y^5z^6+y^4z^7-3y^7z^3+2y^6z^4+3y^5z^5-3y^4z^6-y^3z^7-6y^6z^3-6y^5z^4+5y^4z^5+y^3z^6+3y^6z^2+4y^5z^3-4y^4z^4+4y^3z^5+3y^2z^6+y^5z^2+5y^4z^3-6y^3z^4-6y^2z^5-y^5z-3y^4z^2+3y^3z^3+2y^2z^4-3yz^5+y^4z+y^3z^2+3y^2z^3+6yz^4-y^2z^2-4yz^3+z^4-y^2z-2z^3+y^2+yz+z^2=0.$$

f	$(\alpha,\beta,\gamma,\delta)$	f	$(\alpha,\beta,\gamma,\delta)$	f	$(\alpha,\beta,\gamma,\delta)$	f	$(\alpha,\beta,\gamma,\delta)$
8	$(3,14,6,7)/12$	16	$(1,3,1,4)/4$	16	$(1,10,2,5)/8$	16	$(3,14,6,31)/24$
30	$(1,42,4,17)/30$	30	$(2,13,2,15)/15$	30	$(3,58,6,29)/45$	30	$(3,28,6,59)/45$
42	$(2,19,2,21)/21$	42	$(3,82,6,41)/63$	42	$(3,40,6,83)/63$		

Case 13. $\mathcal{R}(P(x,y,z),P(-x^2,y^2,-z^2),x)$

$$(z^2+1)(y+1)^2(y^6z^4+y^5z^5+y^4z^6-y^6z^3-3y^5z^4-2y^4z^5+y^6z^2+3y^5z^3+3y^4z^4-y^3z^5-y^5z^2+y^4z^3+4y^3z^4+2y^2z^5-2y^4z^2-6y^3z^3-2y^2z^4+2y^4z+4y^3z^2+y^2z^3-yz^4-y^3z+3y^2z^2+3yz^3+z^4-2y^2z-3yz^2-z^3+y^2+yz+z^2)=0.$$

f	$(\alpha,\beta,\gamma,\delta)$	f	$(\alpha,\beta,\gamma,\delta)$	f	$(\alpha,\beta,\gamma,\delta)$	f	$(\alpha,\beta,\gamma,\delta)$
6	$(6,3,4,3)/6$	6	$(1,8,4,3)/6$	8	$(3,14,6,7)/12$	12	$(2,10,3,6)/9$
12	$(1,21,5,8)/15$	12	$(4,9,5,17)/15$	12	$(9,28,10,23)/30$	12	$(3,16,10,41)/30$
12	$(1,2,1,3)/3$	12	$(3,22,6,11)/18$	12	$(3,10,6,23)/18$	16	$(1,3,1,4)/4$
16	$(1,10,2,5)/8$	16	$(3,14,6,31)/24$	60	$(1,14,1,15)/15$	60	$(3,118,6,59)/90$
60	$(3,58,6,119)/90$						

Case 14. $\mathcal{R}(P(x,y,z),P(x^2,-y^2,-z^2),x)$

$$y^8z^8-4y^8z^7-2y^7z^8+8y^8z^6+5y^7z^7-y^6z^8-9y^8z^5-5y^7z^6+9y^6z^7+2y^5z^8+7y^8z^4-24y^6z^6-8y^5z^7+y^4z^8-3y^8z^3+4y^7z^4+35y^6z^5+12y^5z^6-7y^4z^7+y^8z^2-3y^7z^3-30y^6z^4-5y^5z^5+23y^4z^6+3y^3z^7+y^7z^2+18y^6z^3-4y^5z^4-41y^4z^5-6y^3z^6+2y^2z^7-8y^6z^2+8y^5z^3+48y^4z^4+8y^3z^5-8y^2z^6+2y^6z-6y^5z^2-41y^4z^3-4y^3z^4+18y^2z^5+yz^6+3y^5z+23y^4z^2-5y^3z^3-30y^2z^4-3yz^5+z^6-7y^4z+12y^3z^2+35y^2z^3+4yz^4-3z^5+y^4-8y^3z-24y^2z^2+7z^4+2y^3+9y^2z-5yz^2-9z^3-y^2+5yz+8z^2-2y-4z+1=0.$$

f	$(\alpha,\beta,\gamma,\delta)$	f	$(\alpha,\beta,\gamma,\delta)$	f	$(\alpha,\beta,\gamma,\delta)$	f	$(\alpha,\beta,\gamma,\delta)$
8	$(3,14,6,7)/12$	12	$(2,10,3,6)/9$	12	$(1,21,5,8)/15$	12	$(4,9,5,17)/15$
12	$(9,28,10,23)/30$	12	$(3,16,10,41)/30$	12	$(1,2,1,3)/3$	12	$(3,22,6,11)/18$
12	$(3,10,6,23)/18$	16	$(1,3,1,4)/4$	16	$(1,10,2,5)/8$	16	$(3,14,6,31)/24$
20	$(5,32,6,23)/30$	20	$(1,16,6,43)/30$	20	$(1,4,1,5)/5$	20	$(3,38,6,19)/30$
20	$(1,6,2,13)/10$	24	$(1,5,1,6)/6$	24	$(3,46,6,23)/36$	24	$(3,22,6,47)/36$
60	$(1,14,1,15)/15$	60	$(3,118,6,59)/90$	60	$(3,58,6,119)/90$		

Case 15. $\mathcal{R}(P(x,y,z),P(-x^2,-y^2,-z^2),x)$

$$\left(z^2+1\right)(y-1)(y+1)(yz-1)\left(y^2+yz+z^2\right)\left(y^3z-1\right)=0.$$

f	$(\alpha,\beta,\gamma,\delta)$	f	$(\alpha,\beta,\gamma,\delta)$	f	$(\alpha,\beta,\gamma,\delta)$	f	$(\alpha,\beta,\gamma,\delta)$
6	$(6,3,4,3)/6$	6	$(1,8,4,3)/6$	6	$(12,4,6,2)/9$	8	$(3,14,6,7)/12$
	$(4,f-4,4,f)/f$		$(6,4f-4,12,2f-2)/f$		$(6,2f-4,12,4f-2)/f$		

In summary, there are three infinite sequences of rational a^3b-quadrilaterals, and many sporadic rational solutions are merged into these three infinite sequences. The following table combines all the abovementioned solutions, which all admit the simplest 2-layer earth map tilings $T(f\alpha\beta\delta, 2\gamma^{\frac{f}{2}})$ for some even integers $f \geq 6$, together with their various modifications when β is an integer multiple of γ (see Tables 1 and 2). The first two images of Figure 1 are two simple examples with just 10 tiles.

4.2. *Tilings with two degree 3 vertex types (excluding $\alpha\beta\delta$ and $\alpha\gamma\delta$).*

By Lemmas 11 and 12, we obtain seven distinct cases. We solve them following the typical example Case $\{\beta^3,\alpha^4\}$ in Section 3.

Case $\{\alpha^2\beta,\beta^3\}$, $(x=\mathrm{e}^{\mathrm{i}\delta}, y=\mathrm{e}^{\frac{8i\pi}{f}})$

$$\zeta_3x^3-\zeta_3^2x^2y-\left(\zeta_3^2-1\right)x^2+\left(\zeta_3-\zeta_3^2\right)xy-\zeta_3^2x+y=0.$$

This has no good rational angle solution (simply “None” hereafter).

Case $\{\alpha^2\beta, \beta^2\gamma\}$, $(x = \mathrm{e}^{\frac{2\mathrm{i}\delta}{3}}, y = \mathrm{e}^{\frac{4\mathrm{i}\pi}{3f}})$

$$\zeta_3 x^4 y^2 - \zeta_3^2 x^2 y^6 + \zeta_3 x y^5 - 2\zeta_3^2 x^2 y^3 + x^3 y - \zeta_3^2 x^2 + y^4 = 0.$$

None.

Case $\{\alpha^2\beta, \gamma^3\}$, $(x = \mathrm{e}^{2\mathrm{i}\delta}, y = \mathrm{e}^{\frac{4\mathrm{i}\pi}{f}})$

$$-\zeta_3^2 y^4 + \left(\zeta_3 - \zeta_3^2\right) x^2 y - \zeta_3^2 x y^2 - \left(\zeta_3^2 - 1\right) y^3 - \zeta_3^2 x^2 = 0.$$

None.

Case $\{\alpha^2\beta, \gamma\delta^2\}$, $(x = \mathrm{e}^{2\mathrm{i}\delta}, y = \mathrm{e}^{\frac{4\mathrm{i}\pi}{f}})$

$$x y^4 + x y^3 - x y^2 - y^3 - x y - y^2 + y + 1 = 0.$$

None.

Case $\{\beta\delta^2, \beta^2\gamma\}$, $(x = \mathrm{e}^{\frac{\mathrm{i}\delta}{2}}, y = \mathrm{e}^{\frac{2\mathrm{i}\pi}{f}})$

$$x^6 - x^5 y - x^5 + 2x^3 y - x y^2 - x y + y^2 = 0.$$

None.

Case $\{\beta\delta^2, \gamma^3\}$, $(x = \mathrm{e}^{\frac{\mathrm{i}\delta}{2}}, y = \mathrm{e}^{\frac{4\mathrm{i}\pi}{f}})$

$$-\zeta_3^2 x^2 y^2 - (\zeta_3 - 1) x^2 y + \zeta_3^2 x y^2 + \zeta_3^2 x + (\zeta_3 - 1) y - \zeta_3^2 = 0.$$

None.

Case $\{\beta\delta^2, \beta\gamma^2\}$, $(x = \mathrm{e}^{\frac{\mathrm{i}\delta}{2}}, y = \mathrm{e}^{\frac{4\mathrm{i}\pi}{f}})$

$$x^5 y - x^4 y + x^3 y^2 - x^4 - x y^2 + x^2 - x y + y = 0.$$

$f = 16, (\alpha, \beta, \gamma, \delta) = (1, 4, 2, 2)/4.$

These results are summarized in Table 5, and there is only one rational a^3b-monotile admitting two tilings with 16 tiles, shown in Table 1 and the third and fourth images in Figure 1.

Table 5. Tilings with two degree 3 vertex types (excluding $\alpha\beta\delta$ and $\alpha\gamma\delta$).

Vertex	$(\alpha, \beta, \gamma, \delta)$, f	Contradiction
$\alpha^2\beta$, β^3		No solution
$\alpha^2\beta$, $\beta^2\gamma$		
$\alpha^2\beta$, γ^3		
$\alpha^2\beta$, $\gamma\delta^2$		
$\beta\delta^2$, $\beta^2\gamma$		
$\beta\delta^2$, γ^3		
$\beta\delta^2$, $\beta\gamma^2$	$(1, 4, 2, 2)/4$, 16	

4.3. Tilings with a unique degree 3 vertex type (excluding $\alpha\beta\delta$ and $\alpha\gamma\delta$)

By Lemma 13, we obtain $5 + 5 + 9 + 9 = 28$ distinct cases. We solve them following the typical example Case $\{\beta^3, \alpha^4\}$ in Section 3.

Case $\{\beta\gamma^2, \alpha^4\}$, $(x = \mathrm{e}^{\mathrm{i}\gamma}, y = \mathrm{e}^{\frac{4\mathrm{i}\pi}{f}})$

$x^5y^2 - x^4y^2 - x^4y - x^3y + x^2y + xy + x - 1 = 0.$
None.

Case $\{\beta\gamma^2, \alpha^3\delta\}$, $(x = \mathrm{e}^{\frac{2\mathrm{i}\delta}{3}}, y = \mathrm{e}^{\frac{4\mathrm{i}\pi}{f}})$

$\zeta_3x^5y + x^5\zeta_3 - x^6\zeta_3^2 - x^3y^2\zeta_3^2 - x^3y\zeta_3^2 - y^3\zeta_3^2 + xy^3 + xy^2 = 0.$
$f = 36, (\alpha, \beta, \gamma, \delta) = (5, 4, 7, 3)/9.$

Case $\{\beta\gamma^2, \alpha^2\delta^2\}$, $(x = \mathrm{e}^{2\mathrm{i}\delta}, y = \mathrm{e}^{\frac{4\mathrm{i}\pi}{f}})$

$xy^2 + 2y^3 - xy + y^2 - 2x - y = 0.$
None.

Case $\{\beta\gamma^2, \alpha\delta^3\}$, $(x = \mathrm{e}^{2\mathrm{i}\delta}, y = \mathrm{e}^{\frac{4\mathrm{i}\pi}{f}})$

$x^4y^2 + x^3y^3 - x^2y^3 - x^2y^2 - x^2y - x^2 + x + y = 0.$
$f = 10, (\alpha, \beta, \gamma, \delta) = (1, 6, 2, 3)/5, 10.$

Case $\{\beta\gamma^2, \delta^4\}$, $(x = \mathrm{e}^{\mathrm{i}\gamma}, y = \mathrm{e}^{\frac{4\mathrm{i}\pi}{f}})$

$x^3y^2 - x^3y + x^2y - xy^2 + x^2 - xy + y - 1 = 0.$
$f = 16, (\alpha, \beta, \gamma, \delta) = (1, 4, 2, 2)/4.$

Case $\{\beta^3, \alpha^4\}$, $(x = \mathrm{e}^{2\mathrm{i}\delta}, y = \mathrm{e}^{\frac{8\mathrm{i}\pi}{f}})$

$\zeta_{12}^4x^3 + \zeta_{12}^3x^2y - \left(\zeta_{12}^5 - \zeta_{12}\right)x^2 + \left(\zeta_{12}^4 - 1\right)xy + \zeta_{12}^2x + \zeta_{12}y = 0.$
$f = 6, (\alpha, \beta, \gamma, \delta) = (3, 4, 8, 1)/6.$
$f = 8, (\alpha, \beta, \gamma, \delta) = (3, 4, 6, 2)/6.$

Case $\{\beta^3, \alpha^3\delta\}$, $(x = \mathrm{e}^{\frac{2\mathrm{i}\delta}{3}}, y = \mathrm{e}^{\frac{4\mathrm{i}\pi}{f}})$

$\zeta_3x^4 - \zeta_3^2x^3y + \left(\zeta_3 - \zeta_3^2\right)x^2y - \left(\zeta_3^2 - 1\right)x^2 - \zeta_3^2x + y = 0.$
None.

Case $\{\beta^3, \alpha^2\delta^2\}$, $(x = \mathrm{e}^{2\mathrm{i}\delta}, y = \mathrm{e}^{\frac{4\mathrm{i}\pi}{f}})$

$-\left(2\zeta_3^2 - 1\right)xy - x + y - 1 + 2\zeta_3 = 0.$
None.

Case $\{\beta^3, \alpha\delta^3\}$, $(x = \mathrm{e}^{2\mathrm{i}\delta}, y = \mathrm{e}^{\frac{4\mathrm{i}\pi}{f}})$

$\left(\zeta_3 - \zeta_3^2\right)x^4y - \zeta_3x^3y + \zeta_3^2x^2y + \zeta_3^2x^2 - x - \zeta_3^2 + 1 = 0.$
None.

Case $\{\beta^3, \delta^4\}$, $(x = \mathrm{e}^{\mathrm{i}\gamma}, y = \mathrm{e}^{\frac{4\mathrm{i}\pi}{f}})$

$$-\left(\zeta_3^2 - 1\right)x^3 + \left(\zeta_3 + \zeta_3^2\right)x^2y - \left(\zeta_3 + \zeta_3^2\right)xy - (1 - \zeta_3)\,y^2 = 0.$$

None.

Case $\{\alpha^2\beta, \beta^4\}$, $(x = \mathrm{e}^{\mathrm{i}\delta}, y = \mathrm{e}^{\frac{4\mathrm{i}\pi}{f}})$

$$\zeta_8^3x^3 + \zeta_8^2x^2y + \left(\zeta_8^2 + 1\right)x^2 + \left(\zeta_8^3 + \zeta_8\right)xy + \zeta_8x + y = 0.$$

$f = 6, (\alpha, \beta, \gamma, \delta) = (9, 6, 12, 5)/12.$

Case $\{\alpha^2\beta, \beta^3\gamma\}$, $(x = \mathrm{e}^{\frac{2\mathrm{i}\delta}{5}}, y = \mathrm{e}^{\frac{4\mathrm{i}\pi}{5f}})$

$$\zeta_5^2x^7y^2 - \zeta_5^4x^4y^8 + \zeta_5^2x^2y^7 - \zeta_5^4x^4y^3 - \zeta_5^3x^3y^5 + x^5y - \zeta_5^3x^3 + y^6 = 0.$$

None.

Case $\{\alpha^2\beta, \beta^2\gamma^2\}$, $(x = \mathrm{e}^{2\mathrm{i}\delta}, y = \mathrm{e}^{\frac{4\mathrm{i}\pi}{f}})$

$$y^4 + xy^2 + 2y^3 - 2xy - y^2 - x = 0.$$

None.

Case $\{\alpha^2\beta, \beta^2\delta^2\}$, $((x = \mathrm{e}^{\frac{\mathrm{i}\delta}{2}}, y = \mathrm{e}^{\frac{4\mathrm{i}\pi}{f}})$

$$\zeta_4x^7 + x^6y + x^6 + x^4 + \zeta_4x^3y + \zeta_4xy + \zeta_4x + y = 0.$$

None.

Case $\{\alpha^2\beta, \beta\gamma^3\}$, $(x = \mathrm{e}^{2\mathrm{i}\delta}, y = \mathrm{e}^{\frac{4\mathrm{i}\pi}{f}})$

$$-x^2y^8 + y^{10} - x^3y^6 - x^2y^7 + xy^8 + y^9 + x^6y + x^5y^2 - x^4y^3 - x^3y^4 + x^6 - x^4y^2 = 0.$$

$f = 36, (\alpha, \beta, \gamma, \delta) = (15, 6, 10, 7)/18.$

Case $\{\alpha^2\beta, \beta\gamma\delta^2\}$, $(x = \mathrm{e}^{2\mathrm{i}\delta}, y = \mathrm{e}^{\frac{4\mathrm{i}\pi}{f}})$

$$x^2y^4 + x^2y^3 - x^2y^2 - xy^3 - xy - y^2 + y + 1 = 0.$$

None.

Case $\{\alpha^2\beta, \gamma^4\}$, $(x = \mathrm{e}^{2\mathrm{i}\delta}, y = \mathrm{e}^{\frac{4\mathrm{i}\pi}{f}})$

$$\left(\zeta_4^2 + \zeta_4\right)y^4 + 2\zeta_4x^2y + 2\zeta_4xy^2 + 2\zeta_4y^3 + (\zeta_4 + 1)\,x^2 = 0.$$

None.

Case $\{\alpha^2\beta, \gamma^2\delta^2\}$, $(x = \mathrm{e}^{\mathrm{i}\delta}, y = \mathrm{e}^{\frac{2\mathrm{i}\pi}{f}})$

$$-xy^4 + x^3y - x^2y^2 - xy^3 + x^2y + xy^2 - y^3 + x^2 = 0.$$

None.

Case $\{\alpha^2\beta, \delta^4\}$, $(x = \mathrm{e}^{\mathrm{i}\gamma}, y = \mathrm{e}^{\frac{2\mathrm{i}\pi}{f}})$

$$x^3y + x^2y^2 - xy^3 + y^4 + x^3 - x^2y + xy^2 + y^3 = 0.$$

None.

Case $\{\beta\delta^2, \alpha^4\}$, $(x = \mathrm{e}^{\mathrm{i}\gamma}, y = \mathrm{e}^{\frac{4\mathrm{i}\pi}{f}})$

$$x^5 + x^4y + x^2y^3 - xy^4 - x^4 + x^3y + xy^3 + y^4 = 0.$$

None.

Case $\{\beta\delta^2, \alpha^2\beta^2\}$, $(x = \mathrm{e}^{\mathrm{i}\gamma}, y = \mathrm{e}^{\frac{4\mathrm{i}\pi}{f}})$

$$x^7 - x^6y - x^5y^2 + x^2y^5 + x^5y - x^2y^4 - xy^5 + y^6 = 0.$$

None.

Case $\{\beta\delta^2, \alpha^2\beta\gamma\}$, $(x = \mathrm{e}^{2\mathrm{i}\delta}, y = \mathrm{e}^{\frac{4\mathrm{i}\pi}{f}})$

$$x^2y^3 - xy^4 + y^4 - x^2y - y^3 + x^2 - x + y = 0.$$

$f = 16, (\alpha, \beta, \gamma, \delta) = (1, 4, 2, 2)/4.$

Case $\{\beta\delta^2, \alpha^2\gamma^2\}$, $(x = \mathrm{e}^{\mathrm{i}\gamma}, y = \mathrm{e}^{\frac{4\mathrm{i}\pi}{f}})$

$$(y-1)(y+1)(x-1)(x+1)\left(x^2y - xy^2 + xy - x + y\right) = 0.$$

$f = 8, (\alpha, \beta, \gamma, \delta) = (2, 6, 4, 3)/6.$

Case $\{\beta\delta^2, \beta^4\}$, $(x = \mathrm{e}^{\mathrm{i}\gamma}, y = \mathrm{e}^{\frac{4\mathrm{i}\pi}{f}})$

$$(\zeta_4 + 1)x^3 - 2x^2y + 2xy + (\zeta_4 - 1)y^2 = 0.$$

$f = 8, (\alpha, \beta, \gamma, \delta) = (11, 6, 4, 9)/12.$

Case $\{\beta\delta^2, \beta^3\gamma\}$, $(x = \mathrm{e}^{\frac{\mathrm{i}\gamma}{3}}, y = \mathrm{e}^{\frac{4\mathrm{i}\pi}{f}})$

$$\zeta_3x^7y - \zeta_3^2x^9 - \zeta_3x^4y + x^8 + \zeta_3xy^2 - x^5y - \zeta_3^2y^2 + x^2y = 0.$$

None.

Case $\{\beta\delta^2, \beta^2\gamma^2\}$, $(x = \mathrm{e}^{\mathrm{i}\gamma}, y = \mathrm{e}^{\frac{4\mathrm{i}\pi}{f}})$

$$x^3y + x^3 - x^2y + xy^2 + x^2 - xy + y^2 + y = 0.$$

None.

Case $\{\beta\delta^2, \beta\gamma^3\}$, $(x = \mathrm{e}^{\frac{2\mathrm{i}\delta}{3}}, y = \mathrm{e}^{\frac{4\mathrm{i}\pi}{f}})$

$$(x-1)(x+1)(x^2+x+1)(x^2-x+1)(x^4y - x^3y + x^2y^2 - x^2 + xy - y) = 0.$$

None.

Case $\{\beta\delta^2, \gamma^4\}$, $(x = \mathrm{e}^{2\mathrm{i}\delta}, y = \mathrm{e}^{\frac{4\mathrm{i}\pi}{f}})$

$$\zeta_4x^2y^2 - (\zeta_4 - 1)x^2y - \zeta_4xy^2 - x + (\zeta_4 - 1)y + 1 = 0.$$

$f = 16, (\alpha, \beta, \gamma, \delta) = (1, 4, 2, 2)/4.$

These results are summarized in Table 3. There are only two new rational a^3b-monotiles, each admitting a unique tiling with 36 tiles, as shown in Table 1 and the last two images in Figure 1.

5. Conclusions

In conclusion, we have found all rational a^3b-monotiles using more efficient new algebraic number theory tool and verified the previous work [6]. Currently, this method has been applied manually to individual cases. Developing a fully automated approach—possibly through the integration of artificial intelligence—would be a valuable next step.

Author contributions

Jinjin Liang: conceptualization, methodology, formal analysis, investigation, and writing–original draft preparation; Yixi Liao: methodology, validation, formal analysis, investigation, and writing–review and editing; Erxiao Wang: conceptualization, methodology, supervision, project administration, funding acquisition, and writing–review and editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

During the preparation of this work, the authors utilized Gemini for the sole purpose of language polishing and grammatical refinement to enhance the readability of the manuscript. The authors declare that this study is an original work; all mathematical proofs, theoretical derivations, and core research findings were independently developed by the authors without the use of AI. Following the use of the tool, the authors critically reviewed, verified, and edited the content to ensure academic accuracy and take full responsibility for the final version of the published work.

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Conflict of interest

The authors declare that there are no conflicts of interest.

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