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Research article

Lyapunov type inequalities for coupled fractional differential equations under multi-point boundary conditions

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Abstract: In this paper, we derive Lyapunov type inequalities for a coupled system of Caputo fractional differential equations with boundary values. By constructing the associated Green's function and discussing its relevant properties, we provide two distinct proofs based on matrix spectral analysis and Perov's fixed point theorem, respectively. These analyses derive explicit necessary conditions on the coefficient functions for the existence of nontrivial solutions, extending the classical Lyapunov inequality theory to a coupled system framework involving differential and boundary coupling. The findings of this study are therefore valuable, offering new perspectives that enrich the literature.

Keywords: fractional differential system; Lyapunov type inequality; fixed point theorem; matrix spectral analysis; coupled boundary conditions
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1. Introduction

In the literature [1], Lyapunov proved that if the second-order Dirichlet problem

$$\begin{cases} x''(t) + q(t)x(t) = 0, & t \in (a, b), \\ x(a) = x(b) = 0, \end{cases} \tag{1.1}$$

admits a nontrivial solution $x(t)$, then the function $q(t) \in C[a, b]$ satisfies

$$\int_a^b |q(s)|\, ds > \frac{4}{b-a}.$$

The Lyapunov inequalities and their generalizations provide key tools for studying differential equations, with applications in stability criteria, asymptotic theory, and a priori estimates [2–4]. The advantages of fractional calculus in modeling have spurred the extension of this theory from integer order systems to fractional systems, attracting growing interest from researchers in this field.

In [5], Ferreira first established the Lyapunov type inequality for the following Riemann-Liouville fractional boundary value problem (BVP):

$$\begin{cases} (D_{a^+}^{\alpha} x)(t) + q(t)x(t) = 0, \ t \in (a,b), \\ x(a) = x(b) = 0, \end{cases} \tag{1.2}$$

where $1 < \alpha \le 2$, $q(t) \in C[a,b]$. If the BVP (1.2) admits a nontrivial solution, then

$$\int_a^b |q(s)|\, ds > \frac{4^{\alpha-1}\Gamma(\alpha)}{(b-a)^{\alpha-1}}.$$

In [6], Ferreira applied a similar methodology to the Caputo fractional derivative, obtaining the inequality

$$\int_a^b |q(s)|ds > \frac{\alpha^{\alpha}\Gamma(\alpha)}{[(b-a)(\alpha-1)]^{\alpha-1}}.$$

Building on these foundations, the research of Lyapunov type inequalities for fractional BVPs has become an important field. The results on fractional Lyapunov type inequalities fall into two major categories: Generalizations based on various fractional derivatives (e.g., Caputo [7], Caputo-Hadamard [8,9], Hilfer [10], Hilfer-Katugampola [11], Katugampola [12,13], Caputo-Fabrizio [14]), and different types of nonlocal boundary conditions, such as multi-point [15] and integral [16,17] boundary value problems.

Beyond these developments, specific to Lyapunov type inequalities, the study of other fundamental fractional inequalities, such as Hadamard-type inequalities [18,19] and Hermite-Hadamard-type inequalities [20,21], has also seen significant progress. For instance, Ma and Yang [18] established Hadamard-type inequalities via fractional calculus in the framework of exp-convex functions, providing new tools and insights for analyzing the properties of solutions to fractional systems.

As the theory of fractional calculus continues to evolve, researchers have increasingly turned their attention to the qualitative analysis of fractional differential systems, investigating fundamental aspects such as the existence, uniqueness, and stability of solutions [22–24], as well as more advanced dynamical behaviors like center manifold reduction [25]. These diverse lines of inquiry collectively enrich our understanding of fractional models and their applications. In this paper, we aim to contribute to this growing body of knowledge by extending the classical Lyapunov inequality to a general class of coupled fractional differential systems.

The Lyapunov type inequality for fractional coupled systems has also drawn much attention. In [26], Jleli et al. considered the following coupled system:

$$\begin{cases} {}^{C}D_{a^+}^{\alpha} x(t) + f_1(t, x(t), y(t)) = 0, \ t \in (a,b), \\ {}^{C}D_{a^+}^{\beta} y(t) + f_2(t, x(t), y(t)) = 0, \ t \in (a,b), \\ x(a) = x(b) = 0, \ y(a) = y(b) = 0, \end{cases} \tag{1.3}$$

where $1 < \alpha,\beta < 2$, ${}^{C}D_{a^+}^{\alpha}$, and ${}^{C}D_{a^+}^{\beta}$ denote the Caputo fractional derivatives, the nonlinearities f_i : $[a,b] \times \mathbb{R}^2 \to \mathbb{R}$ $(i = 1,2)$ are given functions. Let

$$I_a^b(\alpha, g) = \frac{(\alpha-1)^{\alpha-1}(b-a)^{\alpha-1}}{\alpha^{\alpha}\Gamma(\alpha)} \int_a^b g(s)ds, \quad \text{for } g \in C[a,b]$$

and positive functions $p_{ij}(t)$ for $x_i, y_i \in \mathbb{R}$, such that

$$|f_i(t, x_1, y_1) - f_i(t, x_2, y_2)| \leq p_{i1}(t)|x_1 - x_2| + p_{i2}(t)|y_1 - y_2|, \quad i = 1, 2.$$

If system (1.3) admits a nontrivial solution, then

$$I_a^b(\alpha, p_{11}) + I_a^b(\beta, p_{22}) + \sqrt{\left(I_a^b(\alpha, p_{11}) - I_a^b(\beta, p_{22})\right)^2 + 4I_a^b(\alpha, p_{12})I_a^b(\beta, p_{21})} \geq 2.$$

Zou and Cui [27] considered the following fractional system:

$$\begin{cases} D_{a^+}^{\alpha} x(t) + f_1(t, x(t), y(t)) = 0, \ t \in (a, b), \\ D_{a^+}^{\beta} y(t) + f_2(t, x(t), y(t)) = 0, \ t \in (a, b), \\ x(a) = 0, \ x(b) = \sum\limits_{i=1}^{n} a_{1i} x(\zeta_i) + \sum\limits_{j=1}^{n} a_{2j} y(\eta_j), \\ y(a) = 0, \ y(b) = \sum\limits_{i=1}^{n} a_{3i} x(\zeta_i) + \sum\limits_{j=1}^{n} a_{4j} y(\eta_j), \end{cases} \tag{1.4}$$

where $1 < \alpha, \beta < 2$, $D_{a^+}^{\alpha}$, and $D_{a^+}^{\beta}$ are Riemann-Liouville fractional derivatives, the constants $a_{ij} > 0$ $(i = 1, 2, 3, 4; j = 1, 2, \dots, n)$, $a < \zeta_1 < \zeta_2 < \cdots < \zeta_n < b$, $a < \eta_1 < \eta_2 < \cdots < \eta_n < b$, and the nonlinearities $f_i : [a, b] \times \mathbb{R}^2 \to \mathbb{R}$ $(i = 1, 2)$ are given functions. Following the notation in [27], let

$$J_{ij}(p_{1j}, p_{2j}) = \lambda_{i1} \int_a^b p_{1j}(s)\, ds + \lambda_{i2} \int_a^b p_{2j}(s)\, ds, \quad i, j = 1, 2,$$

where the constants $\lambda_{i1}, \lambda_{i2}$ are positive and depend on the parameters of system (1.4). Moreover, let positive functions $p_{ij}(t)$ for $x_i, y_i \in \mathbb{R}$, such that

$$|f_i(t, x_i, y_i)| \leq p_{i1}(t)|x_i| + p_{i2}(t)|y_i|, \ i = 1, 2.$$

If system (1.4) has a nontrivial solution, then

$$J_{11}(p_{11}, p_{21}) + J_{22}(p_{12}, p_{22}) + \sqrt{(J_{11}(p_{11}, p_{21}) - J_{22}(p_{12}, p_{22}))^2 + 4J_{12}(p_{12}, p_{22})J_{21}(p_{11}, p_{21})} \geq 2.$$

Motivated by the above cited works, we study the following system:

$$\begin{cases} {}^{C}D_{0^+}^{\alpha_1} x(t) + \lambda_1 {}^{C}D_{0^+}^{\beta_1} x(t) + f_1(t, x(t), y(t)) = 0, \ t \in (0, 1), \\ {}^{C}D_{0^+}^{\alpha_2} y(t) + \lambda_2 {}^{C}D_{0^+}^{\beta_2} y(t) + f_2(t, x(t), y(t)) = 0, \ t \in (0, 1), \\ x(0) = 0, \ x(1) = \sum\limits_{i=1}^{n} a_{1i} x(\zeta_i) + \sum\limits_{j=1}^{n} a_{2j} y(\eta_j), \\ y(0) = 0, \ y(1) = \sum\limits_{i=1}^{n} a_{3i} x(\zeta_i) + \sum\limits_{j=1}^{n} a_{4j} y(\eta_j), \end{cases} \tag{1.5}$$

where $1 < \alpha_i \leq 2$, $0 < \beta_i < 1$ $(i = 1, 2)$, ${}^{C}D_{0^+}^{\alpha_i}$, and ${}^{C}D_{0^+}^{\beta_i}$ are the Caputo fractional derivatives, $\lambda_i \in \mathbb{R}$ $(i = 1, 2)$, $a_{ij} > 0$ $(i = 1, 2, 3, 4; j = 1, 2, \dots, n)$, $0 < \zeta_1 < \zeta_2 < \cdots < \zeta_n < 1$, $0 < \eta_1 < \eta_2 < \cdots < \eta_n < 1$, and f_1, f_2 are given functions. The Lyapunov type inequalities for (1.5) are proved via two distinct approaches: One is based on matrix spectral analysis and the other utilizing Perov's fixed point theorem.

We outline the principal contributions of this study as follows: First, our primary contribution is the extension of the traditional Lyapunov inequality theory from single equations to a coupled system framework. Second, we introduce system (1.5), a novel framework that generalizes prior models (1.3)–(1.4) via two crucial innovations: Internal coupling through a differential operator and external coupling via multi-point boundary conditions. In contrast to the Riemann-Liouville type system (1.4), our model employs Caputo fractional derivatives. It is motivated by the fact that Caputo derivatives enable the inclusion of lower-order derivative terms. However, this generalization introduces additional complexity, as the presence of the lower-order terms fundamentally alters the structure of the associated Green's function and its integral estimates. Third, a comprehensive analytical framework is developed for such systems, which is centered on the construction of the associated Green's function. In particular, for different intervals of parameter β_i (e.g., $0 < \beta_i < \alpha_i - 1$ and $\alpha_i - 1 \leq \beta_i < 1$), precise estimates for the integral bounds of the functions $K_{ij}(t, s)$ (see Lemma 2.5 and Lemma 2.6) are rigorously established. Finally, we perform a classification analysis according to the parameter ranges of the lower order derivatives, leading to a series of Lyapunov type inequalities with a unified formulation.

Beyond their theoretical significance, the obtained Lyapunov type inequalities have potential applications in several related areas. For instance, they can provide explicit lower bounds for the eigenvalues of the corresponding fractional differential operators, thereby offering insights into the spectral properties of such coupled systems. Moreover, these inequalities serve as effective tools for establishing uniqueness or non-existence results for solutions when specific nonlinear terms are present; if the inequality is violated, one can immediately conclude the absence of nontrivial solutions. Such applications are particularly relevant in the qualitative analysis of fractional boundary value problems arising in physics and engineering, where understanding solution behavior is of practical importance.

The paper is structured as follows: In Section 2, we introduce fundamental concepts and essential lemmas for the following text. In Section 3, we focuse on establishing Lyapunov type inequalities via matrix analysis. In Section 4, we present the derivation of Lyapunov type inequalities via Perov's fixed point theorem. Finally, two concrete examples are presented to validate the major results.

2. Preliminaries

Definition 2.1. ([28]) The Riemann-Liouville fractional integral operator of order $\alpha > 0$ for a function $f : [0, 1] \to \mathbb{R}$ is defined as

$$I_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} f(s) ds,$$

where $\Gamma(.)$ represents the Euler gamma function.

Definition 2.2. ([28]) The left-sided Caputo fractional derivative of order $\alpha > 0$ for a function $f \in C^n([0, 1])$, where $n = [\alpha] + 1$, is given by

$${}^{C}D_{0^+}^{\alpha} f(t) = \int_0^t \frac{(t - s)^{n-\alpha-1}}{\Gamma(n - \alpha)} f^{(n)}(s) ds.$$

Lemma 2.1. ([29]) Let $f \in L^1(0,1)$, and $\alpha > 0$, $n = [\alpha] + 1$, then

$$I_{0^+}^{\alpha}\, {}^{C}D_{0^+}^{\alpha} f(t) = f(t) + \sum_{k=0}^{n-1} c_k t^k.$$

We give the following formal assumptions, which underlie our subsequent analysis:

(H_1) $\sigma_{ij} \geq 0$ $(i, j = 1, 2)$ and $\sigma = \boldsymbol{\sigma_{11}\sigma_{22} - \sigma_{12}\sigma_{21} > 0}$, where

$$\boldsymbol{\sigma_{11} = 1 - \sum_{i=1}^{n} a_{1i}\zeta_i,\ \sigma_{12} = \sum_{j=1}^{n} a_{2j}\eta_j,\ \sigma_{21} = \sum_{i=1}^{n} a_{3i}\zeta_i,\ \sigma_{22} = 1 - \sum_{j=1}^{n} a_{4j}\eta_j.}$$

Lemma 2.2. Let $\alpha_i \in (1, 2], \beta_i \in (0, 1), \phi_i(t) \in C[0, 1]$ for $i = 1, 2$, then the fractional BVPs

$$\begin{cases} {}^{C}D_{0^+}^{\alpha_1} x(t) + \lambda_1 {}^{C}D_{0^+}^{\beta_1} x(t) + \phi_1(t) = 0,\ t \in (0, 1), \\ {}^{C}D_{0^+}^{\alpha_2} y(t) + \lambda_2 {}^{C}D_{0^+}^{\beta_2} y(t) + \phi_2(t) = 0,\ t \in (0, 1) \\ x(0) = 0,\ x(1) = \sum\limits_{i=1}^{n} a_{1i}x(\zeta_i) + \sum\limits_{j=1}^{n} a_{2j}y(\eta_j) \\ y(0) = 0,\ y(1) = \sum\limits_{i=1}^{n} a_{3i}x(\zeta_i) + \sum\limits_{j=1}^{n} a_{4j}y(\eta_j), \end{cases} \tag{2.1}$$

admits a solution $(x(t), y(t))$ if and only if it satisfies

$$x(t) = \int_0^1 K_{11}(t, s)x(s)ds + \int_0^1 K_{12}(t, s)\phi_1(s)ds + \int_0^1 K_{13}(t, s)y(s)ds + \int_0^1 K_{14}(t, s)\phi_2(s)ds,$$
$$y(t) = \int_0^1 K_{21}(t, s)x(s)ds + \int_0^1 K_{22}(t, s)\phi_1(s)ds + \int_0^1 K_{23}(t, s)y(s)ds + \int_0^1 K_{24}(t, s)\phi_2(s)ds,$$

where

$$K_{11}(t, s) = G_{11}(t, s) + \frac{t}{\sigma}\sum_{i=1}^{n}(\sigma_{22}a_{1i} + \sigma_{12}a_{3i})G_{11}(\zeta_i, s),$$

$$K_{12}(t, s) = G_{12}(t, s) + \frac{t}{\sigma}\sum_{i=1}^{n}(\sigma_{22}a_{1i} + \sigma_{12}a_{3i})G_{12}(\zeta_i, s),$$

$$K_{23}(t, s) = G_{21}(t, s) + \frac{t}{\sigma}\sum_{j=1}^{n}(\sigma_{21}a_{2j} + \sigma_{11}a_{4j})G_{21}(\eta_j, s),$$

$$K_{24}(t, s) = G_{22}(t, s) + \frac{t}{\sigma}\sum_{j=1}^{n}(\sigma_{21}a_{2j} + \sigma_{11}a_{4j})G_{22}(\eta_j, s),$$

$$K_{13}(t, s) = \frac{t}{\sigma}\sum_{j=1}^{n}(\sigma_{22}a_{2j} + \sigma_{12}a_{4j})G_{21}(\eta_j, s),\ K_{14}(t, s) = \frac{t}{\sigma}\sum_{j=1}^{n}(\sigma_{22}a_{2j} + \sigma_{12}a_{4j})G_{22}(\eta_j, s),$$

$$K_{21}(t, s) = \frac{t}{\sigma}\sum_{i=1}^{n}(\sigma_{21}a_{1i} + \sigma_{11}a_{3i})G_{11}(\zeta_i, s),\ K_{22}(t, s) = \frac{t}{\sigma}\sum_{i=1}^{n}(\sigma_{21}a_{1i} + \sigma_{11}a_{3i})G_{12}(\zeta_i, s).$$

Proof. Applying Lemma 2.1 yields that

$$x(t) = -\lambda_1 I_{0^+}^{\alpha_1-\beta_1} x(t) - I_{0^+}^{\alpha_1}\phi_1(t) + c_1 t + c_2,$$
$$y(t) = -\lambda_2 I_{0^+}^{\alpha_2-\beta_2} y(t) - I_{0^+}^{\alpha_2}\phi_2(t) + c_3 t + c_4,$$

for some $c_1, c_2, c_3, c_4 \in \mathbb{R}$. By Definition 2.1, we can derive

$$x(t) = -\frac{\lambda_1}{\Gamma(\alpha_1-\beta_1)}\int_0^t (t-s)^{\alpha_1-\beta_1-1}x(s)ds - \frac{1}{\Gamma(\alpha_1)}\int_0^t (t-s)^{\alpha_1-1}\phi_1(s)ds + c_1 t + c_2, \tag{2.2}$$
$$y(t) = -\frac{\lambda_2}{\Gamma(\alpha_2-\beta_2)}\int_0^t (t-s)^{\alpha_2-\beta_2-1}y(s)ds - \frac{1}{\Gamma(\alpha_2)}\int_0^t (t-s)^{\alpha_2-1}\phi_2(s)ds + c_3 t + c_4. \tag{2.3}$$

Substituting $x(0) = y(0) = 0$ into (2.2) and (2.3), we get $c_2 = c_4 = 0$. Letting $A = x(1)$ and $B = y(1)$ yields

$$c_1 = \frac{\lambda_1}{\Gamma(\alpha_1-\beta_1)}\int_0^1 (1-s)^{\alpha_1-\beta_1-1}x(s)ds + \frac{1}{\Gamma(\alpha_1)}\int_0^1 (1-s)^{\alpha_1-1}\phi_1(s)ds + A,$$
$$c_3 = \frac{\lambda_2}{\Gamma(\alpha_2-\beta_2)}\int_0^1 (1-s)^{\alpha_2-\beta_2-1}y(s)ds + \frac{1}{\Gamma(\alpha_2)}\int_0^1 (1-s)^{\alpha_2-1}\phi_2(s)ds + B.$$

By substituting c_1 and c_3 into equations (2.2) and (2.3), one has

$$\begin{aligned}
x(t) =& -\frac{\lambda_1}{\Gamma(\alpha_1-\beta_1)}\int_0^t (t-s)^{\alpha_1-\beta_1-1}x(s)ds - \frac{1}{\Gamma(\alpha_1)}\int_0^t (t-s)^{\alpha_1-1}\phi_1(s)ds \\
&+ \frac{\lambda_1 t}{\Gamma(\alpha_1-\beta_1)}\int_0^1 (1-s)^{\alpha_1-\beta_1-1}x(s)ds + \frac{t}{\Gamma(\alpha_1)}\int_0^1 (1-s)^{\alpha_1-1}\phi_1(s)ds + At \\
=& \frac{\lambda_1}{\Gamma(\alpha_1-\beta_1)}\left[\int_0^t \left[t(1-s)^{\alpha_1-\beta_1-1} - (t-s)^{\alpha_1-\beta_1-1}\right]x(s)ds + \int_t^1 t(1-s)^{\alpha_1-\beta_1-1}x(s)ds\right] \\
&+ \frac{1}{\Gamma(\alpha_1)}\left[\int_0^t \left[t(1-s)^{\alpha_1-1} - (t-s)^{\alpha_1-1}\right]\phi_1(s)ds + \int_t^1 t(1-s)^{\alpha_1-1}\phi_1(s)ds\right] + At \\
=& \int_0^1 G_{11}(t,s)x(s)ds + \int_0^1 G_{12}(t,s)\phi_1(s)ds + At, \\
y(t) =& -\frac{\lambda_2}{\Gamma(\alpha_2-\beta_2)}\int_0^t (t-s)^{\alpha_2-\beta_2-1}y(s)ds - \frac{1}{\Gamma(\alpha_2)}\int_0^t (t-s)^{\alpha_2-1}\phi_2(s)ds \\
&+ \frac{\lambda_2 t}{\Gamma(\alpha_2-\beta_2)}\int_0^1 (1-s)^{\alpha_2-\beta_2-1}y(s)ds + \frac{t}{\Gamma(\alpha_2)}\int_0^1 (1-s)^{\alpha_2-1}\phi_2(s)ds + Bt \\
=& \frac{\lambda_2}{\Gamma(\alpha_2-\beta_2)}\left[\int_0^t \left[t(1-s)^{\alpha_2-\beta_2-1} - (t-s)^{\alpha_2-\beta_2-1}\right]x(s)ds + \int_t^1 t(1-s)^{\alpha_2-\beta_2-1}x(s)ds\right] \\
&+ \frac{1}{\Gamma(\alpha_2)}\left[\int_0^t \left[t(1-s)^{\alpha_2-1} - (t-s)^{\alpha_2-1}\right]\phi_1(s)ds + \int_t^1 t(1-s)^{\alpha_2-1}\phi_1(s)ds\right] + Bt \\
=& \int_0^1 G_{21}(t,s)y(s)ds + \int_0^1 G_{22}(t,s)\phi_2(s)ds + Bt.
\end{aligned}$$

Therefore, the above equations can be expressed as

$$x(t) = \int_0^1 G_{11}(t,s)x(s)ds + \int_0^1 G_{12}(t,s)\phi_1(s)ds + At, \tag{2.4}$$

$$y(t) = \int_0^1 G_{21}(t,s)y(s)ds + \int_0^1 G_{22}(t,s)\phi_2(s)ds + Bt, \tag{2.5}$$

where

$$G_{i1}(t,s) = \frac{\lambda_i}{\Gamma(\alpha_i-\beta_i)}\begin{cases} t(1-s)^{\alpha_i-\beta_i-1} - (t-s)^{\alpha_i-\beta_i-1}, & 0 \le s \le t \le 1, \\ t(1-s)^{\alpha_i-\beta_i-1}, & 0 \le t \le s \le 1, \end{cases}$$

$$G_{i2}(t,s) = \frac{1}{\Gamma(\alpha_i)}\begin{cases} t(1-s)^{\alpha_i-1} - (t-s)^{\alpha_i-1}, & 0 \le s \le t \le 1, \\ t(1-s)^{\alpha_i-1}, & 0 \le t \le s \le 1. \end{cases}$$

To determine A and B, the functions (2.4) and (2.5) satisfy the boundary conditions of (2.1):

$$\begin{aligned} A = x(1) &= \sum_{i=1}^{n} a_{1i}x(\zeta_i) + \sum_{j=1}^{n} a_{2j}y(\eta_j) \\ &= \sum_{i=1}^{n} a_{1i}\left[\int_0^1 G_{11}(\zeta_i,s)x(s)ds + \int_0^1 G_{12}(\zeta_i,s)\phi_1(s)ds + A\zeta_i\right] \\ &\quad + \sum_{j=1}^{n} a_{2j}\left[\int_0^1 G_{21}(\eta_j,s)y(s)ds + \int_0^1 G_{22}(\eta_j,s)\phi_2(s)ds + B\eta_j\right], \\ B = y(1) &= \sum_{i=1}^{n} a_{3i}x(\zeta_i) + \sum_{j=1}^{n} a_{4j}y(\eta_j) \\ &= \sum_{i=1}^{n} a_{3i}\left[\int_0^1 G_{11}(\zeta_i,s)x(s)ds + \int_0^1 G_{12}(\zeta_i,s)\phi_1(s)ds + A\zeta_i\right] \\ &\quad + \sum_{j=1}^{n} a_{4j}\left[\int_0^1 G_{21}(\eta_j,s)y(s)ds + \int_0^1 G_{22}(\eta_j,s)\phi_2(s)ds + B\eta_j\right]. \end{aligned}$$

The above expressions are equivalent to

$$\begin{pmatrix} \sigma_{11} & -\sigma_{12} \\ -\sigma_{21} & \sigma_{22} \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} C \\ D \end{pmatrix}.$$

where

$$\begin{aligned} C = &\sum_{i=1}^{n} a_{1i}\left[\int_0^1 G_{11}(\zeta_i,s)x(s)ds + \int_0^1 G_{12}(\zeta_i,s)\phi_1(s)ds\right] \\ &+ \sum_{j=1}^{n} a_{2j}\left[\int_0^1 G_{21}(\eta_j,s)y(s)ds + \int_0^1 G_{22}(\eta_j,s)\phi_2(s)ds\right], \end{aligned}$$

$$D=\sum_{i=1}^{n}a_{3i}\left[\int_0^1 G_{11}(\zeta_i,s)x(s)ds+\int_0^1 G_{12}(\zeta_i,s)\phi_1(s)ds\right]$$
$$+\sum_{j=1}^{n}a_{4j}\left[\int_0^1 G_{21}(\eta_j,s)y(s)ds+\int_0^1 G_{22}(\eta_j,s)\phi_2(s)ds\right].$$

Since $\sigma=\sigma_{11}\sigma_{22}-\sigma_{12}\sigma_{21}\neq 0$, it follows that

$$\begin{aligned}
x(t)=&\int_0^1 G_{11}(t,s)x(s)ds+\frac{t}{\sigma}\sum_{i=1}^{n}(\sigma_{22}a_{1i}+\sigma_{12}a_{3i})\int_0^1 G_{11}(\zeta_i,s)x(s)ds\\
&+\frac{t}{\sigma}\sum_{j=1}^{n}\left(\sigma_{22}a_{2j}+\sigma_{12}a_{4j}\right)\int_0^1 G_{21}(\eta_j,s)y(s)ds\\
&+\int_0^1 G_{12}(t,s)\phi_1(s)ds+\frac{t}{\sigma}\sum_{i=1}^{n}(\sigma_{22}a_{1i}+\sigma_{12}a_{3i})\int_0^1 G_{12}(\zeta_i,s)\phi_1(s)ds\\
&+\frac{t}{\sigma}\sum_{j=1}^{n}\left(\sigma_{22}a_{2j}+\sigma_{12}a_{4j}\right)\int_0^1 G_{22}(\eta_j,s)\phi_2(s)ds\\
=&\int_0^1 K_{11}(t,s)x(s)ds+\int_0^1 K_{13}(t,s)y(s)ds\\
&+\int_0^1 K_{12}(t,s)\phi_1(s)ds+\int_0^1 K_{14}(t,s)\phi_2(s)ds,\\
y(t)=&\int_0^1 G_{21}(t,s)y(s)ds+\frac{t}{\sigma}\sum_{j=1}^{n}\left(\sigma_{21}a_{2j}+\sigma_{11}a_{4j}\right)\int_0^1 G_{21}(\eta_j,s)y(s)ds\\
&+\frac{t}{\sigma}\sum_{i=1}^{n}(\sigma_{21}a_{1i}+\sigma_{11}a_{3i})\int_0^1 G_{11}(\zeta_i,s)x(s)ds\\
&+\frac{t}{\sigma}\sum_{i=1}^{n}(\sigma_{21}a_{1i}+\sigma_{11}a_{3i})\int_0^1 G_{12}(\zeta_i,s)\phi_1(s)ds\\
&+\int_0^1 G_{22}(t,s)\phi_2(s)ds+\frac{t}{\sigma}\sum_{j=1}^{n}\left(\sigma_{21}a_{2j}+\sigma_{11}a_{4j}\right)\int_0^1 G_{22}(\eta_j,s)\phi_2(s)ds\\
=&\int_0^1 K_{21}(t,s)x(s)ds+\int_0^1 K_{23}(t,s)y(s)ds\\
&+\int_0^1 K_{22}(t,s)\phi_1(s)ds+\int_0^1 K_{24}(t,s)\phi_2(s)ds.
\end{aligned}$$

Thus, the proof is finished. □

Lemma 2.3. ([6]) Assume that $\beta_i\in(0,\alpha_i-1)$. Then the functions $G_{i1}(t,s)$ and $G_{i2}(t,s)$ from Lemma 2.2 can be bounded as follows:

(*i*) $|G_{i1}(t,s)|\leq\frac{|\lambda_i|(\alpha_i-\beta_i-1)^{\alpha_i-\beta_i-1}}{(\alpha_i-\beta_i)^{\alpha_i-\beta_i}\Gamma(\alpha_i-\beta_i)},\quad i=1,2,$

(*ii*) $|G_{i2}(t,s)|\leq\frac{(\alpha_i-1)^{\alpha_i-1}}{\alpha_i^{\alpha_i}\Gamma(\alpha_i)},\quad i=1,2.$

Lemma 2.4. Assume that $\beta_i \in [\alpha_i - 1, 1)$, then the function $G_{i1}(t,s)$ satisfies the following estimate:

$$\int_0^1 |G_{i1}(t,s)|\,ds \le \frac{3\,|\lambda_i|}{\Gamma(\alpha_i - \beta_i + 1)}, \quad i = 1, 2.$$

Proof. On the basis of Lemma 2.2, one finds that

$$\begin{aligned}\int_0^1 |G_{i1}(t,s)|ds =& \frac{|\lambda_i|}{\Gamma(\alpha_i - \beta_i)}\left(\int_0^t (t-s)^{\alpha_i-\beta_i-1} - t(1-s)^{\alpha_i-\beta_i-1}ds + \int_t^1 t(1-s)^{\alpha_i-\beta_i-1}ds\right)\\ =& \frac{|\lambda_i|}{\Gamma(\alpha_i - \beta_i + 1)}\left[t^{\alpha_i-\beta_i} + t\left((1-t)^{\alpha_i-\beta_i} - 1\right) + t(1-t)^{\alpha_i-\beta_i}\right]\\ =& \frac{|\lambda_i|}{\Gamma(\alpha_i - \beta_i + 1)}\left(t^{\alpha_i-\beta_i} + 2t(1-t)^{\alpha_i-\beta_i} - t\right)\\ \le& \frac{3\,|\lambda_i|}{\Gamma(\alpha_i - \beta_i + 1)}.\end{aligned}$$

The proof is complete. □

Lemma 2.5. The functions $K_{ij}(t,s)$ $(j = 1, 3)$ given in Lemma 2.2 satisfy the following conditions:

$$\mathbf{T_i} = \frac{|\lambda_i|(\alpha_i-\beta_i-1)^{\alpha_i-\beta_i-1}}{(\alpha_i-\beta_i)^{\alpha_i-\beta_i}\Gamma(\alpha_i-\beta_i)},\ \mathbf{W_i} = \frac{(\alpha_i-1)^{\alpha_i-1}}{\alpha_i^{\alpha_i}\Gamma(\alpha_i)},\ \mathbf{V_i} = \frac{3|\lambda_i|}{\Gamma(\alpha_i-\beta_i+1)},\ \mathbf{i = 1, 2},$$

$$\int_0^1 |K_{11}(t,s)|ds \le \begin{cases} T_1\left(1 + \frac{1}{\sigma}\sum\limits_{i=1}^n (\sigma_{22}a_{1i} + \sigma_{12}a_{3i})\right),\ \beta_1 \in (0, \alpha_1 - 1),\\ V_1\left(1 + \frac{1}{\sigma}\sum\limits_{i=1}^n (\sigma_{22}a_{1i} + \sigma_{12}a_{3i})\right),\ \beta_1 \in [\alpha_1 - 1, 1),\end{cases}$$

$$\int_0^1 |K_{21}(t,s)|ds \le \begin{cases} \frac{T_1}{\sigma}\sum\limits_{i=1}^n (\sigma_{21}a_{1i} + \sigma_{11}a_{3i}),\ \beta_1 \in (0, \alpha_1 - 1),\\ \frac{V_1}{\sigma}\sum\limits_{i=1}^n (\sigma_{21}a_{1i} + \sigma_{11}a_{3i}),\ \beta_1 \in [\alpha_1 - 1, 1),\end{cases}$$

$$\int_0^1 |K_{13}(t,s)|ds \le \begin{cases} \frac{T_2}{\sigma}\sum\limits_{j=1}^n (\sigma_{22}a_{2j} + \sigma_{12}a_{4j}),\ \beta_2 \in (0, \alpha_2 - 1),\\ \frac{V_2}{\sigma}\sum\limits_{j=1}^n (\sigma_{22}a_{2j} + \sigma_{12}a_{4j}),\ \beta_2 \in [\alpha_2 - 1, 1),\end{cases}$$

$$\int_0^1 |K_{23}(t,s)|ds \le \begin{cases} T_2\left(1 + \frac{1}{\sigma}\sum\limits_{j=1}^n (\sigma_{21}a_{2j} + \sigma_{11}a_{4j})\right),\ \beta_2 \in (0, \alpha_2 - 1),\\ V_2\left(1 + \frac{1}{\sigma}\sum\limits_{j=1}^n (\sigma_{21}a_{2j} + \sigma_{11}a_{4j})\right),\ \beta_2 \in [\alpha_2 - 1, 1).\end{cases}$$

Proof. Applying Lemma 2.3 with the assumption $\beta_1 \in (0, \alpha_1 - 1)$, we have

$$|K_{11}(t,s)| = \left|G_{11}(t,s) + \frac{t}{\sigma}\sum_{i=1}^n (\sigma_{22}a_{1i} + \sigma_{12}a_{3i})G_{11}(\zeta_i, s)\right|$$

$$
\begin{aligned}
&\leq \left|G_{11}(t,s)\right| + \frac{t}{\sigma}\sum_{i=1}^{n}(\sigma_{22}a_{1i}+\sigma_{12}a_{3i})\left|G_{11}(\zeta_i,s)\right| \\
&\leq \frac{|\lambda_1|(\alpha_1-\beta_1-1)^{\alpha_1-\beta_1-1}}{(\alpha_1-\beta_1)^{\alpha_1-\beta_1}\Gamma(\alpha_1-\beta_1)}\Big[1+\frac{1}{\sigma}\sum_{i=1}^{n}(\sigma_{22}a_{1i}+\sigma_{12}a_{3i})\Big] \\
&= T_1\Big[1+\frac{1}{\sigma}\sum_{i=1}^{n}(\sigma_{22}a_{1i}+\sigma_{12}a_{3i})\Big].
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\int_0^1\left|K_{11}(t,s)\right|ds &= \int_0^1\left|G_{11}(t,s)+\frac{t}{\sigma}\sum_{i=1}^{n}(\sigma_{22}a_{1i}+\sigma_{12}a_{3i})G_{11}(\zeta_i,s)\right|ds \\
&\leq \int_0^1 T_1\Big[1+\frac{1}{\sigma}\sum_{i=1}^{n}(\sigma_{22}a_{1i}+\sigma_{12}a_{3i})\Big]ds \\
&= T_1\Big[1+\frac{1}{\sigma}\sum_{i=1}^{n}(\sigma_{22}a_{1i}+\sigma_{12}a_{3i})\Big].
\end{aligned}
$$

Combining Lemma 2.4 with the assumption $\beta_1 \in [\alpha_1-1, 1)$, we obtain

$$
\begin{aligned}
\int_0^1\left|K_{11}(t,s)\right|ds &= \int_0^1\left|G_{11}(t,s)+\frac{t}{\sigma}\sum_{i=1}^{n}(\sigma_{22}a_{1i}+\sigma_{12}a_{3i})G_{11}(\zeta_i,s)\right|ds \\
&\leq \int_0^1\left|G_{11}(t,s)\right|ds + \int_0^1\left|\frac{t}{\sigma}\sum_{i=1}^{n}(\sigma_{22}a_{1i}+\sigma_{12}a_{3i})G_{11}(\zeta_i,s)\right|ds \\
&\leq \frac{3|\lambda_1|}{\Gamma(\alpha_1-\beta_1+1)}\Big[1+\frac{1}{\sigma}\sum_{i=1}^{n}(\sigma_{22}a_{1i}+\sigma_{12}a_{3i})\Big] \\
&= V_1\Big[1+\frac{1}{\sigma}\sum_{i=1}^{n}(\sigma_{22}a_{1i}+\sigma_{12}a_{3i})\Big].
\end{aligned}
$$

The other inequalities mentioned above can be proven in a similar manner. □

Lemma 2.6. The functions $K_{ij}(t,s)$ $(i = 1,2;\ \ j = 2,4)$ introduced in Lemma 2.2 can be bounded as follows:

$$
\begin{aligned}
|K_{12}(t,s)| &\leq W_1\Big(1+\frac{1}{\sigma}\sum_{i=1}^{n}(\sigma_{22}a_{1i}+\sigma_{12}a_{3i})\Big), \\
|K_{22}(t,s)| &\leq \frac{W_1}{\sigma}\sum_{i=1}^{n}(\sigma_{21}a_{1i}+\sigma_{11}a_{3i}), \quad \beta_1\in(0,\alpha_1), \\
|K_{14}(t,s)| &\leq \frac{W_2}{\sigma}\sum_{j=1}^{n}(\sigma_{22}a_{2j}+\sigma_{12}a_{4j}), \\
|K_{24}(t,s)| &\leq W_2\Big(1+\frac{1}{\sigma}\sum_{j=1}^{n}(\sigma_{21}a_{2j}+\sigma_{11}a_{4j})\Big),
\end{aligned}
$$

where W_i $(i = 1, 2)$ are defined in Lemma 2.5.

Proof. Substituting Lemma 2.2 and Lemma 2.3 into $K_{12}(t, s)$, we get

$$\begin{aligned}\left|K_{12}(t, s)\right| &= \left|G_{12}(t, s) + \frac{t}{\sigma}\sum_{i=1}^{n}(\sigma_{22}a_{1i}+\sigma_{12}a_{3i})G_{12}(\zeta_i, s)\right| \\ &\leq \left|G_{12}(t, s)\right| + \frac{1}{\sigma}\sum_{i=1}^{n}(\sigma_{22}a_{1i}+\sigma_{12}a_{3i})\left|G_{12}(\zeta_i, s)\right| \\ &\leq W_1\Big(1 + \frac{1}{\sigma}\sum_{i=1}^{n}(\sigma_{22}a_{1i}+\sigma_{12}a_{3i})\Big).\end{aligned}$$

The other inequalities mentioned above can be proven in a similar manner. □

Let $X = C[0, 1]$ be equipped with the norm $\|x\| = \max_{t\in[0,1]} |x(t)|$, and let $\mathbb{X} = X \times X$ denote the product space. Define the operator $\mathcal{L} : \mathbb{X} \to \mathbb{X}$ by

$$\mathcal{L}(x, y) = (\mathcal{L}_1(x, y),\ \mathcal{L}_2(x, y)),$$

where

$$\begin{aligned}\mathcal{L}_1(x, y)(t) = &\int_0^1 K_{11}(t, s)x(s)\,ds + \int_0^1 K_{12}(t, s)f_1(s, x(s), y(s))\,ds \\ &+ \int_0^1 K_{13}(t, s)y(s)\,ds + \int_0^1 K_{14}(t, s)f_2(s, x(s), y(s))\,ds, \\ \mathcal{L}_2(x, y)(t) = &\int_0^1 K_{21}(t, s)x(s)\,ds + \int_0^1 K_{22}(t, s)f_1(s, x(s), y(s))\,ds \\ &+ \int_0^1 K_{23}(t, s)y(s)\,ds + \int_0^1 K_{24}(t, s)f_2(s, x(s), y(s))\,ds,\end{aligned}$$

for $(x, y) \in \mathbb{X}$ and $K_{ij}(t, s)$ $(i = 1, 2;\ j = 1, 2, 3, 4)$ are defined in Lemma 2.2.

3. Deriving the Lyapunov type inequality for system (1.5) via matrix analysis

In this section, we present the derivation of Lyapunov type inequalities for system (1.5) using nonnegative matrix analysis. The proof relies primarily on the concept of matrices that converge to zero. We say that a nonnegative square matrix $\mathcal{M}$ converges to zero when $\mathcal{M}^n \to 0$ as $n \to \infty$.

Here, we adopt the following conventions: The real matrix $\mathcal{M} = (a_{ij})_{2\times 2}$ is termed nonnegative (denoted $\mathcal{M} \geq 0$) provided that all its entries satisfy $a_{ij} \geq 0$ for all $i, j = 1, 2$. For two square matrices $\mathcal{M}_1$ and $\mathcal{M}_2$ of the same order, if $\mathcal{M}_1 - \mathcal{M}_2 \geq 0$, we write $\mathcal{M}_1 \geq \mathcal{M}_2$. Nonnegativity and order relations for vectors can be defined similarly.

Let $\mathcal{M}_2^+$ denote nonnegative square matrices. For any $\mathcal{A} \in \mathcal{M}_2^+$, its trace, determinant, and spectral radius are denoted by trace($\mathcal{A}$), det($\mathcal{A}$), and $\rho(\mathcal{A})$, respectively.

Lemma 3.1. ([30]) If a nonnegative matrix $C \in \mathcal{M}_2^+$ satisfies $\rho(C) < 1$, then

$$\lim_{n\to\infty} C^n = 0.$$

Lemma 3.2. ([26]) Let $C \in \mathcal{M}_2^+$. Then

$$\rho(C) = \frac{\text{trace}(C) + \sqrt{[\text{trace}(C)]^2 - 4\det(C)}}{2}.$$

Throughout this section, we make the following assumptions:

(H_2) The functions $f_1, f_2 : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous.

(H_3) There are positive functions $q_{i1}(t), q_{i2}(t) \in X$ for all $t \in [0,1]$, such that

$$\left|f_i(t, x_i, y_i)\right| \leq q_{i1}(t)|x_i| + q_{i2}(t)|y_i|,\ (x_i, y_i) \in \mathbb{R}^2,\ i = 1, 2.$$

For $q_{ij} \in X$ $(i, j = 1, 2)$, we define auxiliary functions

$$\mathbf{F_{ij}(q_{1j}, q_{2j}) = d_{i1} \int_0^1 q_{1j}(s) ds + d_{i2} \int_0^1 q_{2j}(s) ds + e_{ij},\ i, j = 1, 2,}$$

where

$$\mathbf{d_{11}} = \mathbf{W_1}\Big(\mathbf{1} + \frac{\mathbf{1}}{\sigma}\sum_{\mathbf{i=1}}^{\mathbf{n}}(\sigma_{\mathbf{22}}\mathbf{a_{1i}} + \sigma_{\mathbf{12}}\mathbf{a_{3i}})\Big),\ \mathbf{d_{12}} = \frac{\mathbf{W_2}}{\sigma}\sum_{\mathbf{j=1}}^{\mathbf{n}}(\sigma_{\mathbf{22}}\mathbf{a_{2j}} + \sigma_{\mathbf{12}}\mathbf{a_{4j}}),$$

$$\mathbf{d_{21}} = \frac{\mathbf{W_1}}{\sigma}\sum_{\mathbf{i=1}}^{\mathbf{n}}(\sigma_{\mathbf{21}}\mathbf{a_{1i}} + \sigma_{\mathbf{11}}\mathbf{a_{3i}}),\ \mathbf{d_{22}} = \mathbf{W_2}\Big(\mathbf{1} + \frac{\mathbf{1}}{\sigma}\sum_{\mathbf{j=1}}^{\mathbf{n}}\big(\sigma_{\mathbf{21}}\mathbf{a_{2j}} + \sigma_{\mathbf{11}}\mathbf{a_{4j}}\big)\Big).$$

Theorem 3.1. Assume that conditions (H_1)-(H_3) hold. If system (1.5) admits a nontrivial solution, then the following Lyapunov type inequality is valid:

$$\begin{aligned} &F_{11}(q_{11}, q_{21}) + F_{22}(q_{12}, q_{22}) \\ &\quad + \sqrt{[F_{11}(q_{11}, q_{21}) - F_{22}(q_{12}, q_{22})]^2 + 4F_{12}(q_{12}, q_{22})F_{21}(q_{11}, q_{21})} \geq 2. \end{aligned} \tag{3.1}$$

The constants e_{ij} are determined by the parameters β_1 and β_2, as specified in the proof.

Proof. We prove the theorem by contradiction. Suppose that system (1.5) admits a nontrivial solution $(x^*, y^*) \in \mathbb{X}$ and $\beta_1 \in (0, \alpha_1 - 1)$, $\beta_2 \in (0, \alpha_2 - 1)$, but inequality (3.1) is false. That is, we assume

$$\begin{aligned} &F_{11}(q_{11}, q_{21}) + F_{22}(q_{12}, q_{22}) \\ &\quad + \sqrt{[F_{11}(q_{11}, q_{21}) - F_{22}(q_{12}, q_{22})]^2 + 4F_{12}(q_{12}, q_{22})F_{21}(q_{11}, q_{21})} < 2. \end{aligned} \tag{3.2}$$

According to Lemma 2.2, (x^*, y^*) is a nontrivial fixed point of the operator $\mathcal{L}$. It then follows from (H_3), Lemma 2.5, and Lemma 2.6 that

$$\begin{aligned} \left|x^*(t)\right| &= \left|\mathcal{L}_1(x^*, y^*)(t)\right| \\ &\leq \int_0^1 \left|K_{11}(t,s)\right|\left|x^*(s)\right| ds + \int_0^1 \left|K_{12}(t,s)\right|\left|f_1(s, x^*(s), y^*(s))\right| ds \end{aligned}$$

$$
\begin{aligned}
&+\int_0^1 \left|K_{13}(t,s)\right|\left|y^*(s)\right|ds+\int_0^1 \left|K_{14}(t,s)\right|\left|f_2(s,x^*(s),y^*(s))\right|ds\\
\le&\int_0^1 \left|K_{11}(t,s)\right|\left\|x^*\right\|ds+\int_0^1 \left|K_{12}(t,s)\right|\left(q_{11}(s)\left|x^*(s)\right|+q_{12}(s)\left|y^*(s)\right|\right)ds\\
&+\int_0^1 \left|K_{13}(t,s)\right|\left\|y^*\right\|ds+\int_0^1 \left|K_{14}(t,s)\right|\left(q_{21}(s)\left|x^*(s)\right|+q_{22}(s)\left|y^*(s)\right|\right)ds\\
\le&T_1\Big[1+\frac{1}{\sigma}\sum_{i=1}^{n}(\sigma_{22}a_{1i}+\sigma_{12}a_{3i})\Big]\left\|x^*\right\|+\frac{T_2}{\sigma}\sum_{j=1}^{n}(\sigma_{22}a_{2j}+\sigma_{12}a_{4j})\left\|y^*\right\|\\
&+W_1\Big[1+\sum_{i=1}^{n}(\sigma_{22}a_{1i}+\sigma_{12}a_{3i})\Big]\Big(\int_0^1 q_{11}(s)ds\left\|x^*\right\|+\int_0^1 q_{12}(s)ds\left\|y^*\right\|\Big)\\
&+\frac{W_2}{\sigma}\sum_{j=1}^{n}(\sigma_{22}a_{2j}+\sigma_{12}a_{4j})\Big(\int_0^1 q_{21}(s)ds\left\|x^*\right\|+\int_0^1 q_{22}(s)ds\left\|y^*\right\|\Big)\\
=&\Big[T_1\Big(1+\frac{1}{\sigma}\sum_{i=1}^{n}(\sigma_{22}a_{1i}+\sigma_{12}a_{3i})\Big)+d_{11}\int_0^1 q_{11}(s)ds+d_{12}\int_0^1 q_{21}(s)ds\Big]\left\|x^*\right\|\\
&+\Big[\frac{T_2}{\sigma}\sum_{j=1}^{n}(\sigma_{22}a_{2j}+\sigma_{12}a_{4j})+d_{11}\int_0^1 q_{12}(s)ds+d_{12}\int_0^1 q_{22}(s)ds\Big]\left\|y^*\right\|\\
=&F_{11}(q_{11},q_{21})\left\|x^*\right\|+F_{12}(q_{12},q_{22})\left\|y^*\right\|,
\end{aligned}
$$

thus

$$
\left\|x^*\right\|\le F_{11}(q_{11},q_{21})\left\|x^*\right\|+F_{12}(q_{12},q_{22})\left\|y^*\right\|. \tag{3.3}
$$

Similarly, using (H_3), Lemma 2.5 and Lemma 2.6, we have

$$
\left\|y^*\right\|\le F_{21}(q_{11},q_{21})\left\|x^*\right\|+F_{22}(q_{12},q_{22})\left\|y^*\right\|, \tag{3.4}
$$

where

$$
\begin{aligned}
F_{21}(q_{11},q_{21})&=\frac{T_1}{\sigma}\sum_{i=1}^{n}(\sigma_{21}a_{1i}+\sigma_{11}a_{3i})+d_{21}\int_0^1 q_{11}(s)ds+d_{22}\int_0^1 q_{21}(s)ds,\\
F_{22}(q_{12},q_{22})&=T_2\Big(1+\frac{1}{\sigma}\sum_{j=1}^{n}(\sigma_{21}a_{2j}+\sigma_{11}a_{4j})\Big)+d_{21}\int_0^1 q_{12}(s)ds+d_{22}\int_0^1 q_{22}(s)ds.
\end{aligned}
$$

It follows from (3.3) and (3.4) that

$$
\begin{pmatrix}||x^*||\\ ||y^*||\end{pmatrix}\le\begin{pmatrix}F_{11}&F_{12}\\F_{21}&F_{22}\end{pmatrix}\begin{pmatrix}||x^*||\\ ||y^*||\end{pmatrix}.
$$

Hence, for all $n\in\mathbb{N}$, we get

$$
\begin{pmatrix}||x^*||\\ ||y^*||\end{pmatrix}\le\begin{pmatrix}F_{11}&F_{12}\\F_{21}&F_{22}\end{pmatrix}^n\begin{pmatrix}||x^*||\\ ||y^*||\end{pmatrix}.
$$

Next, using Lemmas 3.1 and 3.2, and (3.2) yields that

$$\|x^*\| = \|y^*\| = 0.$$

This contradicts the nontriviality of (x^*, y^*), thus establishing the desired result. We note that the constants e_{ij} in the definitions of F_{ij} depend on the parameter ranges of β_1 and β_2.

Case 1: $\beta_1 \in (0, \alpha_1 - 1)$ and $\beta_2 \in [\alpha_2 - 1, 1)$.

$$\mathbf{e_{11}} = \mathbf{T_1}\Big(\mathbf{1} + \frac{\mathbf{1}}{\sigma}\sum_{\mathbf{i=1}}^{\mathbf{n}}(\sigma_{\mathbf{22}}\mathbf{a_{1i}} + \sigma_{\mathbf{12}}\mathbf{a_{3i}})\Big),\ \mathbf{e_{12}} = \frac{\mathbf{V_2}}{\sigma}\sum_{\mathbf{j=1}}^{\mathbf{n}}(\sigma_{\mathbf{22}}\mathbf{a_{2j}} + \sigma_{\mathbf{12}}\mathbf{a_{4j}}),$$
$$\mathbf{e_{21}} = \frac{\mathbf{T_1}}{\sigma}\sum_{\mathbf{i=1}}^{\mathbf{n}}(\sigma_{\mathbf{21}}\mathbf{a_{1i}} + \sigma_{\mathbf{11}}\mathbf{a_{3i}}),\ \mathbf{e_{22}} = \mathbf{V_2}\Big(\mathbf{1} + \frac{\mathbf{1}}{\sigma}\sum_{\mathbf{j=1}}^{\mathbf{n}}(\sigma_{\mathbf{21}}\mathbf{a_{2j}} + \sigma_{\mathbf{11}}\mathbf{a_{4j}})\Big).$$

Case 2: $\beta_1 \in [\alpha_1 - 1, 1)$ and $\beta_2 \in (0, \alpha_2 - 1)$.

$$\mathbf{e_{11}} = \mathbf{V_1}\Big(\mathbf{1} + \frac{\mathbf{1}}{\sigma}\sum_{\mathbf{i=1}}^{\mathbf{n}}(\sigma_{\mathbf{22}}\mathbf{a_{1i}} + \sigma_{\mathbf{12}}\mathbf{a_{3i}})\Big),\ \mathbf{e_{12}} = \frac{\mathbf{T_2}}{\sigma}\sum_{\mathbf{j=1}}^{\mathbf{n}}(\sigma_{\mathbf{22}}\mathbf{a_{2j}} + \sigma_{\mathbf{12}}\mathbf{a_{4j}}),$$
$$\mathbf{e_{21}} = \frac{\mathbf{V_1}}{\sigma}\sum\nolimits_{\mathbf{i=1}}^{\mathbf{n}}(\sigma_{\mathbf{21}}\mathbf{a_{1i}} + \sigma_{\mathbf{11}}\mathbf{a_{3i}}),\ \mathbf{e_{22}} = \mathbf{T_2}\Big(\mathbf{1} + \frac{\mathbf{1}}{\sigma}\sum_{\mathbf{j=1}}^{\mathbf{n}}(\sigma_{\mathbf{21}}\mathbf{a_{2j}} + \sigma_{\mathbf{11}}\mathbf{a_{4j}})\Big).$$

Case 3: $\beta_1 \in [\alpha_1 - 1, 1)$ and $\beta_2 \in [\alpha_2 - 1, 1)$.

$$\mathbf{e_{11}} = \mathbf{V_1}\Big(\mathbf{1} + \frac{\mathbf{1}}{\sigma}\sum\nolimits_{\mathbf{i=1}}^{\mathbf{n}}(\sigma_{\mathbf{22}}\mathbf{a_{1i}} + \sigma_{\mathbf{12}}\mathbf{a_{3i}})\Big),\ \mathbf{e_{12}} = \frac{\mathbf{V_2}}{\sigma}\sum_{\mathbf{j=1}}^{\mathbf{n}}(\sigma_{\mathbf{22}}\mathbf{a_{2j}} + \sigma_{\mathbf{12}}\mathbf{a_{4j}}),$$
$$\mathbf{e_{21}} = \frac{\mathbf{V_1}}{\sigma}\sum\nolimits_{\mathbf{i=1}}^{\mathbf{n}}(\sigma_{\mathbf{21}}\mathbf{a_{1i}} + \sigma_{\mathbf{11}}\mathbf{a_{3i}}),\ \mathbf{e_{22}} = \mathbf{V_2}\Big(\mathbf{1} + \frac{\mathbf{1}}{\sigma}\sum_{\mathbf{j=1}}^{\mathbf{n}}(\sigma_{\mathbf{21}}\mathbf{a_{2j}} + \sigma_{\mathbf{11}}\mathbf{a_{4j}})\Big).$$

The other three cases can be proved analogously. The proof is now complete. □

4. Deriving the Lyapunov type inequality for system (1.5) via Perov's fixed point theorem

In this section, we establish Lyapunov type inequalities for system (1.5) via Perov's fixed point theorem. Next, we recall some concepts that will be used subsequently.

The space $\mathbb{R}^N$ is endowed with the standard partial order $\leq_{\mathbb{R}^N}$ defined component-wise: For vectors $\mu = (\mu_1, \ldots, \mu_N)$ and $\nu = (\nu_1, \ldots, \nu_N)$, we have $\mu \leq_{\mathbb{R}^N} \nu$ by definition if and only if $\mu_i \leq \nu_i$ for each $i = 1, \ldots, N$. The zero vector in $\mathbb{R}^N$ is denoted by $0_{\mathbb{R}^N}$.

Definition 4.1. ([26]) Let X be a nonempty set. A mapping $d : \mathbb{X} \to \mathbb{R}^N$ is called a vector-valued metric if it satisfies the following axioms for every $\mu_1, \mu_2, \mu_3 \in X$:

(i) $d(\mu_1, \mu_2) \geq_{\mathbb{R}^N} 0_{\mathbb{R}^N}$;

(ii) $d(\mu_1, \mu_2) = d(\mu_2, \mu_1)$;

(iii) $d(\mu_1, \mu_2) = 0_{\mathbb{R}^N} \Leftrightarrow \mu_1 = \mu_2$;

(iv) $d(\mu_1, \mu_2) \leq_{\mathbb{R}^N} d(\mu_1, \mu_3) + d(\mu_3, \mu_2)$.

Then (X, d) is said to be a generalized metric space.

We equip the product space $\mathbb{X}$ with the vector-valued metric $d : \mathbb{X} \times \mathbb{X} \to \mathbb{R}^2$ given by

$$d((\mu_1, \nu_1), (\mu_2, \nu_2)) = (\|\mu_1 - \mu_2\|, \|\nu_1 - \nu_2\|)^T.$$

It is straightforward to verify that $(\mathbb{X}, d)$ satisfies Definition 4.1 and is a complete generalized metric space.

Lemma 4.1. ([31]) Let (X, d) be a complete generalized metric space. Suppose a mapping $\mathcal{L} : X \to X$ satisfies, for all $x_1, x_2 \in X$

$$d(\mathcal{L}x_1, \mathcal{L}x_2) \leq_{\mathbb{R}^N} \mathcal{A}\, d(x_1, x_2),$$

where $\mathcal{A} \in M_N(\mathbb{R}_+)$ and $\rho(\mathcal{A}) < 1$, then $\mathcal{L}$ admits a unique fixed point.

The following assumptions are required for our main result.

(H_4) For all $t \in [0, 1]$, $f_1(t, \Theta, \Theta) = f_2(t, \Theta, \Theta) = \Theta$, where Θ is the zero function.

(H_5) There are positive functions $p_{i1}(t), p_{i2}(t) \in X$, such as

$$|f_i(t, x_1, y_1) - f_i(t, x_2, y_2)| \leq p_{i1}(t)\|x_1 - x_2\| + p_{i2}(t)\|y_1 - y_2\|, \ i = 1, 2,$$

for all $t \in [0, 1]$, $((x_1, y_1), (x_2, y_2)) \in \mathbb{X} \times \mathbb{X}$.

For $p_{ij}(t) \in X$ $(i, j = 1, 2)$, let the auxiliary function

$$\mathbf{J_{ij}(p_{1j}, p_{2j}) = d_{i1} \int_0^1 p_{1j}(s)ds + d_{i2} \int_0^1 p_{2j}(s)ds + e_{ij}, \ \ i, j = 1, 2.}$$

Theorem 4.1. Assume that conditions (H_1), (H_4), and (H_5) hold. If system (1.5) admits a nontrivial solution, then

$$\begin{aligned} &J_{11}(p_{11}, p_{21}) + J_{22}(p_{12}, p_{22}) \\ &\quad + \sqrt{[J_{11}(p_{11}, p_{21}) - J_{22}(p_{12}, p_{22})]^2 + 4J_{12}(p_{12}, p_{22})J_{21}(p_{11}, p_{21})} \geq 2. \end{aligned} \tag{4.1}$$

The constants e_{ij} in F_{ij} depend on the ranges of β_1 and β_2, as detailed in the proof.

Proof. Let $(\widehat{x}, \widehat{y}) \in \mathbb{X}$ be a nontrivial solution of system (1.5), which by Lemma 2.2 corresponds to a fixed point of the operator $\mathcal{L}$. Assuming that $\beta_i \in (0, \alpha_i - 1)$ $(i = 1, 2)$ and (4.1) does not hold, then

$$\begin{aligned} &J_{11}(p_{11}, p_{21}) + J_{22}(p_{12}, p_{22}) \\ &\quad + \sqrt{[J_{11}(p_{11}, p_{21}) - J_{22}(p_{12}, p_{22})]^2 + 4J_{12}(p_{12}, p_{22})J_{21}(p_{11}, p_{21})} < 2. \end{aligned} \tag{4.2}$$

Using Lemma 2.5 and Lemma 2.6, for all $(x_i, y_i) \in \mathbb{X}(i = 1, 2)$, we obtain

$$\left|\mathcal{L}_1(x_1, y_1)(t) - \mathcal{L}_1(x_2, y_2)(t)\right|$$

$$
\begin{aligned}
&= \left| \int_0^1 K_{11}(t,s)[x_1(s) - x_2(s)]ds + \int_0^1 K_{12}(t,s)[f_1(s,x_1(s),y_1(s)) - f_1(s,x_2(s),y_2(s))]ds \right. \\
&\quad \left. + \int_0^1 K_{13}(t,s)[y_1(s) - y_2(s)]ds + \int_0^1 K_{14}(t,s)[f_2(s,x_1(s),y_1(s)) - f_2(s,x_2(s),y_2(s))]ds \right| \\
&\leq \int_0^1 \left|K_{11}(t,s)\right| \left\|x_1 - x_2\right\| ds + \int_0^1 \left|K_{12}(t,s)\right| \left|f_1(s,x_1(s),y_1(s)) - f_1(s,x_2(s),y_2(s))\right| ds \\
&\quad + \int_0^1 \left|K_{13}(t,s)\right| \left\|y_1 - y_2\right\| ds + \int_0^1 \left|K_{14}(t,s)\right| \left|f_2(s,x_1(s),y_1(s)) - f_2(s,x_2(s),y_2(s))\right| ds \\
&\leq \int_0^1 \left|K_{11}(t,s)\right| \left\|x_1 - x_2\right\| ds + \int_0^1 \left|K_{12}(t,s)\right| \Big[p_{11}(t)\left\|x_1 - x_2\right\| + p_{12}(t)\left\|y_1 - y_2\right\|\Big] ds \\
&\quad + \int_0^1 \left|K_{13}(t,s)\right| \left\|y_1 - y_2\right\| ds + \int_0^1 \left|K_{14}(t,s)\right| \Big[p_{21}(t)\left\|x_1 - x_2\right\| + p_{22}(t)\left\|y_1 - y_2\right\|\Big] ds \\
&\leq T_1\Big[1 + \frac{1}{\sigma}\sum_{i=1}^{n}(\sigma_{22}a_{1i} + \sigma_{12}a_{3i})\Big]\left\|x_1 - x_2\right\| + \frac{V_2\left\|y_1 - y_2\right\|}{\sigma}\sum_{j=1}^{n}\Big(\sigma_{22}a_{2j} + \sigma_{12}a_{4j}\Big) \\
&\quad + W_1\Big[1 + \frac{1}{\sigma}\sum_{i=1}^{n}(\sigma_{22}a_{1i} + \sigma_{12}a_{3i})\Big]\Big(\int_0^1 p_{11}(s)ds\left\|x_1 - x_2\right\| + \int_0^1 p_{12}(s)ds\left\|y_1 - y_2\right\|\Big) \\
&\quad + \frac{W_2}{\sigma}\sum_{j=1}^{n}\Big(\sigma_{22}a_{2j} + \sigma_{12}a_{4j}\Big)\Big(\int_0^1 p_{21}(s)ds\left\|x_1 - x_2\right\| + \int_0^1 p_{22}(s)ds\left\|y_1 - y_2\right\|\Big) \\
&= \Big[e_{11} + d_{11}\int_0^1 p_{11}(s)ds + d_{12}\int_0^1 p_{21}(s)ds\Big]\left\|x_1 - x_2\right\| \\
&\quad + \Big[e_{12} + d_{11}\int_0^1 p_{12}(s)ds + d_{12}\int_0^1 p_{22}(s)ds\Big]\left\|y_1 - y_2\right\| \\
&= J_{11}(p_{11},p_{21})\left\|x_1 - x_2\right\| + J_{12}(p_{12},p_{22})\left\|y_1 - y_2\right\|.
\end{aligned}
$$

Thus, we have

$$
\left\|\mathcal{L}_1(x_1,y_1) - \mathcal{L}_1(x_2,y_2)\right\| \leq J_{11}(p_{11},p_{21})\left\|x_1 - x_2\right\| + J_{12}(p_{12},p_{22})\left\|y_1 - y_2\right\|. \tag{4.3}
$$

Similarly, using Lemma 2.5 and Lemma 2.6, for all $t \in [0,1]$, we have

$$
\left\|\mathcal{L}_2(x_1,y_1) - \mathcal{L}_2(x_2,y_2)\right\| \leq J_{21}(p_{11},p_{21})\left\|x_1 - x_2\right\| + J_{22}(p_{12},p_{22})\left\|y_1 - y_2\right\|. \tag{4.4}
$$

It follows from (4.3) and (4.4) , we deduce that

$$
\begin{pmatrix} ||\mathcal{L}_1(x_1,y_1) - \mathcal{L}_1(x_2,y_2)|| \\ ||\mathcal{L}_2(x_1,y_1) - \mathcal{L}_2(x_2,y_2)|| \end{pmatrix} \leq \begin{pmatrix} J_{11}(p_{11},p_{21}) & J_{12}(p_{12},p_{22}) \\ J_{21}(p_{11},p_{21}) & J_{22}(p_{12},p_{22}) \end{pmatrix} \begin{pmatrix} ||x_1 - x_2|| \\ ||y_1 - y_2|| \end{pmatrix}.
$$

Let $\mathcal{A} = (\mathbf{J_{ij}})_{\mathbf{2\times 2}}$. Using Lemma 3.2 and (4.2), we deduce that

$$
\begin{aligned}
\rho(\mathcal{A}) &= J_{11}(p_{11},p_{21}) + J_{22}(p_{12},p_{22}) \\
&\quad + \sqrt{[J_{11}(p_{11},p_{21}) - J_{22}(p_{12},p_{22})]^2 + 4J_{12}(p_{12},p_{22})J_{21}(p_{11},p_{21})} < 1.
\end{aligned}
$$

Therefore, Lemma 4.1 combined with $\rho(\mathcal{A}) < 1$ guarantees that the operator $\mathcal{L}$ admits a unique nonzero fixed point $(\widehat{x}, \widehat{y})$. However, the condition (H_4) states that $(0, 0)$ is a fixed point of $\mathcal{L}$ as well, contradicting the above and thereby establishing (4.1).

The constants e_{ij} in the definitions of J_{ij} are determined by the parameter ranges of β_1 and β_2, as follows:

Case 1: $\beta_1 \in (0, \alpha_1 - 1)$ and $\beta_2 \in [\alpha_2 - 1, 1)$.

$$e_{11} = T_1\Big(1 + \frac{1}{\sigma} \sum\nolimits_{i=1}^{n} (\sigma_{22} a_{1i} + \sigma_{12} a_{3i})\Big),\ e_{12} = \frac{V_2}{\sigma} \sum_{j=1}^{n} (\sigma_{22} a_{2j} + \sigma_{12} a_{4j}),$$
$$e_{21} = \frac{T_1}{\sigma} \sum\nolimits_{i=1}^{n} (\sigma_{21} a_{1i} + \sigma_{11} a_{3i}),\ e_{22} = V_2\Big(1 + \frac{1}{\sigma} \sum_{j=1}^{n} (\sigma_{21} a_{2j} + \sigma_{11} a_{4j})\Big).$$

Case 2: $\beta_1 \in [\alpha_1 - 1, 1)$ and $\beta_2 \in (0, \alpha_2 - 1)$.

$$e_{11} = V_1\Big(1 + \frac{1}{\sigma} \sum\nolimits_{i=1}^{n} (\sigma_{22} a_{1i} + \sigma_{12} a_{3i})\Big),\ e_{12} = \frac{T_2}{\sigma} \sum_{j=1}^{n} (\sigma_{22} a_{2j} + \sigma_{12} a_{4j}),$$
$$e_{21} = \frac{V_1}{\sigma} \sum\nolimits_{i=1}^{n} (\sigma_{21} a_{1i} + \sigma_{11} a_{3i}),\ e_{22} = T_2\Big(1 + \frac{1}{\sigma} \sum_{j=1}^{n} (\sigma_{21} a_{2j} + \sigma_{11} a_{4j})\Big).$$

Case 3: $\beta_1 \in [\alpha_1 - 1, 1)$ and $\beta_2 \in [\alpha_2 - 1, 1)$.

$$e_{11} = V_1\Big(1 + \frac{1}{\sigma} \sum\nolimits_{i=1}^{n} (\sigma_{22} a_{1i} + \sigma_{12} a_{3i})\Big),\ e_{12} = \frac{V_2}{\sigma} \sum_{j=1}^{n} (\sigma_{22} a_{2j} + \sigma_{12} a_{4j}),$$
$$e_{21} = \frac{V_1}{\sigma} \sum\nolimits_{i=1}^{n} (\sigma_{21} a_{1i} + \sigma_{11} a_{3i}),\ e_{22} = V_2\Big(1 + \frac{1}{\sigma} \sum_{j=1}^{n} (\sigma_{21} a_{2j} + \sigma_{11} a_{4j})\Big).$$

The other three cases can be proved analogously. The proof is now complete. □

5. Example

Example 5.1. Consider the following Caputo fractional differential system:

$$\begin{cases} {}^{C}D_{0^+}^{\frac{5}{3}} x(t) - \frac{1}{3}\, {}^{C}D_{0^+}^{\frac{1}{2}} x(t) + t\, x(t) + \frac{t^2}{2}\, y(t) = 0, \quad t \in (0, 1), \\ {}^{C}D_{0^+}^{\frac{7}{4}} y(t) - \frac{1}{4}\, {}^{C}D_{0^+}^{\frac{1}{3}} y(t) + \frac{t}{3}\, x(t) + \frac{t}{4}\, y(t) = 0, \quad t \in (0, 1), \\ x(0) = 0, \quad x(1) = \frac{1}{6}\, x\left(\frac{1}{3}\right) + \frac{1}{8}\, y\left(\frac{2}{3}\right), \\ y(0) = 0, \quad y(1) = \frac{1}{9}\, x\left(\frac{1}{3}\right) + \frac{1}{10}\, y\left(\frac{2}{3}\right), \end{cases} \tag{5.1}$$

where $\alpha_1 = 5/3, \beta_1 = 1/2, \lambda_1 = -1/3, \alpha_2 = 7/4, \beta_2 = 1/3$, and $\lambda_2 = -1/4$. The boundary coefficients are $a_{11} = 1/6, a_{21} = 1/8, a_{31} = 1/9$, and $a_{41} = 1/10$ with $\zeta_1 = 1/3, \eta_1 = 2/3$. By calculation, we obtain

$$\sigma_{11} \approx 0.9444, \quad \sigma_{12} \approx 0.08333, \quad \sigma_{21} \approx 0.03704, \quad \sigma_{22} \approx 0.9333, \quad \sigma \approx 0.8784 > 0,$$
$$T_1 \approx 0.2228, \quad T_2 \approx 0.1196, \quad W_1 \approx 0.3610, \text{ and } \quad W_2 \approx 0.3294.$$

Since $\beta_1 \in (0, \alpha_1 - 1), \beta_2 \in (0, \alpha_2 - 1)$, the constants e_{ij} and d_{ij} appearing in Theorems 3.1 and 4.1 are given by

$$d_{11} \approx 0.4287, \quad d_{12} \approx 0.04689, \quad d_{21} \approx 0.04566, \quad d_{22} \approx 0.3666,$$
$$e_{11} \approx 0.2646, \quad e_{12} \approx 0.01701, \quad e_{21} \approx 0.02819, \text{ and } \quad e_{22} \approx 0.1331.$$

Application of Theorem 3.1 (matrix spectral analysis).

Set

$$q_{11}(t) = t, \quad q_{12}(t) = \frac{t^2}{2}, \quad q_{21}(t) = \frac{t}{3}, \quad q_{22}(t) = \frac{t}{4}.$$

Using $F_{ij}(q_{1j}, q_{2j}) = d_{i1} \int_0^1 q_{1j}(s)ds + d_{i2} \int_0^1 q_{2j}(s)ds + e_{ij}$, we compute

$$F_{11}\left(t, \frac{t}{3}\right) \approx 0.4868, \quad F_{12}\left(\frac{t^2}{2}, \frac{t}{4}\right) \approx 0.09432,$$
$$F_{21}\left(t, \frac{t}{3}\right) \approx 0.1121, \quad F_{22}\left(\frac{t^2}{2}, \frac{t}{4}\right) \approx 0.1865.$$

Evidently, assumptions (H_1)–(H_3) hold. Then

$$F_{11}\left(t, \frac{t}{3}\right) + F_{22}\left(\frac{t^2}{2}, \frac{t}{4}\right) + \sqrt{\left[F_{11}\left(t, \frac{t}{3}\right) - F_{22}\left(\frac{t^2}{2}, \frac{t}{4}\right)\right]^2 + 4F_{21}\left(t, \frac{t}{3}\right)F_{21}\left(t, \frac{t}{3}\right)} \approx 1.0372 < 2.$$

Therefore, the only solution to system (5.1) is $(0, 0)$. It can be concluded that the Lyapunov inequality in Theorem 3.1 holds.

Application of Theorem 4.1 (Perov's fixed point theorem).

Because $f_1(t, 0, 0) = f_2(t, 0, 0) = 0$, the condition (H_4) holds. Moreover, f_1 and f_2 are Lipschitz continuous:

$$|f_1(t, x_1, y_1) - f_1(t, x_2, y_2)| \le t|x_1 - x_2| + \frac{t^2}{2}|y_1 - y_2|,$$
$$|f_2(t, x_1, y_1) - f_2(t, x_2, y_2)| \le \frac{t}{3}|x_1 - x_2| + \frac{t}{4}|y_1 - y_2|.$$

Set

$$p_{11}(t) = t, \quad p_{12}(t) = \frac{t^2}{2}, \quad p_{21}(t) = \frac{t}{3}, \quad p_{22}(t) = \frac{t}{4},$$

which coincide with the $q_{ij}(t)$ $(i, j = 1, 2)$ used above. Consequently,

$$J_{11} = F_{11},\ J_{12} = F_{12},\ J_{21} = F_{21},\ J_{22} = F_{22},$$

and

$$J_{11} + J_{22} + \sqrt{(J_{11} - J_{22})^2 + 4J_{12}J_{21}} \approx 1.0372 < 2.$$

Hence, Theorem 4.1 also implies that the only solution of (5.1) is the zero solution.

Remark. In this example, both methods yield exactly the same numerical value for the Lyapunov type inequality, demonstrating their consistency. The matrix spectral analysis (Theorem 3.1) provides a direct algebraic criterion, while Perov's fixed point approach (Theorem 4.1) is based on a contraction principle in a generalized metric space and can be extended to more general nonlinearities. The agreement of the two approaches confirms the robustness of the obtained necessary conditions.

Example 5.2. Consider the Caputo fractional coupled system:

$$\begin{cases} {}^{C}D_{0^+}^{\frac{3}{2}}x(t) - \frac{1}{5}\,{}^{C}D_{0^+}^{\frac{2}{5}}x(t) + \frac{t}{5}\arctan(x(t)) + \frac{t^2}{10}\sin(y(t)) = 0, & t \in (0,1), \\ {}^{C}D_{0^+}^{\frac{5}{3}}y(t) - \frac{1}{6}\,{}^{C}D_{0^+}^{\frac{5}{6}}y(t) + \frac{t}{7}x(t) + \frac{t}{8}y(t) = 0, & t \in (0,1), \\ x(0) = 0, \quad x(1) = \frac{1}{7}\,x\left(\frac{1}{4}\right) + \frac{1}{9}\,y\left(\frac{3}{4}\right), & \\ y(0) = 0, \quad y(1) = \frac{1}{11}\,x\left(\frac{1}{4}\right) + \frac{1}{13}\,y\left(\frac{3}{4}\right), & \end{cases} \tag{5.2}$$

where $\alpha_1 = 3/2, \beta_1 = 2/5, \lambda_1 = -1/5, \alpha_2 = 5/3, \beta_2 = 5/6, \lambda_2 = -1/6$, and $a_{11} = 1/5$, $a_{21} = 1/6$, $a_{31} = 1/7$, and $a_{41} = 1/8$, with $\zeta_1 = 1/4$ and $\eta_1 = 1/2$. By calculation, we obtain

$$\begin{aligned} &\sigma_{11} \approx 0.9643, \quad \sigma_{12} \approx 0.08333, \quad \sigma_{21} \approx 0.02273, \quad \sigma_{22} \approx 0.9423, \quad \sigma \approx 0.9067 > 0, \\ &T_1 \approx 0.1505, \quad V_2 \approx 0.5180, \quad W_1 \approx 0.4344, \text{ and } \quad W_2 \approx 0.3610. \end{aligned}$$

Since $\beta_1 \in (\alpha_1 - 1, \alpha_1)$, and $\beta_2 \in (\alpha_2 - 1, \alpha_2)$, the constants e_{ij} and d_{ij} appearing in Theorems 3.1 and 4.1 are given by

$$\begin{aligned} &d_{11} \approx 0.5025, \quad d_{12} \approx 0.04425, \quad d_{21} \approx 0.04355, \quad d_{22} \approx 0.3914, \\ &e_{11} \approx 0.1741, \quad e_{12} \approx 0.06348, \quad e_{21} \approx 0.01509, \text{ and } \quad e_{22} \approx 0.5618. \end{aligned}$$

Application of Theorem 3.1 (matrix spectral analysis)

Set

$$q_{11}(t) = \frac{t}{5}, \quad q_{12}(t) = \frac{t^2}{10}, \quad q_{21}(t) = \frac{t}{7}, \quad q_{22}(t) = \frac{t}{8}.$$

Using $F_{ij}(q_{1j}, q_{2j}) = d_{i1}\int_0^1 q_{1j}(s)ds + d_{i2}\int_0^1 q_{2j}(s)ds + e_{ij}$, we compute

$$F_{11}\left(\frac{t}{5}, \frac{t}{7}\right) \approx 0.2275, \quad F_{12}\left(\frac{t^2}{10}, \frac{t}{8}\right) \approx 0.08300,$$

$$F_{21}\left(\frac{t}{5},\frac{t}{7}\right)\approx 0.04741,\quad F_{22}\left(\frac{t^2}{10},\frac{t}{8}\right)\approx 0.5877.$$

Evidently, assumptions (H_1)–(H_3) hold. Then

$$\begin{aligned}&F_{11}\left(\frac{t}{5},\frac{t}{7}\right)+F_{22}\left(\frac{t^2}{10},\frac{t}{8}\right)\\&+\sqrt{\left[F_{11}\left(\frac{t}{5},\frac{t}{7}\right)-F_{22}\left(\frac{t^2}{10},\frac{t}{8}\right)\right]^2+4F_{12}\left(\frac{t^2}{10},\frac{t}{8}\right)F_{21}\left(\frac{t}{5},\frac{t}{7}\right)}\approx 1.1966<2.\end{aligned}$$

Therefore, the only solution to system (5.2) is $(0,0)$. It can be concluded that the Lyapunov inequality in Theorem 3.1 holds.

Application of Theorem 4.1 (Perov's fixed point theorem)

Clearly $f_1(t,0,0)=f_2(t,0,0)=0$, so (H_4) holds. Moreover,

$$\begin{aligned}|f_1(t,x_1,y_1)-f_1(t,x_2,y_2)|&\le\frac{t}{5}|x_1-x_2|+\frac{t^2}{10}|y_1-y_2|,\\|f_2(t,x_1,y_1)-f_2(t,x_2,y_2)|&\le\frac{t}{7}|x_1-x_2|+\frac{t}{8}|y_1-y_2|.\end{aligned}$$

Set

$$p_{11}(t)=\frac{t}{5},\quad p_{12}(t)=\frac{t^2}{10},\quad p_{21}(t)=\frac{t}{7},\quad p_{22}(t)=\frac{t}{8},$$

which coincide with the $q_{ij}(t)$ $(i,j=1,2)$ used above. Consequently,

$$J_{11}=F_{11},\ J_{12}=F_{12},\ J_{21}=F_{21},\ J_{22}=F_{22},$$

and

$$J_{11}+J_{22}+\sqrt{(J_{11}-J_{22})^2+4J_{12}J_{21}}\approx 1.1966<2.$$

Hence, Theorem 4.1 also implies that the only solution of (5.1) is the zero solution.

6. Conclusions

In this paper, we study a Caputo fractional differential coupled system with multi-point coupled boundary conditions. By employing matrix spectral analysis methods and Perov's fixed point theorem, we establish Lyapunov type inequalities for this system. To our knowledge, this is the first work to derive such inequalities for coupled system with multi-term fractional derivatives, thereby extending the classical theory to a more general framework. Specifically, we address different value ranges of β_i (for instance, $0<\beta_i<\alpha_i-1, \alpha_i-1<\beta_i<1$) and provide the corresponding Lyapunov type inequalities. Finally, several promising directions remain for future work on fractional Lyapunov type inequalities for BVPs, such as extensions to Caputo-Fabrizio fractional differential equations under multi-point coupled boundary conditions, or to equations with anti-periodic boundary conditions.

Author contributions

Shuangqiao Chen: Conceptualization, Formal analysis, Investigation, Writing–original draft; Zhanbing Bai and Suiming Shang: Methodology, Validation, Writing–review and editing, Funding acquisition. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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