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Research article

Lie biderivations on a generalized matrix algebra

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Abstract: In this paper, we investigate the structure of Lie biderivations on a generalized matrix algebra $\mathcal{G}$. Although results on Lie biderivations are well-established for triangular algebras, the extension to the broader class of generalized matrix algebras has remained largely unexplored, and our work fills this gap. By leveraging the faithful bimodule structure of $\mathcal{G}$, we prove that, under mild conditions, every Lie biderivation on $\mathcal{G}$ can be decomposed into the sum of a biderivation and a central mapping. As a direct application, we extend this result to obtain an analogous decomposition for Lie biderivations on full matrix algebras.

Keywords: generalized matrix algebra; Lie biderivation; extremal biderivation; biderivation
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1. Introduction

The investigation of derivation-type maps constitutes a fundamental research theme in the theory of algebraic structures. Within this domain, the notions of biderivations and Lie biderivations have proven particularly fruitful, providing powerful tools for the analysis of functional identities and the structural properties of operator algebras. Let R be a commutative ring with identity, and let $\mathcal{A}$ be an R-algebra. Recall that the Lie product is given by $[x, y] = xy - yx$ for all $x, y \in \mathcal{A}$, and the center is $Z(\mathcal{A}) = \{x \in \mathcal{A} \mid xy = yx, \forall y \in \mathcal{A}\}$. A bilinear map $D : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is called a biderivation (resp. Lie biderivation) if it acts as a derivation (resp. Lie derivation) in each argument:

$$D(xy, z) = D(x, z)y + xD(y, z) \text{ and } D(x, yz) = D(x, y)z + yD(x, z)$$

$$(resp. D([x, y], z) = [D(x, z), y] + [x, D(y, z)] \text{ and } D(x, [y, z]) = [D(x, y), z] + [y, D(x, z)])$$

for all $x, y, z \in \mathcal{A}$. Prominent examples include inner biderivations of the form $\theta(x, y) = \lambda[x, y]$ for some $\lambda \in Z(\mathcal{A})$ (the centrality of λ is essential for θ to be a well-defined biderivation) and extremal biderivations $\Delta(x, y) = [x, [y, a]]$ defined for an element $a \notin Z(\mathcal{A})$ satisfying $[[x, y], a] = 0$ for all $x, y \in \mathcal{A}$.

The study of Jordan and Lie biderivations on triangular algebras has evolved along two primary methodological lines: the use of faithful bimodule structures and the application of the maximum left ring of quotients. A seminal work by Benkovič [1] initiated the systematic study of biderivations on triangular algebras employing bimodule theory. Concurrently, Wang Yu [2] independently derived analogous results via the maximum left ring of quotients, demonstrating that every biderivation decomposes into the sum of an extremal and an inner biderivation under mild conditions. This foundational work was extended by Eremita [3,4], who further characterized functional identities of degree 2 within both frameworks. Subsequent contributions by Ren et al. [6] delineated Jordan biderivations using bimodule properties, and a result later reformulated by Liu [7] employed the ring of quotients approach. Parallel advances in Lie biderivations were achieved by Liang et al. [8], who established decomposition theorems via bimodule theory. These findings were subsequently generalized to Lie n-biderivations through techniques of the maximum left ring of quotients by Alghazzawi et al. [9] and contemporaneous work by Liang et al. [10].

Notably, triangular algebras represent a specialized subclass of the more general framework of generalized matrix algebras. Consequently, a significant gap persists in the literature: although the methodologies are well-established for triangular algebras, their application to characterize Lie biderivations on generalized matrix algebras remains largely unexplored. Addressing this gap, the present work examines the decomposition structure of Lie biderivations on generalized matrix algebras through the lens of bimodule theory (see Theorem 3.1). As an application, we derive a corresponding characterization of Lie biderivations on full matrix algebras.

2. Preliminaries

This section will provide some definitions and preliminary lemmas.

Let $\mathcal{A}$ be an algebra, $\emptyset \neq S \subseteq \mathcal{A}$. S is called an algebraic ideal of $\mathcal{A}$ if S is both an ideal of $\mathcal{A}$ as a ring and a submodule of $\mathcal{A}$. S is called a central ideal of $\mathcal{A}$ if S is an algebraic ideal of $\mathcal{A}$, and $S \subseteq Z(\mathcal{A})$.

Let R be a commutative ring with unity, $\mathcal{A}$ and $\mathcal{B}$ be two algebras over R and $\mathcal{M}$ and $\mathcal{N}$ be an $(\mathcal{A}, \mathcal{B})$-bimodule and a $(\mathcal{B}, \mathcal{A})$-bimodule, respectively, and let $\Phi : \mathcal{M} \otimes_{\mathcal{B}} \mathcal{N} \to \mathcal{A}$ and $\Psi : \mathcal{N} \otimes_{\mathcal{A}} \mathcal{M} \to \mathcal{B}$ be two homomorphisms satisfying the following commutative diagrams:

$$\begin{array}{ccc} \mathcal{M} \underset{\mathcal{B}}{\otimes} \mathcal{N} \underset{\mathcal{A}}{\otimes} \mathcal{M} & \xrightarrow{\Phi_{MN} \otimes I_{M}} & \mathcal{A} \underset{\mathcal{A}}{\otimes} \mathcal{M} \\ \downarrow I_{M} \otimes \Psi_{MN} & & \downarrow \cong \\ \mathcal{M} \underset{\mathcal{B}}{\otimes} \mathcal{B} & \xrightarrow{\cong} & \mathcal{M}, \end{array}$$

and

$$\begin{array}{ccc} \mathcal{N} \underset{\mathcal{A}}{\otimes} \mathcal{M} \underset{\mathcal{B}}{\otimes} \mathcal{N} & \xrightarrow{\Psi_{NM} \otimes I_{N}} & \mathcal{B} \underset{\mathcal{B}}{\otimes} \mathcal{N} \\ \downarrow I_{N} \otimes \Phi_{MN} & & \downarrow \cong \\ \mathcal{N} \underset{\mathcal{A}}{\otimes} \mathcal{A} & \xrightarrow{\cong} & \mathcal{N}. \end{array}$$

Then, the set

$$\mathcal{G} = \mathcal{G}(\mathcal{A}, \mathcal{M}, \mathcal{N}, \mathcal{B}) = \begin{pmatrix} \mathcal{A} & \mathcal{M} \\ \mathcal{N} & \mathcal{B} \end{pmatrix} = \left\{ \begin{pmatrix} a & m \\ n & b \end{pmatrix} \middle| \; a \in \mathcal{A}, b \in \mathcal{B}, m \in \mathcal{M}, n \in \mathcal{N} \right\}$$

forms an algebra over R under the usual matrix addition, scalar multiplication, and matrix multiplication. Such algebra is said to be a generalized matrix algebra over R [11]. When $\mathcal{N}$=0, $\mathcal{G} = \mathcal{G}(\mathcal{A}, \mathcal{M}, \mathcal{N}, \mathcal{B})$ is called a triangular algebra over R and is denoted by $\mathcal{T} = \mathcal{T}(\mathcal{A}, \mathcal{M}, \mathcal{B})$. Obviously, the full matrix algebra $M_n(R)(n \geq 2)$ and any triangular algebra $\mathcal{T} = \mathcal{T}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ are both examples of generalized matrix algebras over R.

Define two natural projections $\pi_{\mathcal{A}} : \mathcal{G} \to \mathcal{A}$ and $\pi_{\mathcal{B}} : \mathcal{G} \to \mathcal{B}$ as

$$\pi_{\mathcal{A}} : \begin{pmatrix} a & m \\ n & b \end{pmatrix} \mapsto a, \text{ and } \pi_{\mathcal{B}} : \begin{pmatrix} a & m \\ n & b \end{pmatrix} \mapsto b.$$

Let $1_{\mathcal{A}}$ (resp. $1_{\mathcal{B}}$) be the identity element of the algebra $\mathcal{A}$ (resp. $\mathcal{B}$), and let I be the identity of the generalized matrix algebra $\mathcal{G}$. Set

$$f_1 = \begin{pmatrix} 1_{\mathcal{A}} & 0 \\ 0 & 0 \end{pmatrix}, f_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1_{\mathcal{B}} \end{pmatrix},$$

and

$$\mathcal{G}_{11} = f_1\mathcal{G}f_1, \mathcal{G}_{12} = f_1\mathcal{G}f_2, \mathcal{G}_{21} = f_2\mathcal{G}f_1, \mathcal{G}_{22} = f_2\mathcal{G}f_2.$$

Then, $\mathcal{G} = \mathcal{G}_{11} \oplus \mathcal{G}_{12} \oplus \mathcal{G}_{21} \oplus \mathcal{G}_{22}$, $\mathcal{G}_{11}$, and $\mathcal{G}_{22}$ are subalgebras of $\mathcal{G}$ which are isomorphic to $\mathcal{A}$ and $\mathcal{B}$, respectively, $\mathcal{G}_{12}$ is a $(\mathcal{G}_{11}, \mathcal{G}_{22})$-bimodule which is isomorphic to $\mathcal{M}$, and $\mathcal{G}_{21}$ is a $(\mathcal{G}_{22}, \mathcal{G}_{11})$-bimodule which is isomorphic to $\mathcal{N}$.

Lemma 2.1. *[11] Let $\mathcal{G} = \mathcal{G}(\mathcal{A}, \mathcal{M}, \mathcal{N}, \mathcal{B})$ be a generalized matrix algebra and $\mathcal{M}$ a faithful $(\mathcal{A}, \mathcal{B})$-bimodule. Then,*

$$Z(\mathcal{G}) = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \middle| \; am = mb, na = bn, \forall m \in \mathcal{M}, n \in \mathcal{N} \right\}.$$

Furthermore, there exists a unique algebra isomorphism $\eta : \pi_{\mathcal{A}}(Z(\mathcal{G})) \to \pi_{\mathcal{B}}(Z(\mathcal{G}))$ such that $am = m\eta(a)$, $na = \eta(a)n$ for all $m \in \mathcal{M}$ and $n \in \mathcal{N}$.

Remark 2.2. *In Lemma 2.1, if $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in Z(\mathcal{G})$, then $b = \eta(a)$. In fact, by $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in Z(\mathcal{G})$, we have $a \in \pi_{\mathcal{A}}(Z(\mathcal{G}))$ and $am = mb$, $na = bn$ for all $m \in \mathcal{M}$ and $n \in \mathcal{N}$. On the other hand, because $a \in \pi_{\mathcal{A}}(Z(\mathcal{G}))$, we have $\eta : \pi_{\mathcal{A}}(Z(\mathcal{G})) \to \pi_{\mathcal{B}}(Z(\mathcal{G}))$ such that $am = m\eta(a)$, $na = \eta(a)n$ for all $m \in \mathcal{M}$ and $n \in \mathcal{N}$, and so $mb = am = m\eta(a)$ for all $m \in \mathcal{M}$. Using the faithfulness of $\mathcal{M}$, we have $b = \eta(a)$.*

Lemma 2.3. *[12] Let $\mathcal{G} = \mathcal{G}(\mathcal{A}, \mathcal{M}, \mathcal{N}, \mathcal{B})$ be a generalized matrix algebra and $\mathcal{M}$ a faithful $(\mathcal{A}, \mathcal{B})$-bimodule such that $\pi_{\mathcal{A}}(Z(\mathcal{G})) = Z(\mathcal{A})$, $\pi_{\mathcal{B}}(Z(\mathcal{G})) = Z(\mathcal{B})$. For $a \in \mathcal{A}, b \in \mathcal{B}$, if $am = mb$ for any $m \in \mathcal{M}$, then*

$$a \in Z(\mathcal{A}), b \in Z(\mathcal{B}), \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in Z(\mathcal{G}).$$

Lemma 2.4. *Let $\mathcal{G} = \mathcal{G}(\mathcal{A}, \mathcal{M}, \mathcal{N}, \mathcal{B})$ be a generalized matrix algebra and L a Lie biderivation on $\mathcal{G}$. Then,* *(1)* $L(0, x) = 0, L(x, 0) = 0$ *for any* $x \in \mathcal{G}$*;*

(2) $L(I, x) \in Z(\mathcal{G}), L(x, I) \in Z(\mathcal{G})$ *for any* $x \in \mathcal{G}$*;*

(3) $f_1L(f_1, f_1)f_2 = -f_1L(f_1, f_2)f_2 = -f_1L(f_2, f_1)f_2 = f_1L(f_2, f_2)f_2$*;*
$f_2L(f_2, f_2)f_1 = -f_2L(f_2, f_1)f_1 = -f_2L(f_1, f_2)f_1 = f_2L(f_1, f_1)f_1$*;*

(The displayed equalities exhibit a symmetric pattern between the components involving f_1 *and* f_2*.)*

(4) For any $x, y \in \mathcal{G}$*, if* $[f_1, L(x, y)] = 0$*, then* $L(x, y) = f_1L(x, y)f_1 + f_2L(x, y)f_2$*.*

Proof: (1) Because L is a Lie derivation with respect to the first component, we have

$$L(0, x) = L([0, 0], x) = [L(0, x), 0] + [0, L(0, x)] = 0$$

for any $x \in \mathcal{G}$. Similarly, we can prove that $L(x, 0) = 0$ for any $x \in \mathcal{G}$.

(2) Because L is a Lie derivation with respect to the first component, we have

$$0 = L(0, x) = L([I, y], x) = [L(I, x), y] + [I, L(y, x)] = [L(I, x), y]$$

for any $x, y \in \mathcal{G}$. By the arbitrariness of element y, we have $L(I, x) \in Z(\mathcal{G})$. Similarly, we can prove that $L(x, I) \in Z(\mathcal{G})$ for any $x \in \mathcal{G}$.

(3) By (2), we have $L(I, f_1) \in Z(\mathcal{G})$. Then $f_1L(I, f_1)f_2 = 0$. Because $I = f_1 + f_2$, we have

$$f_1L(f_1, f_1)f_2 = f_1L(I - f_2, f_1)f_2 = f_1L(I, f_1)f_2 - f_1L(f_2, f_1)f_2 = -f_1L(f_2, f_1)f_2.$$

Similarly, by (2), we have

$$L(f_1, I) \in Z(\mathcal{G}), L(f_2, I) \in Z(\mathcal{G}),$$

and so

$$f_1L(f_1, f_1)f_2 = -f_1L(f_1, f_2)f_2, f_1L(f_2, f_2)f_2 = -f_1L(f_2, f_1)f_2.$$

Then,

$$f_1L(f_1, f_1)f_2 = -f_1L(f_1, f_2)f_2 = -f_1L(f_2, f_1)f_2 = f_1L(f_2, f_2)f_2.$$

Similarly, we can prove

$$f_2L(f_2, f_2)f_1 = -f_2L(f_2, f_1)f_1 = -f_2L(f_1, f_2)f_1 = f_2L(f_1, f_1)f_1.$$

(4) By $[f_1, L(x, y)] = 0$, we have $f_1L(x, y) = L(x, y)f_1$. Because $f_1f_2 = f_2f_1 = 0$, we get $f_1L(x, y)f_2 = 0, f_2L(x, y)f_1 = 0$, and so

$$L(x, y) = f_1L(x, y)f_1 + f_2L(x, y)f_2.$$

□

Lemma 2.5. *[7, Lemma 3.1] Let* $\mathcal{A}$ *be an algebra and* L *a Lie biderivation on* $\mathcal{A}$*. Then,*

$$[[u, v], L(x, y)] + [[x, y], L(u, v)] = [[u, y], L(x, v)] + [[x, v], L(u, y)] \tag{2.1}$$

for any $x, y, u, v \in \mathcal{A}$*.*

Throughout this paper, it is assumed that all algebras over a commutative ring R are 2-torsion free (i.e., $\forall x \in \mathcal{A}, 2x = 0$ implies $x = 0$), and the $(\mathcal{A}, \mathcal{B})$-bimodule $\mathcal{M}$ is faithful.

3. Main results and proofs

Theorem 3.1. *Let $\mathcal{G} = \mathcal{G}(\mathcal{A}, \mathcal{M}, \mathcal{N}, \mathcal{B})$ be a generalized matrix algebra. Suppose that*
(i) $\pi_{\mathcal{A}}(Z(\mathcal{G})) = Z(\mathcal{A}), \pi_{\mathcal{B}}(Z(\mathcal{G})) = Z(\mathcal{B})$;
(ii) the algebra $\mathcal{A}$ or the algebra $\mathcal{B}$ does not contain nonzero central ideals when $\mathcal{N} \neq 0$.
Then, any Lie biderivation $L : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ is of the form

$$L(x, y) = D(x, y) + \tau(x, y)$$

for all $x, y \in \mathcal{G}$, where $D : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ is a biderivation, and $\tau : \mathcal{G} \times \mathcal{G} \to Z(\mathcal{G})$ is a bilinear map vanishing at commutators.

Condition (ii) of Theorem 3.1, which requires that $\mathcal{A}$ or $\mathcal{B}$ contains no nonzero central ideals when $\mathcal{N} \neq 0$, is naturally satisfied in many important cases. For example, every noncommutative unital prime algebra possessing a nontrivial idempotent contains no nonzero central ideals. The full matrix algebra $M_n(\mathcal{A})$ ($n \geq 2$) likewise has no nonzero central ideal. Furthermore, a generalized matrix algebra $\mathcal{G}$ with a faithful bimodule $\mathcal{M}$ itself has no central ideals. To streamline the proof of Theorem 3.1 and enhance readability, we break the argument into the following lemmas.

Lemma 3.2. *Let $\mathcal{G} = \mathcal{G}(\mathcal{A}, \mathcal{M}, \mathcal{N}, \mathcal{B})$ be a generalized matrix algebra such that $\pi_{\mathcal{A}}(Z(\mathcal{G})) = Z(\mathcal{A}), \pi_{\mathcal{B}}(Z(\mathcal{G})) = Z(\mathcal{B})$, and $L : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ is a Lie biderivation. Suppose that the algebra $\mathcal{A}$ or the algebra $\mathcal{B}$ does not contain nonzero central ideals when $\mathcal{N} \neq 0$. If $L(f_1, f_1) \notin Z(\mathcal{G})$, then $L = \Delta + L_1$, where Δ is an extremal biderivation on $\mathcal{G}$, and L_1 is a Lie biderivation on $\mathcal{G}$ satisfying $L_1(f_1, f_1) \in Z(\mathcal{G})$.*

Proof strategy: The main idea is to extract an extremal biderivation from L when $L(f_1, f_1) \notin Z(\mathcal{G})$. We first show that the diagonal part of $L(f_1, f_1)$ is in $Z(\mathcal{G})$. Then, using the condition that $\mathcal{A}$ or $\mathcal{B}$ has no nonzero central ideals, we prove that certain off-diagonal actions vanish (e.g., $y_{12} \cdot f_2 L(f_1, f_1) f_1 = 0$). This allows us to define an extremal biderivation $\Delta(x, y) = [x, [y, L(f_1, f_1)]]$. The remainder Lie biderivation $L_1 = L - \Delta$ then satisfies $L_1(f_1, f_1) \in Z(\mathcal{G})$.

Proof: First, we prove $f_1 L(f_1, f_1) f_1 + f_2 L(f_1, f_1) f_2 \in Z(\mathcal{G})$.

Because L is a Lie biderivation on $\mathcal{G}$, it follows that

$$\begin{aligned} L(f_1, y_{12}) = L(f_1, [f_1, y_{12}]) &= [L(f_1, f_1), y_{12}] + [f_1, L(f_1, y_{12})] \\ &= L(f_1, f_1) y_{12} - y_{12} L(f_1, f_1) + f_1 L(f_1, y_{12}) - L(f_1, y_{12}) f_1 \end{aligned} \tag{3.1}$$

for any $y_{12} \in \mathcal{G}_{12}$.

Multiplying the left by f_1 and the right by f_2 in Eq (3.1), we get

$$f_1 L(f_1, f_1) f_1 \cdot y_{12} = y_{12} \cdot f_2 L(f_1, f_1) f_2.$$

By Lemma 2.3, we have $f_1 L(f_1, f_1) f_1 + f_2 L(f_1, f_1) f_2 \in Z(\mathcal{G})$.

Next, we prove that Δ is an extremal biderivation on $\mathcal{G}$, where Δ is defined by $\Delta(x, y) = [x, [y, L(f_1, f_1)]]$ for $x, y \in \mathcal{G}$.

To do this, we need prove $[[x, y], L(f_1, f_1)] = 0$ for all $x, y \in \mathcal{G}$.

Utilizing Eq (2.1) and the condition $L(f_1, f_1) \notin Z(\mathcal{G})$, one obtains

$$[[x_{11}, y_{11}], L(f_1, f_1)] = 0, [[x_{22}, y_{22}], L(f_1, f_1)] = 0 \tag{3.2}$$

for any $x_{11}, y_{11} \in \mathcal{G}_{11}, x_{22}, y_{22} \in \mathcal{G}_{22}$.

For any $x_{12}, y_{12} \in \mathcal{G}_{12}$, according to the definition of Lie biderivation, we have

$$\begin{aligned} L(x_{12}, y_{12}) &= L([f_1, x_{12}], y_{12}) = [L(f_1, y_{12}), x_{12}] + [f_1, L(x_{12}, y_{12})] \\ &= L(f_1, y_{12})x_{12} - x_{12}L(f_1, y_{12}) + f_1L(x_{12}, y_{12}) - L(x_{12}, y_{12})f_1. \end{aligned}$$

Multiplying the above equation by f_1 on the left and by f_2 on the right, we have

$$f_1L(f_1, y_{12})f_1 \cdot x_{12} = x_{12} \cdot f_2L(f_1, y_{12})f_2.$$

By Lemma 2.3, we have

$$f_1L(f_1, y_{12})f_1 \in Z(\mathcal{G}_{11}), f_2L(f_1, y_{12})f_2 \in Z(\mathcal{G}_{22}), f_1L(f_1, y_{12})f_1 + f_2L(f_1, y_{12})f_2 \in Z(\mathcal{G}).$$

By Lemma 2.1 and Remark 2.2, we have $f_2L(f_1, y_{12})f_2 = \eta(f_1L(f_1, y_{12})f_1)$, where η is as in Lemma 2.1.

In the following, we will prove that $y_{12} \cdot f_2L(f_1, f_1)f_1 = 0$ for any $y_{12} \in \mathcal{G}_{12}$.

If $\mathcal{N} = 0$, then the conclusion clearly holds. If $\mathcal{N} \neq 0$, then $\mathcal{G}_{11}$ or $\mathcal{G}_{22}$ does not contain nonzero central ideals. Without losing generality, we assume that $\mathcal{G}_{11}$ does not contain nonzero central ideals.

Multiplying the left by f_1 and the right by f_1 in Eq (3.1), we have

$$-y_{12} \cdot f_2L(f_1, f_1)f_1 = f_1L(f_1, y_{12})f_1 \in Z(\mathcal{G}_{11}). \tag{3.3}$$

Then, $\mathcal{G}_{12} \cdot f_2L(f_1, f_1)f_1$ is a central ideal of $\mathcal{G}_{11}$. Because $\mathcal{G}_{11}$ does not contain a nonzero central ideal, we have $\mathcal{G}_{12} \cdot f_2L(f_1, f_1)f_1 = \{0\}$, and so

$$y_{12} \cdot f_2L(f_1, f_1)f_1 = 0 \tag{3.4}$$

for any $y_{12} \in \mathcal{G}_{12}$.

Multiplying the left by f_2 and the right by f_2 in Eq (3.1), we have

$$f_2L(f_1, f_1)f_1 \cdot y_{12} = f_2L(f_1, y_{12})f_2.$$

From $f_1L(f_1, y_{12})f_1 + f_2L(f_1, y_{12})f_2 \in Z(\mathcal{G})$, we have $f_2L(f_1, y_{12})f_2 = \eta(f_1L(f_1, y_{12})f_1)$. By Eqs (3.3) and (3.4), we find

$$f_2L(f_1, f_1)f_1 \cdot y_{12} = \eta(f_1L(f_1, y_{12})f_1) = \eta(-y_{12} \cdot f_2L(f_1, f_1)f_1) = \eta(0) = 0,$$

and so

$$[y_{12}, f_2L(f_1, f_1)f_1] = y_{12} \cdot f_2L(f_1, f_1)f_1 - f_2L(f_1, f_1)f_1 \cdot y_{12} = 0.$$

Because $f_1L(f_1, f_1)f_1 + f_2L(f_1, f_1)f_2 \in Z(\mathcal{G})$, it follows that

$$[y_{12}, L(f_1, f_1)] = [y_{12}, f_1L(f_1, f_1)f_2 + f_2L(f_1, f_1)f_1] = [y_{12}, f_2L(f_1, f_1)f_1] = 0 \tag{3.5}$$

for any $y_{12} \in \mathcal{G}_{12}$.

By a similar discussion, one can obtain

$$[y_{21}, L(f_1, f_1)] = 0 \tag{3.6}$$

for any $y_{21} \in \mathcal{G}_{21}$.

From Jacobi's identity and Eqs (3.5) and (3.6), we have

$$[[x_{12}, y_{21}], L(f_1, f_1)] = [x_{12}, [y_{21}, L(f_1, f_1)]] + [y_{21}, [L(f_1, f_1), x_{12}]] = 0,$$

$$[[x_{21}, y_{12}], L(f_1, f_1)] = [x_{21}, [y_{12}, L(f_1, f_1)]] + [y_{12}, [L(f_1, f_1), x_{21}]] = 0$$

for any $x_{12}, y_{12} \in \mathcal{G}_{12}, x_{21}, y_{21} \in \mathcal{G}_{21}$.

Then for any $x, y \in \mathcal{G}$, we have

$$\begin{aligned}
[[x, y], L(f_1, f_1)] &= [f_1[x, y]f_1 + f_1[x, y]f_2 + f_2[x, y]f_1 + f_2[x, y]f_2, L(f_1, f_1)] \\
&= [f_1[x, y]f_1 + f_2[x, y]f_2, L(f_1, f_1)](By(3.5)(3.6)) \\
&= [[x_{11}, y_{11}] + [x_{12}, y_{21}] + [x_{21}, y_{12}] + [x_{22}, y_{22}], L(f_1, f_1)] \\
&= [[x_{12}, y_{21}] + [x_{21}, y_{12}], L(f_1, f_1)](By(3.2)) \\
&= 0,
\end{aligned}$$

where $x_{ij} = f_i x f_j, y_{ij} = f_i y f_j, i, j \in \{1, 2\}$, and so $\Delta(x, y) = [x, [y, L(f_1, f_1)]]$ is an extremal biderivation on $\mathcal{G}$.

Set $L_1 = L - \Delta$. Then, it is easy to verify that L_1 is a Lie biderivation on $\mathcal{G}$, and

$$\begin{aligned}
L_1(f_1, f_1) = L(f_1, f_1) - \Delta(f_1, f_1) &= L(f_1, f_1) - [f_1, [f_1, L(f_1, f_1)]] \\
&= L(f_1, f_1) - (f_1 L(f_1, f_1) f_2 + f_2 L(f_1, f_1) f_1) \\
&= f_1 L(f_1, f_1) f_1 + f_2 L(f_1, f_1) f_2 \in Z(\mathcal{G}).
\end{aligned}$$

Lemma 3.3. *Let $\mathcal{G} = \mathcal{G}(\mathcal{A}, \mathcal{M}, \mathcal{N}, \mathcal{B})$ be a generalized matrix algebra satisfying $\pi_{\mathcal{A}}(Z(\mathcal{G})) = Z(\mathcal{A}), \pi_{\mathcal{B}}(Z(\mathcal{G})) = Z(\mathcal{B})$. If $L : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ is a Lie biderivation on $\mathcal{G}$ such that $L(f_1, f_1) \in Z(\mathcal{G})$, then for any $x_{ij}, y_{ij} \in \mathcal{G}_{ij}, i, j \in \{1, 2\}$,*

$$\begin{aligned}
&(i) L(x_{11}, y_{12}) = f_1 L(x_{11}, y_{12}) f_2 = x_{11} L(f_1, y_{12}), L(x_{22}, y_{12}) = f_1 L(x_{22}, y_{12}) f_2 = -L(f_1, y_{12}) x_{22}, \\
&\quad L(x_{12}, y_{11}) = f_1 L(x_{12}, y_{11}) f_2 = y_{11} L(x_{12}, f_1), L(x_{12}, y_{22}) = f_1 L(x_{12}, y_{22}) f_2 = -L(x_{12}, f_1) y_{22}; \\
&(ii) L(x_{11}, y_{21}) = f_2 L(x_{11}, y_{21}) f_1 = L(f_1, y_{21}) x_{11}, L(x_{22}, y_{21}) = f_2 L(x_{22}, y_{21}) f_1 = -x_{22} L(f_1, y_{21}), \\
&\quad L(x_{21}, y_{11}) = f_2 L(x_{21}, y_{11}) f_1 = L(x_{21}, f_1) y_{11}, L(x_{21}, y_{22}) = f_2 L(x_{21}, y_{22}) f_1 = -y_{22} L(x_{21}, f_1); \\
&(iii) L(x_{11}, y_{22}) \in Z(\mathcal{G}), L(x_{22}, y_{11}) \in Z(\mathcal{G}); \\
&(iv) L(x_{11}, y_{11}) = f_1 L(x_{11}, y_{11}) f_1 + f_2 L(x_{11}, y_{11}) f_2, \textit{and } f_2 L(x_{11}, y_{11}) f_2 \in Z(\mathcal{G}_{22}), \\
&\quad L(x_{22}, y_{22}) = f_1 L(x_{22}, y_{22}) f_1 + f_2 L(x_{22}, y_{22}) f_2, \textit{and } f_1 L(x_{22}, y_{22}) f_1 \in Z(\mathcal{G}_{11}); \\
&(v) L(x_{12}, y_{12}) = f_1 L(x_{12}, y_{12}) f_2, L(x_{21}, y_{21}) = f_2 L(x_{21}, y_{21}) f_1; \\
&(vi) L(x_{12}, y_{21}) = [L(f_1, y_{21}), x_{12}] = [y_{21}, L(x_{12}, f_1)], L(x_{21}, y_{12}) = [L(x_{21}, f_1), y_{12}] = [x_{21}, L(f_1, y_{12})].
\end{aligned}$$

Proof: First, we prove that $L(x_{11}, f_1), L(x_{22}, f_1) \in Z(\mathcal{G})$ for any $x_{11} \in \mathcal{G}_{11}, x_{22} \in \mathcal{G}_{22}$. Because L is a Lie biderivation on $\mathcal{G}$, we have

$$L([a, b], u) = [L(a, u), b] + [a, L(b, u)], \tag{3.7}$$

$$L(a, [u, v]) = [L(a, u), v] + [u, L(a, v)] \tag{3.8}$$

for any $a, b, u, v \in \mathcal{G}$.

Taking $a = f_1, b = x_{11}, u = f_1$ in Eq (3.7), we have

$$L([f_1, x_{11}], f_1) = [L(f_1, f_1), x_{11}] + [f_1, L(x_{11}, f_1)]. \tag{3.9}$$

By Lemma 2.4 (1), we have $L([f_1, x_{11}], f_1) = L(0, f_1) = 0$. By the condition $L(f_1, f_1) \in Z(\mathcal{G})$ and Eq (3.9), we have $[f_1, L(x_{11}, f_1)] = 0$, and by Lemma 2.4 (4), we have

$$L(x_{11}, f_1) = f_1 L(x_{11}, f_1) f_1 + f_2 L(x_{11}, f_1) f_2.$$

Taking $a = x_{11}, u = f_1, v = y_{12} \in \mathcal{G}_{12}$ in Eq (3.8), we get

$$L(x_{11}, [f_1, y_{12}]) = [L(x_{11}, f_1), y_{12}] + [f_1, L(x_{11}, y_{12})],$$

that is,

$$L(x_{11}, y_{12}) = L(x_{11}, f_1)y_{12} - y_{12}L(x_{11}, f_1) + f_1 L(x_{11}, y_{12}) f_2 - f_2 L(x_{11}, y_{12}) f_1. \tag{3.10}$$

Multiplying the left by f_1 and the right by f_2 in Eq (3.10), we have

$$f_1 L(x_{11}, f_1) f_1 \cdot y_{12} = y_{12} \cdot f_2 L(x_{11}, f_1) f_2$$

for any $y_{12} \in \mathcal{G}_{12}$. By Lemma 2.3, we have $f_1 L(x_{11}, f_1) f_1 + f_2 L(x_{11}, f_1) f_2 \in Z(\mathcal{G})$. This means that $L(x_{11}, f_1) \in Z(\mathcal{G})$ for any $x_{11} \in \mathcal{G}_{11}$.

Similarly, we can prove that $L(x_{22}, f_1) \in Z(\mathcal{G})$ for any $x_{22} \in \mathcal{G}_{22}$.

Next, we prove (i) and (ii).

Utilizing Eq (3.10) and $L(x_{11}, f_1) \in Z(\mathcal{G})$, one obtains

$$L(x_{11}, y_{12}) = f_1 L(x_{11}, y_{12}) f_2 - f_2 L(x_{11}, y_{12}) f_1.$$

Multiplying the above equation by f_2 on the left and by f_1 on the right, we have

$$f_2 L(x_{11}, y_{12}) f_1 = -f_2 L(x_{11}, y_{12}) f_1.$$

Then, $f_2 L(x_{11}, y_{12}) f_1 = 0$, and so

$$L(x_{11}, y_{12}) = f_1 L(x_{11}, y_{12}) f_2. \tag{3.11}$$

In particular, we have $L(f_1, y_{12}) = f_1 L(f_1, y_{12}) f_2$.

Taking $a = f_1, b = x_{11}, u = y_{12}$ in Eq (3.7), by Lemma 2.4 (1), we get

$$[L(f_1, y_{12}), x_{11}] + [f_1, L(x_{11}, y_{12})] = L([f_1, x_{11}], y_{12}) = L(0, y_{12}) = 0,$$

and so $[f_1, L(x_{11}, y_{12})] = -[L(f_1, y_{12}), x_{11}]$.

By Eq (3.11), we have

$$f_1 L(x_{11}, y_{12}) f_2 = [f_1, L(x_{11}, y_{12})] = -[L(f_1, y_{12}), x_{11}] = x_{11} L(f_1, y_{12}),$$

and so $L(x_{11}, y_{12}) = f_1 L(x_{11}, y_{12}) f_2 = x_{11} L(f_1, y_{12})$.

Taking $a = x_{22}, u = f_1, v = y_{12}$ in Eq (3.8), we have

$$L(x_{22}, [f_1, y_{12}]) = [L(x_{22}, f_1), y_{12}] + [f_1, L(x_{22}, y_{12})].$$

By $L(x_{22}, f_1) \in Z(\mathcal{G})$, we obtain $L(x_{22}, [f_1, y_{12}]) = [f_1, L(x_{22}, y_{12})]$. Then,

$$\begin{aligned} L(x_{22}, y_{12}) &= L(x_{22}, [f_1, y_{12}]) = [f_1, L(x_{22}, y_{12})] = f_1 L(x_{22}, y_{12}) - L(x_{22}, y_{12}) f_1 \\ &= f_1 L(x_{22}, y_{12})(f_1 + f_2) - (f_1 + f_2) L(x_{22}, y_{12}) f_1 \\ &= f_1 L(x_{22}, y_{12}) f_2 - f_2 L(x_{22}, y_{12}) f_1. \end{aligned}$$

Multiplying the above equation by f_2 on the left and by f_1 on the right, we have

$$f_2 L(x_{22}, y_{12}) f_1 = -f_2 L(x_{22}, y_{12}) f_1.$$

Then, $f_2 L(x_{22}, y_{12}) f_1 = 0$, and so

$$L(x_{22}, y_{12}) = f_1 L(x_{22}, y_{12}) f_2. \tag{3.12}$$

Taking $a = f_1, b = x_{22}, u = y_{12}$ in Eq (3.7), by Lemma 2.4 (1), we get

$$[L(f_1, y_{12}), x_{22}] + [f_1, L(x_{22}, y_{12})] = L([f_1, x_{22}], y_{12}) = L(0, y_{12}) = 0.$$

Then, $[f_1, L(x_{22}, y_{12})] = -[L(f_1, y_{12}), x_{22}]$. By Eq (3.12), we have

$$f_1 L(x_{22}, y_{12}) f_2 = [f_1, L(x_{22}, y_{12})] = -[L(f_1, y_{12}), x_{22}] = -L(f_1, y_{12}) x_{22},$$

and so

$$L(x_{22}, y_{12}) = f_1 L(x_{22}, y_{12}) f_2 = -L(f_1, y_{12}) x_{22}.$$

Similarly, we can prove

$$L(x_{12}, y_{11}) = f_1 L(x_{12}, y_{11}) f_2 = y_{11} L(x_{12}, f_1), L(x_{12}, y_{22}) = f_1 L(x_{12}, y_{22}) f_2 = -L(x_{12}, f_1) y_{22}.$$

Thus, (i) holds.

Taking $a = x_{11}, u = y_{21} \in \mathcal{G}_{21}, v = f_1$ in Eq (3.8), we have

$$L(x_{11}, [y_{21}, f_1]) = [L(x_{11}, y_{21}), f_1] + [y_{21}, L(x_{11}, f_1)].$$

By $L(x_{11}, f_1) \in Z(\mathcal{G})$, we get $L(x_{11}, [y_{21}, f_1]) = [L(x_{11}, y_{21}), f_1]$. Then,

$$\begin{aligned} L(x_{11}, y_{21}) &= L(x_{11}, [y_{21}, f_1]) = [L(x_{11}, y_{21}), f_1] = L(x_{11}, y_{21}) f_1 - f_1 L(x_{11}, y_{21}) \\ &= (f_1 + f_2) L(x_{11}, y_{21}) f_1 - f_1 L(x_{11}, y_{21})(f_1 + f_2) \\ &= f_2 L(x_{11}, y_{21}) f_1 - f_1 L(x_{11}, y_{21}) f_2. \end{aligned}$$

Multiplying the above equation by f_1 on the left and by f_2 on the right, we have

$$f_1 L(x_{11}, y_{21}) f_2 = -f_1 L(x_{11}, y_{21}) f_2.$$

Then, $f_1 L(x_{11}, y_{21}) f_2 = 0$, and so

$$L(x_{11}, y_{21}) = f_2 L(x_{11}, y_{21}) f_1. \tag{3.13}$$

In particular, $L(f_1, y_{21}) = f_2 L(f_1, y_{21}) f_1$.

Taking $a = f_1, b = x_{11}, u = y_{21}$ in Eq (3.7), by Lemma 2.4 (1), we get

$$[L(f_1, y_{21}), x_{11}] + [f_1, L(x_{11}, y_{21})] = L([f_1, x_{11}], y_{21}) = L(0, y_{21}) = 0,$$

and so $[f_1, L(x_{11}, y_{21})] = -[L(f_1, y_{21}), x_{11}]$.

By Eq (3.13), we have

$$f_2 L(x_{11}, y_{21}) f_1 = -[f_1, L(x_{11}, y_{21})] = [L(f_1, y_{21}), x_{11}] = L(f_1, y_{21}) x_{11},$$

and so $L(x_{11}, y_{21}) = f_2 L(x_{11}, y_{21}) f_1 = L(f_1, y_{21}) x_{11}$.

By a similar process, one can obtain

$$L(x_{21}, y_{11}) = f_2 L(x_{21}, y_{11}) f_1 = L(x_{21}, f_1) y_{11}, L(x_{22}, y_{21}) = f_2 L(x_{22}, y_{21}) f_1 = -x_{22} L(f_1, y_{21}),$$

and

$$L(x_{21}, y_{22}) = f_2 L(x_{21}, y_{22}) f_1 = -y_{22} L(x_{21}, f_1).$$

Thus, (ii) holds.

In the following, we prove (iii) and (iv).

Taking $a = x_{11}, u = f_1, v = y_{22}$ in Eq (3.8), we have

$$L(x_{11}, [f_1, y_{22}]) = [L(x_{11}, f_1), y_{22}] + [f_1, L(x_{11}, y_{22})].$$

Because $L(x_{11}, [f_1, y_{22}]) = L(x_{11}, 0) = 0$, by $L(x_{11}, f_1) \in Z(\mathcal{G})$, we obtain $[f_1, L(x_{11}, y_{22})] = 0$. By Lemma 2.4 (4), we have $L(x_{11}, y_{22}) = f_1 L(x_{11}, y_{22}) f_1 + f_2 L(x_{11}, y_{22}) f_2$.

Taking $x = x_{11}, y = y_{22}, u = x_{12}, v = f_1$ in Eq (2.1), we have

$$[[x_{12}, f_1], L(x_{11}, y_{22})] + [[x_{11}, y_{22}], L(x_{12}, f_1)] = [[x_{12}, y_{22}], L(x_{11}, f_1)] + [[x_{11}, f_1], L(x_{12}, y_{22})].$$

By $L(x_{11}, f_1) \in Z(\mathcal{G})$ and the above equation, we get $[[x_{12}, f_1], L(x_{11}, y_{22})] = 0$, that is, $f_1 L(x_{11}, y_{22}) f_1 \cdot x_{12} = x_{12} \cdot f_2 L(x_{11}, y_{22}) f_2$, and by Lemma 2.3, we have

$$L(x_{11}, y_{22}) = f_1 L(x_{11}, y_{22}) f_1 + f_2 L(x_{11}, y_{22}) f_2 \in Z(\mathcal{G}).$$

Similarly, we can prove that $L(x_{22}, y_{11}) = f_1 L(x_{22}, y_{11}) f_1 + f_2 L(x_{22}, y_{11}) f_2 \in Z(\mathcal{G})$. Thus, (iii) holds.

Taking $a = x_{11}, u = f_1, v = y_{11}$ in Eq (3.8), we get

$$L(x_{11}, [f_1, y_{11}]) = [L(x_{11}, f_1), y_{11}] + [f_1, L(x_{11}, y_{11})].$$

Because $L(x_{11}, [f_1, y_{11}]) = L(x_{11}, 0) = 0$, by $L(x_{11}, f_1) \in Z(\mathcal{G})$, we obtain $[f_1, L(x_{11}, y_{11})] = 0$. By Lemma 2.4 (4), we have

$$L(x_{11}, y_{11}) = f_1 L(x_{11}, y_{11}) f_1 + f_2 L(x_{11}, y_{11}) f_2.$$

Taking $a = x_{11}, u = y_{22}, v = y_{11}$ in Eq (3.8), we get

$$L(x_{11}, [y_{22}, y_{11}]) = [L(x_{11}, y_{22}), y_{11}] + [y_{22}, L(x_{11}, y_{11})].$$

By $L(x_{11}, y_{22}) \in Z(\mathcal{G})$, we have $[y_{22}, L(x_{11}, y_{11})] = 0$. Then,

$$f_2 L(x_{11}, y_{11}) f_2 \cdot y_{22} = y_{22} \cdot f_2 L(x_{11}, y_{11}) f_2,$$

and so $f_2 L(x_{11}, y_{11}) f_2 \in Z(\mathcal{G}_{22})$.

Similarly, we can prove

$$L(x_{22}, y_{22}) = f_1 L(x_{22}, y_{22}) f_1 + f_2 L(x_{22}, y_{22}) f_2, \text{and } f_1 L(x_{22}, y_{22}) f_1 \in Z(\mathcal{G}_{11}).$$

Thus, (iv) holds.

Finally , we prove (v) and (vi).

Taking $a = x_{12}, u = f_1, v = y_{12}$ in Eq (3.8), we get

$$L(x_{12}, [f_1, y_{12}]) = [L(x_{12}, f_1), y_{12}] + [f_1, L(x_{12}, y_{12})]. \tag{3.14}$$

By $L(x_{12}, f_1) = f_1 L(x_{12}, f_1) f_2$, we have $[L(x_{12}, f_1), y_{12}] = 0$. Because $L(x_{12}, [f_1, y_{12}]) = L(x_{12}, y_{12})$, Eq (3.14) reduces to

$$L(x_{12}, y_{12}) = [f_1, L(x_{12}, y_{12})] = f_1 L(x_{12}, y_{12}) f_2 - f_2 L(x_{12}, y_{12}) f_1.$$

Multiplying the above equation by f_2 on the left and by f_1 on the right, we have

$$f_2 L(x_{12}, y_{12}) f_1 = -f_2 L(x_{12}, y_{12}) f_1.$$

Then, $f_2 L(x_{12}, y_{12}) f_1 = 0$, and so $L(x_{12}, y_{12}) = f_1 L(x_{12}, y_{12}) f_2$.

Similarly, we can prove $L(x_{21}, y_{21}) = f_2 L(x_{21}, y_{21}) f_1$. Thus, (v) holds.

Taking $a = f_1, b = x_{12}, u = y_{21}$ in Equation (3.7), we get

$$L([f_1, x_{12}], y_{21}) = [L(f_1, y_{21}), x_{12}] + [f_1, L(x_{12}, y_{21})],$$

that is,

$$L(x_{12}, y_{21}) = [L(f_1, y_{21}), x_{12}] + f_1 L(x_{12}, y_{21}) f_2 - f_2 L(x_{12}, y_{21}) f_1. \tag{3.15}$$

Multiplying the above equation by f_2 on the left and by f_1 on the right, we have

$$f_2 L(x_{12}, y_{21}) f_1 = -f_2 L(x_{12}, y_{21}) f_1,$$

and so $f_2 L(x_{12}, y_{21}) f_1 = 0$.

Taking $a = x_{12}, u = y_{21}, v = f_1$ in Eq (3.8), we get

$$L(x_{12}, [y_{21}, f_1]) = [L(x_{12}, y_{21}), f_1] + [y_{21}, L(x_{12}, f_1)],$$

that is,

$$L(x_{12}, y_{21}) = f_2 L(x_{12}, y_{21}) f_1 - f_1 L(x_{12}, y_{21}) f_2 + [y_{21}, L(x_{12}, f_1)]. \tag{3.16}$$

Multiplying the above equation by f_1 on the left and by f_2 on the right, we have

$$f_1L(x_{12}, y_{21})f_2 = -f_1L(x_{12}, y_{21})f_2,$$

and so $f_1L(x_{12}, y_{21})f_2 = 0$.

By Eqs (3.15) and (3.16), we have

$$L(x_{12}, y_{21}) = [L(f_1, y_{21}), x_{12}] = [y_{21}, L(x_{12}, f_1)].$$

Similarly, we can prove $L(x_{21}, y_{12}) = [L(x_{21}, f_1), y_{12}] = [x_{21}, L(f_1, y_{12})]$. Thus, (vi) holds. □

Proof of Theorem 3.1: If $L(f_1, f_1) \notin Z(\mathcal{G})$, Lemma 3.2 extracts an extremal biderivation $\Delta(x, y) = [x, [y, L(f_1, f_1)]]$ from L, leaving a residual Lie biderivation L_1 with $L_1(f_1, f_1) \in Z(\mathcal{G})$. Therefore, without loss of generality, we may assume that $L(f_1, f_1) \in Z(\mathcal{G})$ in the following proof.

To isolate the central component of the Lie biderivation L that vanishes on commutators, we now construct a map $\tau : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ as follows:

$$\tau(x, y) = \eta^{-1}(f_2L(x_{11}, y_{11})f_2) + f_2L(x_{11}, y_{11})f_2 + f_1L(x_{22}, y_{22})f_1 + \eta(f_1L(x_{22}, y_{22})f_1) + L(x_{11}, y_{22}) + L(x_{22}, y_{11}),$$

and we define

$$D(x, y) = L(x, y) - \tau(x, y)$$

for any $x, y \in \mathcal{G}$, where $x_{ij} = f_ixf_j, y_{ij} = f_iyf_j, i, j \in \{1, 2\}$. Clearly, τ and D are bilinear maps on $\mathcal{G}$, and by Lemma 3.3 (iii) and (iv), $\tau(x, y) \in Z(\mathcal{G})$.

In the following, we prove that D is a biderivation on $\mathcal{G}$, and τ is a bilinear map vanishing at commutators.

First, we prove that D is a biderivation on $\mathcal{G}$. We divide the proof into two steps.

Step 1. For any $x, y, z \in \mathcal{G}$, we have

$$D(xy, z_{11}) = D(x, z_{11})y + xD(y, z_{11}), D(xy, z_{22}) = D(x, z_{22})y + xD(y, z_{22}),$$

where $x_{ij} = f_ixf_j, y_{ij} = f_iyf_j, z_{ij} = f_izf_j \in \mathcal{G}_{ij}, i, j \in \{1, 2\}$.

From Lemma 3.3 and the definition of D, we have

$$D(x_{11}, z_{11}) \in \mathcal{G}_{11}, D(x_{12}, z_{11}) \in \mathcal{G}_{12}, D(x_{21}, z_{11}) \in \mathcal{G}_{21}, D(x_{22}, z_{11}) = 0.$$

For any $u, v, w \in \mathcal{G}$, by the definition of D and $\tau(x, y) \in Z(\mathcal{G})$, we have

$$\begin{aligned} D([u, v], w) &= L([u, v], w) - \tau([u, v], w) = [L(u, w), v] + [u, L(v, w)] - \tau([u, v], w) \\ &= [D(u, w), v] + [u, D(v, w)] - \tau([u, v], w), \end{aligned}$$

and so

$$D([u, v], w) - [D(u, w), v] - [u, D(v, w)] \in Z(\mathcal{G}). \tag{3.17}$$

Taking $u = x_{11}, v = y_{12}, w = z_{11}$ in Eq (3.17), we get

$$D([x_{11}, y_{12}], z_{11}) - [D(x_{11}, z_{11}), y_{12}] - [x_{11}, D(y_{12}, z_{11})] \in Z(\mathcal{G}).$$

Notice that $D(x_{11}, z_{11}) \in \mathcal{G}_{11}, D(y_{12}, z_{11}) \in \mathcal{G}_{12}$, so we have

$$D(x_{11}y_{12}, z_{11}) - D(x_{11}, z_{11})y_{12} - x_{11}D(y_{12}, z_{11}) \in Z(\mathcal{G}).$$

Because $D(x_{11}y_{12}, z_{11}) - D(x_{11}, z_{11})y_{12} - x_{11}D(y_{12}, z_{11}) \in \mathcal{G}_{12}$, we have

$$D(x_{11}y_{12}, z_{11}) - D(x_{11}, z_{11})y_{12} - x_{11}D(y_{12}, z_{11}) \in \mathcal{G}_{12} \cap Z(\mathcal{G}) = \{0\},$$

and so

$$D(x_{11}y_{12}, z_{11}) = D(x_{11}, z_{11})y_{12} + x_{11}D(y_{12}, z_{11}). \tag{3.18}$$

Taking $u = x_{12}, v = y_{22}, w = z_{11}$ in Eq (3.17), we get

$$D([x_{12}, y_{22}], z_{11}) - [D(x_{12}, z_{11}), y_{22}] - [x_{12}, D(y_{22}, z_{11})] \in Z(\mathcal{G}).$$

Notice that $D(y_{22}, z_{11}) = 0, D(x_{12}, z_{11}) \in \mathcal{G}_{12}$, so we have $D(x_{12}y_{22}, z_{11}) - D(x_{12}, z_{11})y_{22} \in Z(\mathcal{G})$. Because $D(x_{12}y_{22}, z_{11}) - D(x_{12}, z_{11})y_{22} \in \mathcal{G}_{12}$, we have $D(x_{12}y_{22}, z_{11}) = D(x_{12}, z_{11})y_{22}$. By $D(y_{22}, z_{11}) = 0$, we get

$$D(x_{12}y_{22}, z_{11}) = D(x_{12}, z_{11})y_{22} + x_{12}D(y_{22}, z_{11}). \tag{3.19}$$

Similarly, we can prove

$$D(x_{21}y_{11}, z_{11}) = D(x_{21}, z_{11})y_{11} + x_{21}D(y_{11}, z_{11}), D(x_{22}y_{21}, z_{11}) = D(x_{22}, z_{11})y_{21} + x_{22}D(y_{21}, z_{11}). \tag{3.20}$$

By Eq (3.18), we have

$$D(x_{11}y_{11}a_{12}, z_{11}) = D(x_{11}y_{11}, z_{11})a_{12} + x_{11}y_{11}D(a_{12}, z_{11}),$$

and

$$\begin{aligned} D(x_{11}y_{11}a_{12}, z_{11}) &= D(x_{11}, z_{11})y_{11}a_{12} + x_{11}D(y_{11}a_{12}, z_{11}) \\ &= D(x_{11}, z_{11})y_{11}a_{12} + x_{11}D(y_{11}, z_{11})a_{12} + x_{11}y_{11}D(a_{12}, z_{11}) \end{aligned}$$

for any $a_{12} \in \mathcal{G}_{12}$, and so $D(x_{11}y_{11}, z_{11})a_{12} = (D(x_{11}, z_{11})y_{11} + x_{11}D(y_{11}, z_{11}))a_{12}$. Because $\mathcal{M}$ is a faithful $(\mathcal{A}, \mathcal{B})$-bimodule, we have

$$D(x_{11}y_{11}, z_{11}) = D(x_{11}, z_{11})y_{11} + x_{11}D(y_{11}, z_{11}). \tag{3.21}$$

Taking $u = x_{12}, v = y_{21}, w = z_{11}$ in (3.17), we get

$$D([x_{12}, y_{21}], z_{11}) - [D(x_{12}, z_{11}), y_{21}] - [x_{12}, D(y_{21}, z_{11})] \in Z(\mathcal{G}),$$

that is,

$$(D(x_{12}y_{21}, z_{11}) - D(x_{12}, z_{11})y_{21} - x_{12}D(y_{21}, z_{11})) - (D(y_{21}x_{12}, z_{11}) - D(y_{21}, z_{11})x_{12} - y_{21}D(x_{12}, z_{11})) \in Z(\mathcal{G}).$$

By Lemma 2.1 and Remark 2.2, we have

$$D(x_{12}y_{21}, z_{11}) - D(x_{12}, z_{11})y_{21} - x_{12}D(y_{21}, z_{11}) \in Z(\mathcal{G}_{11}),$$

and

$$D(y_{21}x_{12}, z_{11}) - D(y_{21}, z_{11})x_{12} - y_{21}D(x_{12}, z_{11}) = \eta(D(x_{12}y_{21}, z_{11}) - D(x_{12}, z_{11})y_{21} - x_{12}D(y_{21}, z_{11})).$$

If $\mathcal{N} = 0$, then $D(x_{12}y_{21}, z_{11}) - D(x_{12}, z_{11})y_{21} - x_{12}D(y_{21}, z_{11}) = 0$ clearly holds. If $\mathcal{N} \neq 0$, then $\mathcal{G}_{11}$ or $\mathcal{G}_{22}$ does not contain nonzero central ideals. Without losing generality, we assume that $\mathcal{G}_{11}$ does not contain nonzero central ideals.

For any $a_{11} \in \mathcal{G}_{11}$, by Eq (3.21), we have

$$D(a_{11}x_{12}y_{21}, z_{11}) = D(a_{11}, z_{11})x_{12}y_{21} + a_{11}D(x_{12}y_{21}, z_{11}),$$

and so $a_{11}D(x_{12}y_{21}, z_{11}) = D(a_{11}x_{12}y_{21}, z_{11}) - D(a_{11}, z_{11})x_{12}y_{21}$. Then,

$$\begin{aligned}
&a_{11}(D(x_{12}y_{21}, z_{11}) - D(x_{12}, z_{11})y_{21} - x_{12}D(y_{21}, z_{11}))\\
&= a_{11}D(x_{12}y_{21}, z_{11}) - a_{11}D(x_{12}, z_{11})y_{21} - a_{11}x_{12}D(y_{21}, z_{11})\\
&= (D(a_{11}x_{12}y_{21}, z_{11}) - D(a_{11}, z_{11})x_{12}y_{21}) - a_{11}D(x_{12}, z_{11})y_{21} - a_{11}x_{12}D(y_{21}, z_{11})\\
&= D(a_{11}x_{12}y_{21}, z_{11}) - (D(a_{11}, z_{11})x_{12} + a_{11}D(x_{12}, z_{11}))y_{21} - a_{11}x_{12}D(y_{21}, z_{11})\\
&= D(a_{11}x_{12}y_{21}, z_{11}) - D(a_{11}x_{12}, z_{11})y_{21} - a_{11}x_{12}D(y_{21}, z_{11}) \ (By\ (3.18))\\
&\in Z(\mathcal{G}_{11}).
\end{aligned}$$

Then, $\mathcal{G}_{11}(D(x_{12}y_{21}, z_{11}) - D(x_{12}, z_{11})y_{21} - x_{12}D(y_{21}, z_{11}))$ is a central ideal of $\mathcal{G}_{11}$. Because $\mathcal{G}_{11}$ does not contain a nonzero central ideal, we have

$$D(x_{12}y_{21}, z_{11}) - D(x_{12}, z_{11})y_{21} - x_{12}D(y_{21}, z_{11}) = 0,$$

and so

$$\begin{aligned}
D(y_{21}x_{12}, z_{11}) - D(y_{21}, z_{11})x_{12} - y_{21}D(x_{12}, z_{11}) &= \eta(D(x_{12}y_{21}, z_{11}) - D(x_{12}, z_{11})y_{21} - x_{12}D(y_{21}, z_{11}))\\
&= \eta(0) = 0,
\end{aligned}$$

that is,

$$D(x_{12}y_{21}, z_{11}) = D(x_{12}, z_{11})y_{21} + x_{12}D(y_{21}, z_{11}), D(y_{21}x_{12}, z_{11}) = D(y_{21}, z_{11})x_{12} + y_{21}D(x_{12}, z_{11}). \quad (3.22)$$

Because $D(x_{ij}, z_{11}) \in \mathcal{G}_{ij}$, we have that

$$D(x_{ij}y_{sk}, z_{11}) = D(x_{ij}, z_{11})y_{sk} + x_{ij}D(y_{sk}, z_{11})$$

holds for $j \neq s$. From Eqs (3.18)–(3.22) and $D(y_{22}, z_{11}) = 0$, we have

$$D(x_{ij}y_{sk}, z_{11}) = D(x_{ij}, z_{11})y_{sk} + x_{ij}D(y_{sk}, z_{11}),$$

where $i, j, s, k \in \{1, 2\}$. Thus,

$$D(xy, z_{11}) = D(x, z_{11})y + xD(y, z_{11}).$$

Similarly, we can prove $D(xy, z_{22}) = D(x, z_{22})y + xD(y, z_{22})$. Thus, Step 1 holds.
Step 2. For any $x, y, z \in \mathcal{G}$, we have

$$D(xy, z_{12}) = D(x, z_{12})y + xD(y, z_{12}), D(xy, z_{21}) = D(x, z_{21})y + xD(y, z_{21}),$$

where $x_{ij} = f_i x f_j, y_{ij} = f_i y f_j, z_{ij} = f_i z f_j \in \mathcal{G}_{ij}, i, j \in \{1, 2\}$.

From Lemma 3.3 and the definition of D, we have

$$D(x_{11}, z_{12}) = x_{11}L(f_1, y_{12}) \in \mathcal{G}_{12}, D(x_{12}, z_{12}) \in \mathcal{G}_{12},$$
$$D(x_{22}, z_{12}) = -L(f_1, y_{12})x_{22} \in \mathcal{G}_{12}, D(x_{21}, z_{12}) = [x_{21}, L(f_1, z_{12})].$$

To prove $D(xy, z_{12}) = D(x, z_{12})y + xD(y, z_{12})$, it is sufficient to show that

$$D(x_{ij}y_{sk}, z_{12}) = D(x_{ij}, z_{12})y_{sk} + x_{ij}D(y_{sk}, z_{12}),$$

where $i, j, s, k \in \{1, 2\}$.

Because $D(x_{11}, z_{12}), D(x_{12}, z_{12}), D(x_{22}, z_{12}) \in \mathcal{G}_{12}$, we have

$$D(x_{12}y_{11}, z_{12}) = 0 = D(x_{12}, z_{12})y_{11} + x_{12}D(y_{11}, z_{12}), \tag{3.23}$$
$$D(x_{12}y_{12}, z_{12}) = 0 = D(x_{12}, z_{12})y_{12} + x_{12}D(y_{12}, z_{12}), \tag{3.24}$$
$$D(x_{22}y_{12}, z_{12}) = 0 = D(x_{22}, z_{12})y_{12} + x_{22}D(y_{12}, z_{12}), \tag{3.25}$$
$$D(x_{22}y_{11}, z_{12}) = 0 = D(x_{22}, z_{12})y_{11} + x_{22}D(y_{11}, z_{12}). \tag{3.26}$$

Because $D(x_{11}, z_{12}) = x_{11}L(f_1, z_{12}) \in \mathcal{G}_{12}$, we have $D(x_{11}, z_{12})y_{11} = 0$, and so

$$D(x_{11}y_{11}, z_{12}) = x_{11}y_{11}L(f_1, z_{12}) = x_{11}D(y_{11}, z_{12}) = D(x_{11}, z_{12})y_{11} + x_{11}D(y_{11}, z_{12}). \tag{3.27}$$

Similarly, because $D(x_{22}, z_{12}) = -L(f_1, z_{12})x_{22}$ and $x_{22}D(y_{22}, z_{12}) = 0$, we have

$$D(x_{22}y_{22}, z_{12}) = -L(f_1, z_{12})x_{22}y_{22} = D(x_{22}, z_{12})y_{22} = D(x_{22}, z_{12})y_{22} + x_{22}D(y_{22}, z_{12}). \tag{3.28}$$

Because $D(x_{11}, z_{12})y_{22} + x_{11}D(y_{22}, z_{12}) = x_{11}L(f_1, z_{12})y_{22} - x_{11}L(f_1, z_{12})y_{22} = 0$, we have

$$D(x_{11}y_{22}, z_{12}) = 0 = D(x_{11}, z_{12})y_{22} + x_{11}D(y_{22}, z_{12}). \tag{3.29}$$

From $D(x_{11}, z_{12}) = x_{11}L(f_1, z_{12}) \in \mathcal{G}_{12}$ and $D(y_{21}, z_{12}) = [y_{21}, L(f_1, z_{12})]$, we have

$$D(x_{11}, z_{12})y_{21} + x_{11}D(y_{21}, z_{12}) = x_{11}L(f_1, y_{12})y_{21} + x_{11}[y_{21}, L(f_1, z_{12})] = 0,$$

and so

$$D(x_{11}y_{21}, z_{12}) = 0 = D(x_{11}, z_{12})y_{21} + x_{11}D(y_{21}, z_{12}). \tag{3.30}$$

Similarly, because

$$D(x_{21}, z_{12})y_{22} + x_{21}D(y_{22}, z_{12}) = [x_{21}, L(f_1, z_{12})]y_{22} - x_{21}L(f_1, z_{12})y_{22} = 0,$$

we have

$$D(x_{21}y_{22}, z_{12}) = 0 = D(x_{21}, z_{12})y_{22} + x_{21}D(y_{22}, z_{12}). \tag{3.31}$$

Because $D(x_{21}, z_{12})y_{21} + x_{21}D(y_{21}, z_{12}) = [x_{21}, L(f_1, z_{12})]y_{21} + x_{21}[y_{21}, L(f_1, z_{12})] = 0$, we have

$$D(x_{21}y_{21}, z_{12}) = 0 = D(x_{21}, z_{12})y_{21} + x_{21}D(y_{21}, z_{12}). \tag{3.32}$$

Because $D(x_{22}, z_{12}) = -L(f_1, z_{12})x_{22}$ and $D(y_{21}, z_{12}) = [y_{21}, L(f_1, z_{12})]$, we have

$$\begin{aligned} D(x_{22}y_{21}, z_{12}) &= [x_{22}y_{21}, L(f_1, z_{12})] = x_{22}y_{21}L(f_1, z_{12}) - L(f_1, z_{12})x_{22}y_{21} \\ &= -L(f_1, z_{12})x_{22}y_{21} + x_{22}[y_{21}, L(f_1, z_{12})] \\ &= D(x_{22}, z_{12})y_{21} + x_{22}D(y_{21}, z_{12}). \end{aligned} \tag{3.33}$$

Similarly, because $D(y_{11}, z_{12}) = y_{11}L(f_1, z_{12})$, we have

$$\begin{aligned} D(x_{21}y_{11}, z_{12}) &= [x_{21}y_{11}, L(f_1, z_{12})] = x_{21}y_{11}L(f_1, z_{12}) - L(f_1, z_{12})x_{21}y_{11} \\ &= x_{21}D(y_{11}, z_{12}) + [x_{21}, L(f_1, z_{12})]y_{11} \\ &= D(x_{21}, z_{12})y_{11} + x_{21}D(y_{11}, z_{12}). \end{aligned} \tag{3.34}$$

In the following, we will prove $D(y_{12}, z_{12})x_{21} = 0, x_{21}D(y_{12}, z_{12}) = 0$.

If $\mathcal{N} = 0$, then the conclusion clearly holds. If $\mathcal{N} \neq 0$, then $\mathcal{G}_{11}$ or $\mathcal{G}_{22}$ does not contain nonzero central ideals. Without losing generality, we assume that $\mathcal{G}_{11}$ does not contain nonzero central ideals.

Taking $u = x_{21}, v = y_{12}, w = z_{12}$ in Eq (3.17), we get

$$D([x_{21}, y_{12}], z_{12}) - [D(x_{21}, z_{12}), y_{12}] - [x_{21}, D(y_{12}, z_{12})] \in Z(\mathcal{G}),$$

Notice that $D([x_{21}, y_{12}], z_{12}) - [D(x_{21}, z_{12}), y_{12}] \in \mathcal{G}_{12}$, so we have $[x_{21}, D(y_{12}, z_{12})] \in Z(\mathcal{G})$. By Lemma 2.1 and Remark 2.1, we have

$$-D(y_{12}, z_{12})x_{21} \in Z(\mathcal{G}_{11}), x_{21}D(y_{12}, z_{12}) \in Z(\mathcal{G}_{22}),$$

and

$$x_{21}D(y_{12}, z_{12}) = \eta(-D(y_{12}, z_{12})x_{21}),$$

and so $D(y_{12}, z_{12})\mathcal{G}_{21}$ is a central ideal of $\mathcal{G}_{11}$. Because $\mathcal{G}_{11}$ does not contain a nonzero central ideal, we have $D(y_{12}, z_{12})\mathcal{G}_{21} = \{0\}$. This implies

$$D(y_{12}, z_{12})x_{21} = 0,$$

and

$$x_{21}D(y_{12}, z_{12}) = \eta(-D(y_{12}, z_{12})x_{21}) = \eta(0) = 0.$$

Because $D(x_{21}y_{12}, z_{12}) = -L(f_1, z_{12})x_{21}y_{12} = [x_{21}, L(f_1, z_{12})]y_{12} = D(x_{21}, z_{12})y_{12}$, we have

$$D(x_{21}y_{12}, z_{12}) = D(x_{21}, z_{12})y_{12} + x_{21}D(y_{12}, z_{12}). \tag{3.35}$$

Because $D(x_{12}y_{21}, z_{12}) = x_{12}y_{21}L(f_1, z_{12}) = x_{12}[y_{21}, L(f_1, z_{12})] = x_{12}D(y_{21}, z_{12})$, we have

$$D(x_{12}y_{21}, z_{12}) = D(x_{12}, z_{12})y_{21} + x_{12}D(y_{21}, z_{12}). \tag{3.36}$$

Taking $u = x_{11}, v = y_{12}, w = z_{12}$ in Eq (3.17), we get

$$D([x_{11}, y_{12}], z_{12}) - [D(x_{11}, z_{12}), y_{12}] - [x_{11}, D(y_{12}, z_{12})] \in Z(\mathcal{G}).$$

Notice that $D(x_{11}, z_{12}), D(y_{12}, z_{12}) \in \mathcal{G}_{12}$; the above expression simplifies to

$$D(x_{11}y_{12}, z_{12}) - x_{11}D(y_{12}, z_{12}) \in Z(\mathcal{G}).$$

Because $D(x_{11}y_{12}, z_{12}) - x_{11}D(y_{12}, z_{12}) \in \mathcal{G}_{12}$, we have

$$D(x_{11}y_{12}, z_{12}) = x_{11}D(y_{12}, z_{12}),$$

and so

$$D(x_{11}y_{12}, z_{12}) = D(x_{11}, z_{12})y_{12} + x_{11}D(y_{12}, z_{12}). \tag{3.37}$$

By a similar process, it can be shown that

$$D(x_{12}y_{22}, z_{12}) = D(x_{12}, z_{12})y_{22} + x_{12}D(y_{22}, z_{12}). \tag{3.38}$$

From Eqs (3.23)–(3.38), it immediately follows that

$$D(xy, z_{12}) = D(x, z_{12})y + xD(y, z_{12}).$$

By a similar process, it can be shown that $D(xy, z_{21}) = D(x, z_{21})y + xD(y, z_{21})$.

By Steps 1 and 2, D is a derivation with respect to the first component.

In an analogous manner, we can prove that D is a derivation with respect to the second component. Hence, D is a biderivation on $\mathcal{G}$.

Second, we prove that τ is a bilinear map vanishing at commutators.

For any $x, y, z \in \mathcal{G}$, we have

$$\begin{aligned}\tau([x, y], z) &= L([x, y], z) - D([x, y], z)\\ &= [L(x, z), y] + [x, L(y, z)] - D([x, y], z)\\ &= [D(x, z), y] + [x, D(y, z)] - D([x, y], z)\\ &= 0.\end{aligned}$$

Similarly, we can prove $\tau(x, [y, z]) = 0$. Thus, τ is a bilinear map vanishing at commutators. □

Let $\mathcal{A}$ be an algebra with unity and $M_n(\mathcal{A})$ the algebra of all $n \times n$ matrices over $\mathcal{A}$. Then, $M_n(\mathcal{A})(n \geq 2)$ can be represented as a generalized matrix algebra of the form

$$\begin{pmatrix} \mathcal{A} & M_{1\times(n-1)}(\mathcal{A}) \\ M_{(n-1)\times 1}(\mathcal{A}) & M_{(n-1)\times(n-1)}(\mathcal{A}) \end{pmatrix}.$$

It is easy to verify that $Z(M_n(\mathcal{A})) = Z(\mathcal{A}) \cdot I_n$ and $M_n(\mathcal{A})$ does not contain any nonzero central ideal $(n \geq 2)$. In view of Theorem 5.1 in [13], every biderivation of $M_n(\mathcal{A})$ is inner. By Theorem 3.1, we have the following.

Corollary 3.4. *Let $\mathcal{A}$ be an algebra with unity and $M_n(\mathcal{A})(n \geq 3)$ the algebra of all $n \times n$ matrices over $\mathcal{A}$. Then, each Lie biderivation L on $M_n(\mathcal{A})$ is of the form*

$$L(X, Y) = \lambda[X, Y] + \tau(X, Y)$$

for all $X, Y \in M_n(\mathcal{A})$, where $\lambda \in Z(\mathcal{A})$ and $\tau : M_n(\mathcal{A}) \times M_n(\mathcal{A}) \to Z(\mathcal{A}) \cdot I_n$ is a bilinear map vanishing at commutators.

4. Conclusions

In this paper, we have systematically investigated the structural properties of Lie biderivations on generalized matrix algebra $\mathcal{G}$. Leveraging the faithful bimodule structure of $\mathcal{G}$, we proved that under suitable mild conditions—specifically, when $\pi_{\mathcal{A}}(Z(\mathcal{G})) = Z(\mathcal{A})$ and $\pi_{\mathcal{B}}(Z(\mathcal{G})) = Z(\mathcal{B})$, and either $\mathcal{A}$ or $\mathcal{B}$ contains no nonzero central ideals when the bimodule is nontrivial—every Lie biderivation $L : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ can be decomposed into the sum of a biderivation and a central bilinear mapping vanishing at all commutators. As a direct and significant application, we extend this characterization to full matrix algebras $M_n(\mathcal{A})$ ($n \geq 3$).

Author contributions

Jinhong Zhuang: Methodology, formal analysis, investigation, writing original draft; Yanpin Chen: Conceptualization, methodology, validation, funding acquisition; Yijia Tan: Methodology, validation, writing review and editing, supervision. All authors contributed equally to this work.

Use of Generative-AI tools declaration

The authors declare that no generative AI tools were used in the development of this manuscript.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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