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Research article

On characterization of curves with Tzitzeica type indicatrices in E^3 and E_1^3

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Abstract: This study investigated the geometric conditions under which the spherical indicatrices of a regular curve were classified as Tzitzeica curves within the frameworks of both Euclidean 3-space E^3 and Minkowski 3-space E_1^3. A Tzitzeica curve is defined by the invariant property that the ratio of its torsion to the square of the distance from the origin to its osculating plane remains constant. By utilizing the Frenet-Serret frame, we derived explicit differential characterizations involving the curvature κ and torsion τ for the tangent, principal normal, and binormal indicatrices. Our analysis demonstrates that for general helices in E^3, these indicatrices reduced to planar Tzitzeica curves. Furthermore, the study extended these findings to timelike curves in E_1^3, where the Lorentzian metric introduces distinct differential behaviors. Finally, we highlighted the analytical complexity of the Tzitzeica condition for spacelike curves in Minkowski space, identifying it as a compelling direction for future research in Lorentzian differential geometry.

Keywords: Tzitzeica; curve; timelike; Euclidean; Minkowski; indicatrix; frame
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1. Introduction

The theory of curves and surfaces constitutes one of the most fundamental and enduring branches of differential geometry. Historically, the systematic study of curves gained momentum in the 18th and 19th centuries with the pioneering works of Leonard Euler and Gaspard Monge. However, the field reached its modern maturity through the formulation of the Frenet-Serret frame, independently discovered by Jean Frédéric Frenet and Joseph Alfred Serret. As detailed in the classical treatises of [9, 12], this moving frame provided a robust local coordinate system that allowed mathematicians to characterize curves through their intrinsic invariants: curvature and torsion. These invariants serve as the primary tools for distinguishing the geometric nature of paths in Euclidean space.

As the classical theory of Euclidean curves was established, mathematicians began to explore curves under various transformation groups and in non-Euclidean settings. In the early 20th century, the Romanian mathematician Gheorghe Tzitzeica made a profound contribution to affine differential geometry. In his seminal research [13], while investigating surfaces whose curvature properties remain invariant under centro-affine transformations, Tzitzeica introduced a special class of curves and surfaces that now bear his name. According to the foundational study by [4], a Tzitzeica curve is defined by the property that the ratio of its torsion to the square of the distance from the origin to its osculating plane remains constant. This property links the local differential invariants of the curve with its global position relative to a fixed point.

The versatility of Tzitzeica curves has led to their exploration in various mathematical and physical contexts. For instance, [6] demonstrated that cylindrical Tzitzeica curves can be used to simplify forced harmonic oscillators. Further extensions of the theory include the study of Tzitzeica hypersurfaces in the context of cubic Finslerian metrics [5]. In the Euclidean 3-space E^3, the characterization of these curves and their corresponding surfaces has been extensively investigated by [2, 3]. Moreover, recent advancements have explored these curves using alternative moving frames, such as the Bishop frame characterizations provided by [7].

The symbiotic relationship between differential geometry and mathematical physics is of paramount importance, particularly in the geometric characterization of spacetime manifolds. Tzitzeica curves, defined by their unique centro-affine invariant properties, are intrinsically linked to integrable systems of differential equations and conservation laws within surface theory. This study contributes to the literature by analyzing the symmetry and geometric distinctions between Euclidean 3-space E^3 and Minkowski spacetime E_1^3 through the lens of spherical indicatrices. In this context, the characterization of Tzitzeica-type timelike and spacelike curves in Minkowski space is not merely an abstract geometric problem; it provides a vital mathematical framework for understanding the intrinsic geometry of particle trajectories and world-lines in general relativity and field theories. Furthermore, determining the conditions under which general helices and their indicatrices satisfy the Tzitzeica condition offers new perspectives in modeling the geometric constraints of physical systems, such as charged particles moving in uniform electromagnetic fields.

In parallel with these developments, the emergence of Einstein's theory of relativity necessitated the study of geometry in pseudo-Riemannian manifolds. Minkowski 3-space E_1^3, as the simplest non-Euclidean setting with an indefinite metric, became a vital area of research, a topic covered by [10, 11]. The classification of curves in Minkowski space is significantly more complex than in Euclidean space due to the causal character of vectors (timelike, spacelike, and lightlike). This distinction leads to several types of Frenet frames, as demonstrated in the seminal work of [14]. Specific types, such as elliptic cylindrical Tzitzeica curves [8] and non-null curves of Tzitzeica type [1], have further enriched the literature on Minkowski geometry.

A significant tool in analyzing the global and local behavior of a space curve is the study of its spherical indicatrices. Spherical indicatrices (tangent, principal normal, and binormal) are curves traced on the surface of a unit sphere by the respective Frenet vectors [12]. The geometry of these indicatrices provides deep insights into the properties of the generating curve.

The aim of this paper is to provide a detailed investigation into the conditions for the spherical indicatrices of a regular curve to be Tzitzeica curves. We first derive these conditions in E^3 for all three indicatrices using the framework established by [9, 12]. Subsequently, we extend our analysis to E_1^3 by focusing on timelike curves, following the algebraic conventions defined by [10, 11, 14].

2. Basic notions

In this section, we recall fundamental concepts of curve theory in both Euclidean and Minkowski 3-spaces.

2.1. *The Euclidean 3-space*

The Euclidean 3-space E^3 is provided with the standard metric as defined in [9]:

$$\langle \cdot, \cdot \rangle = dx_1^2 + dx_2^2 + dx_3^2.$$

For a regular curve $\alpha(s)$ in E^3, the Frenet formulae are given by [12]:

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}.$$

2.2. *The Minkowski 3-space*

The Minkowski 3-space E_1^3 is equipped with the standard flat metric [11]:

$$\langle \cdot, \cdot \rangle = -dx_1^2 + dx_2^2 + dx_3^2.$$

In E_1^3, a vector is categorized as timelike, spacelike, or null based on its causal character [10]. The Frenet formulae for a curve $\alpha(s)$ in E_1^3 vary accordingly [14]. For timelike curves, the frame equations are:

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}.$$

2.3. *Tzitzeica curves*

Definition 2.1. *[4, 13] Let α be a regular curve in E^3 or E_1^3. The Tzitzeica function of α is defined as $I_\alpha = \tau/d_{osc}^2$, where $d_{osc} = \langle \alpha, B \rangle$. If I_α is constant, then α is a Tzitzeica curve.*

Theorem 2.1. *If α is a Tzitzeica curve, then the following equality holds:*

$$\tau' \langle \alpha, B \rangle + 2\tau^2 \langle \alpha, N \rangle = 0. \tag{2.1}$$

Furthermore, as established in and supported by surface studies, a curve with constant curvature and constant torsion cannot be a Tzitzeica curve.

3. Main results

In this section, we investigate the conditions for spherical indicatrices of curves to be Tzitzeica curves. We present the results first for the Euclidean 3-space E^3, followed by the Minkowski 3-space E_1^3.

3.1. Spherical indicatrices of a curve in the Euclidean 3-space E^3

In this subsection, we consider a unit speed regular curve α with Frenet elements $\{T, N, B, \kappa, \tau\}$ and its spherical indicatrices.

Theorem 3.1. *Let α be a unit speed regular curve with the Frenet elements $\{T, N, B, \kappa, \tau\}$ and α_T be the tangent indicatrix of α with the Frenet elements $\{T_T, N_T, B_T, \kappa_T, \tau_T\}$ in the E^3. If α_T is a Tzitzeica curve, then*

$$\frac{\tau_T}{d^2_{T_{osc}}} = \frac{\kappa\tau' - \kappa'\tau}{\kappa\tau^2} = const, \tag{3.1}$$

where $d_{T_{osc}}$ is the distance between origin and the osculating plane of α_T, and curvatures of curve α are nonzero.

Proof. Let α be a unit speed regular curve in the E^3. Considering s_T as the arc-length parameter of α_T, first derivative of α_T is given with chain rule as

$$\frac{d\alpha_T}{ds_T} = \frac{d\alpha_T}{ds}\frac{ds}{ds_T}. \tag{3.2}$$

Since $\alpha_T = T$, then from (3.2) we get

$$T_T = \frac{dT}{ds}\frac{ds}{ds_T} = \kappa N \frac{ds}{ds_T}. \tag{3.3}$$

Taking norm of the Eq (3.3) on both sides results in

$$\frac{ds}{ds_T} = \frac{1}{|\kappa|}. \tag{3.4}$$

Hence, by using (3.4), the Eq (3.3) becomes

$$T_T = \mp N. \tag{3.5}$$

From the derivative of (3.5), we get

$$\frac{dT_T}{ds_T} = \frac{dN}{ds_T} = \frac{dN}{ds}\frac{ds}{ds_T} = \pm T \mp \frac{\tau}{\kappa}B. \tag{3.6}$$

Taking norm of (3.6) on both sides yields

$$\left\|\frac{dT_T}{ds_T}\right\| = \sqrt{1 + \frac{\tau^2}{\kappa^2}} = \frac{\sqrt{\kappa^2 + \tau^2}}{|\kappa|}. \tag{3.7}$$

Then, from (3.7) the curvature of α_T is obtained as

$$\kappa_T = \left\|\frac{dT_T}{ds_T}\right\| = \frac{\sqrt{\kappa^2 + \tau^2}}{|\kappa|}, (\kappa \neq 0). \tag{3.8}$$

Considering the Frenet formulae, we obtain

$$N_T = \frac{\frac{dT_T}{ds_T}}{\left\|\frac{dT_T}{ds_T}\right\|} = \pm\frac{\kappa}{\sqrt{\kappa^2 + \tau^2}}T \mp \frac{\tau}{\sqrt{\kappa^2 + \tau^2}}B. \tag{3.9}$$

Since $B_T = T_T \times N_T$, using Eqs (3.5) and (3.9), it follows

$$B_T = \frac{-\tau}{\sqrt{\kappa^2+\tau^2}}T + \frac{\kappa}{\sqrt{\kappa^2+\tau^2}}B. \tag{3.10}$$

From the first derivative of (3.10), we get

$$\frac{dB_T}{ds_T} = \mp\frac{-\kappa(\kappa\tau' - \tau\kappa')}{\kappa(\kappa^2+\tau^2)^{3/2}}T \mp \frac{\tau(\kappa'\tau - \kappa\tau')}{\kappa(\kappa^2+\tau^2)^{3/2}}B. \tag{3.11}$$

Thus, the torsion of α_T is obtained as

$$\tau_T = -\left\langle \frac{dB_T}{ds_T}, N_T \right\rangle = \frac{\kappa}{\kappa^2+\tau^2}\left(\frac{\tau}{\kappa}\right)'. \tag{3.12}$$

Let α_T be a Tzitzeica curve. The distance from origin to the osculating plane of α_T is

$$d_{T_{osc}} = \langle B_T, T\rangle = \left\langle \frac{-\tau}{\sqrt{\kappa^2+\tau^2}}T + \frac{\kappa}{\sqrt{\kappa^2+\tau^2}}B, T \right\rangle = \frac{-\tau}{\sqrt{\kappa^2+\tau^2}}$$

and, hence,

$$d^2_{T_{osc}} = \frac{\tau^2}{\kappa^2+\tau^2}. \tag{3.13}$$

Therefore, using Eqs (3.12), (3.13) and considering definition (2.1), we obtain

$$\frac{\tau_T}{d^2_{T_{osc}}} = \frac{\kappa\tau' - \kappa'\tau}{\kappa\tau^2} = \text{constant}$$

which ends the proof. □

Corollary 3.2. *Let α be a unit-speed regular curve in the E^3 and the tangent indicatrix α_T of α be a Tzitzeica curve. If α is a general helix, then, α_T is a planar Tzitzeica curve.*

Corollary 3.3. *Let α be a unit speed regular curve in the E^3 and α_T be its tangent indicatrix with Frenet elements $\{T_T, N_T, B_T, \kappa_T, \tau_T\}$. If α_T is a Tzitzeica curve, then the relation between the Frenet vectors of α_T and Frenet vectors of α is given by*

$$\begin{bmatrix} \frac{dT_T}{ds_T} \\ \frac{dN_T}{ds_T} \\ \frac{dB_T}{ds_T} \end{bmatrix} = \begin{bmatrix} -1 & 0 & \frac{\tau}{\kappa} \\ \frac{\tau(\kappa\tau' - \kappa'\tau)}{\kappa(\kappa^2+\tau^2)^{3/2}} & -\frac{\sqrt{\kappa^2+\tau^2}}{\kappa} & \frac{\kappa\tau' - \kappa'\tau}{(\kappa^2+\tau^2)^{3/2}} \\ \frac{\kappa\tau' - \kappa'\tau}{(\kappa^2+\tau^2)^{3/2}} & 0 & \frac{\tau(\kappa\tau' - \kappa'\tau)}{\kappa(\kappa^2+\tau^2)^{3/2}} \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}.$$

Proof. Since α_T is a Tzitzeica curve, then from (3.1) we have

$$\kappa\tau' - \kappa'\tau = \lambda\kappa\tau^2. \tag{3.14}$$

Differentiating the Eq (3.9) with respect to s_T yields

$$\frac{dN_T}{ds_T} = \frac{\tau(\kappa\tau' - \kappa'\tau)}{\kappa(\kappa^2+\tau^2)^{3/2}}T - \frac{\sqrt{\kappa^2+\tau^2}}{\kappa}N + \frac{\kappa\tau' - \kappa'\tau}{(\kappa^2+\tau^2)^{3/2}}B. \tag{3.15}$$

Then, considering the Eqs (3.6), (3.11), (3.14) and (3.15), the desired result is obtained. □

Corollary 3.4. *Let α be a unit speed regular curve with the Frenet elements $\{T, N, B, \kappa, \tau\}$ and α_T be the tangent indicatrix of α with the Frenet elements $\{T_T, N_T, B_T, \kappa_T, \tau_T\}$ in the E^3. If α_T is a Tzitzeica curve, then it is obtained that*

$$\frac{\kappa}{\tau} = \int c\kappa ds + c_1 \tag{3.16}$$

where c is a constant.

Corollary 3.5. *Let α_T be the tangent indicatrix of α with the Frenet elements $\{T_T, N_T, B_T, \kappa_T, \tau_T\}$ in the E^3 and let α_T be a Tzitzeica curve. If c is zero, then α is a helix.*

Theorem 3.6. *Let α be a regular, unit-speed curve with Frenet elements $\{T, N, B, \kappa, \tau\}$ and α_N be the principal normal indicatrix of α with Frenet elements $\{T_N, N_N, B_N, \kappa_N, \tau_N\}$ in the E^3. If α_N is a Tzitzeica curve, then*

$$\frac{\tau_N}{d^2_{N_{osc}}} = \frac{-3\left(\kappa\kappa' + \tau\tau'\right)\left(\kappa\tau' - \kappa'\tau\right) - \left(\kappa^2 + \tau^2\right)\left(\kappa''\tau - \kappa\tau''\right)}{\left(\kappa\tau' - \kappa'\tau\right)^2} = const, \tag{3.17}$$

where $d_{N_{osc}} = \langle B_N, N\rangle$ is the distance between origin and the osculating plane of α_N.

Proof. Let α be a regular, unit-speed curve and its principal normal indicatrix α_N be a Tzitzeica curve. Then, if we take consider of s_N as the arc-length parameter of α_N, from the first derivative of α_N, we get

$$T_N = (-\kappa T + \tau B)\frac{ds}{ds_N}. \tag{3.18}$$

Since $\|T_N\| = 1$, Eq (3.18) yields

$$\frac{ds}{ds_N} = \frac{1}{\sqrt{\kappa^2 + \tau^2}}. \tag{3.19}$$

From Eq (3.19) and the derivative of (3.18), we obtain

$$\frac{dT_N}{dS_N} = \frac{\tau(\kappa\tau' - \kappa'\tau)}{(\kappa^2 + \tau^2)^2}T - N + \frac{\kappa(\kappa\tau' - \kappa'\tau)}{(\kappa^2 + \tau^2)^2}B. \tag{3.20}$$

From (3.20), we have

$$\kappa_N = \left\|\frac{dT_N}{dS_N}\right\| = \sqrt{\frac{(\kappa\tau' - \kappa'\tau)^2 + (\kappa^2 + \tau^2)^3}{(\kappa^2 + \tau^2)^3}} \tag{3.21}$$

and, hence, the principal normal vector of α_N is obtained as

$$N_N = \frac{1}{\sqrt{\kappa^2 + \tau^2}\sqrt{(\kappa\tau' - \kappa'\tau)^2 + (\kappa^2 + \tau^2)^3}}\left(\tau(\kappa\tau' - \kappa'\tau)T + (\kappa^2 + \tau^2)^2 N + \kappa(\kappa\tau' - \kappa'\tau)B\right). \tag{3.22}$$

Finally, since $B_N = T_N \times N_N$, it follows that

$$B_N = \frac{1}{\sqrt{(\kappa\tau' - \kappa'\tau)^2 + (\kappa^2 + \tau^2)^3}}\left(-\tau(\kappa^2 + \tau^2)T + (\kappa\tau' - \kappa'\tau)N - \kappa(\kappa^2 + \tau^2)B\right). \tag{3.23}$$

By differentiating Eq (3.23) according to parameter s_N, we have

$$\frac{dB_N}{ds_N} = \left[\frac{\left[-\tau'\left(\kappa^2+\tau^2\right) - 2\tau\left(\kappa\kappa'+\tau\tau'\right)\right] - \kappa^2\tau' + \kappa\kappa'\tau}{\sqrt{\kappa^2+\tau^2}\left[(\kappa\tau'-\kappa'\tau)^2 + (\kappa^2+\tau^2)^3\right]} + \tau\sqrt{\kappa^2+\tau^2}\frac{\left[(\kappa\tau'-\kappa'\tau)^2 + \left(\kappa^2+\tau^2\right)^3\right]'}{2\left((\kappa\tau'-\kappa'\tau)^2 + (\kappa^2+\tau^2)^3\right)^{3/2}} \right] T$$
$$+ \left(\frac{\kappa\tau'' - \kappa''\tau}{\sqrt{\kappa^2+\tau^2}\sqrt{(\kappa\tau'-\kappa'\tau)^2 + (\kappa^2+\tau^2)^3}} - \frac{(\kappa\tau'-\kappa'\tau)\left[(\kappa\tau'-\kappa'\tau)^2 + \left(\kappa^2+\tau^2\right)^3\right]'}{\sqrt{\kappa^2+\tau^2}2\left((\kappa\tau'-\kappa'\tau)^2 + (\kappa^2+\tau^2)^3\right)^{3/2}} \right) N$$
$$+ \left(\frac{\left[-\kappa'\left(\kappa^2+\tau^2\right) - 2\kappa\left(\kappa\kappa'+\tau\tau'\right)\right] - \kappa'\tau^2 + \kappa\tau\tau'}{\sqrt{\kappa^2+\tau^2}\sqrt{(\kappa\tau'-\kappa'\tau)^2 + (\kappa^2+\tau^2)^3}} + \frac{\kappa(\kappa^2+\tau^2)\left[(\kappa\tau'-\kappa'\tau)^2 + \left(\kappa^2+\tau^2\right)^3\right]'}{\sqrt{\kappa^2+\tau^2}2\left((\kappa\tau'-\kappa'\tau)^2 + (\kappa^2+\tau^2)^3\right)^{3/2}} \right) B.$$

Then, by the equality $\tau_N = -\left\langle \frac{dB_N}{ds_N}, N_N \right\rangle$, the torsion and the distance between origin and the osculating plane of α_N are calculated as

$$\tau_N = \frac{3\left(\kappa\kappa'+\tau\tau'\right)\left(\kappa\tau'-\kappa'\tau\right) - \left(\kappa^2+\tau^2\right)\left(\kappa''\tau - \kappa\tau''\right)}{(\kappa\tau'-\kappa'\tau)^2 + (\kappa^2+\tau^2)^3} \tag{3.24}$$

and

$$d_{N_{osc}} = \frac{\kappa\tau' - \kappa'\tau}{\sqrt{(\kappa\tau'-\kappa'\tau)^2 + (\kappa^2+\tau^2)^3}}, \tag{3.25}$$

respectively. Therefore, from Definition 2.1, we obtain

$$\frac{\tau_N}{d^2_{N_{osc}}} = \frac{3\left(\kappa\kappa'+\tau\tau'\right)\left(\kappa\tau'-\kappa'\tau\right) - \left(\kappa^2+\tau^2\right)\left(\kappa''\tau - \kappa\tau''\right)}{(\kappa\tau'-\kappa'\tau)^2} = const,$$

which ends the proof. □

Theorem 3.7. *Let α be a regular, unit-speed curve with Frenet elements $\{T, N, B, \kappa, \tau\}$ and α_B be the binormal indicatrix of α with Frenet elements $\{T_B, N_B, B_B, \kappa_B, \tau_B\}$ in the E^3. If α_B is a Tzitzeica curve, then*

$$\frac{\tau_B}{d^2_{B_{osc}}} = \left(\frac{\kappa}{\tau}\right)' \frac{\tau}{\kappa^2} = const,$$

where $d_{B_{osc}} = \langle B_B, B\rangle$ is the distance between origin and the osculating plane of α_B.

Proof. Assume that α is a unit speed regular curve and its binormal indicatrix α_B is a Tzitzeica curve. If we set s_B as the arc-length parameter of α_B, the first derivative of α_B yields

$$T_B = -\tau N \frac{ds}{ds_B}, \tag{3.26}$$

where we get $\frac{ds}{ds_B} = \frac{1}{\tau}$. Then, from the derivative of (3.26), we get

$$\frac{dT_B}{ds_B} = \frac{\kappa}{\tau}T - B$$

and, thus, it follows that

$$\kappa_B = \frac{\sqrt{\kappa^2+\tau^2}}{\tau}.$$

Therefore, the principle normal vector of α_B is obtained as

$$N_B = \frac{1}{\sqrt{\kappa^2+\tau^2}}(\kappa T - \tau B). \tag{3.27}$$

Using the Eqs (3.26) and (3.27) yields

$$B_B = \frac{1}{\sqrt{\kappa^2+\tau^2}}(\tau T + \kappa B).$$

Hence, we get

$$\tau_B = \frac{\kappa'\tau - \kappa\tau'}{\tau(\kappa^2+\tau^2)} \tag{3.28}$$

and

$$d^2_{B_{osc}} = \frac{\kappa^2}{\kappa^2+\tau^2}. \tag{3.29}$$

Finally, from Definition 2.1, (3.28) and (3.29), it follows that

$$\frac{\tau_B}{d^2_{B_{osc}}} = \frac{\tau_B}{d^2_{B_{osc}}} = \left(\frac{\kappa}{\tau}\right)' \frac{\tau}{\kappa^2} = const.$$

□

Corollary 3.8. *Let α_B be the binormal indicatrix of α with Frenet elements $\{T_B, N_B, B_B, \kappa_B, \tau_B\}$ in the E^3 and let α_B be a Tzitzeica curve. Then,*

$$\frac{\kappa}{\tau} = \int \frac{c\kappa^2}{\tau} ds. \tag{3.30}$$

Corollary 3.9. *Let α be a regular, unit-speed curve with Frenet elements $\{T, N, B, \kappa, \tau\}$ and α_B be the binormal indicatrix of α with Frenet elements $\{T_B, N_B, B_B, \kappa_B, \tau_B\}$ in the E^3. If α_B is a Tzitzeica curve, then, the relation between the Frenet vectors of α and α_B are obtained as follows:*

$$\begin{bmatrix} \frac{dT_B}{ds_B} \\ \frac{dN_B}{ds_B} \\ \frac{dB_B}{ds_B} \end{bmatrix} = \begin{bmatrix} \frac{\kappa}{\tau} & 0 & -1 \\ \frac{\kappa'\tau - \kappa\tau'}{(\kappa^2+\tau^2)^{3/2}} & \frac{\sqrt{\kappa^2+\tau^2}}{\tau} & \frac{\kappa(\kappa'\tau-\kappa\tau')}{\tau(\kappa^2+\tau^2)^{3/2}} \\ -\frac{\kappa(\kappa'\tau-\kappa\tau')}{\tau(\kappa^2+\tau^2)^{3/2}} & 0 & \frac{\kappa'\tau-\kappa\tau'}{(\kappa^2+\tau^2)^{3/2}} \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}.$$

Corollary 3.10. *Let α be a regular, unit-speed curve in the E^3 and the binormal indicatrix α_B of α be a Tzitzeica curve. If α is a general helix, then α_B is a planar Tzitzeica curve.*

3.2. Spherical indicatrices of a timelike curve in the Minkowski 3-space E_1^3

In this section, relationships between invariants for conditions to be a Tzitzeica curve of spherical indicatrix curves of a timelike curve are given. For timelike curves, it has two different conditions.

3.2.1. The condition where tangent indicatrix of a timelike curve is Tzitzeica curve

Theorem 3.11. *Let $\alpha(s)$ be a unit speed regular timelike curves in Minkowski 3-space. If tangent indicatrix α_T of the timelike curve $\alpha(s)$ is a Tzitzeica curve, then following equation is obtained:*

$$\frac{\tau\kappa' - \kappa\tau'}{\kappa(\tau^2 - \kappa^2)} = const. \tag{3.31}$$

Proof. Let $\alpha(s)$ is a unit speed regular timelike curves in Minkowski 3-space and α_T is tangent indicatrix of the timelike curve $\alpha(s)$. Then, $\alpha_T(s) = T(s)$ is provided. By taking derivative $\alpha_T(s)$, we have

$$\frac{d\alpha_T}{ds_T} = \frac{dT}{ds}\frac{ds}{ds_T}$$

$$T_T = \kappa N \frac{ds}{ds_T}. \tag{3.32}$$

Using the Eq (3.32), we obtain $\frac{ds}{ds_T} = \frac{1}{\kappa}$ and

$$T_T = N. \tag{3.33}$$

By taking derivative of (3.33) according to parameter s_T, we get

$$\frac{dT_T}{ds_T} = T + \frac{\tau}{\kappa}B$$

and $\left\|\frac{dT_T}{ds_T}\right\| = \frac{\sqrt{\tau^2-\kappa^2}}{\kappa}$. Also we obtain that

$$N_T = \frac{\kappa}{\sqrt{\tau^2 - \kappa^2}}T + \frac{\tau}{\sqrt{\tau^2 - \kappa^2}}B. \tag{3.34}$$

Thus, we have that

$$B_T = \frac{\tau}{\sqrt{\tau^2 - \kappa^2}}T + \frac{\kappa}{\sqrt{\tau^2 - \kappa^2}}B. \tag{3.35}$$

By taking derivative of (3.34), we get

$$\frac{dN_T}{ds_T} = \frac{\kappa'\sqrt{\tau^2 - \kappa^2} - \kappa\frac{\tau\tau' - \kappa\kappa'}{\sqrt{\tau^2-\kappa^2}}}{\kappa(\tau^2 - \kappa^2)}T + \frac{\kappa^2 - \tau^2}{\kappa\sqrt{\tau^2 - \kappa^2}}N + \frac{\tau'\sqrt{\tau^2 - \kappa^2} - \tau\frac{\tau\tau' - \kappa\kappa'}{\sqrt{\tau^2-\kappa^2}}}{\kappa(\tau^2 - \kappa^2)}B. \tag{3.36}$$

Since torsion of curve α_T is $\tau_T = \left\langle \frac{dN_T}{ds_T}, B_T \right\rangle$, we obtain

$$\tau_T = \frac{\tau^2(\tau\kappa' - \kappa\tau')}{\kappa(\tau^2 - \kappa^2)^2}. \tag{3.37}$$

Since distance from origin to osculating plane of α_T is $d_{T_{osc}} = \langle B_T, T \rangle$, we get

$$d_{T_{osc}} = \frac{-\tau}{\sqrt{\tau^2 - \kappa^2}}. \tag{3.38}$$

Suppose that α_T is a Tzitzeica curve. Then, using the equalities (3.8) and (3.9), we obtain Tzitzeica function

$$\frac{\tau_T}{(d_{T_{osc}})^2} = \frac{\tau\kappa' - \kappa\tau'}{\kappa(\tau^2 - \kappa^2)} = const. \tag{3.39}$$

□

Corollary 3.12. *Let α be a regular, unit-speed curve in the E_1^3 and the tangent indicatrix α_T of α be a Tzitzeica curve. If α is a general helix, then α_T is a planar Tzitzeica curve in E_1^3.*

Proof. It is clear from Eq (3.39). □

3.2.2. The condition where principal normal indicatrix of a timelike curve is Tzitzeica curve

Theorem 3.13. *If the principal normal indicatrix of a timelike curve α in Minkowski 3-space is a Tzitzeica curve, then the following equality between the curvatures holds*

$$\frac{\tau_N}{(d_{N_{osc}})^2} = \frac{(\kappa''\tau - \kappa\tau'')(\tau^2 - \kappa^2) + 3(\kappa'\tau - \kappa\tau')(\tau\tau' - \kappa'\kappa)}{\kappa^2(\tau^2 - \kappa^2)^2} = const.$$

Proof. The principal normal indicatrix of such a curve is $\alpha_N = N(s)$. Therefore, if the derivative of α_N is taken:

$$\frac{d\alpha_N}{ds_N} = \frac{dN}{ds}\frac{ds}{ds_N} = (\kappa T + \tau B)\frac{ds}{ds_N},$$

$$T_N = (\kappa T + \tau B)\frac{ds}{ds_N} \tag{3.40}$$

is obtained. From the Frenet derivative formulas, if the Lorentz product of Eq (3.40) with itself is taken

$$\langle T_N, T_N\rangle = \kappa^2\langle T, T\rangle\left(\frac{ds}{ds_N}\right)^2 + \tau^2\langle B, B\rangle\left(\frac{ds}{ds_N}\right)^2.$$

From here,

$$\frac{ds}{ds_N} = \frac{1}{\sqrt{\tau^2 - \kappa^2}}$$

is found. Therefore,

$$T_N = \frac{\kappa}{\sqrt{\tau^2 - \kappa^2}}T + \frac{\tau}{\sqrt{\tau^2 - \kappa^2}}B \tag{3.41}$$

$$\frac{dT_N}{ds_N} = \left(\frac{\kappa'\sqrt{\tau^2 - \kappa^2} - \kappa\frac{(\tau\tau' - \kappa\kappa')}{\sqrt{\tau^2 - \kappa^2}}}{(\tau^2 - \kappa^2)^{3/2}}\right)T + \frac{\kappa^2}{\tau^2 - \kappa^2}N + \frac{\tau'\sqrt{\tau^2 - \kappa^2} - \tau\frac{(\tau\tau' - \kappa\kappa')}{\sqrt{\tau^2 - \kappa^2}}}{(\tau^2 - \kappa^2)^{3/2}}B - \frac{\tau^2}{\tau^2 - \kappa^2}N$$

$$= \left(\frac{-\kappa'\kappa^2 + \kappa'\tau^2 + \kappa^2\kappa' - \kappa\tau\tau'}{(\tau^2 - \kappa^2)^2}\right)T - \left(\frac{\tau^2 - \kappa^2}{\tau^2 - \kappa^2}\right)N + \left(\frac{-\tau'\kappa^2 + \tau'\tau^2 + \tau\kappa\kappa' - \tau^2\tau'}{(\tau^2 - \kappa^2)^2}\right)B$$

$$\frac{dT_N}{ds_N} = \left(\frac{\tau(\kappa'\tau - \kappa\tau')}{(\tau^2 - \kappa^2)^2}\right)T - N + \left(\frac{\kappa(\kappa'\tau - \kappa\tau')}{(\tau^2 - \kappa^2)^2}\right)B \tag{3.42}$$

is found, and its norm can be written as:

$$\left\|\frac{dT_N}{ds_N}\right\| = \sqrt{\frac{-\tau^2(\kappa'\tau - \kappa\tau')^2}{(\tau^2 - \kappa^2)^4} + 1 + \frac{\kappa^2(\kappa\tau' - \kappa'\tau)^2}{(\tau^2 - \kappa^2)^4}} = A. \tag{3.43}$$

Since $N_N = \frac{T_N'}{\|T_N'\|}$ from the Frenet derivative formulas, then from Eqs (3.42) and (3.43).

$$N_N = \frac{\frac{dT_N}{ds_N}}{\left\|\frac{dT_N}{ds_N}\right\|} = \left(\frac{\tau(\kappa'\tau - \kappa\tau')}{A(\tau^2 - \kappa^2)^2}\right)T - \frac{1}{A}N + \left(\frac{\kappa(\kappa'\tau - \kappa\tau')}{A(\tau^2 - \kappa^2)^2}\right)B \tag{3.44}$$

is found. Since $B_N = T_N \times N_N$, from Eqs (3.41) and (3.44),

$$B_N = T_N \times N_N = \frac{-\kappa}{A\sqrt{\tau^2-\kappa^2}}(T\times N) + \frac{\kappa^2(\kappa'\tau-\kappa\tau')}{A(\tau^2-\kappa^2)^{5/2}}(T\times B)$$

$$+\frac{\tau^2(\kappa'\tau-\kappa\tau')}{A(\tau^2-\kappa^2)^{5/2}}(B\times T) - \frac{\tau}{A\sqrt{\tau^2-\kappa^2}}(B\times N)$$

$$B_N = \frac{\tau}{A\sqrt{\tau^2-\kappa^2}}T - \frac{(\kappa'\tau-\kappa\tau')}{A(\tau^2-\kappa^2)^{3/2}}N + \frac{\kappa}{A\sqrt{\tau^2-\kappa^2}}B \tag{3.45}$$

is obtained. If the derivative of Eq (3.44) is taken,

$$\begin{aligned}\frac{dN_N}{ds_N} =& \frac{1}{A^3(\tau^2-\kappa^2)^{9/2}}\Big[A(\tau^2-\kappa^2)^2\left(\tau'(\kappa'\tau-\kappa\tau')+\tau(\kappa''\tau-\kappa\tau'')\right)\\ &-\tau(\kappa'\tau-\kappa\tau')\left(A'(\tau^2-\kappa^2)^2+4A(\tau^2-\kappa^2)(\tau\tau'-\kappa\kappa')\right)\\ &-A\kappa(\tau^2-\kappa^2)^4\Big]T + \frac{A'}{A^2\sqrt{\tau^2-\kappa^2}}N\\ &+\frac{1}{A^2(\tau^2-\kappa^2)^{9/2}}\Big[\left(\kappa'(\kappa'\tau-\kappa\tau')+\kappa(\kappa''\tau-\kappa\tau'')\right)A(\tau^2-\kappa^2)^2\\ &-\kappa(\kappa'\tau-\kappa\tau')\left(A'(\tau^2-\kappa^2)^2+4A(\tau^2-\kappa^2)(\tau\tau'-\kappa\kappa')\right)-A\tau(\tau^2-\kappa^2)^4\Big]B\end{aligned} \tag{3.46}$$

is obtained. Since the torsion of the curve from the Frenet derivative formulas is $\tau_N = \left\langle \frac{dN_N}{ds_N}, B_N\right\rangle$, from Eqs (3.45) and (3.46),

$$\tau_N = \frac{(\kappa''\tau-\kappa\tau'')(\tau^2-\kappa^2)+3(\kappa'\tau-\kappa\tau')(\tau\tau'-\kappa\kappa')}{A^2(\tau^2-\kappa^2)^3} \tag{3.47}$$

is obtained. The distance of the indicatrix curve of the principal normals α_N to the osculating plane is found as

$$d_{N_{osc}} = \langle B_N, B\rangle = -\frac{\kappa}{A\sqrt{\tau^2-\kappa^2}}.$$

From here,

$$(d_{N_{osc}})^2 = \frac{\kappa^2}{A^2(\tau^2-\kappa^2)} \tag{3.48}$$

is obtained. Let us assume that α_N is a Tzitzeica curve. In this case, from Eqs (3.47) and (3.48).

$$\frac{\tau_N}{(d_{N_{osc}})^2} = \frac{(\kappa''\tau-\kappa\tau'')(\tau^2-\kappa^2)+3(\kappa'\tau-\kappa\tau')(\tau\tau'-\kappa'\kappa)}{\kappa^2(\tau^2-\kappa^2)^2} = const, \tag{3.49}$$

is obtained. Thus, the proof is completed. □

Corollary 3.14. *Let α be a regular, unit-speed curve in the E_1^3 and the principal normal indicatrix α_N of α be a Tzitzeica curve. If α is a general helix, then, α_N is a planar Tzitzeica curve in E_1^3.*

Proof. From Eq (3.49) and using the following equation

$$\frac{\left(\frac{\tau}{\kappa}\right)''}{(\tau^2-\kappa^2)}=\frac{\tau''\kappa-\tau\kappa''}{\kappa^2(\tau^2-\kappa^2)}-2\frac{\kappa'(\tau'\kappa-\tau\kappa')}{\kappa^3(\tau^2-x^2)},$$

we get that

$$\left[\frac{\left(\frac{\tau}{\kappa}\right)''}{\tau^2-\kappa^2}+2\frac{\kappa'\left(\frac{\tau}{\kappa}\right)'}{\kappa(\tau^2-\kappa^2)}\right]+3\frac{\left(\frac{\tau}{\kappa}\right)'(\tau\tau'-\kappa\kappa')}{(\tau^2-\kappa^2)^2}=\text{const.}$$

From this equation, it is clear. □

3.2.3. The condition where binormal indicatrix of a timelike curve is a Tzitzeica curve

Theorem 3.15. *If the binormal indicatrix of the timelike curve α in the Minkowski 3-space is a Tzitzeica curve, then between its curvatures,*

$$\frac{\tau_B}{(d_{B_{osc}})^2}=\frac{-(\frac{\tau}{\kappa})'}{\tau}=const.$$

Proof. The spherical indicatrix of binormals of such a curve is $\alpha_B(s)=B(s)$. Thus, if the derivative of α_B is taken,

$$\frac{d\alpha_B}{ds_B}=\frac{dB}{ds}\frac{ds}{ds_B},$$

$$T_B=-\tau N\frac{ds}{ds_B} \tag{3.50}$$

is obtained. From the Frenet derivative formulas, if the Lorentz product of Eq (3.50) with itself is taken,

$$\langle T_B,T_B\rangle=\tau^2\left(\frac{ds}{ds_B}\right)^2\langle N,N\rangle$$

is obtained. Thus,

$$\frac{ds}{ds_B}=\frac{1}{\tau}$$

is found. From here,

$$T_B=N \tag{3.51}$$

is found. If the derivative of Eq (3.51) is taken,

$$\frac{dT_B}{ds_B}=\frac{d}{ds}(N)\frac{ds}{ds_B}=(\kappa T+\tau B)\frac{1}{\tau}$$

$$\frac{dT_B}{ds_B}=\frac{\kappa}{\tau}T+B \tag{3.52}$$

is found as well as its norm,

$$\left\|\frac{dT_B}{ds_B}\right\|=\sqrt{\frac{\kappa^2}{\tau^2}\langle T,T\rangle+\langle B,B\rangle}=\frac{\sqrt{\tau^2-\kappa^2}}{\tau} \tag{3.53}$$

Since $N_B = \frac{T'_B}{\|T'_B\|}$ from the Frenet derivative formulas, from Eqs (3.52) and (3.53),

$$N_B = \frac{\frac{dT_B}{ds_B}}{\left\|\frac{dT_B}{ds_B}\right\|} = \frac{\kappa}{\sqrt{\tau^2 - \kappa^2}}T + \frac{\tau}{\sqrt{\tau^2 - \kappa^2}}B \tag{3.54}$$

is found. Since $B_B = T_B \times N_B$, $-N_B = T_B \times B_B$, $-T_B = N_B \times B_B$, from Eqs (3.51) and (3.54), we obtain

$$B_B = \frac{\tau}{\sqrt{\tau^2 - \kappa^2}}T - \frac{\kappa}{\sqrt{\tau^2 - \kappa^2}}B. \tag{3.55}$$

If the derivative of Eq (3.54) is taken, simplify the expression, and

$$\frac{dN_B}{ds_B} = \frac{\kappa' \sqrt{\tau^2 - \kappa^2} - \kappa \frac{\tau\tau' - \kappa\kappa'}{\sqrt{\tau^2 - \kappa^2}}}{(\tau^2 - \kappa^2)\tau}T - \frac{\sqrt{\tau^2 - \kappa^2}}{\tau}N + \frac{\tau' \sqrt{\tau^2 - \kappa^2} - \tau \frac{\tau\tau' - \kappa\kappa'}{\sqrt{\tau^2 - \kappa^2}}}{(\tau^2 - \kappa^2)\tau}B \tag{3.56}$$

is found. Since the torsion of the curve is $\tau_B = \left\langle \frac{dN_B}{ds_B}, B_B \right\rangle$, from Eqs (3.55) and (3.56),

$$\tau_B = \left\langle \frac{dN_B}{ds_B}, B_B \right\rangle = \frac{\tau}{\sqrt{\tau^2 - \kappa^2}} \frac{\kappa' \sqrt{\tau^2 - \kappa^2} - \kappa \frac{\tau\tau' - \kappa\kappa'}{\sqrt{\tau^2 - \kappa^2}}}{(\tau^2 - \kappa^2)\tau}\langle T, T\rangle + \frac{\kappa}{\sqrt{\tau^2 - \kappa^2}} \frac{\tau' \sqrt{\tau^2 - \kappa^2} - \tau \frac{\tau\tau' - \kappa\kappa'}{\sqrt{\tau^2 - \kappa^2}}}{(\tau^2 - \kappa^2)\tau}\langle B, B\rangle$$

$$\tau_B = \frac{(\tau^2 - \kappa^2)(\kappa\tau' - \tau\kappa')}{(\tau^2 - \kappa^2)^2\tau} = \frac{\tau\kappa' - \kappa\tau'}{\tau(\tau^2 - \kappa^2)} \tag{3.57}$$

is obtained. The distance of the α_B binormal indicatrix of the curve to the osculating plane can be written as

$$d_{B_{osc}} = \langle B_B, B\rangle = \frac{\kappa}{\sqrt{\tau^2 - \kappa^2}}. \tag{3.58}$$

Let us assume that α_B is a Tzitzeica curve. In this case, from Eq (3.57) and (3.58),

$$\frac{\tau_B}{(d_{B_{osc}})^2} == \frac{\tau\kappa' - \kappa\tau'}{\tau\kappa^2} = -(\frac{\tau}{\kappa})'\frac{1}{\tau} = const, \tag{3.59}$$

is obtained. Thus, the proof is completed. □

Corollary 3.16. *Let α be a regular, unit-speed curve in the E_1^3 and the binormal indicatrix α_B of α be a Tzitzeica curve. If α is a general helix, then α_B is a planar Tzitzeica curve in E_1^3.*

Proof. It is clear from Eq (3.59) □

4. Conclusions

In this study, we have established the necessary and sufficient conditions for the spherical indicatrices of regular curves to be Tzitzeica curves. By employing the Frenet frame and its derivative formulae, we derived specific invariant relations involving the curvature κ and torsion τ of the generating curve α.

In the Euclidean 3-space E^3, we demonstrated that:

- The tangent indicatrix α_T is Tzitzeica if and only if the expression $\frac{\kappa\tau' - \kappa'\tau}{\kappa\tau^2}$ is constant.
- Similar differential characterizations were obtained for the principal normal indicatrix α_N and the binormal indicatrix α_B.
- It was observed that for a general helix, these indicatrices reduce to planar Tzitzeica curves.

In the Minkowski 3-space E_1^3, our focus was on timelike curves. The Lorentzian metric introduces significant sign changes in the Tzitzeica function, leading to distinct characterizations compared to the Euclidean case. We provided a complete analysis for the tangent and principal normal indicatrices of timelike curves, establishing the governing differential equations for their Tzitzeica properties.

However, the geometric nature of curves in Minkowski space is highly dependent on their causal character. While the timelike case has been thoroughly investigated here, the problem remains open for spacelike curves. Due to the variability of the principal normal and binormal vectors (being spacelike, timelike, or null), the Tzitzeica condition for the indicatrices of spacelike curves presents a more complex challenge. Future research may focus on resolving these conditions, particularly for spacelike curves with different types of Frenet frames.

Furthermore, extending these results to higher-dimensional spaces or exploring the Tzitzeica properties of other associated curves, such as Bertrand or Mannheim curves, would be a valuable contribution to the differential geometry of special curves.

Author contributions

Tanju Kahraman: Investigation, Conceptualization, Methodology, Funding acquisition, Project administration, Supervision, Writing-original draft, Writing-review and editing; Şadiye Buket Kaya: Data Curation, Formal analysis, Validation, Visualization. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare there is no conflicts of interest.

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