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Research article

Advancements in intuitionistic fuzzy rough graphs

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Abstract: Rough sets and intuitionistic fuzzy (IF) sets are two separate mathematical frameworks designed to model and manage incomplete or uncertain knowledge. By integrating these models, an IF rough framework is constructed, offering enhanced expressiveness and flexibility for representing and processing incomplete data within information systems. In this paper, we introduce a new hybrid model utilizing minimal IF neighborhoods. This model, based on any two IF binary relations defined on a non-empty universe, leads to the development of two novel IF graph approximation spaces aimed at reducing the boundary region of fuzzy uncertainty and increasing the precision degree of the fuzzy approximations. Furthermore, key results pertaining to both types of IF graph approximations are established. The relationships between the existing IF approximation methods are derived, and comparisons are made to demonstrate that the proposed approaches are more general than previous models. Finally, we explore an application of these IF graph approximation spaces in decision-making contexts and propose an algorithm to facilitate solving such problems.

Keywords: rough relations; intuitionistic fuzzy rough digraphs; minimal neighborhoods; decision-making
Mathematics Subject Classification: 03E72, 05C72, 05C99, 57M15

Symbol	Description
$G^\star$	Simple directed graph
X	Set of vertices (nodes) over $G^\star$
Y	Set of edges over $G^\star$
$x, x_1, x_2, x_3, \ldots$	Vertices (nods) of $G^\star$
$y, y_1, y_2, y_3, \ldots$	Edges of $G^\star$

x_ix_j	Edge connects the two vertices x_i and x_j with $i, j \in \{1, 2, 3, ...\}$
μ_X, μ_Y	Fuzzy membership functions of vertices and edges, respectively
$R = (R^+, R^-)$	IF relation on X, where R^+, R^- membership and non-membership degree, respectively
$W = (W^+, W^-)$	IF relation on Y, where W^+, W^- membership and non-membership degree, respectively
$A = (\rho^+, \rho^-)$ and $A_\sigma = (\sigma^+, \sigma^-)$	IF sets on X and Y, respectively
$T_{\underline{R}(A)}, F_{\underline{R}(A)}$	Membership and non-membership degree in $\underline{R}$, respectively
$T_{\overline{R}(A)}, F_{\overline{R}(A)}$	Membership and non-membership degree in $\overline{R}$, respectively
$T_{\underline{W}(A_\sigma)}, F_{\underline{W}(A_\sigma)}$	Membership and non-membership degree in $\underline{W}$, respectively
$T_{\overline{W}(A_\sigma)}, F_{\overline{W}(A_\sigma)}$	Membership and non-membership degree in $\overline{W}$, respectively
$[RA]_G = \left([\underline{R}A]_G, [\overline{R}A]_G\right)$	IF rough set in X
$[WA_\sigma]_G = \left([\underline{W}A_\sigma]_G, [\overline{W}A_\sigma]_G\right)$	IF rough set in Y
$G = (\underline{G}, \overline{G}) = ([RA]_G, [WA_\sigma]_G)$	IF rough graph of $G^\star$
$\underline{G} = \left([\underline{R}A]_G, [\underline{W}A_\sigma]_G\right)$	Lower approximate IF graph of G
$\overline{G} = \left([\overline{R}A]_G, [\overline{W}A_\sigma]_G\right)$	Upper approximate IF graph of G

1. Introduction

Pawlak [1] established rough set theory as a formal framework for modeling and managing missing information. A certain topological configuration is required for this theory to work. A topological rough set is an immensely significant generalization of a rough set since it bridges the gap between topological academics and those who are interested in the application of topology theory. This theory extends classical set theory [2,3], particularly to facilitate the study of intelligent systems characterized by insufficient and uncertain data. In many instances, the equivalence relations present within a universe are inadequate for defining the upper and lower approximation operators. Since its inception, numerous mathematicians, logicians, and researchers have shown interest in further developing the theory, exploring its extensions, and applying it across various fields. Such applications are prevalent in areas like expert systems, data mining, and machine learning [4–6]. The theory fundamentally depends on specific topological structures, and because of its ability to connect topological concepts with practical applications, the notion of a topological rough set constitutes a significant and broader generalization of the traditional rough set model.

Graph theory is an essential tool for researching various network topologies and analyzing pairwise interactions between entities [7]. Specifically, considering the connections between options when addressing decision-making issues yields more useful outcomes. Graph theory and general topology are closely related mathematical fields. This relationship includes the construction of topologies on the set of vertices and edges of a graph. Directed and undirected graphs have been used in numerous studies to design various topologies (see [8–10]), with many of these constructions rooted in the theory of simple undirected graphs. A relation defined on a graph acts as a bridge between graph theory and topological structures, allowing the graph to gain new types of topological structures. Many set-based decision-making problems utilize concepts like rough sets, rough fuzzy sets, generalized rough fuzzy sets, soft rough fuzzy sets, and IF soft rough sets [11–13]. Notably, there is a one-to-one correspondence between the labeled topologies on n points and the labeled transitive directed graph with n points [14]. The initial definition of a fuzzy graph was presented in [15] using Zadeh's fuzzy relations [16]. A more comprehensive explanation and the introduction of fuzzy graph theory, which involved studying fuzzy relations on fuzzy sets, is credited to [14] in 1975. This work developed relationships related to the characteristics of trees, different graphs, and path graphs. The concepts of fuzzy cut, nodes, and fuzzy bridges were later introduced in [17]. Fuzzy topological analysis of the pizza graph was discussed in [18]. Furthermore, specific results on rough fuzzy digraphs and their applications to decision-making problems are available in [11, 19, 20]. It is extremely challenging to address complicated situations with a single type of uncertainty technique because of the limitations of human understanding. As a result, creating hybrid models that combine the benefits of several distinct mathematical models for handling uncertainty is essential. The concept of IF rough graphs was introduced in [21]. The authors of [22] considered operations on IF graphs. Recently, many extensions of hybrid models were introduced in [23, 24].

The motivation for this paper stems from the need to address limitations in existing IF rough graph approximation methods [21, 23]. While previous approaches have contributed to handling uncertainty in graph-based decision-making, they often produce larger boundary regions and lower accuracy measures, which translates to greater ambiguity in classification and decision outcomes. The proposed model introduces a novel hybrid framework that leverages minimal IF neighborhoods derived from two IF binary relations to define new lower and upper approximation operators. This approach demonstrably minimizes the IF boundary region more effectively than [21, 23], achieving higher precision in approximations.

This paper is structured as follows: Section 1 presents the introduction, providing the necessary background on rough sets, IF sets, and their integration into hybrid models, along with a review of relevant literature and the motivation for this work. Section 2 establishes the foundational preliminaries and essential definitions related to graphs, IF sets, and approximation spaces. The core of the paper lies in Section 3, where we introduce our novel hybrid model based on minimal IF neighborhoods; here, we formally define the new lower and upper approximation operators for IF rough graphs, establish their fundamental properties, and demonstrate their generality compared with existing approaches. Section 4 explores fundamental operations on these new IF rough graphs, including complement, union, intersection, and join, proving that the resulting structures maintain the integrity of the proposed model. In Section 5, we delve into the structural characteristics of these graphs by defining key concepts such as order and size and introducing classifications like regular, irregular, and neighborly irregular IF rough graphs. Section 6 showcases the practical utility of our framework through a detailed application in decision-making, presenting a real-world example of selecting bridal embroidery and providing an efficient algorithm to solve such problems. Finally, Section 7 concludes the paper by summarizing the key contributions, discussing limitations, and outlining promising directions for future research.

Key contributions and findings of the research include:

(1) Establishing vital results for both types of IF graph approximations.
(2) Inducing the relationships between the present IF approximations.
(3) Presenting comparisons with preceding IF graph approximations [21, 23] and demonstrating that the current approximations are more general.
(4) Inducing IF topologies of rough IF graphs.
(5) Considering applications of IF graph approximation spaces.
(6) Presenting an efficient algorithm to solve decision-making problems.

2. Preliminaries

The study primarily focuses on simple directed graphs, which may or may not include loops. The foundational definitions and notations used are:

- The unqualified term “graph” is used as an abbreviation for a “simple undirected graph with or without loops”.
- X is a non-empty set (the domain), $\mathbf{I}$ represents the unit interval $[0,1]$, $IF(X)$ is the class of all IF sets in X.
- $\mathbf{I}^X$ is the class of all fuzzy sets in X, $IFG(X)$ is the class of all IF graphs in X.

All other terminology not explicitly defined in the manuscript can be found in external sources [9, 14, 17].

A pair $\mu_{G^\star} = (\mu_X, \mu_Y) \in \mathbf{I}^{G^\star}$ is said to be a fuzzy graph over a non-empty set X. Here, μ_X and μ_Y are respectively called fuzzy vertex and fuzzy edge membership functions of the fuzzy graph $\mu_{G^\star} = (\mu_X, \mu_Y)$. In the operator $\mu_X : X \longrightarrow \mathbf{I}$, the value $\mu_X(x)$ is called the degree of the membership of a vertex x in μ_X for each $x \in X$. Again, in the operator $\mu_Y : Y \longrightarrow \mathbf{I}$, the value $\mu_Y(y)$ is called the degree of the membership of an edge y in μ_Y for each $y \in Y$, where $Y \subseteq X \times X$. An IF set B in X is an object of the form $B = \{(x, \mu(x), \nu(x)) \mid x \in X\}$, where $\mu : X \to \mathbf{I}$ and $\nu : X \to \mathbf{I}$ are membership and non-membership functions, respectively, and for all $x \in X$, $0 \leq \mu(x) + \nu(x) \leq 1$. The empty and full IF sets are denoted, respectively by $(\underline{0}, \underline{1}), (\underline{1}, \underline{0})$. Let X, X^* be two sets of universes; an IF relation in $X \times X^*$ is an IF set R given as $R = \{((x, x^*), R^+(x, x^*) = \mu(x, x^*), R^-(x, x^*) = \nu(x, x^*)) \mid (x, x^*) \in X \times X^*\}$ where $\mu(x, x^*)$ and $\nu(x, x^*)$ are membership and non-membership degrees, such that for all $(x, x^*) \in X \times X^*, 0 \leq \mu(x, x^*) + \nu(x, x^*) \leq 1$. The complement of an IF set $B = \{(x, \mu(x), \nu(x)) \mid x \in X\}$ is denoted by $B^c = \{(x, \nu(x), \mu(x)) \mid x \in X\}$. An IF graph on a non-empty set X is defined to be a pair $G = (B, C)$ such that

(i) $\mu_B : X \to \mathbf{I}$ and $\nu_B : X \to \mathbf{I}$ denote the degree of membership and non-membership of each element $x \in X$, respectively, such that $\mu_B(x) + \nu_B(x) \leq 1$,

(ii) the functions $\mu_C : Y \subseteq X \times X \to \mathbf{I}$ and $\nu_C : Y \subseteq X \times X \to \mathbf{I}$ are defined by $\mu_C(x, x') \leq \min\{\mu_B(x), \mu_B(x')\}, \nu_C(x, x') \leq \max\{\nu_B(x), \nu_B(x')\}$ such that $0 \leq \mu_C(x, x') + \nu_C(x, x') \leq 1, \forall (x, x') \in Y$. We call B the IF vertex set and C the IF edge set. $G = (B, C)$ is an IF graph.

Let X be a non-empty and finite universe of discourse and R be an IF relation on X; the pair (X, R) is called an IF approximation space. For any $A \in IF(X)$, the upper and lower approximations of A w.r.t. (X, R), denoted by $\overline{R}(A)$ and $\underline{R}(A)$, are two IF sets and are, respectively, defined as follows:

$$\overline{R}(A) = \left\{\left\langle x, \mu_{\overline{R}(A)}(x), \gamma_{\overline{R}(A)}(x)\right\rangle \mid x \in X\right\},$$
$$\underline{R}(A) = \left\{\left\langle x, \mu_{\underline{R}(A)}(x), \gamma_{\underline{R}(A)}(x)\right\rangle \mid x \in X\right\},$$

where

$$\mu_{\overline{R}(A)}(x) = \bigvee_{x' \in X}\left[\mu_R(x, x') \wedge \mu_A(x)\right], \quad \gamma_{\overline{R}(A)}(x) = \bigwedge_{x' \in X}\left[\gamma_R(x, x') \vee \gamma_A(x')\right];$$
$$\mu_{\underline{R}(A)}(x) = \bigwedge_{x' \in X}\left[\gamma_R(x, x') \vee \mu_A(x')\right], \quad \gamma_{\underline{R}(A)}(x) = \bigvee_{x' \in X}\left[\mu_R(x, x') \wedge \gamma_A(x')\right].$$

$\overline{R}(A)$ and $\underline{R}(A)$ are respectively called the upper and lower approximations of A w.r.t. (X, R). The pair $(\underline{R}(A), \overline{R}(A))$ is called the IF rough set of A w.r.t. (X, R), and $\overline{R}, \underline{R} : IF(X) \to IF(X)$ are referred to as upper and lower IF rough approximation operators, respectively. Similarly, the lower and upper approximations of IF sets can be extended to IF graphs (see [21, 23]).

3. IF rough relations

This section introduces the IF rough graph, a new hybrid model that uses minimal IF neighborhoods and two IF binary relations to define lower and upper approximations. This approach aims to minimize the boundary region and improve accuracy. The section establishes several properties and demonstrates that this new model is more generalized and flexible than previous models.

Definition 3.1. *Assume that X is a non-empty finite set, and R is an IF relation on X. Then, for any point $x \in X$, define the fuzzy sets xR^+, xR^- and $R^+x, R^-x \in \mathbf{I}^X$ as follows:*

$$xR^+(x') = R^+(x,x'), xR^-(x') = R^-(x,x'), \quad and \quad R^+x(x') = R^+(x',x), R^-x(x') = R^-(x',x) \forall x' \in X$$

Define for any point $x \in X$, the fuzzy sets $<x>R^+, <x>R^-, R^+<x>, R^-<x> \in \mathbf{I}^X$ as follows:

$$<x>R^+ = \bigwedge_{x' \in X : R^+(x',x)>0} x'R^+ \ \ and \ \ <x>R^- = \bigvee_{x' \in X : R^-(x',x)>0} x'R^-.$$

$$R^+<x> = \bigwedge_{x' \in X, R^+(x,x')>0} R^+x' \ \ and \ \ R^-<x> = \bigvee_{x' \in X, R^-(x,x')>0} R^-x'.$$

For any point $x \in X$, define the minimal fuzzy neighborhoods $R^+<x>R^+, R^-<x>R^- : X \to \mathbf{I}$ as follows:

$$R^+<x>R^+ = <x>R^+ \wedge R^+<x> \ \ and \ \ R^-<x>R^- = <x>R^- \vee R^-<x>.$$

For any point $x \in X$, the pair $(R^+<x>R^+, R^-<x>R^-)$ is said to be the minimal IF neighborhood.

Definition 3.2. *Assume that X is a non-empty finite universal set, $Y \subseteq X \times X$ and $W = (W^+, W^-)$ an IF relation on Y. Then, for any point $y \in Y$, define the IF sets $yW^+, yW^-, W^+y, W^-y \in \mathbf{I}^Y$ as follows:*

$$yW^+(y') = W^+(y,y'), yW^-(y') = W^-(y,y') \quad and \quad W^+y(y') = W^+(y',y), W^-y(y') = W^-(y',y) \forall y' \in Y$$

Define for any $y \in Y$, the fuzzy sets $<y>W^+$, $<y>W^-$, $W^+<y>$, $W^-<y> \in \mathbf{I}^Y$ as follows:

$$<y>W^+ = \bigwedge_{y' \in Y : W^+(y',\ y)>0} y'W^+ \ \ and \ \ <y>W^- = \bigvee_{y' \in Y : W^-(y',y)>0} y'W^-.$$

$$W^+<y> = \bigwedge_{y' \in Y, W^+(y,y')>0} W^+y'. \ and \ W^-<y> = \bigvee_{y' \in Y, W^-(y,y')>0} W^-y'.$$

For any $y \in Y$, define the minimal fuzzy neighborhoods $W^+<y>W^+$, $W^-<y>W^- : Y \to \mathbf{I}$ as follows:

$$W^+<y>W^+ = <y>W^+ \wedge W^+<y> \ \ and \ \ W^-<y>W^- = <y>W^- \vee W^-<y>.$$

For any $y \in Y$, the pair $(W^+<y>W^+,\ W^-<y>W^-)$ is said to be the minimal IF neighborhood.

Note that: Definition 3.1 focuses on introducing minimal IF neighborhoods on the vertex set X, while Definition 3.2 extends this concept to the edge set Y.

Definition 3.3. *Assume that X is a non-empty set and $R = (R^+, R^-)$ an IF relation on X. Assume that $A = (\rho^+, \rho^-)$ is an IF set in X. The lower and upper IF rough approximations of A are denoted by $\underline{R}A$ and $\overline{R}A$, respectively. $\underline{R}A$ and $\overline{R}A$ are two IF rough sets defined as follows: $\underline{R}A = \left(T_{\underline{R}(A)}, F_{\underline{R}(A)}\right)$ and $\overline{R}A = \left(T_{\overline{R}(A)}, F_{\overline{R}(A)}\right)$ are defined as IF sets in X such that, for all $x \in X$,*

$$T_{\underline{R}(A)}(x) = \bigwedge_{x' \in X} [R^- < x > R^-(x') \vee \rho^+(x')], \quad F_{\underline{R}(A)}(x) = \bigvee_{x' \in X} [R^+ < x > R^+(x') \wedge \rho^-(x')],$$
$$T_{\overline{R}(A)}(x) = \bigvee_{x' \in X} [R^+ < x > R^+(x') \wedge \rho^+(x')], \quad F_{\overline{R}(A)}(x) = \bigwedge_{x' \in X} [R^- < x > R^-(x') \vee \rho^-(x')].$$

Definition 3.4. *Assume that X is a non-empty set and $R = (R^+, R^-)$ an IF relation on X. Assume that $A = (\rho^+, \rho^-)$ is an IF set in X and $RA = \left(\underline{R}A, \overline{R}A\right)$ an IF rough set. Assume that $Y \subseteq X \times X$ and $W = (W^+, W^-)$ an IF relation on Y such that for all $x_{ij}, x_{kl} \in Y$*

$$W^+\left(x_{ij}, x_{kl}\right) \leq R^+(x_i, x_k) \wedge R^+\left(x_j, x_l\right), \quad W^-\left(x_{ij}, x_{kl}\right) \leq R^-(x_i, x_k) \vee R^-\left(x_j, x_l\right).$$

Assume that $A_\sigma = (\sigma^+, \sigma^-)$ is an IF set in Y such that for all $x_{ij} \in Y$

$$\sigma^+\left(x_{ij}\right) \leq \min\left\{\rho^+(x_i), \rho^+\left(x_j\right)\right\}, \quad \sigma^-\left(x_{ij}\right) \leq \max\left\{\rho^-(x_i), \rho^-\left(x_j\right)\right\}.$$

Then the upper and lower approximations of A_σ, denoted by $\overline{W}A_\sigma = \left(T_{\overline{W}(A_\sigma)}, F_{\overline{W}(A_\sigma)}\right)$ and $\underline{W}A_\sigma = \left(T_{\underline{W}(A_\sigma)}, F_{\underline{W}(A_\sigma)}\right)$, respectively, are defined as IF sets in Y such that, for all $x_{ij} \in Y$,

$$T_{\overline{W}(A_\sigma)}\left(x_{ij}\right) = \bigvee_{x_{kl} \in Y} \left[W^+ < x_{ij} > W^+(x_{kl}) \wedge \sigma^+(x_{kl})\right], \quad F_{\overline{W}(A_\sigma)}\left(x_{ij}\right) = \bigwedge_{x_{kl} \in Y} \left[W^- < x_{ij} > W^-(x_{kl}) \vee \sigma^-(x_{kl})\right],$$
$$T_{\underline{W}(A_\sigma)}\left(x_{ij}\right) = \bigwedge_{x_{kl} \in Y} \left[W^- < x_{ij} > W^-(x_{kl}) \vee \sigma^+(x_{kl})\right], \quad F_{\underline{W}(A_\sigma)}\left(x_{ij}\right) = \bigvee_{x_{kl} \in Y} \left[W^+ < x_{ij} > W^+(x_{kl}) \wedge \sigma^-(x_{kl})\right].$$

A pair $WA_\sigma = \left(\underline{W}A_\sigma, \overline{W}A_\sigma\right)$ is called an IF rough relation on an IF rough set $RA = \left(\underline{R}A, \overline{R}A\right)$ if and only if $T_{\overline{W}(A_\sigma)} \neq T_{\underline{W}(A_\sigma)}, F_{\overline{W}(A_\sigma)} \neq F_{\underline{W}(A_\sigma)}$.

Example 3.5. *Assume that $A = \{(x_1, 0.3, 0.1), (x_2, 0.9, 0.1), (x_3, 0.7, 0.0)\}$ is an IF set on $X = \{x_1, x_2, x_3\}$ and R an IF relation on X given in Table 1.*

Table 1. The IF relation R on X in Example 3.5.

R	x_1	x_2	x_3
x_1	$(1.0, 0.0)$	$(0.3, 0.4)$	$(0.6, 0.2)$
x_2	$(0.3, 0.4)$	$(1.0, 0.0)$	$(0.5, 0.5)$
x_3	$(0.6, 0.2)$	$(0.5, 0.5)$	$(1.0, 0.0)$

By computing, $x_1R^+ = \{1.0, 0.3, 0.6\}$, $x_2R^+ = \{0.3, 1.0, 0.5\}$, $x_3R^+ = \{0.6, 0.5, 1.0\}$, $x_1R^- = \{0.0, 0.4, 0.2\}$, $x_2R^- = \{0.4, 0.0, 0.5\}$, $x_3R^- = \{0.2, 0.5, 0.0\}$. So, $< x_1 > R^+ =< x_2 > R^+ =< x_3 > R^+ = \{0.3, 0.3, 0.5\}$, $< x_1 > R^- = \{0.4, 0.5, 0.5\}$, $< x_2 > R^- = \{0.4, 0.5, 0.5\}$, $< x_3 > R^- = \{0.4, 0.5, 0.5\}$. Similarly, $R^+x_1 = \{1.0, 0.3, 0.6\}$, $R^+x_2 = \{0.3, 1.0, 0.5\}$, $R^+x_3 = \{0.6, 0.5, 1.0\}$, $R^-x_1 = \{0.0, 0.4, 0.2\}$, $R^-x_2 = \{0.4, 0.0, 0.5\}$, $R^-x_3 = \{0.2, 0.5, 0.0\}$. Thus, $R^+ < x_1 >= R^+ < x_2 >= R^+ < x_3 >= \{0.3, 0.3, 0.5\}$,

$R^- < x_1 >= \{0.4, 0.5, 0.5\}$, $R^- < x_2 >= \{0.4, 0.5, 0.5\}$,$R^- < x_3 >= \{0.4, 0.5, 0.5\}$. *Therefore,* $R^+ < x_1 > R^+ = R^+ < x_2 > R^+ = R^+ < x_3 > R^+ = \{0.3, 0.3, 0.5\}$, $R^- < x_1 > R^- = \{0.4, 0.5, 0.5\}$, $R^- < x_2 > R^- = \{0.5, 0.5, 0.5\}$, $R^- < x_3 > R^- = \{0.4, 0.5, 0.5\}$. *This implies that* $RA = \left(\underline{R}A, \overline{R}A\right)$ *is an IF rough set, where* $\underline{R}A$ *and* $\overline{R}A$ *are lower and upper approximations of A, respectively, as given below:*

$$\underline{R}A = \{(x_1, 0.4, 0.1), (x_2, 0.4, 0.1), (x_3, 0.4, 0.1)\},$$
$$\overline{R}A = \{(x_1, 0.5, 0.4), (x_2, 0.5, 0.4), (x_3, 0.5, 0.4)\}.$$

Assume that $Y = \{x_{12}, x_{13}, x_{23}, x_{31}, x_{32}, x_{33}\} \subseteq X \times X$. *Assume that W is an IF relation on Y defined in Table 2.*

Table 2. The IF relation W on X in Example 3.5.

W	x_{12}	x_{13}	x_{23}	x_{31}	x_{32}	x_{33}
x_{12}	(0.0, 0.0)	(0.5, 0.4)	(0.1, 0.3)	(0.2, 0.2)	(0.0, 0.3)	(0.1, 0.1)
x_{13}	(0.2, 0.4)	(0.0, 0.0)	(0.1, 0.2)	(0.0, 0.4)	(0.0, 0.2)	(0.0, 0.3)
x_{23}	(0.1, 0.4)	(0.1, 0.1)	(0.0, 0.0)	(0.5, 0.3)	(0.5, 0.4)	(0.5, 0.5)
x_{31}	(0.1, 0.3)	(0.5, 0.2)	(0.2, 0.1)	(0.0, 0.0)	(0.0, 0.0)	(0.1, 0.2)
x_{32}	(0.0, 0.0)	(0.0, 0.0)	(0.0, 0.0)	(0.0, 0.0)	(1.0, 0.0)	(0.5, 0.5)
x_{33}	(0.4, 0.0)	(0.3, 0.0)	(0.4, 0.0)	(0.01, 0.0)	(0.4, 0.5)	(1.0, 0.0)

Assume that $A_\sigma = \{(x_{12}, 0.2, 0.01), (x_{13}, 0.1, 0.02), (x_{23}, 0.2, 0.02), (x_{31}, 0.0, 0.0), (x_{32}, 0.0, 0.1), (x_{33}, 0.1, 0.1)\}$ *is an IF set on Y. By computing,*

$x_{12}W^+ = \{0.0, 0.5, 0.1, 0.2, 0.0, 0.1\}$, $x_{13}W^+ = \{0.2, 0.0, 0.1, 0.0, 0.0, 0.0\}$, $x_{23}W^+ = \{0.1, 0.1, 0.0, 0.5, 0.5, 0.5\}$, $x_{31}W^+ = \{0.1, 0.5, 0.2, 0.0, 0.0, 0.1\}$, $x_{32}W^+ = \{0.0, 0.0, 0.0, 0.0, 1.0, 0.5\}$, $x_{33}W^+ = \{0.4, 0.3, 0.4, 0.01, 0.4, 1.0\}$, $x_{12}W^- = \{0.0, 0.4, 0.3, 0.2, 0.3, 0.1\}$, $x_{13}W^- = \{0.4, 0.0, 0.2, 0.4, 0.2, 0.3\}$, $x_{23}W^- = \{0.4, 0.1, 0.0, 0.3, 0.4, 0.5\}$, $x_{31}W^- = \{0.3, 0.2, 0.1, 0.0, 0.0, 0.2\}$, $x_{32}W^- = \{0.0, 0.0, 0.0, 0.0, 0.0, 0.5\}$, $x_{33}W^- = \{0.0, 0.0, 0.0, 0.0, 0.5, 0.0\}$. *So,* $< x_{12} > W^+ = \{0.1, 0.0, 0.0, 0.0, 0.0, 0.0\}$, $< x_{13} > W^+ = \{0.0, 0.1, 0.0, 0.0, 0.0, 0.0\}$, $< x_{23} > W^+ = \{0.0, 0.0, 0.1, 0.0, 0.0, 0.0\}$, $< x_{31} > W^+ = \{0.0, 0.0, 0.0, 0.0.1, 0.0, 0.0\}$, $< x_{32} > W^+ = \{0.0, 0.0, 0.0, 0.0, 0.4, 0.0\}$, $< x_{33} > W^+ = \{0.0, 0.0, 0.0, 0.0, 0.0, 0.1\}$, $< x_{12} > W^- = \{0.4, 0.4, 0.3, 0.4, 0.5, 0.5\}$, $< x_{13} > W^- = \{0.4, 0.4, 0.3, 0.4, 0.5, 0.5\}$, $< x_{23} > W^- = \{0.4, 0.4, 0.3, 0.4, 0.5, 0.5\}$, $< x_{31} > W^- = \{0.4, 0.4, 0.3, 0.4, 0.5, 0.5\}$, $< x_{32} > W^- = \{0.4, 0.4, 0.3, 0.4, 0.5, 0.5\}$, $< x_{33} > W^- = \{0.4, 0.4, 0.3, 0.4, 0.5, 0.5\}$. *Similarly,* $W^+x_{12} = \{0.0, 0.2, 0.1, 0.1, 0.0, 0.4\}$, $W^+x_{13} = \{0.5, 0.0, 0.1, 0.5, 0.0, 0.3\}$, $W^+x_{23} = \{0.1, 0.1, 0.0, 0.2, 0.0, 0.4\}$, $W^+x_{31} = \{0.2, 0.0, 0.5, 0.0, 0.0, 0.01\}$, $W^+x_{32} = \{0.0, 0.0, 0.5, 0.0, 1.0, 0.4\}$, $W^+x_{33} = \{0.1, 0.0, 0.5, 0.1, 0.5, 1.0\}$, $W^-x_{12} = \{0.0, 0.4, 0.4, 0.3, 0.0, 0.0\}$, $W^-x_{13} = \{0.4, 0.0, 0.1, 0.2, 0.0, 0.0\}$, $W^-x_{23} = \{0.3, 0.2, 0.0, 0.1, 0.0, 0.0\}$, $W^-x_{31} = \{0.2, 0.4, 0.3, 0.0, 0.0, 0.0\}$, $W^-x_{32} = \{0.3, 0.2, 0.4, 0.0, 0.0, 0.5\}$, $W^-x_{33} = \{0.1, 0.3, 0.5, 0.2, 0.5, 0.0\}$. *So,* $W^+ < x_{12} >= \{0.1, 0.0, 0.0, 0.0, 0.0, 0.01\}$, $W^+ < x_{13} >= \{0.0, 0.1, 0.0, 0.0, 0.0, 0.01\}$, $W^+ < x_{23} >= \{0.0, 0.0, 0.1, 0.0, 0.0, 0.01\}$, $W^+ < x_{31} >= \{0.0, 0.0, 0.0, 0.1, 0.0, 0.01\}$, $W^+ < x_{32} >= \{0.0, 0.0, 0.0, 0.0, 0.5, 0.01\}$, $W^+ < x_{33} >= \{0.0, 0.0, 0.0, 0.0, 0.0, 0.01\}$, $W^- < x_{12} >= \{0.4, 0.4, 0.5, 0.3, 0.5, 0.5\}$, $W^- < x_{13} >= \{0.4, 0.4, 0.5, 0.3, 0.5, 0.5\}$, $W^- < x_{23} >= \{0.4, 0.4, 0.5, 0.3, 0.5, 0.5\}$, $W^- < x_{31} >= \{0.4, 0.4, 0.5, 0.3, 0.5, 0.5\}$, $W^- < x_{32} >= \{0.4, 0.4, 0.5, 0.3, 0.5, 0.5\}$, $W^- < x_{33} >= \{0.4, 0.4, 0.5, 0.3, 0.5, 0.5\}$. *Therefore,* $W^+ < x_{12} > W^+ = \{0.1, 0.0, 0.0, 0.0, 0.0, 0.0\}$, $W^+ < x_{13} > W^+ = \{0.0, 0.1, 0.0, 0.0, 0.0, 0.0\}$,

$W^+ < x_{23} > W^+ = \{0.0, 0.0, 0.1, 0.0, 0.0, 0.0\}$, $W^+ < x_{31} > W^+ = \{0.0, 0.0, 0.0, 0.1, 0.0, 0.0\}$, $W^+ < x_{32} > W^+ = \{0.0, 0.0, 0.0, 0.0, 0.4, 0.0\}$, $W^+ < x_{33} > W^+ = \{0.0, 0.0, 0.0, 0.0, 0.0, 0.01\}$, $W^- < x_{12} > W^- = \{0.4, 0.4, 0.3, 0.3, 0.5, 0.5\}$, $W^- < x_{13} > W^- = \{0.4, 0.4, 0.3, 0.3, 0.5, 0.5\}$, $W^- < x_{23} > W^- = \{0.4, 0.4, 0.3, 0.3, 0.5, 0.5\}$, $W^- < x_{31} > W^- = \{0.4, 0.4, 0.3, 0.3, 0.5, 0.5\}$, $W^- < x_{32} > W^- = \{0.4, 0.4, 0.3, 0.3, 0.5, 0.5\}$, $W^- < x_{33} > W^- = \{0.4, 0.4, 0.5, 0.4, 0.5, 0.5\}$. *Then by definition, we have*

$$\begin{aligned}
T_{\underline{W}(A_\sigma)}(x_{12}) =& (0.4 \vee 0.2) \wedge (0.4 \vee 0.1) \wedge (0.5 \vee 0.2) \wedge (0.4 \vee 0.0) \wedge (0.5 \vee 0.0) \wedge (0.5 \vee 0.1) \\
=& 0.4 \wedge 0.4 \wedge 0.5 \wedge 0.4 \wedge 0.5 \wedge 0.5 = 0.4, \\
F_{\underline{W}(A_\sigma)}(x_{12}) =& (0.1 \wedge 0.01) \vee (0.0 \wedge 0.02) \vee (0.0 \wedge 0.02) \vee (0.0 \wedge 0.0) \vee (0.0 \wedge 0.1) \vee (0.0 \wedge 0.1) \\
=& 0.01 \vee 0.0 \vee 0.0 \vee 0.0 \vee 0.0 \vee 0.01 = 0.01, \\
T_{\overline{W}(A_\sigma)}(x_{12}) =& (0.1 \wedge 0.2) \vee (0.0 \wedge 0.1) \vee (0.0 \wedge 0.2) \vee (0.0 \wedge 0.0) \vee (0.0 \wedge 0.0) \vee (0.0 \wedge 0.1) \\
=& 0.1 \vee 0.0 \vee 0.0 \vee 0.0 \vee 0.0 \vee 0.0 = 0.1, \\
F_{\overline{W}(A_\sigma)}(x_{12}) =& (0.4 \vee 0.01) \wedge (0.4 \vee 0.02) \wedge (0.5 \vee 0.02) \wedge (0.4 \vee 0.0) \wedge (0.5 \vee 0.1) \wedge (0.5 \vee 0.1) \\
=& 0.4 \wedge 0.4 \wedge 0.5 \wedge 0.4 \wedge 0.5 \wedge 0.5 = 0.4.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\underline{W}A_\sigma(x_{13}) = (0.4, 0.02),\ \overline{W}A_\sigma(x_{13}) = (0.1, 0.4),\ \underline{W}A_\sigma(x_{23}) = (0.4, 0.02),\ \overline{W}A_\sigma(x_{23}) = (0.1, 0.4), \\
&\underline{W}A_\sigma(x_{31}) = (0.4, 0.0),\ \overline{W}A_\sigma(x_{31}) = (0.0, 0.4),\ \underline{W}A_\sigma(x_{32}) = (0.4, 0.1),\ \overline{W}A_\sigma(x_{32}) = (0.0, 0.4), \\
&\underline{W}A_\sigma(x_{33}) = (0.4, 0.01),\ \overline{W}A_\sigma(x_{33}) = (0.01, 0.4).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\underline{W}A_\sigma =& \{(x_{12}, 0.4, 0.01), (x_{13}, 0.3, 0.02), (x_{23}, 0.4, 0.02),\ (x_{31}, 0.4, 0.0), (x_{32}, 0.4, 0.1), (x_{33}, 0.4, 0.01)\}, \\
\overline{W}A_\sigma =& \{(x_{12}, 0.1, 0.4), (x_{13}, 0.1, 0.4), (x_{23}, 0.1, 0.4),\ (x_{31}, 0.0, 0.4), (x_{32}, 0.0, 0.4), (x_{33}, 0.01, 0.4)\}.
\end{aligned}$$

Hence, $WA_\sigma = \left(\underline{W}A_\sigma, \overline{W}A_\sigma\right)$ *is an IF rough relation on X.*

Lemma 3.6. *Assume that* (X, R) *is called an IF approximation space and* $WA_\sigma = \left(\underline{W}A_\sigma, \overline{W}A_\sigma\right)$ *is an IF rough relation on the IF rough set* $RA = \left(\underline{R}A, \overline{R}A\right)$. *Then, the following properties hold:*

(1) $\overline{R}A = (\underline{R}A^c)^c$ *and* $\overline{W}A_\sigma = \left(\underline{W}A_\sigma^c\right)^c$,
(2) $\overline{R}(\underline{0}, \underline{1}) = (\underline{0}, \underline{1})$ *and* $\overline{W}(\underline{0}, \underline{1}) = (\underline{0}, \underline{1})$,
(3) $\overline{R}(\underline{1}, \underline{0}) = (\underline{1}, \underline{0})$ *and* $\overline{W}(\underline{1}, \underline{0}) = (\underline{1}, \underline{0})$,
(4) $A \leq A_1$ *and* $A_\sigma \leq A_{\sigma_1}$ *implies that* $\underline{\underline{R}}A \leq \underline{\underline{R}}A_1$ *and* $\underline{\underline{W}}A_\sigma \leq \underline{\underline{W}}A_{\sigma_1}$,
(5) $A \leq A_1$ *and* $A_\sigma \leq A_{\sigma_1}$ *implies that* $\overline{\overline{R}}A \leq \overline{\overline{R}}A_1$ *and* $\overline{\overline{W}}A_\sigma \leq \overline{\overline{W}}A_{\sigma_1}$,
(6) $\overline{R}(A \wedge A_1) \leq \overline{R}A \wedge \overline{R}A_1$ *and* $\overline{W}(A_\sigma \wedge A_{\sigma_1}) \leq \overline{W}A_\sigma \wedge \overline{W}A_{\sigma_1}$,
(7) $\underline{R}(A \vee A_1) \geq \underline{R}A \vee \underline{R}A_1$ *and* $\underline{W}(A_\sigma \vee A_{\sigma_1}) \geq \underline{W}A_\sigma \vee \underline{W}A_{\sigma_1}$,
(8) $\underline{\underline{R}}(A \wedge A_1) \leq \underline{\underline{R}}A \wedge \underline{\underline{R}}A_1$ *and* $\underline{\underline{W}}(A_\sigma \wedge A_{\sigma_1}) \leq \underline{\underline{W}}A_\sigma \wedge \underline{\underline{W}}A_{\sigma_1}$,
(9) $\overline{\overline{R}}(A \vee A_1) \geq \overline{\overline{R}}A \vee \overline{\overline{R}}A_1$, *and* $\overline{\overline{W}}(A_\sigma \vee A_{\sigma_1}) \geq \overline{\overline{W}}A_\sigma \vee \overline{\overline{W}}A_{\sigma_1}$.

Proof. We only prove the first part of property (1), and the remaining properties directly follow from Definitions 3.3 and 3.4:

(1) The lower and upper IF rough approximations of A, $\underline{R}A = \left(T_{\underline{R}(A)}, F_{\underline{R}(A)}\right)$ and $\overline{R}A = \left(T_{\overline{R}(A)}, F_{\overline{R}(A)}\right)$, are defined by:

$$T_{\underline{R}(A)}(x) = \bigwedge_{x' \in X} [R^- < x > R^-(x') \vee \rho^+(x')], \quad F_{\underline{R}(A)}(x) = \bigvee_{x' \in X} [R^+ < x > R^+(x') \wedge \rho^-(x')],$$
$$T_{\overline{R}(A)}(x) = \bigvee_{x' \in X} [R^+ < x > R^+(x') \wedge \rho^+(x')], \quad F_{\overline{R}(A)}(x) = \bigwedge_{x' \in X} [R^- < x > R^-(x') \vee \rho^-(x')].$$

The UP IF rough approximations of A with respect to R are defined by

$$\overline{R}A(x) = \left(T_{\overline{R}(A)}(x) = \bigvee_{x' \in X} [R^+ < x > R^+(x') \wedge \rho^+(x')], F_{\overline{R}(A)}(x) = \bigwedge_{x' \in X} [R^- < x > R^-(x') \vee \rho^-(x')]\right).$$

Taking the complement yields

$$\left(\overline{R}A\right)^c(x) = \left(F_{\overline{R}(A)}(x), T_{\overline{R}(A)}(x)\right)$$

$$= \left(F_{\overline{R}(A)}(x) = \bigwedge_{x' \in X} [R^- < x > R^-(x') \vee \rho^-(x')], T_{\overline{R}(A)}(x) = \bigvee_{x' \in X} [R^+ < x > R^+(x') \wedge \rho^+(x')]\right).$$

Now consider $A^c = (\rho^-, \rho^+)$. The lower IF rough approximations of A^c with respect to R are defined by

$$\underline{R}A^c(x) = \left(T_{\underline{R}(A^c)}(x), F_{\underline{R}(A^c)}(x)\right)$$
$$= \left(T_{\underline{R}(A^c)}(x) = \bigwedge_{x' \in X} [R^- < x > R^-(x') \vee \rho^-(x')], F_{\underline{R}(A^c)}(x) = \bigvee_{x' \in X} [R^+ < x > R^+(x') \wedge \rho^+(x')]\right).$$

Hence, $\overline{R}A = (\underline{R}A^c)^c$.

□

3.1. IF rough graphs

Definition 3.7. *An IF rough graph G on a non-empty set X is a 4-ordered tuple ($R, [RA]_G, W, [WA_\sigma]_G$) such that*

(1) R is an IF relation on the set X,
(2) $[RA]_G = \left([\underline{R}A]_G, [\overline{R}A]_G\right) = \left(\underline{R}A \wedge A, \overline{R}A \vee A\right)$ is an IF rough set in X,
(3) W is an IF relation on $Y \subseteq X \times X$,
(4) $[WA_\sigma]_G = \left([\underline{W}A_\sigma]_G, [\overline{W}A_\sigma]_G\right) = \left(\underline{W}A_\sigma \wedge A_\sigma, \overline{W}A_\sigma \vee A_\sigma\right)$ is an IF rough set in Y.

Thus, the pair $G = (\underline{G}, \overline{G}) = ([RA]_G, [WA_\sigma]_G)$ is an IF rough graph, where $\underline{G} = \left([\underline{R}A]_G, [\underline{W}A_\sigma]_G\right)$ and $\overline{G} = \left([\overline{R}A]_G, [\overline{W}A_\sigma]_G\right)$ are lower and upper approximate IF graphs of G such that $\forall x_i, x_j \in X$,

$$T_{[\underline{W}(A_\sigma)]_G}\left(x_{ij}\right) \le \min\left\{T_{[\underline{R}(A)]_G}(x_i), T_{[\underline{R}(A)]_G}\left(x_j\right)\right\}, \quad F_{[\underline{W}(A_\sigma)]_G}\left(x_{ij}\right) \le \max\left\{F_{[\underline{R}(A)]_G}(x_i), F_{[\underline{R}(A)]_G}\left(x_j\right)\right\},$$
$$T_{[\overline{W}(A_\sigma)]_G}\left(x_{ij}\right) \le \min\left\{T_{[\overline{R}(A)]_G}(x_i), T_{[\overline{R}(A)]_G}\left(x_j\right)\right\}, \quad F_{[\overline{W}(A_\sigma)]_G}\left(x_{ij}\right) \le \max\left\{F_{[\overline{R}(A)]_G}(x_i), F_{[\overline{R}(A)]_G}\left(x_j\right)\right\}.$$

Example 3.8. *Assume that* $A = \{(x_1, 0.6, 0.1), (x_2, 0.3, 0.3), (x_3, 0.5, 0.4), (x_4, 0.4, 0.5), (x_5\ , \ 0.2, 0.2)\}$ *is an IF set on* $X = \{x_1, x_2, x_3, x_4, x_5\}$*, and* R *is an IF relation on* X *given in Table 3.*

Table 3. The IF relation R on X in Example 3.8.

R	x_1	x_2	x_3	x_4	x_5
x_1	(0.6, 0.1)	(0.5, 0.2)	(0.4, 0.3)	(0.3, 0.2)	(0.9, 0.1)
x_2	(0.4, 0.2)	(0.9, 0.1)	(1.0, 0.0)	(0.7, 0.1)	(0.1, 0.1)
x_3	(0.8, 0.1)	(0.1, 0.8)	(0.0, 0.0)	(0.2, 0.0)	(0.3, 0.3)
x_4	(0.5, 0.1)	(0.6, 0.2)	(0.6, 0.4)	(0.0, 0.4)	(0.0, 0.9)
x_5	(0.3, 0.4)	(0.1, 0.0)	(0.2, 0.3)	(0.6, 0.1)	(0.0, 1.0)

Assume that $Y = \{x_{12}, x_{25}, x_{31}, x_{43}, x_{54}\} \subseteq X \times X$*, and* W *is an IF relation on* Y *given in Table 4.*

Table 4. The IF relation W on X in Example 3.8.

W	x_{12}	x_{25}	x_{31}	x_{43}	x_{54}
x_{12}	(0.4, 0.0)	(0.1, 0.1)	(0.4, 0.2)	(0.3, 0.2)	(0.7, 0.1)
x_{25}	(0.1, 0.0)	(0.0, 0.6)	(0.3, 0.3)	(0.1, 0.0)	(0.1, 0.0)
x_{31}	(0.3, 0.1)	(0.1, 0.1)	(0.0, 0.0)	(0.2, 0.0)	(0.3, 0.1)
x_{43}	(0.1, 0.0)	(0.0, 0.3)	(0.3, 0.2)	(0.0, 0.2)	(0.0, 0.0)
x_{54}	(0.3, 0.0)	(0.0, 0.0)	(0.2, 0.1)	(0.6, 0.3)	(0.0, 1.0)

Assume that $A_\sigma = \{(x_{12}, 0.3, 0.1), (x_{25}, 0.2, 0.3), (x_{31}, 0.5, 0.1), (x_{43}, 0.4, 0.1), (x_{54}, 0.1\ , \ 0.3)\}$ *is an IF set on* Y*. Then, the upper and lower approximation relations are calculated as:*

By computing, $< x_1 > R^+ =< x_2 > R^+ = \{0.3, 0.1, 0.0, 0.0, 0.0\}, < x_3 > R^+ = \{0.3, 0.1, 0.2, 0.0, 0.0\}$, $< x_4 > R^+ = \{0.3, 0.1, 0.0, 0.2, 0.0\}$, $< x_5 > R^+ = \{0.3, 0.1, 0.0, 0.0, 0.1\}$, $< x_1 > R^- = \{0.4, 0.8, 0.4, 0.4, 1.0\}$, $< x_2 > R^- = \{0.4, 0.8, 0.4, 0.4, 1.0\}$, $< x_3 > R^- = \{0.4, 0.8, 0.4, 0.4, 1.0\}$, $< x_4 > R^- = \{0.4, 0.8, 0.4, 0.4, 1.0\}$, $< x_5 > R^- = \{0.4, 0.8, 0.4, 0.4, 1.0\}$. *Similarly,* $R^+ < x_1 >= \{0.3, 0.1, 0.0, 0.0, 0.0\}$, $R^+ < x_2 >= \{0.3, 0.1, 0.0, 0.0, 0.0\}$, $R^+ < x_3 >= \{0.3, 0.1, 0.1, 0.0, 0.0\}$, $R^+ < x_4 >= \{0.3, 0.1, 0.0, 0.5, 0.0\}$, $R^+ < x_5 >= \{0.3, 0.1, 0.0, 0.0, 0.1\}$, $R^- < x_1 >= \{0.3, 0.2, 0.8, 0.9, 1.0\}$, $R^- < x_2 > \{0.3, 0.2, 0.8, 0.9, 1.0\}$, $R^- < x_3 >= \{0.3, 0.2, 0.8, 0.9, 1.0\}$, $R^- < x_4 >= \{0.3, 0.2, 0.8, 0.9, 1.0\}$, $R^- < x_5 >= \{0.3, 0.2, 0.8, 0.9, 1.0\}$. *Therefore,* $R^+ < x_1 > R^+ = R^+ < x_2 > R^+ = \{0.3, 0.1, 0.0, 0.0, 0.0\}$, $R^+ < x_3 > R^+ = \{0.3, 0.1, 0.1, 0.0, 0.0\}$, $R^+ < x_4 > R^+ = \{0.3, 0.1, 0.0, 0.2, 0.0\}$, $R^+ < x_5 > R^+ = \{0.3, 0.1, 0.0, 0.0, 0.1\}$, $R^- < x_1 > R^- = \{0.4, 0.8, 0.8, 0.9, 1.0\}$, $R^- < x_2 > R^- = \{0.4, 0.8, 0.8, 0.9, 1.0\}$, $R^- < x_3 > R^- = \{0.4, 0.8, 0.8, 0.9, 1.0\}$, $R^- < x_4 > R^- = \{0.4, 0.8, 0.8, 0.9, 1.0\}$, $R^- < x_5 > R^- = \{0.4, 0.8, 0.8, 0.9, 1.0\}$. *This implies that* $[RA]_G = \left([\underline{R}A]_G, [\overline{R}A]_G\right)$ *is an IF rough set, where* $[\underline{R}(A)]_G$ *and* $[\overline{R}A]_G$ *are lower and upper approximations of X, respectively, given below:*

$$[\underline{R}A]_G = \{(x_1, 0.6, 0.1), (x_2, 0.3, 0.3), (x_3, 0.5, 0.4), (x_4, 0.4, 0.5), (x_5, 0.2, 0.2)\},$$
$$[\overline{R}A]_G = \{(x_1, 0.6, 0.1), (x_2, 0.3, 0.3), (x_3, 0.5, 0.4), (x_4, 0.4, 0.4), (x_5, 0.2, 0.2)\}.$$

By computing, $< x_{12} > W^+ = \{0.1, 0.0, 0.0, 0.0, 0.0\}$, $< x_{25} > W^+ = \{0.1, 0.1, 0.0, 0.0, 0.0\}$, $< x_{31} > W^+ = \{0.1, 0.0, 0.2, 0.0, 0.0\}$, $< x_{43} > W^+ = \{0.1, 0.0, 0.0, 0.0.1, 0.0\}$, $< x_{54} > W^+ = \{0.1, 0.0, 0.0, 0.0, 0.1\}$, $< x_{12} > W^- = \{0.1, 0.6, 0.3, 0.3, 0.1\}$, $< x_{25} > W^- = \{0.1, 0.6, 0.3, 0.3, 0.1\}$,

$< x_{31} > W^{-} = \{0.1, 0.6, 0.3, 0.3, 0.1\}$, $< x_{43} > W^{-} = \{0.1, 0.6, 0.3, 0.3, 0.1\}$, $< x_{54} > W^{-} = \{0.1, 0.6, 0.3, 0.3, 0.1\}$. *Similarly,* $W^{+} < x_{12} >= \{0.1, 0.0, 0.0, 0.0, 0.0\}$, $W^{+} < x_{25} >= \{0.1, 0.1, 0.0, 0.0, 0.0\}$, $W^{+} < x_{31} >= \{0.1, 0.0, 0.1, 0.0, 0.0\}$, $W^{+} < x_{43} >= \{0.1, 0.0, 0.0, 0.1, 0.0\}$, $W^{+} < x_{54} >= \{0.1, 0.0, 0.0, 0.0, 0.2\}$, $W^{-} < x_{12} >= \{0.2, 0.6, 0.1, 0.3, 0.3\}$, $W^{-} < x_{25} >= \{0.2, 0.6, 0.1, 0.3, 0.3\}$, $W^{-} < x_{31} >= \{0.2, 0.6, 0.1, 0.3, 0.3\}$, $W^{-} < x_{43} >= \{0.2, 0.6, 0.1, 0.3, 0.3\}$, $W^{-} < x_{54} >= \{0.2, 0.6, 0.1, 0.3, 0.3\}$. *Therefore,* $W^{+} < x_{12} > W^{+} = \{0.1, 0.0, 0.0, 0.0, 0.0\}$, $W^{+} < x_{25} > W^{+} = \{0.1, 0.1, 0.0, 0.0, 0.0\}$, $W^{+} < x_{31} > W^{+} = \{0.1, 0.0, 0.1, 0.0, 0.0\}$, $W^{+} < x_{43} > W^{+} = \{0.1, 0.0, 0.0, 0.1, 0.0\}$, $W^{+} < x_{54} > W^{+} = \{0.1, 0.0, 0.0, 0.0, 0.1\}$, $W^{-} < x_{12} > W^{-} = \{0.2, 0.6, 0.3, 0.3, 0.3\}$, $W^{-} < x_{25} > W^{-} = \{0.2, 0.6, 0.3, 0.3, 0.3\}$, $W^{-} < x_{31} > W^{-} = \{0.2, 0.6, 0.3, 0.3, 0.3\}$, $W^{-} < x_{43} > W^{-} = \{0.2, 0.6, 0.3, 0.3, 0.3\}$, $W^{-} < x_{54} > W^{-} = \{0.2, 0.6, 0.3, 0.3, 0.3\}$. *Then, we have*

$$
\begin{aligned}
T_{[\overline{W}(A_\sigma)]_G}(x_{12}) &= 0.3 \vee [(0.1 \wedge 0.3) \vee (0.0 \wedge 0.2) \vee (0.0 \wedge 0.5) \vee (0.0 \wedge 0.4) \vee (0.0 \wedge 0.1)] \\
&= 0.3 \vee [0.1 \vee 0.0 \vee 0.0 \vee 0.0 \vee 0.0 \vee 0.0] = 0.3, \\
F_{[\overline{W}(A_\sigma)]_G}(x_{12}) &= 0.1 \wedge [(0.2 \vee 0.1) \wedge (0.6 \vee 0.3) \wedge (0.3 \vee 0.1) \wedge (0.3 \vee 0.1) \wedge (0.3 \vee 0.3)] \\
&= 0.1 \wedge [0.2 \wedge 0.6 \wedge 0.3 \wedge 0.3 \wedge 0.3] = 0.1, \\
T_{[\underline{W}(A_\sigma)]_G}(x_{12}) &= 0.3 \wedge [(0.2 \vee 0.3) \wedge (0.6 \vee 0.2) \wedge (0.3 \vee 0.5) \wedge (0.3 \vee 0.4) \wedge (0.3 \vee 0.1)] \\
&= 0.3 \wedge [0.3 \wedge 0.6 \wedge 0.5 \wedge 0.4 \wedge 0.3] = 0.3, \\
F_{[\underline{W}(A_\sigma)]_G}(x_{12}) &= 0.1 \vee [(0.1 \wedge 0.1) \vee (0.0 \wedge 0.3) \vee (0.0 \wedge 0.1) \vee (0.0 \wedge 0.1) \vee (0.0 \wedge 0.3)] \\
&= 0.1 \vee [0.1 \vee 0.0 \vee 0.0 \vee 0.0 \vee 0.0] = 0.1.
\end{aligned}
$$

Similarly,

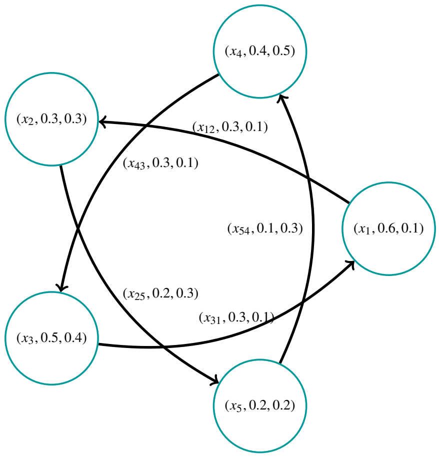
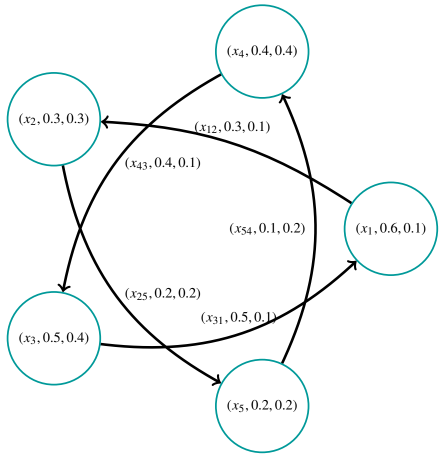
$$
\begin{aligned}
&[\underline{W}A_\sigma]_G(x_{25}) = (0.2, 0.3), \quad [\overline{W}A_\sigma]_G(x_{25}) = (0.2, 0.2), \\
&[\underline{W}A_\sigma]_G(x_{31}) = (0.3, 0.1), \quad [\overline{W}A_\sigma]_G(x_{31}) = (0.5, 0.1), \\
&[\underline{W}A_\sigma]_G(x_{54}) = (0.3, 0.1), \quad [\overline{W}A_\sigma]_G(x_{54}) = (0.4, 0.1), \\
&[\underline{W}A_\sigma]_G(x_{43}) = (0.1, 0.3), \quad [\overline{W}A_\sigma]_G(x_{43}) = (0.1, 0.2).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
[\overline{W}A_\sigma]_G &= \{(x_{12}, 0.3, 0.1), (x_{25}, 0.2, 0.2), (x_{31}, 0.5, 0.1), (x_{43}, 0.4, 0.1), (x_{54}, 0.1, 0.2)\}, \\
[\underline{W}A_\sigma]_G &= \{(x_{12}, 0.3, 0.1), (x_{25}, 0.2, 0.3), (x_{31}, 0.3, 0.1), (x_{43}, 0.3, 0.1), (x_{54}, 0.1, 0.3)\}.
\end{aligned}
$$

As a result, $([\underline{W}A_\sigma]_G, [\overline{W}A_\sigma]_G)$ *is an IF rough relation on the set X. This implies that* $\underline{G} = \left([\underline{R}A]_G, [\underline{W}A_\sigma]_G\right)$ *and* $\overline{G} = \left([\overline{R}A]_G, [\overline{W}A_\sigma]_G\right)$ *are IF graphs, as shown in Table 5.*

Table 5. IF rough graph $G = (\underline{G}, \overline{G})$ in Example 3.8.

The IF lower approximation $\underline{G}$	The IF upper approximation $\overline{G}$
	

Lemma 3.9. *Assume that* $G = (\underline{G}, \overline{G})$ *is an IF rough graph. The lower approximation* $\underline{G} = \left([\underline{R}A]_G, [\underline{W}A_\sigma]_G\right)$ *and the upper approximation* $\overline{G} = \left([\overline{R}A]_G, [\overline{W}A_\sigma]_G\right)$ *of the IF graph G such that*

(1) $\left([\overline{R}A]_G\right)^c = [\underline{R}\,(A^c)]_G$ *and* $\left([\overline{W}A_\sigma]_G\right)^c = [\underline{W}\,(A^c_\sigma)]_G$,
(2) $([\underline{R}A]_G)^c = [\overline{R}\,(A^c)]_G$ *and* $\left([\underline{W}A_\sigma]_G\right)^c = [\overline{W}\,(A^c_\sigma)]_G$,
(3) $[\underline{R}A]_G \leq A \leq [\overline{R}A]_G$ *and* $[\underline{W}A_\sigma]_G \leq A_\sigma \leq [\overline{W}A_\sigma]_G$,
(4) $[\underline{R}(\underline{0}, \underline{1})]_G = [\overline{R}(\underline{0}, \underline{1})]_G = (\underline{0}, \underline{1})$ *and* $[\underline{W}(\underline{0}, \underline{1})]_G = [\overline{W}(\underline{0}, \underline{1})]_G = (\underline{0}, \underline{1})$,
(5) $[\underline{R}(\underline{1}, \underline{0})]_G = [\overline{R}(\underline{1}, \underline{0})]_G = (\underline{1}, \underline{0})$ *and* $[\underline{W}(\underline{1}, \underline{0})]_G = [\overline{W}(\underline{1}, \underline{0})]_G = (\underline{1}, \underline{0})$,
(6) $[\underline{R}(A \vee A_1)]_G \geq [\underline{R}A]_G \vee [\underline{R}A_1]_G$ *and* $[\underline{W}(A_\sigma \vee A_{\sigma_1})]_G \geq [\underline{W}A_\sigma]_G \vee [\underline{W}A_{\sigma 1}]_G$,
(7) $[\overline{R}(A \wedge A_1)]_G \leq [\overline{R}A]_G \wedge [\overline{R}A_1]_G$ *and* $[\overline{W}(A_\sigma \wedge A_{\sigma_1})]_G \leq [\underline{W}A_\sigma]_G \vee [\underline{W}A_{\sigma 1}]_G$,
(8) $A \leq A_1$ *and* $A_\sigma \leq A_{\sigma_1}$ *implies that* $[\underline{R}A]_G \leq [\underline{R}A_1]_G$ *and* $[\underline{W}A_\sigma]_G \leq [\underline{W}A_{\sigma 1}]_G$,
(9) $A \leq A_1$ *and* $A_\sigma \leq A_{\sigma_1}$ *implies that* $[\overline{R}A]_G \leq [\overline{R}A_1]_G$ *and* $[\overline{W}A_\sigma]_G \leq [\overline{W}A_{\sigma 1}]_G$,
(10) $[\overline{R}(A \vee A_1)]_G \geq [\overline{R}A]_G \vee [\overline{R}A_1]_G$ *and* $[\overline{W}(A_\sigma \vee A_{\sigma_1})]_G \geq [\overline{W}A_\sigma]_G \vee [\overline{W}A_{\sigma 1}]_G$,
(11) $[\underline{R}(A \wedge A_1)]_G \leq [\underline{R}A]_G \wedge [\underline{R}A_1]_G$ *and* $[\underline{W}(A_\sigma \wedge A_{\sigma_1})]_G \leq [\underline{W}A_\sigma]_G \wedge [\underline{W}A_{\sigma 1}]_G$,
(12) $[\overline{R}\,([\underline{R}A]_G)]_G \geq [\underline{R}A]_G \geq [\underline{R}\,([\underline{R}A]_G)]_G$ *and* $[\overline{R}\left([\underline{W}A_\sigma]_G\right)]_G \geq [\underline{W}A_\sigma]_G \geq [\underline{W}\left([\underline{W}A_\sigma]_G\right)]_G$,
(13) $[\underline{R}\left([\overline{R}A]_G\right)]_G \leq [\overline{R}A]_G \leq [\overline{R}\left([\overline{R}A]_G\right)]_G$ *and* $[\underline{R}\left([\overline{W}A_\sigma]_G\right)]_G \leq [\overline{W}A_\sigma]_G \leq [\overline{W}\left([\overline{W}A_\sigma]_G\right)]_G$.

Proof. The proof is straightforward from Definitions 3.3, 3.4, 3.7, and Lemma 3.6. □

Lemma 3.10. *Assume that* $G = (\underline{G}, \overline{G})$ *is an IF rough graph and* $\underline{G} = \left([\underline{R}A]_G, [\underline{W}A_\sigma]_G\right)$, $\overline{G} = \left([\overline{R}A]_G, [\overline{W}A_\sigma]_G\right)$ *are the lower and upper approximations of the IF graph G, respectively. For G and* G^*, *we have*

(1) $\underline{G} \leq G \leq \overline{G}$,
(2) $\underline{0_X} = \overline{0_X} = 0_X$ *and* $\underline{1_X} = \overline{1_X} = 1_X$,
(3) $\underline{(G \vee G^*)} \geq \underline{G} \vee \underline{G^*}$,
(4) $\overline{(G \wedge G^*)} \leq \overline{G} \wedge \overline{G^*}$,
(5) $G \leq G^*$ *implies that* $\underline{G} \leq \underline{G^*}$ *and* $\overline{G} \leq \overline{G^*}$,

(6) $\overline{(G \vee G^*)} \geq \overline{G} \vee \overline{G^*}$,

(7) $\underline{(G \wedge G^*)} \leq \underline{G} \wedge \underline{G^*}$,

(8) $\underline{\left(\overline{G}\right)}^c = \underline{(G^c)}$ *and* $\left(\underline{G}\right)^c = \overline{(G^c)}$,

(9) $\overline{\left(\underline{G}\right)} \geq \underline{G} \geq \underline{\left(\underline{G}\right)}$,

(10) $\underline{\left(\overline{G}\right)} \leq \overline{G} \leq \overline{\left(\overline{G}\right)}$.

Proof. The proof is straightforward from Definition 3.7 and Lemma 3.9. □

Remark 3.11. *It is important to note that the proposed IF lower and upper approximations preserve almost all fundamental properties of classical rough sets. In particular, they satisfy key characteristics such as inclusion, monotonicity, boundary consistency, and the relationships between lower, upper, and boundary regions. This demonstrates that our model remains theoretically consistent with the classical rough set framework while providing a more flexible representation of uncertainty through IF information.*

Now we define the IF boundary region and accuracy measure in direct analogy with their classical rough set counterparts. Consequentially, we can obtain high accuracy and the best possible decisions.

Definition 3.12. *Assume that* $G = (\underline{G}, \overline{G})$ *is an IF rough graph and* $\underline{G} = \left([\underline{R}A]_G, [\underline{W}A_\sigma]_G\right)$, $\overline{G} = \left([\overline{R}A]_G, [\overline{W}A_\sigma]_G\right)$ *are the lower and upper approximations of the IF graph G, respectively. Then, we define the vertex IF boundary region* $BND(V)_G$ *and the edge IF boundary region* $BND(E)_G$ *by* $BND(V)_G = [\overline{R}A]_G - [\underline{R}A]_G$ *and* $BND(E)_G = [\overline{R}A_\sigma]_G - [\underline{R}A_\sigma]_G$, *where the operation* $[\overline{R}A]_G - [\underline{R}A]_G$ *is given by using the ordinary difference between their membership values. In this case, the membership degree of an element in* $BND(V)_G$ *is determined by subtracting the membership value of that element in* $[\underline{R}A]_G$ *from its membership value in* $[\overline{R}A]_G$. *Similarly, the non-membership degree can be adjusted consistently so that the basic condition of intuitionistic fuzzy sets is preserved. The same applies when calculating* $BND(E)_G$.

Definition 3.13. *Assume that* $G = (\underline{G}, \overline{G})$ *is an IF rough graph and* $\underline{G} = \left([\underline{R}A]_G, [\underline{W}A_\sigma]_G\right)$, $\overline{G} = \left([\overline{R}A]_G, [\overline{W}A_\sigma]_G\right)$ *are the lower and upper approximations of the IF graph G, respectively. Then, we define the vertex accuracy measure* $ACC(V)_G$ *and the edge accuracy measure* $ACC(E)_G$ *by:*

$$ACC(V)_G = \frac{\left(\sum_{x \in X} \left\{T_{[\underline{R}A]_G}(x) + (1 - F_{[\underline{R}A]_G}(x)\right\}\right)}{\left(\sum_{x \in X} \left\{T_{[\overline{R}A]_G}(x) + (1 - F_{[\overline{R}A]_G}(x)\right\}\right)},$$

$$ACC(E)_G = \frac{\left(\sum_{x_{ij} \in Y} \left\{T_{[\underline{R}A_\sigma]_G}(x) + (1 - F_{[\underline{R}A_\sigma]_G}\left(x_{ij}\right)\right\}\right)}{\left(\sum_{x_{ij} \in Y} \left\{T_{[\overline{R}A_\sigma]_G}\left(x_{ij}\right) + (1 - F_{[\overline{R}A_\sigma]_G}\left(x_{ij}\right)\right\}\right)},$$

where $0 \leq ACC(V)_G \leq 1$ *and* $0 \leq ACC(E)_G \leq 1$. *In the case that the IF lower approximation is greater than the IF upper approximation, we consider the accuracy measure equal to zero. Therefore, the accuracy value remains positive, indicating that there is always at least some degree of certainty in describing the IF rough set, even in the presence of uncertainty and vagueness.*

Example 3.14. *Consider the IF relations R on a set* $X = \{x_1, x_2, x_3\}$ *as given in Table 6.*

Table 6. The IF relation R on X in Example 3.5.

R	x_1	x_2	x_3
x_1	$(0.7, 0.3)$	$(0.3, 0.4)$	$(0.6, 0.2)$
x_2	$(0.3, 0.4)$	$(0.6, 0.4)$	$(0.5, 0.5)$
x_3	$(0.6, 0.2)$	$(0.5, 0.5)$	$(0.8, 0.5)$

By computing, $x_1R^+ = \{7.0, 0.3, 0.6\}$, $x_2R^+ = \{0.3, 6.0, 0.5\}$, $x_3R^+ = \{0.6, 0.5, 8.0\}$, $x_1R^- = \{0.3, 0.4, 0.2\}$, $x_2R^- = \{0.4, 0.4, 0.5\}$, $x_3R^- = \{0.2, 0.5, 0.5\}$. *So,* $< x_1 > R^+ =< x_2 > R^+ =< x_3 > R^+ = \{0.3, 0.3, 0.5\}$, $< x_1 > R^- = \{0.4, 0.5, 0.5\}$, $< x_2 > R^- = \{0.4, 0.5, 0.5\}$, $< x_3 > R^- = \{0.4, 0.5, 0.5\}$. *Similarly,* $R^+x_1 = \{0.7, 0.3, 0.6\}$, $R^+x_2 = \{0.3, 0.6, 0.5\}$, $R^+x_3 = \{0.6, 0.5, 0.8\}$, $R^-x_1 = \{0.3, 0.4, 0.2\}$, $R^-x_2 = \{0.4, 0.4, 0.5\}$, $R^-x_3 = \{0.2, 0.5, 0.5\}$. *Thus,* $R^+ < x_1 >= R^+ < x_2 >= R^+ < x_3 >= \{0.3, 0.3, 0.5\}$, $R^- < x_1 >= \{0.4, 0.5, 0.5\}$, $R^- < x_2 >= \{0.4, 0.5, 0.5\}$, $R^- < x_3 >= \{0.4, 0.5, 0.5\}$. *Therefore,* $R^+ < x_1 > R^+ = R^+ < x_2 > R^+ = R^+ < x_3 > R^+ = \{0.3, 0.3, 0.5\}$, $R^- < x_1 > R^- = \{0.4, 0.5, 0.5\}$, $R^- < x_2 > R^- = \{0.5, 0.5, 0.5\}$, $R^- < x_3 > R^- = \{0.4, 0.5, 0.5\}$.

Table 7 is presented to compare the IF boundary regions and the accuracy measures obtained by our method with those produced by previous approaches [21, 23].

Table 7. Comparison of IF boundary and accuracy measures.

	Previous techniques [21,23]		**Our proposed technique in Definition 3.7**	
$A = (\rho^+, \rho^-)$	$\mathrm{BND}(V)_G$	$\mathrm{ACC}(V)_G$	$\mathrm{BND}(V)_G$	$\mathrm{ACC}(V)_G$
$\{(x_1, 0.1, 0.1), (x_2, 0.1, 0.1), (x_3, 0.1, 0.1)\}$	$\{(x_1, 0.0, 0.1), (x_2, 0.0, 0.1), (x_3, 0.0, 0.1)\}$	0.0	$\{(x_1, 0.0, 0.0), (x_2, 0.0, 0.0), (x_3, 0.0, 0.0)\}$	1.0
$\{(x_1, 0.2, 0.1), (x_2, 0.2, 0.1), (x_3, 0.2, 0.1)\}$	$\{(x_1, 0.0, 0.1), (x_2, 0.0, 0.1), (x_3, 0.0, 0.1)\}$	0.0	$\{(x_1, 0.0, 0.0), (x_2, 0.0, 0.0), (x_3, 0.0, 0.0)\}$	1.0
$\{(x_1, 0.3, 0.1), (x_2, 0.3, 0.1), (x_3, 0.3, 0.1)\}$	$\{(x_1, 0.0, 0.1), (x_2, 0.0, 0.1), (x_3, 0.0, 0.1)\}$	0.0	$\{(x_1, 0.0, 0.0), (x_2, 0.0, 0.0), (x_3, 0.0, 0.0)\}$	1.0
$\{(x_1, 0.4, 0.1), (x_2, 0.4, 0.1), (x_3, 0.4, 0.1)\}$	$\{(x_1, 0.0, 0.1), (x_2, 0.0, 0.1), (x_3, 0.0, 0.1)\}$	0.0	$\{(x_1, 0.0, 0.0), (x_2, 0.0, 0.0), (x_3, 0.0, 0.0)\}$	1.0
$\{(x_1, 0.5, 0.1), (x_2, 0.5, 0.1), (x_3, 0.5, 0.1)\}$	$\{(x_1, 0.0, 0.1), (x_2, 0.0, 0.1), (x_3, 0.0, 0.1)\}$	0.0	$\{(x_1, 0.0, 0.0), (x_2, 0.0, 0.0), (x_3, 0.0, 0.0)\}$	1.0
$\{(x_1, 0.6, 0.1), (x_2, 0.6, 0.1), (x_3, 0.6, 0.1)\}$	$\{(x_1, 0.0, 0.1), (x_2, 0.0, 0.1), (x_3, 0.0, 0.1)\}$	0.0	$\{(x_1, 0.0, 0.0), (x_2, 0.0, 0.0), (x_3, 0.0, 0.0)\}$	1.0
$\{(x_1, 0.7, 0.1), (x_2, 0.7, 0.1), (x_3, 0.7, 0.1)\}$	$\{(x_1, 0.0, 0.1), (x_2, 0.0, 0.1), (x_3, 0.0, 0.1)\}$	0.0	$\{(x_1, 0.0, 0.0), (x_2, 0.0, 0.0), (x_3, 0.0, 0.0)\}$	1.0
$\{(x_1, 0.8, 0.1), (x_2, 0.8, 0.1), (x_3, 0.0, 0.1)\}$	$\{(x_1, 0.0, 0.1), (x_2, 0.0, 0.1), (x_3, 0.0, 0.0)\}$	0.0	$\{(x_1, 0.0, 0.0), (x_2, 0.0, 0.0), (x_3, 0.0, 0.0)\}$	1.0
$\{(x_1, 0.9, 0.1), (x_2, 0.9, 0.1), (x_3, 0.9, 0.1)\}$	$\{(x_1, 0.0, 0.1), (x_2, 0.0, 0.1), (x_3, 0.0, 0.1)\}$	0.0	$\{(x_1, 0.0, 0.0), (x_2, 0.0, 0.0), (x_3, 0.0, 0.0)\}$	1.0

The comparison in Table 7 shows that our method yields an IF boundary region equal to $\{(x_1, 0.0, 0.0), (x_2, 0.0, 0.0), (x_3, 0.0, 0.0)\}$ *for every IF set in the table, whereas the previous method produces* $\{(x_1, 0.0, 0.1), (x_2, 0.0, 0.1), (x_3, 0.0, 0.1)\}$. *A boundary of* $\{(x_1, 0.0, 0.0), (x_2, 0.0, 0.0), (x_3, 0.0, 0.0)\}$ *indicates that the lower and upper approximations coincide, meaning that the classification is exact and free from uncertainty. In contrast, the boundary value* $\{(x_1, 0.0, 0.1), (x_2, 0.0, 0.1), (x_3, 0.0, 0.1)\}$ *reflects the presence of a small degree of residual uncertainty between the approximations. Therefore, our method achieves higher precision and clearer decision boundaries, demonstrating its superiority over the existing approach in reducing ambiguity within IF rough set analysis. Similarly, Table 7 indicates that our technique achieves an IF accuracy value of* 1.0, *while the previous methods [21, 23] yield an accuracy of* 0.0. *An accuracy equal to* 1.0 *signifies perfect classification, meaning that all elements are assigned to their appropriate regions without any misclassifications or uncertainty. In contrast, an accuracy of* 0.0 *reflects a complete inability of the previous methods [21, 23] to correctly approximate the set. This clearly demonstrates that our technique provides significantly more reliable, precise, and effective results than the existing approaches. It is worth noting that a similar table can be easily constructed to compare the IF boundary regions and accuracy measures for the set of edges.*

Remark 3.15. *Assume that* $G = (\underline{G}, \overline{G})$ *is an IF rough graph on an IF set X and* $\underline{G} = \left([\underline{R}A]_G, [\underline{W}A_\sigma]_G\right)$, $\overline{G} = \left([\overline{R}A]_G, [\overline{W}A_\sigma]_G\right)$ *are the lower and upper approximations of the IF graph G, respectively. Then*

(1) As in the usual case, whenever R and W are reflexive IF relations on G, then the equality holds in both (10) and (11) in Lemma 3.9, and thus the equality holds in both (6) and (7) in Lemma 3.10.

(2) As in the usual case, if R and W are reflexive and transitive IF relations, then $\underline{\left(\underline{G}\right)} = \left(\underline{G}\right)$ *and* $\overline{\left(\overline{G}\right)} = \left(\overline{G}\right)$. *Then, an IF topology* τ_G *on an IF set X is generated by the following:*

$$\tau_X = \left\{G \in IFG(X) : G = \underline{G}\right\} \quad or \quad \tau_X = \left\{G \in IFG(X) : [G]^c = \overline{([G]^c)}\right\}.$$

In the following, weaker definitions than Definitions 3.3 and 3.4 will be defined.

Definition 3.16. *Assume that X is a non-empty set and* $R = (R^+, R^-)$ *an IF relation on X. Assume that* $A = (\rho^+, \rho^-)$ *is an IF set in X. The lower and upper IF rough approximations of A are denoted by* $(\underline{R}A)^*$ *and* $(\overline{R}A)^*$, *respectively.* $(\underline{R}A)^*$ *and* $(\overline{R}A)^*$ *are two IF rough sets defined as follows:* $(\underline{R}A)^* = \left(T_{(\underline{R}(A))^*}, F_{(\underline{R}(A))^*}\right)$ *and* $(\overline{R}A)^* = \left(T_{(\overline{R}(A))^*}, F_{(\overline{R}(A))^*}\right)$, *defined as IF sets in X such that, for all* $x \in X$,

$$T_{(\underline{R}(A))^*}(x) = \bigwedge_{x' \in X} [< x > R^-(x') \vee \sigma^+(x')],$$
$$F_{(\underline{R}(A))^*}(x) = \bigvee_{x' \in X} [< x > R^+(x') \wedge \sigma^-(x')],$$
$$T_{(\overline{R}(A))^*}(x) = \bigvee_{x' \in X} [< x > R^+(x') \wedge \sigma^+(x')],$$
$$F_{(\overline{R}(A))^*}(x) = \bigwedge_{x' \in X} [< x > R^-(x') \vee \sigma^-(x')].$$

Definition 3.17. *Assume that X is a non-empty set and* $R = (R^+, R^-)$ *an IF relation on X. Assume that* $A = (\rho^+, \rho^-)$ *is an IF set in X and* $RA = \left((\underline{R}A)^*, (\overline{R}A)^*\right)$ *an IF rough set. Assume that* $Y \subseteq X \times X$ *and* $W = (W^+, W^-)$ *an IF relation on Y such that for all* $x_{ij}, x_{kl} \in Y$

$$W^+\left(x_{ij}, x_{kl}\right) \leq R^+(x_i, x_k) \wedge R^+\left(x_j, x_l\right), \qquad W^-\left(x_{ij}, x_{kl}\right) \leq R^-(x_i, x_k) \vee R^-\left(x_j, x_l\right).$$

Assume that $A_\sigma = (\sigma^+, \sigma^-)$ *is an IF set in Y such that for all* $x_{ij} \in Y$

$$\sigma^+\left(x_{ij}\right) \leq \min\left\{\rho^+(x_i), \rho^+\left(x_j\right)\right\}, \qquad \sigma^-\left(x_{ij}\right) \leq \max\left\{\rho^-(x_i), \rho^-\left(x_j\right)\right\}.$$

Then the upper and lower approximations of A_σ, *denoted by* $(\overline{W}(A_\sigma)^* = \left(T_{(\overline{W}(A_\sigma))^*}, F_{(\overline{W}(A_\sigma))^*}\right)$ *and* $\underline{W}A_\sigma = \left(T_{(\underline{W}(A_\sigma))^*}, F_{(\underline{W}(A_\sigma))^*}\right)$, *respectively, are defined as IF sets in Y such that, for all* $x_{ij} \in Y$,

$$T_{(\overline{W}(A_\sigma))^*}\left(x_{ij}\right) = \bigvee_{x_{kl} \in Y} \left[< x_{ij} > W^+(x_{kl}) \wedge \sigma^+(x_{kl})\right],$$
$$F_{(\overline{W}(A_\sigma))^*}\left(x_{ij}\right) = \bigwedge_{x_{kl} \in Y} \left[< x_{ij} > W^-(x_{kl}) \vee \sigma^-(x_{kl})\right],$$
$$T_{(\underline{W}(A_\sigma))^*}\left(x_{ij}\right) = \bigwedge_{x_{kl} \in Y} \left[< x_{ij} > W^-(x_{kl}) \vee \sigma^+(x_{kl})\right],$$
$$F_{(\underline{W}(A_\sigma))^*}\left(x_{ij}\right) = \bigvee_{x_{kl} \in Y} \left[< x_{ij} > W^+(x_{kl}) \wedge \sigma^-(x_{kl})\right].$$

A pair $(WA_\sigma)^* = \left((\underline{W}A_\sigma)^*, (\overline{W}(A_\sigma))^*\right)$ *is called an IF rough relation on an IF rough set* $(RA)^* = \left((\underline{R}A)^*, (\overline{R}A)^*\right)$ *if and only if* $T_{(\overline{W}(A_\sigma))^*} \neq T_{(\underline{W}(A_\sigma))^*}, F_{(\overline{W}(A_\sigma))^*} \neq F_{(\underline{W}(A_\sigma))^*}$. *All the results given in this section are satisfied exactly; the only main difference comes from the fact that* $R^+ < x > R^+ \leq < x > R^+, R^- < x > R^- \geq < x > R^-$ *and* $W^+ < x > W^+ \leq < x > W^+, W^- < x > W^- \geq < x > W^-$ *for all vertexes and edges of the produced graphs on the set X.*

Lemma 3.18. *Assume that* (X, R) *is an IF approximation space, and* $WA_\sigma = \left(\underline{W}A_\sigma, \overline{W}A_\sigma\right)$ *is an IF rough relation on the IF rough set* $RA = \left(\underline{R}A, \overline{R}A\right)$. *Then,*

(1) $[\underline{R}A]^* \leq \underline{R}A$, $[\underline{W}A_\sigma]^* \leq \underline{W}A_\sigma$ *and so* $([\underline{R}A]_G)^* \leq [\underline{R}A]_G$, $([\underline{W}A_\sigma]_G)^* \leq [\underline{W}A_\sigma]_G$.
(2) $\overline{R}A \leq [\overline{R}A]^*$, $\overline{W}A_\sigma \leq [\overline{W}A_\sigma]^*$ *and so* $[\overline{R}A]_G \leq ([\overline{R}A]_G)^*$, $[\overline{W}A_\sigma]_G \leq ([\overline{W}A_\sigma]_G)^*$.

Proof. The proof is straightforward from the fact that $R^+ < x > R^+ \leq < x > R^+, R^- < x > R^- \leq < x > R^-$ and $W^+ < x > W^+ \leq < x > W^+, W^- < x > W \leq < x > W$. □

Example 3.19. *In Example 3.8,* $[RA]_G = \left([\underline{R}A]_G, [\overline{R}A]_G\right)$ *is given by:*

$$[\underline{R}A]_G = \{(x_1, 0.6, 0.1), (x_2, 0.3, 0.3), (x_3, 0.5, 0.4), (x_4, 0.4, 0.5), (x_5, 0.2, 0.2)\},$$
$$[\overline{R}A]_G = \{(x_1, 0.6, 0.1), (x_2, 0.3, 0.3), (x_3, 0.5, 0.4), (x_4, 0.4, 0.4), (x_5, 0.2, 0.2)\}.$$

On the other hand, $([RA]_G)^* = \left(([\underline{R}A]_G)^*, ([\overline{R}A]_G)^*\right)$ *is given by:*

$$([\underline{R}A]_G)^* = \{(x_1, 0.4, 0.1), (x_2, 0.3, 0.3), (x_3, 0.4, 0.4), (x_4, 0.4, 0.5), (x_5, 0.2, 0.2)\},$$
$$([\overline{R}A]_G)^* = \{(x_1, 0.6, 0.1), (x_2, 0.3, 0.3), (x_3, 0.5, 0.4), (x_4, 0.4, 0.4), (x_5, 0.3, 0.2)\}.$$

Hence, $([\underline{R}A]_G)^* \leq [\underline{R}A]_G$ $([\overline{R}A]_G)^* \nleq [\overline{R}A]_G$. *Any one can add similar examples to illustrate Lemma 3.18.*

Remark 3.20. *Let* $(R, [RA]_G, W, [WA_\sigma]_G)$ *be an IF rough graph on* G. *It should be noted from Lemma 3.18 that Definitions 3.3, 3.4 decrease the upper approximation and increase the lower approximation. This new approach is more general than many previous approaches. As a special case:*

(1) If we have equivalence IF relation on the set X, then Definition 3.3 will be the IF of the main definition given by Pawlak [1].
(2) If we have symmetric IF relations R on the set X, then Definition 3.3 will be the IF of the definition given by Allam [25].
(3) If we have reflexive and symmetric IF relation R on the set X, then Definition 3.3 will be the IF of the definition given by Kandil [26].
(4) If every IF set $B = \{(x, \mu(x), \nu(x)) \mid x \in X\}$ *is equal to* $\{(x, \mu(x), 1 - \mu(x)) \mid x \in X\}$ *and every IF relation R is equal to* $\{((x, x^*), \mu(x, x^*), 1 - \mu(x, x^*)) \mid (x, x^*) \in X \times X\}$, *the same for W, then Definitions 3.3, 3.4 will coincide with the definition given in [11] in the case of the trivial fuzzy ideal* $\mathfrak{L} = \mathfrak{L}_0$.

4. Certain operations on IF rough graphs

This section focuses on defining fundamental operations for IF rough graphs, including the complement, union, intersection, and join, and establishes that the resulting structures are also IF rough graphs.

Definition 4.1. *Assume that* $G = (\underline{G}, \overline{G})$ *is an IF rough graph. The* μ *complement of* G*, denoted by* $G^{\mu} = \left(\underline{G}^{\mu}, \overline{G}^{\mu}\right)$*, is an IF rough graph, where* $\underline{G}^{\mu} = \left([\underline{R}A]_G{}^{\mu}, [\underline{W}A_{\sigma}]_G{}^{\mu}\right)$ *and* $\overline{G}^{\mu} = \left([\overline{R}A]_G{}^{\mu}, [\overline{W}A_{\sigma}]_G{}^{\mu}\right)$ *are IF graphs such that*

(i) $\forall x \in [RA]_G$

$$T_{[\underline{R}(A)]_G{}^{\mu}}(x) = T_{[\underline{R}(A)]_G}(x), \quad F_{[\underline{R}(A)]_G{}^{\mu}}(x) = F_{[\underline{R}(A)]_G}(x),$$
$$T_{[\overline{R}(A)]_G{}^{\mu}}(x) = T_{[\overline{R}(A)]_G}(x), \quad F_{[\overline{R}(A)]_G{}^{\mu}}(x) = F_{[\overline{R}(A)]_G}(x),$$

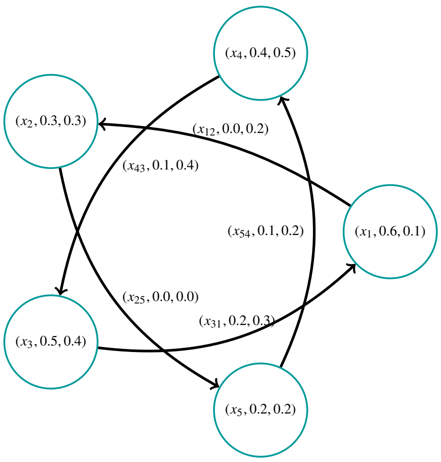
(ii) $\forall x_{ij} \in [WA_{\sigma}]_G$

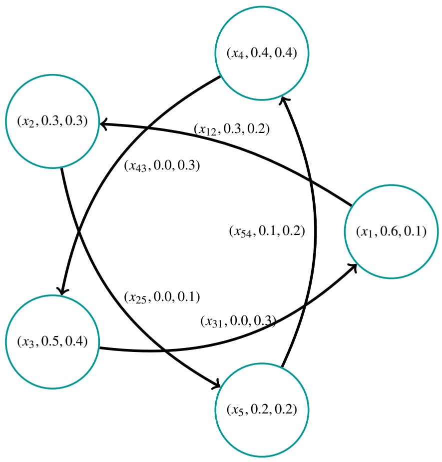
$$T_{[\underline{W}A_{\sigma}]_G{}^{\mu}}\left(x_{ij}\right) = \min\left\{T_{[\underline{R}A]_G}\left(x_i\right), T_{[\underline{R}A]_G}(x_j)\right\} - T_{[\underline{W}A_{\sigma}]_G}\left(x_{ij}\right),$$
$$F_{[\underline{W}A_{\sigma}]_G{}^{\mu}}\left(x_{ij}\right) = \max\left\{F_{[\underline{R}A]_G}\left(x_i\right), F_{[\underline{R}A]_G}(x_j)\right\} - F_{[\underline{W}A_{\sigma}]_G}\left(x_{ij}\right),$$
$$T_{[\overline{W}A_{\sigma}]_G{}^{\mu}}\left(x_{ij}\right) = \min\left\{T_{[\overline{R}A]_G}\left(x_i\right), T_{[\overline{R}A]_G}(x_j)\right\} - T_{[\overline{W}A_{\sigma}]_G}\left(x_{ij}\right),$$
$$F_{[\overline{W}A_{\sigma}]_G{}^{\mu}}\left(x_{ij}\right) = \max\left\{F_{[\overline{R}A]_G}\left(x_i\right), F_{[\overline{R}A]_G}(x_j)\right\} - F_{[\overline{W}A_{\sigma}]_G}\left(x_{ij}\right).$$

Example 4.2. *Consider the IF rough graph* G *as shown in Table 5. Thus,* μ*-complement of* G *is* $G^{\mu} = \left(\underline{G}^{\mu}, \overline{G}^{\mu}\right)$*, where* $\underline{G}^{\mu} = \left([\underline{R}A]_G{}^{\mu}, [\underline{W}A_{\sigma}]_G{}^{\mu}\right)$ *and* $\overline{G}^{\mu} = \left([\overline{R}A]_G{}^{\mu}, [\overline{W}A_{\sigma}]_G{}^{\mu}\right)$ *are IF graphs as shown in Table 8.*

Table 8. IF rough graph $G^{\mu} = \left(\underline{G}^{\mu}, \overline{G}^{\mu}\right)$ in Example 4.2.

The IF lower approximation $\underline{G}^{\mu}$	The IF upper approximation $\overline{G}^{\mu}$





Definition 4.3. *Assume that* $G = (\underline{G}, \overline{G})$ *is an IF rough graph. The complement of* G*, denoted by* $\underline{G'} = (\underline{G}', \overline{G}')$*, is an IF rough graph, where* $\underline{G'} = \left([\underline{R'}A]_G, [\underline{W'}A_{\sigma}]_G\right)$ *and* $\overline{G}' = \left([\overline{R}'A]_G, [\overline{W}'A_{\sigma}]_G\right)$ *are IF graphs such that*

(i) $\forall x \in [RA]_G$

$$T_{[\underline{R'}A]_G}(x) = T_{[\underline{R}(A)]_G}(x), \quad F_{[\underline{R'}A]_G}(x) = F_{[\underline{R}(A)]_G}(x),$$
$$T_{[\overline{R}'A]_G}(x) = T_{[\overline{R}(A)]_G}(x), \quad F_{[\overline{R}'A]_G}(x) = F_{[\overline{R}(A)]_G}(x),$$

(ii) $\forall x_i, x_j \in [RA]_G$

$$T_{[\underline{W'}A_\sigma]_G}\left(x_{ij}\right) = \min\left\{T_{[\underline{R}A]_G}\left(x_i\right), T_{[\underline{R}A]_G}(x_j)\right\} - T_{[\underline{W}A_\sigma]_G}\left(x_{ij}\right),$$
$$F_{[\underline{W'}A_\sigma]_G}\left(x_{ij}\right) = \max\left\{F_{[\underline{R}A]_G}\left(x_i\right), F_{[\underline{R}A]_G}(x_j)\right\} - F_{[\underline{W}A_\sigma]_G}\left(x_{ij}\right),$$
$$T_{[\overline{W'}A_\sigma]_G}\left(x_{ij}\right) = \min\left\{T_{[\overline{R}A]_G}\left(x_i\right), T_{[\overline{R}A]_G}(x_j)\right\} - T_{[\overline{W}A_\sigma]_G}\left(x_{ij}\right),$$
$$F_{[\overline{W'}A_\sigma]_G}\left(x_{ij}\right) = \max\left\{F_{[\overline{R}A]_G}\left(x_i\right), F_{[\overline{R}A]_G}(x_j)\right\} - F_{[\overline{W}A_\sigma]_G}\left(x_{ij}\right).$$

Example 4.4. *Consider the IF rough graph G as shown in Table 9.*

Table 9. IF rough graph $G = \left(\underline{G}, \overline{G}\right)$ in Example 4.4.

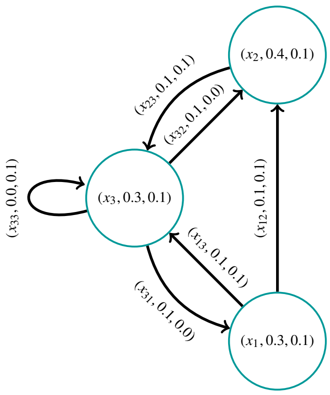
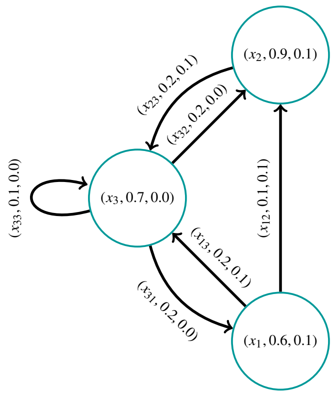
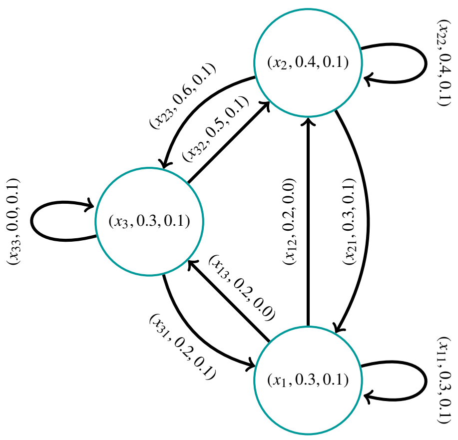
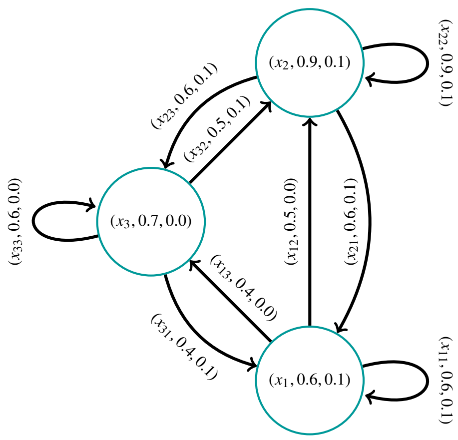
The IF lower approximation $\underline{G}$	The IF upper approximation $\overline{G}$
	

Table 10. IF rough graph $G' = \left(\underline{G'}\overline{G}'\right)$ in Example 4.4.

The IF lower approximation $\underline{G'}$	The IF upper approximation $\overline{G}'$
	

The complement of G, denoted by $\underline{G}' = (\underline{G}', \overline{G}')$*, is an IF rough graph, where* $\underline{G}' = \left([\underline{R}'A]_G, [\underline{W}'A_\sigma]_G\right)$ *and* $\overline{G}' = \left([\overline{R}'A]_G, [\overline{W}'A_\sigma]_G\right)$ *are IF graphs, as shown in Table 10.*

Remark 4.5. *In Definition 4.1, in case there is an edge* x_{ij} *between two vertices "*x_i*" and "*x_j*", there must be an edge between "*x_i*" and "*x_j*" in each approximation graph of G. Whereas in Definition 4.3, in case there is an edge* x_{ij} *between two vertices "*x_i*" and "*x_j*", then there may or may not be an edge between "*x_i*" and "*x_j*" in any approximation graph of G.*

Definition 4.6. *Assume that* $G_1 = \left(\underline{G}_1, \overline{G}_1\right)$ *and* $G_2 = \left(\underline{G}_2, \overline{G}_2\right)$ *IF rough graphs on X. The union of* G_1 *and* G_2 *is an IF rough graph* $G = G_1 \cup G_2 = \left(\underline{G}_1 \cup \underline{G}_2, \overline{G}_1 \cup \overline{G}_2\right)$*, where*

$$\underline{G}_1 \cup \underline{G}_2 = \left([\underline{R}A_1]_{G_1} \cup [\underline{R}A_2]_{G_2}, [\underline{W}A_{\sigma_1}]_{G_1} \cup [\underline{W}A_{\sigma_2}]_{G_2}\right)$$

and

$$\overline{G}_1 \cup \overline{G}_2 = \left([\overline{R}A_1]_{G_1} \cup [\overline{R}A_2]_{G_2}, [\overline{W}A_{\sigma_1}]_{G_1} \cup [\overline{W}A_{\sigma_2}]_{G_2}\right)$$

are IF graphs, respectively, such that

(i) if $x \in [RA_1]_{G1}$ *but* $x \notin [RA_2]_{G2}$

$$\begin{aligned}
T_{[\underline{R}(A_1)]_{G_1} \cup [\underline{R}_i(A_2)]_{G_2}}(x) &= T_{[\underline{R}(A_1)]_{G_1}}(x),\\
F_{[\underline{R}(A_1)]_{G_1} \cup [\underline{R}(A_2)]_{G_2}}(x) &= F_{[\underline{R}(A_1)]_{G_1}}(x),\\
T_{[\overline{R}(A_1)]_{G_1} \cup [\overline{R}(A_2)]_{G_2}}(x) &= T_{[\overline{R}(A_1)]_{G_1}}(x),\\
F_{[\overline{R}(A_1)]_{G_1} \cup [\overline{R}(A_2)]_{G_2}}(x) &= F_{[\overline{R}(A_1)]_{G_1}}(x),
\end{aligned}$$

(ii) if $x \in [RA_2]_{G2}$ *but* $x \notin [RA_1]_{G1}$

$$\begin{aligned}
T_{[\underline{R}(A_1)]_{G_1} \cup [\underline{R}_i(A_2)]_{G_2}}(x) &= T_{[\underline{R}(A_2)]_{G_2}}(x),\\
F_{[\underline{R}(A_1)]_{G_1} \cup [\underline{R}(A_2)]_{G_2}}(x) &= F_{[\underline{R}(A_2)]_{G_2}}(x),\\
T_{[\overline{R}(A_1)]_{G_1} \cup [\overline{R}(A_2)]_{G_2}}(x) &= T_{[\overline{R}(A_2)]_{G_2}}(x),\\
F_{[\overline{R}(A_1)]_{G_1} \cup [\overline{R}(A_2)]_{G_2}}(x) &= F_{[\overline{R}(A_2)]_{G_2}}(x),
\end{aligned}$$

(iii) if $x \in [RA]_{G1}$ *and* $x \in [RA]_{G2}$

$$\begin{aligned}
T_{[\underline{R}(A_1)]_{G_1} \cup [R(A_2)]_{G_2}}(x) &= \max\left\{T_{[\underline{R}(A_1)]_{G_1}}(x), T_{[\underline{R}(A_2)]_{G_2}}(x)\right\},\\
F_{[\underline{R}(A_1)]_{G_1} \cup [\underline{R}(A_2)]_{G_2}}(x) &= \min\left\{F_{[\underline{R}(A_1)]_{G_1}}(x), F_{[\underline{R}(A_2)]_{G_2}}(x)\right\},\\
T_{[\overline{R}(A_1)]_{G_1} \cup [\overline{R}(A_2)]_{G_2}}(x) &= \max\left\{T_{[\overline{R}(A_1)]_{G_1}}(x), T_{[\overline{R}(A_2)]_{G_2}}(x)\right\},\\
F_{[\overline{R}(A_1)]_{G_1} \cup [\overline{R}(A_2)]_{G_2}}(x) &= \min\left\{F_{[\overline{R}(A_1)]_{G_1}}(x), F_{[\overline{R}(A_2)]_{G_2}}(x)\right\},
\end{aligned}$$

(iv) if $x_{ij} \in [WA_{\sigma_1}]_{G_1}$ *but* $x_{ij} \notin [WA_{\sigma_2}]_{G_2}$

$$\begin{aligned}
T_{[\underline{W}(A_{\sigma_1})]_{G_1} \cup [\underline{W}(A_{\sigma_2})]_{G_2}}\left(x_{ij}\right) &= T_{[\underline{W}(A_{\sigma_1})]_{G_1}}\left(x_{ij}\right),\\
F_{[\underline{W}(A_{\sigma_1})]_{G_1} \cup [\underline{W}(A_{\sigma_2})]_{G_2}}\left(x_{ij}\right) &= F_{[\underline{W}(A_{\sigma_1})]_{G_1}}\left(x_{ij}\right),\\
T_{[\overline{W}(A_{\sigma_2})]_{G_1} \cup [\overline{W}(A_{\sigma_2})]_{G_2}}\left(x_{ij}\right) &= T_{[\overline{W}(A_{\sigma_2})]_{G_1}}\left(x_{ij}\right),\\
F_{[\overline{W}(A_{\sigma_2})]_{G_1} \cup [\overline{W}(A_{\sigma_2})]_{G_2}}\left(x_{ij}\right) &= F_{[\overline{W}(A_{\sigma_2})]_{G_1}}\left(x_{ij}\right),
\end{aligned}$$

(v) if $x_{ij} \in [WA_{\sigma_2}]_{G_2}$ *but* $x_{ij} \notin [WA_{\sigma_1}]_{G_1}$

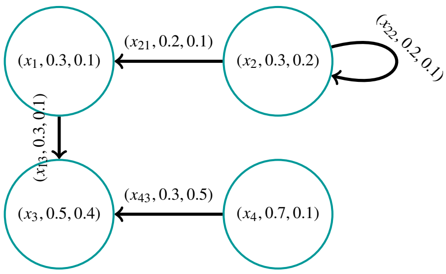
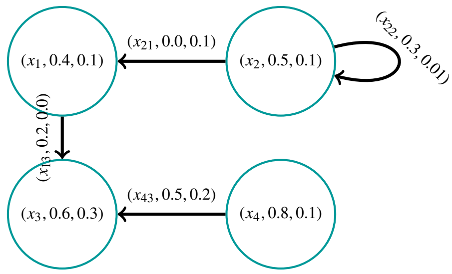
$$\begin{aligned}
T_{[\underline{W}(A_{\sigma_1})]_{G_1} \cup [\underline{W}(A_{\sigma_2})]_{G_2}}\left(x_{ij}\right) &= T_{[\underline{W}(A_{\sigma_2})]_{G_2}}\left(x_{ij}\right),\\
F_{[\underline{W}(A_{\sigma_1})]_{G_1} \cup [\underline{W}(A_{\sigma_2})]_{G_2}}\left(x_{ij}\right) &= F_{[\underline{W}(A_{\sigma_2})]_{G_2}}\left(x_{ij}\right),\\
T_{[\overline{W}(A_{\sigma_1})]_{G_1} \cup [\overline{W}(A_{\sigma_2})]_{G_2}}\left(x_{ij}\right) &= T_{[\overline{W}(A_{\sigma_2})]_{G_2}}\left(x_{ij}\right),\\
F_{[\overline{W}(A_{\sigma_1})]_{G_1} \cup [\overline{W}(A_{\sigma_2})]_{G_2}}\left(x_{ij}\right) &= F_{[\overline{W}(A_{\sigma_2})]_{G_2}}\left(x_{ij}\right),
\end{aligned}$$

(vi) if $x_{ij} \in [WA_{\sigma_1}]_{G_1}$ *and* $x_{ij} \in [WA_{\sigma_2}]_{G_2}$

$$\begin{aligned}
T_{[\underline{W}(A_{\sigma_1})]_{G_1} \cup [\underline{W}(A_{\sigma_2})]_{G_2}}\left(x_{ij}\right) &= \max\left\{T_{[\underline{W}(A_{\sigma_1})]_{G_1}}\left(x_{ij}\right), T_{[\underline{W}(A_{\sigma_2})]_{G_2}}\left(x_{ij}\right)\right\},\\
F_{[\underline{W}(A_{\sigma_1})]_{G_1} \cup [\underline{W}(A_{\sigma_2})]_{G_2}}\left(x_{ij}\right) &= \min\left\{F_{[\underline{W}(A_{\sigma_1})]_{G_1}}\left(x_{ij}\right), F_{[\underline{W}(A_{\sigma_2})]_{G_2}}\left(x_{ij}\right)\right\},\\
T_{[\overline{W}(A_{\sigma_1})]_{G_1} \cup [\overline{W}(A_{\sigma_2})]_{G_2}}\left(x_{ij}\right) &= \max\left\{T_{[\overline{W}(A_{\sigma_1})]_{G_1}}\left(x_{ij}\right), T_{[\overline{W}(A_{\sigma_2})]_{G_2}}\left(x_{ij}\right)\right\},\\
F_{[\overline{W}(A_{\sigma_1})]_{G_1} \cup [\overline{W}(A_{\sigma_2})]_{G_2}}\left(x_{ij}\right) &= \min\left\{F_{[\overline{W}(A_{\sigma_1})]_{G_1}}\left(x_{ij}\right), F_{[\overline{W}(A_{\sigma_2})]_{G_2}}\left(x_{ij}\right)\right\},
\end{aligned}$$

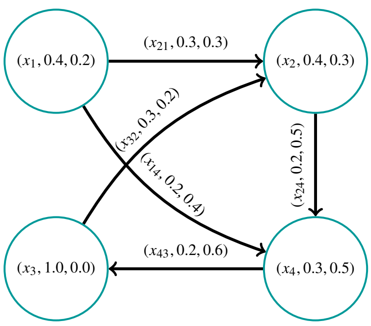
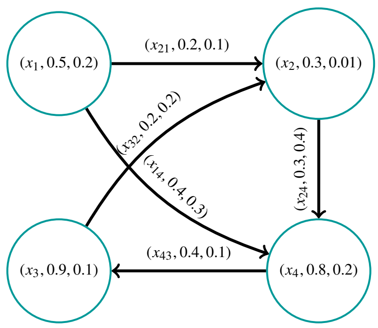
Example 4.7. *Assume that* $X = \{x_1, x_2, x_3, x_4\}$ *is a set. Consider* $G_1 = \left(\underline{G}_1, \overline{G}_1\right)$ *and* $G_2 = \left(\underline{G}_2, \overline{G}_2\right)$ *are two IF rough graphs on X, where* $\underline{G}_1 = \left([\underline{R}A_1]_{G_1}, [\underline{W}A_{\sigma_1}]_{G_1}\right)$ *and* $\overline{G}_1 = \left([\overline{R}A_1]_{G_1}, [\overline{W}A_{\sigma_1}]_{G_1}\right)$ *are IF graphs, as shown in Table 11.*

Table 11. IF rough graph $G_1 = \left(\underline{G}_1, \overline{G}_1\right)$ in Example 4.7.

The IF lower approximation $\underline{G}_1$	The IF upper approximation $\overline{G}_1$
	

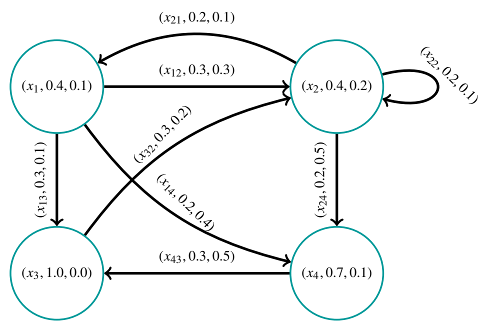
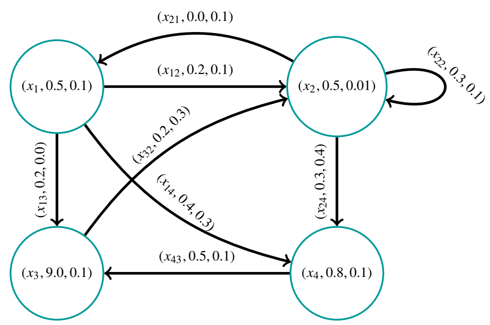
$\underline{G}_2 = \left([\underline{R}A_2]_{G_2}, [\underline{W}A_{\sigma_2}]_{G_2}\right)$ *and* $\overline{G}_2 = \left([\overline{R}A_2]_{G_2}, [\overline{W}A_{\sigma_2}]_{G_2}\right)$ *are also IF graphs as shown in Table 12.*

Table 12. IF rough graph $G_2 = \left(\underline{G}_2, \overline{G}_2\right)$ in Example 4.7.

The IF lower approximation $\underline{G}_2$	The IF upper approximation $\overline{G}_2$
	

The union of G_1 *and* G_2 *is a* $G = G_1 \cup G_2 = \left(\underline{G}_1 \cup \underline{G}_2, \overline{G}_1 \cup \overline{G}_2\right)$, *where* $\underline{G}_1 \cup \underline{G}_2 = \left([\underline{R}A_1]_{G_1} \cup [\underline{R}A_2]_{G_2}, [\underline{W}A_{\sigma_1}]_{G_1} \cup [\underline{W}A_{\sigma_2}]_{G_2}\right)$ *and* $\overline{G}_1 \cup \overline{G}_2 = \left([\overline{R}A_1]_{G_1} \cup [\overline{R}A_2]_{G_2}, [\overline{W}A_{\sigma_1}]_{G_1} \cup [\overline{W}A_{\sigma_2}]_{G_2}\right)$ *are IF graphs as shown in Table 13.*

Table 13. IF rough graph $G_1 \cup G_2 = \left(\underline{G}_1 \cup \underline{G}_2, \overline{G}_1 \cup \overline{G}_2\right)$ in Example 4.7.

$\underline{G}_1 \cup \underline{G}_2$	$\overline{G}_1 \cup \overline{G}_2$
	

Definition 4.8. *Assume that* $G_1 = \left(\underline{G}_1, \overline{G}_1\right)$ *and* $G_2 = \left(\underline{G}_2, \overline{G}_2\right)$ *IF rough graphs on X. The intersection of* G_1 *and* G_2 *is an IF rough graph* $G = G_1 \cap G_2 = \left(\underline{G}_1 \cap \underline{G}_2, \overline{G}_1 \cap \overline{G}_2\right)$, *where* $\underline{G}_1 \cap \underline{G}_2 = \left([\underline{R}A_1]_{G_1} \cap [\underline{R}A_2]_{G_2}, [\underline{W}A_{\sigma_1}]_{G_1} \cap [\underline{W}A_{\sigma_2}]_{G_2}\right)$ *and* $\overline{G}_1 \cap \overline{G}_2 = \left([\overline{R}A_1]_{G_1} \cap [\overline{R}A_2]_{G_2}, [\overline{W}A_{\sigma_1}]_{G_1} \cap [\overline{W}A_{\sigma_2}]_{G_2}\right)$ *are IF graphs, respectively, such that*

(i) if $x \in [RA_1]_{G1}$ *but* $x \notin [RA_2]_{G2}$

$$
\begin{aligned}
T_{[\underline{R}(A_1)]_{G_1} \cap [\underline{R}_i(A_2)]_{G_2}}(x) &= T_{[\underline{R}(A_1)]_{G_1}}(x), \\
F_{[\underline{R}(A_1)]_{G_1} \cap [\underline{R}(A_2)]_{G_2}}(x) &= F_{[\underline{R}(A_1)]_{G_1}}(x), \\
T_{[\overline{R}(A_1)]_{G_1} \cap [\overline{R}(A_2)]_{G_2}}(x) &= T_{[\overline{R}(A_1)]_{G_1}}(x), \\
F_{[\overline{R}(A_1)]_{G_1} \cap [\overline{R}(A_2)]_{G_2}}(x) &= F_{[\overline{R}(A_1)]_{G_1}}(x),
\end{aligned}
$$

(ii) if $x \in [RA_2]_{G2}$ *but* $x \notin [RA_1]_{G1}$

$$
\begin{aligned}
T_{[\underline{R}(A_1)]_{G_1} \cap [\underline{R}_i(A_2)]_{G_2}}(x) &= T_{[\underline{R}(A_2)]_{G_2}}(x), \\
F_{[\underline{R}(A_1)]_{G_1} \cap [\underline{R}(A_2)]_{G_2}}(x) &= F_{[\underline{R}(A_2)]_{G_2}}(x), \\
T_{[\overline{R}(A_1)]_{G_1} \cap [\overline{R}(A_2)]_{G_2}}(x) &= T_{[\overline{R}(A_2)]_{G_2}}(x), \\
F_{[\overline{R}(A_1)]_{G_1} \cap [\overline{R}(A_2)]_{G_2}}(x) &= F_{[\overline{R}(A_2)]_{G_2}}(x),
\end{aligned}
$$

(iii) if $x \in [RA]_{G1}$ *and* $x \in [RA]_{G2}$

$$
\begin{aligned}
T_{[\underline{R}(A_1)]_{G_1} \cap [R(A_2)]_{G_2}}(x) &= \min\left\{T_{[\underline{R}(A_1)]_{G_1}}(x), T_{[\underline{R}(A_2)]_{G_2}}(x)\right\}, \\
F_{[\underline{R}(A_1)]_{G_1} \cap [\underline{R}(A_2)]_{G_2}}(x) &= \max\left\{F_{[\underline{R}(A_1)]_{G_1}}(x), F_{[\underline{R}(A_2)]_{G_2}}(x)\right\}, \\
T_{[\overline{R}(A_1)]_{G_1} \cap [\overline{R}(A_2)]_{G_2}}(x) &= \min\left\{T_{[\overline{R}(A_1)]_{G_1}}(x), T_{[\overline{R}(A_2)]_{G_2}}(x)\right\}, \\
F_{[\overline{R}(A_1)]_{G_1} \cap [\overline{R}(A_2)]_{G_2}}(x) &= \max\left\{F_{[\overline{R}(A_1)]_{G_1}}(x), F_{[\overline{R}(A_2)]_{G_2}}(x)\right\},
\end{aligned}
$$

(iv) if $x_{ij} \in [WA_{\sigma_1}]_{G_1}$ *but* $x_{ij} \notin [WA_{\sigma_2}]_{G_2}$

$$
\begin{aligned}
T_{[\underline{W}(A_{\sigma_1})]_{G_1}\cap[\underline{W}(A_{\sigma_2})]_{G_2}}\left(x_{ij}\right) &= T_{[\underline{W}(A_{\sigma_1})]_{G_1}}\left(x_{ij}\right),\\
F_{[\underline{W}(A_{\sigma_1})]_{G_1}\cap[\underline{W}(A_{\sigma_2})]_{G_2}}\left(x_{ij}\right) &= F_{[\underline{W}(A_{\sigma_1})]_{G_1}}\left(x_{ij}\right),\\
T_{[\overline{W}(A_{\sigma_2})]_{G_1}\cap[\overline{W}(A_{\sigma_2})]_{G_2}}\left(x_{ij}\right) &= T_{[\overline{W}(A_{\sigma_2})]_{G_1}}\left(x_{ij}\right),\\
F_{[\overline{W}(A_{\sigma_2})]_{G_1}\cap[\overline{W}(A_{\sigma_2})]_{G_2}}\left(x_{ij}\right) &= F_{[\overline{W}(A_{\sigma_2})]_{G_1}}\left(x_{ij}\right),
\end{aligned}
$$

(v) if $x_{ij} \in [WA_{\sigma_2}]_{G_2}$ *but* $x_{ij} \notin [WA_{\sigma_1}]_{G_1}$

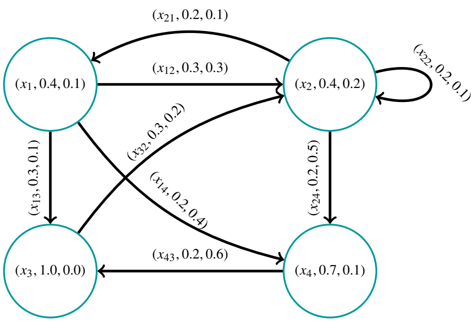
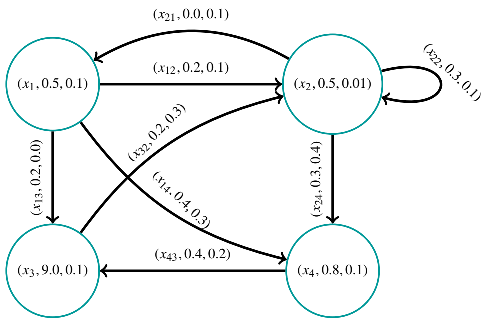
$$
\begin{aligned}
T_{[\underline{W}(A_{\sigma_1})]_{G_1}\cap[\underline{W}(A_{\sigma_2})]_{G_2}}\left(x_{ij}\right) &= T_{[\underline{W}(A_{\sigma_2})]_{G_2}}\left(x_{ij}\right),\\
F_{[\underline{W}(A_{\sigma_1})]_{G_1}\cap[\underline{W}(A_{\sigma_2})]_{G_2}}\left(x_{ij}\right) &= F_{[\underline{W}(A_{\sigma_2})]_{G_2}}\left(x_{ij}\right),\\
T_{[\overline{W}(A_{\sigma_1})]_{G_1}\cap[\overline{W}(A_{\sigma_2})]_{G_2}}\left(x_{ij}\right) &= T_{[\overline{W}(A_{\sigma_2})]_{G_2}}\left(x_{ij}\right),\\
F_{[\overline{W}(A_{\sigma_1})]_{G_1}\cap[\overline{W}(A_{\sigma_2})]_{G_2}}\left(x_{ij}\right) &= F_{[\overline{W}(A_{\sigma_2})]_{G_2}}\left(x_{ij}\right),
\end{aligned}
$$

(vi) if $x_{ij} \in [WA_{\sigma_1}]_{G_1}$ *and* $x_{ij} \in [WA_{\sigma_2}]_{G_2}$

$$
\begin{aligned}
T_{[\underline{W}(A_{\sigma_1})]_{G_1}\cap[\underline{W}(A_{\sigma_2})]_{G_2}}\left(x_{ij}\right) &= \min\left\{T_{[\underline{W}(A_{\sigma_1})]_{G_1}}\left(x_{ij}\right), T_{[\underline{W}(A_{\sigma_2})]_{G_2}}\left(x_{ij}\right)\right\},\\
F_{[\underline{W}(A_{\sigma_1})]_{G_1}\cap[\underline{W}(A_{\sigma_2})]_{G_2}}\left(x_{ij}\right) &= \max\left\{F_{[\underline{W}(A_{\sigma_1})]_{G_1}}\left(x_{ij}\right), F_{[\underline{W}(A_{\sigma_2})]_{G_2}}\left(x_{ij}\right)\right\},\\
T_{[\overline{W}(A_{\sigma_1})]_{G_1}\cap[\overline{W}(A_{\sigma_2})]_{G_2}}\left(x_{ij}\right) &= \min\left\{T_{[\overline{W}(A_{\sigma_1})]_{G_1}}\left(x_{ij}\right), T_{[\overline{W}(A_{\sigma_2})]_{G_2}}\left(x_{ij}\right)\right\},\\
F_{[\overline{W}(A_{\sigma_1})]_{G_1}\cap[\overline{W}(A_{\sigma_2})]_{G_2}}\left(x_{ij}\right) &= \max\left\{F_{[\overline{W}(A_{\sigma_1})]_{G_1}}\left(x_{ij}\right), F_{[\overline{W}(A_{\sigma_2})]_{G_2}}\left(x_{ij}\right)\right\},
\end{aligned}
$$

Example 4.9. *Consider the two IF rough graphs* G_1 *and* G_2 *as shown in Tables 11 and 12, respectively. The intersection of* G_1 *and* G_2 *is* $G = G_1 \cap G_2 = \left(\underline{G}_1 \cap \underline{G}_2, \overline{G}_1 \cap \overline{G}_2\right)$, *where* $\underline{G}_1 \cap \underline{G}_2 = \left([\underline{R}A]_{G1} \cap [\underline{R}A]_{G2}, \underline{W}A_{\sigma_1} \cap \underline{W}A_{\sigma_2}\right)$ *and* $\overline{G}_1 \cap \overline{G}_2 = \left([\overline{R}A]_{G1} \cap [\overline{R}A]_{G2}, \overline{W}A_{\sigma_1} \cap \overline{W}A_{\sigma_2}\right)$ *is an IF rough graph as shown in Table 14.*

Table 14. IF rough graph $G_1 \cap G_2 = \left(\underline{G}_1 \cap \underline{G}_2, \overline{G}_1 \cap \overline{G}_2\right)$ in Example 4.9.

$\underline{G}_1 \cap \underline{G}_2$	$\overline{G}_1 \cap \overline{G}_2$
	

Definition 4.10. *Assume that* $G_1 = \left(\underline{G}_1, \overline{G}_1\right)$ *and* $G_2 = \left(\underline{G}_2, \overline{G}_2\right)$ *IF rough graphs on* X. *The join of* G_1 *and* G_2 *is an IF rough graph* $G = G_1 + G_2 = \left(\underline{G}_1 + \underline{G}_2, \overline{G}_1 + \overline{G}_2\right)$,

where $\underline{G}_1 + \underline{G}_2 = \left([\underline{R}A_1]_{G1} + [\underline{R}A_2]_{G2}, [\underline{W}A_{\sigma_1}]_{G1} + [\underline{W}A_{\sigma_2}]_{G2}\right)$ *and* $\overline{G}_1 + \overline{G}_2 = \left([\overline{R}A_1]_{G1} + [\overline{R}A_2]_{G2}, [\overline{W}A_{\sigma_1}]_{G1} + [\overline{W}A_{\sigma_2}]_{G2}\right)$ *are IF graphs, respectively, such that*

(i) if $x \in [RA_1]_{G1}$ *but* $x \notin [RA_2]_{G2}$

$$\begin{aligned} T_{[\underline{R}(A_1)]_{G_1}+[\underline{R}_i(A_2)]_{G_2}}(x) &= T_{[\underline{R}(A_1)]_{G_1}}(x), \\ F_{[\underline{R}(A_1)]_{G_1}+[\underline{R}(A_2)]_{G_2}}(x) &= F_{[\underline{R}(A_1)]_{G_1}}(x), \\ T_{[\overline{R}(A_1)]_{G_1}+[\overline{R}(A_2)]_{G_2}}(x) &= T_{[\overline{R}(A_1)]_{G_1}}(x), \\ F_{[\overline{R}(A_1)]_{G_1}+[\overline{R}(A_2)]_{G_2}}(x) &= F_{[\overline{R}(A_1)]_{G_1}}(x), \end{aligned}$$

(ii) if $x \in [RA_2]_{G2}$ *but* $x \notin [RA_1]_{G1}$

$$\begin{aligned} T_{[\underline{R}(A_1)]_{G_1}+[\underline{R}_i(A_2)]_{G_2}}(x) &= T_{[\underline{R}(A_2)]_{G_2}}(x), \\ F_{[\underline{R}(A_1)]_{G_1}+[\underline{R}(A_2)]_{G_2}}(x) &= F_{[\underline{R}(A_2)]_{G_2}}(x), \\ T_{[\overline{R}(A_1)]_{G_1}+[\overline{R}(A_2)]_{G_2}}(x) &= T_{[\overline{R}(A_2)]_{G_2}}(x), \\ F_{[\overline{R}(A_1)]_{G_1}+[\overline{R}(A_2)]_{G_2}}(x) &= F_{[\overline{R}(A_2)]_{G_2}}(x), \end{aligned}$$

(iii) if $x \in [RA]_{G1}$ *and* $x \in [RA]_{G2}$

$$\begin{aligned} T_{[\underline{R}(A_1)]_{G_1}+[R(A_2)]_{G_2}}(x) &= \max\left\{T_{[\underline{R}(A_1)]_{G_1}}(x), T_{[\underline{R}(A_2)]_{G_2}}(x)\right\}, \\ F_{[\underline{R}(A_1)]_{G_1}+[\underline{R}(A_2)]_{G_2}}(x) &= \min\left\{F_{[\underline{R}(A_1)]_{G_1}}(x), F_{[\underline{R}(A_2)]_{G_2}}(x)\right\}, \\ T_{[\overline{R}(A_1)]_{G_1}+[\overline{R}(A_2)]_{G_2}}(x) &= \max\left\{T_{[\overline{R}(A_1)]_{G_1}}(x), T_{[\overline{R}(A_2)]_{G_2}}(x)\right\}, \\ F_{[\overline{R}(A_1)]_{G_1}+[\overline{R}(A_2)]_{G_2}}(x) &= \min\left\{F_{[\overline{R}(A_1)]_{G_1}}(x), F_{[\overline{R}(A_2)]_{G_2}}(x)\right\}, \end{aligned}$$

(iv) if $x_{ij} \in [WA_{\sigma_1}]_{G_1}$ *but* $x_{ij} \notin [WA_{\sigma_2}]_{G_2}$

$$\begin{aligned} T_{[\underline{W}(A_{\sigma_1})]_{G_1}+[\underline{W}(A_{\sigma_2})]_{G_2}}\left(x_{ij}\right) &= T_{[\underline{W}(A_{\sigma_1})]_{G_1}}\left(x_{ij}\right), \\ F_{[\underline{W}(A_{\sigma_1})]_{G_1}+[\underline{W}(A_{\sigma_2})]_{G_2}}\left(x_{ij}\right) &= F_{[\underline{W}(A_{\sigma_1})]_{G_1}}\left(x_{ij}\right), \\ T_{[\overline{W}(A_{\sigma_2})]_{G_1}+[\overline{W}(A_{\sigma_2})]_{G_2}}\left(x_{ij}\right) &= T_{[\overline{W}(A_{\sigma_2})]_{G_1}}\left(x_{ij}\right), \\ F_{[\overline{W}(A_{\sigma_2})]_{G_1}+[\overline{W}(A_{\sigma_2})]_{G_2}}\left(x_{ij}\right) &= F_{[\overline{W}(A_{\sigma_2})]_{G_1}}\left(x_{ij}\right), \end{aligned}$$

(v) if $x_{ij} \in [WA_{\sigma_2}]_{G_2}$ *but* $x_{ij} \notin [WA_{\sigma_1}]_{G_1}$

$$\begin{aligned} T_{[\underline{W}(A_{\sigma_1})]_{G_1}+[\underline{W}(A_{\sigma_2})]_{G_2}}\left(x_{ij}\right) &= T_{[\underline{W}(A_{\sigma_2})]_{G_2}}\left(x_{ij}\right), \\ F_{[\underline{W}(A_{\sigma_1})]_{G_1}+[\underline{W}(A_{\sigma_2})]_{G_2}}\left(x_{ij}\right) &= F_{[\underline{W}(A_{\sigma_2})]_{G_2}}\left(x_{ij}\right), \\ T_{[\overline{W}(A_{\sigma_1})]_{G_1}+[\overline{W}(A_{\sigma_2})]_{G_2}}\left(x_{ij}\right) &= T_{[\overline{W}(A_{\sigma_2})]_{G_2}}\left(x_{ij}\right), \\ F_{[\overline{W}(A_{\sigma_1})]_{G_1}+[\overline{W}(A_{\sigma_2})]_{G_2}}\left(x_{ij}\right) &= F_{[\overline{W}(A_{\sigma_2})]_{G_2}}\left(x_{ij}\right), \end{aligned}$$

(vi) if $x_{ij} \in [WA_{\sigma_1}]_{G_1}$ *and* $x_{ij} \in [WA_{\sigma_2}]_{G_2}$

$$\begin{aligned}
T_{[\underline{W}(A_{\sigma_1})]_{G_1}+[\underline{W}(A_{\sigma_2})]_{G_2}}\left(x_{ij}\right) &= \max\left\{T_{[\underline{W}(A_{\sigma_1})]_{G_1}}\left(x_{ij}\right), T_{[\underline{W}(A_{\sigma_2})]_{G_2}}\left(x_{ij}\right)\right\},\\
F_{[\underline{W}(A_{\sigma_1})]_{G_1}+[\underline{W}(A_{\sigma_2})]_{G_2}}\left(x_{ij}\right) &= \min\left\{F_{[\underline{W}(A_{\sigma_1})]_{G_1}}\left(x_{ij}\right), F_{[\underline{W}(A_{\sigma_2})]_{G_2}}\left(x_{ij}\right)\right\},\\
T_{[\overline{W}(A_{\sigma_1})]_{G_1}+[\overline{W}(A_{\sigma_2})]_{G_2}}\left(x_{ij}\right) &= \max\left\{T_{[\overline{W}(A_{\sigma_1})]_{G_1}}\left(x_{ij}\right), T_{[\overline{W}(A_{\sigma_2})]_{G_2}}\left(x_{ij}\right)\right\},\\
F_{[\overline{W}(A_{\sigma_1})]_{G_1}+[\overline{W}(A_{\sigma_2})]_{G_2}}\left(x_{ij}\right) &= \min\left\{F_{[\overline{W}(A_{\sigma_1})]_{G_1}}\left(x_{ij}\right), F_{[\overline{W}(A_{\sigma_2})]_{G_2}}\left(x_{ij}\right)\right\},
\end{aligned}$$

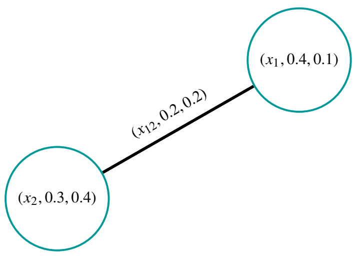
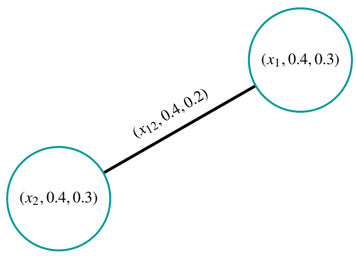
(vii) if $x_{ij} \in \tilde{Y}$

$$\begin{aligned}
T_{[\underline{W}(A_{\sigma_1})]_{G_1}+[\underline{W}(A_{\sigma_2})]_{G_2}}\left(x_{ij}\right) &= \min\left\{T_{[\underline{W}(A_{\sigma_1})]_{G_1}}\left(x_{ij}\right), T_{[\underline{W}(A_{\sigma_2})]_{G_2}}\left(x_{ij}\right)\right\},\\
F_{[\underline{W}(A_{\sigma_1})]_{G_1}+[\underline{W}(A_{\sigma_2})]_{G_2}}\left(x_{ij}\right) &= \max\left\{F_{[\underline{W}(A_{\sigma_1})]_{G_1}}\left(x_{ij}\right), F_{[\underline{W}(A_{\sigma_2})]_{G_2}}\left(x_{ij}\right)\right\},\\
T_{[\overline{W}(A_{\sigma_1})]_{G_1}+[\overline{W}(A_{\sigma_2})]_{G_2}}\left(x_{ij}\right) &= \min\left\{T_{[\overline{W}(A_{\sigma_1})]_{G_1}}\left(x_{ij}\right), T_{[\overline{W}(A_{\sigma_2})]_{G_2}}\left(x_{ij}\right)\right\},\\
F_{[\overline{W}(A_{\sigma_1})]_{G_1}+[\overline{W}(A_{\sigma_2})]_{G_2}}\left(x_{ij}\right) &= \max\left\{F_{[\overline{W}(A_{\sigma_1})]_{G_1}}\left(x_{ij}\right), F_{[\overline{W}(A_{\sigma_2})]_{G_2}}\left(x_{ij}\right)\right\},
\end{aligned}$$

where $\tilde{Y}$ *is the set of edges connecting vertices of* $[RA]_{G1}$ *and* $[RA]_{G2}$.

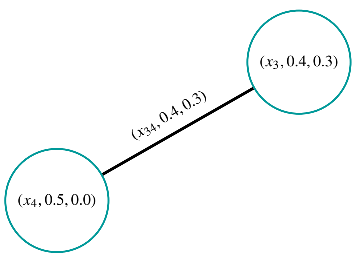
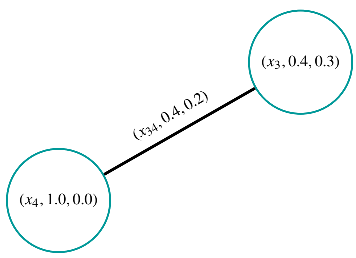
Example 4.11. *Assume that* $X = \{x_1, x_2, x_3, x_4\}$ *is a set. Consider* $G_1 = \left(\underline{G}_1, \overline{G}_1\right)$ *and* $G_2 = \left(\underline{G}_2, \overline{G}_2\right)$ *are two IF rough graphs on X, where* $\underline{G}_1 = \left([\underline{R}A_1]_{G1}, [\underline{W}A_{\sigma_1}]_{G1}\right)$ *and* $\overline{G}_1 = \left([\overline{R}A_1]_{G1}, [\overline{W}A_{\sigma_1}]_{G1}\right)$ *are IF graphs as shown in Table 15.*

Table 15. IF rough graph $G_1 = \left(\underline{G}_1, \overline{G}_1\right)$ in Example 4.11.

<table>
<tr><th>The IF lower approximation $\underline{G}_1$</th><th>The IF upper approximation $\overline{G}_1$</th></tr>
<tr><td>
</td><td>
</td></tr>
</table>

$\underline{G}_2 = \left([\underline{R}A_2]_{G2}, [\underline{W}A_{\sigma_2}]_{G2}\right)$ *and* $\overline{G}_2 = \left([\overline{R}A_2]_{G2}, [\overline{W}A_{\sigma_2}]_{G2}\right)$ *are also IF graphs as shown in Table 16.*

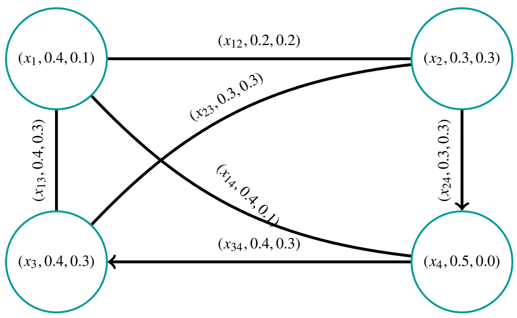
Table 16. IF rough graph $G_2 = \left(\underline{G}_2, \overline{G}_2\right)$ in Example 4.11.

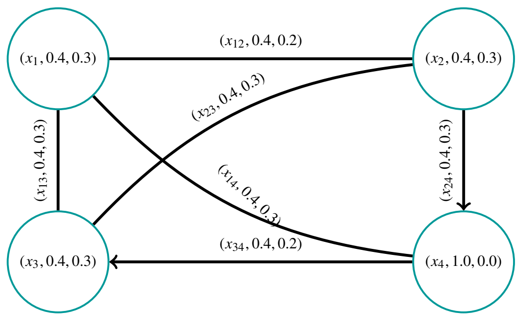
<table>
<tr><th>The IF lower approximation $\underline{G}_2$</th><th>The IF upper approximation $\overline{G}_2$</th></tr>
<tr><td>
</td><td>
</td></tr>
</table>

The join of G_1 *and* G_2 *is a* $G = G_1 + G_2 = \left(\underline{G}_1 + \underline{G}_2, \overline{G}_1 \cup \overline{G}_2\right)$, *where* $\underline{G}_1 + \underline{G}_2 = \left([\underline{R}A_1]_{G_1} + [\underline{R}A_2]_{G_2}, [\underline{W}A_{\sigma_1}]_{G_1} + [\underline{W}A_{\sigma_2}]_{G_2}\right)$ *and* $\overline{G}_1 + \overline{G}_2 = \left([\overline{R}A_1]_{G_1} + [\overline{R}A_2]_{G_2}, [\overline{W}A_{\sigma_1}]_{G_1} + [\overline{W}A_{\sigma_2}]_{G_2}\right)$ *are IF graphs as shown in Table 17.*

Table 17. IF rough graph $G_1 + G_2 = \left(\underline{G}_1 + \underline{G}_2, \overline{G}_1 + \overline{G}_2\right)$ in Example 4.11.

$\underline{G}_1 + \underline{G}_2$	$\overline{G}_1 + \overline{G}_2$





Remark 4.12. *For any two IF rough graphs* G_1 *and* G_2*, the union, intersection, and join are IF rough graphs.*

5. Regular and irregular IF rough graphs

This section defines key structural metrics, specifically the order and size of an IF rough graph. It also introduces the concept of an effective edge, defined by the condition that the edge's membership and non- membership values in the upper approximation equal the minimum of the respective values of its incident vertices in the lower approximation.

Definition 5.1. *Let* $G = (\underline{G}, \overline{G}) = ([RA]_G, [WA_\sigma]_G)$ *be an IF rough graph on a non-empty set X, where* $\underline{G} = \left([\underline{R}A]_G, [\underline{W}A_\sigma]_G\right)$ *and* $\overline{G} = \left([\overline{R}A]_G, [\overline{W}A_\sigma]_G\right)$ *are lower and upper approximate IF graphs of G. The order of G is given by*

$$\mathbf{O}(G) = \left(\sum_{x\in X}\left\{T_{[\underline{R}A]_G}(x) + T_{[\overline{R}A]_G}(x)\right\}, \sum_{x\in X}\left\{F_{[\underline{R}A]_G}(x) + F_{[\overline{R}A]_G}(x)\right\}\right)$$

and the size of G is given by

$$\mathbf{S}(G) = \left(\sum_{x_{ij}\in Y}\left\{T_{[\underline{W}(A_\sigma)]_G}\left(x_{ij}\right) + T_{[\overline{W}(A_\sigma)]_G}\left(x_{ij}\right)\right\}, \sum_{x_{ij}\in Y}\left\{F_{[\underline{W}(A_\sigma)]_G}\left(x_{ij}\right) + F_{[\overline{W}(A_\sigma)]_G}\left(x_{ij}\right)\right\}\right).$$

Definition 5.2. *In an IF rough graph* $G = (\underline{G}, \overline{G}) = ([RA]_G, [WA_\sigma]_G)$*, the sum of upper and lower approximations of arcs directed away from the vertex* x_i *is said to be the out degree of vertex* $x_i \in X$*, and is denoted by*

$$\text{od}(x_i) = \left(\sum_{\forall x_j\in X}\left\{T_{[\underline{W}(A_\sigma)]_G}\left(x_{ij}\right) + T_{[\overline{W}(A_\sigma)]_G}\left(x_{ij}\right)\right\}, \sum_{\forall x_j\in X}\left\{F_{[\underline{W}(A_\sigma)]_G}\left(x_{ij}\right) + F_{[\overline{W}(A_\sigma)]_G}\left(x_{ij}\right)\right\}\right).$$

Definition 5.3. *In an IF rough graph* $G = (\underline{G}, \overline{G}) = ([RA]_G, [WA_\sigma]_G)$*, the sum of upper and lower approximations of arcs directed to the vertex* x_i *is called indegree of vertex* $x_i \in X$ *and is defined by*

$$id(x_i) = \left(\sum_{\forall x_j \in X} \left\{ T_{[\underline{W}(A_\sigma)]_G}\left(x_{ji}\right) + T_{[\overline{W}(A_\sigma)]_G}\left(x_{ji}\right) \right\}, \sum_{\forall x_j \in X} \left\{ F_{[\underline{W}(A_\sigma)]_G}\left(x_{ji}\right) + F_{[\overline{W}(A_\sigma)]_G}\left(x_{ji}\right) \right\} \right).$$

Definition 5.4. *In an IF rough graph* $G = (\underline{G}, \overline{G}) = ([RA]_G, [WA_\sigma]_G)$*, the degree of a vertex* $x_i \in X$ *is defined as*

$$\begin{aligned} d(x_i) &= \mathrm{od}(x_i) + id(x_i) \\ &= \left(\sum_{\forall x_i \in X} \left\{ T_{[\underline{W}(A_\sigma)]_G}\left(x_{ij}\right) + T_{[\overline{W}(A_\sigma)]_G}\left(x_{ij}\right) + T_{[\underline{W}(A_\sigma)]_G}\left(x_{ji}\right) + T_{[\overline{W}(A_\sigma)]_G}\left(x_{ji}\right) \right\}, \right. \\ &\quad \left. \sum_{\forall x_i \in X} \left\{ F_{[\underline{W}(A_\sigma)]_G}\left(x_{ij}\right) + F_{[\overline{W}(A_\sigma)]_G}\left(x_{ij}\right) + F_{[\underline{W}(A_\sigma)]_G}\left(x_{ji}\right) + F_{[\overline{W}(A_\sigma)]_G}\left(x_{ji}\right) \right\} \right). \end{aligned}$$

The ordered pair (od(x_i *), id(* x_i *)) is called the degree pair of* x_i*.*

Definition 5.5. *In an IF rough graph* $G = (\underline{G}, \overline{G}) = ([RA]_G, [WA_\sigma]_G)$*, the total degree of a vertex* $x_i \in X$ *is defined as*

$$\begin{aligned} td(x_i) &= d(x_i) + [\underline{R}A]_G(x_i) + [\overline{R}A]_G(x_i) \\ &= \left(\sum_{\forall x_i \in X} \left\{ T_{[\underline{W}(A_\sigma)]_G}\left(x_{ij}\right) + T_{[\overline{W}(A_\sigma)]_G}\left(x_{ij}\right) + T_{[\underline{W}(A_\sigma)]_G}\left(x_{ji}\right) + T_{[\overline{W}(A_\sigma)]_G}\left(x_{ji}\right) \right\} \right. \\ &\quad + T_{[\underline{R}(A)]_G}(x_i) + T_{[\underline{R}(A)]_G}(x_i), \sum_{\forall x_i, x_j \in X} \left\{ F_{[\underline{W}(A_\sigma)]_G}\left(x_{ij}\right) + F_{[\overline{W}(A_\sigma)]_G}\left(x_{ij}\right) \right. \\ &\quad \left. \left. + F_{[\underline{W}(A_\sigma)]_G}\left(x_{ji}\right) + F_{[\overline{W}(A_\sigma)]_G}\left(x_{ji}\right) \right\} + F_{[\underline{R}(A)]_G}(x_i) + F_{[\underline{R}(A)]_G}(x_i) \right). \end{aligned}$$

Definition 5.6. *In an IF rough graph* $G = (\underline{G}, \overline{G}) = ([RA]_G, [WA_\sigma]_G)$*, the minimum degree of G is* $\theta(G) = \min_{x_i \in X}\{d(x_i)\}$*, that is,*

$$\begin{aligned} \theta(G) = &\left(\min_{x_i \in X} \left\{ \sum_{\forall x_j \in X} \left(T_{[\underline{W}(A_\sigma)]_G}\left(x_{ij}\right) + T_{[\overline{W}(A_\sigma)]_G}\left(x_{ij}\right) + T_{[\underline{W}(A_\sigma)]_G}\left(x_{ji}\right) + T_{[\overline{W}(A_\sigma)]_G}\left(x_{ji}\right) \right) \right\}, \right. \\ &\left. \max_{x_i \in X} \left\{ \sum_{\forall x_j \in X} \left(F_{[\underline{W}(A_\sigma)]_G}\left(x_{ij}\right) + F_{[\overline{W}(A_\sigma)]_G}\left(x_{ij}\right) + F_{[\underline{W}(A_\sigma)]_G}\left(x_{ji}\right) + T_{[\overline{W}(A_\sigma)]_G}\left(x_{ji}\right) \right) \right\} \right). \end{aligned}$$

The maximum degree of G is $\vartheta(G) = \max_{x_i \in X}\{d(x_i)\}$*, that is,*

$$\begin{aligned} \vartheta(G) = &\left(\max_{x_i \in X} \left\{ \sum_{\forall x_j \in X} \left(T_{[\underline{W}(A_\sigma)]_G}\left(x_{ij}\right) + T_{[\overline{W}(A_\sigma)]_G}\left(x_{ij}\right) + T_{[\underline{W}(A_\sigma)]_G}\left(x_{ji}\right) + T_{[\overline{W}(A_\sigma)]_G}\left(x_{ji}\right) \right) \right\}, \right. \\ &\left. \min_{x_i \in X} \left\{ \sum_{\forall x_j \in X} \left(F_{[\underline{W}(A_\sigma)]_G}\left(x_{ij}\right) + F_{[\overline{W}(A_\sigma)]_G}\left(x_{ij}\right) + F_{[\underline{W}(A_\sigma)]_G}\left(x_{ji}\right) + T_{[\overline{W}(A_\sigma)]_G}\left(x_{ji}\right) \right) \right\} \right). \end{aligned}$$

Definition 5.7. *In an IF rough graph $G = (\underline{G}, \overline{G}) = ([RA]_G, [WA_\sigma]_G)$, an edge $x_{ij} \in Y$ is called an effective edge if*

$$T_{[\underline{W}(A_\sigma)]_G}\left(x_{ij}\right) = T_{[\underline{R}(A)]_G}\left(x_i\right) \wedge T_{[\underline{R}(A)]_G}\left(x_j\right),$$
$$F_{[\underline{W}(A_\sigma)]_G}\left(x_{ij}\right) = F_{[\underline{R}(A)]_G}\left(x_i\right) \wedge F_{[\underline{R}(A)]_G}\left(x_j\right),$$
$$T_{[\overline{W}(A_\sigma)]_G}\left(x_{ij}\right) = T_{[\overline{R}(A)]_G}\left(x_i\right) \wedge T_{[\overline{R}(A)]_G}\left(x_j\right),$$
$$F_{[\overline{W}(A_\sigma)]_G}\left(x_{ij}\right) = F_{[\overline{R}(A)]_G}\left(x_i\right) \wedge F_{[\overline{R}(A)]_G}\left(x_j\right).$$

Definition 5.8. *In an IF rough graph $G = (\underline{G}, \overline{G}) = ([RA]_G, [WA_\sigma]_G)$, a node $x_i \in X$ is called a busy node if*

$$T_{[\underline{R}(A)]_G}\left(x_i\right) \leq \min\left\{\sum_{\forall x_j \in X} T_{[\underline{W}(A_\sigma)]_G}\left(x_{ij}\right), \sum_{\forall x_j \in X} T_{[\underline{W}(A_\sigma)]_G}\left(x_{ji}\right)\right\},$$
$$F_{[\underline{R}(A)]_G}\left(x_i\right) \geq \max\left\{\sum_{\forall x_j \in X} F_{[\underline{W}(A_\sigma)]_G}\left(x_{ij}\right), \sum_{\forall x_j \in X} F_{[\underline{W}(A_\sigma)]_G}\left(x_{ji}\right)\right\},$$
$$T_{[\overline{R}(A)]_G}\left(x_i\right) \leq \min\left\{\sum_{\forall x_j \in X} T_{[\overline{W}(A_\sigma)]_G}\left(x_{ij}\right), \sum_{\forall x_j \in X} T_{[\overline{W}(A_\sigma)]_G}\left(x_{ji}\right)\right\},$$
$$F_{[\overline{R}(A)]_G}\left(x_i\right) \geq \max\left\{\sum_{\forall x_j \in X} F_{[\overline{W}(A_\sigma)]_G}\left(x_{ij}\right), \sum_{\forall x_j \in X} F_{[\overline{W}(A_\sigma)]_G}\left(x_{ji}\right)\right\}.$$

Otherwise, it is called a free node.

Definition 5.9. *In an IF rough graph $G = (\underline{G}, \overline{G}) = ([RA]_G, [WA_\sigma]_G)$ is called regular if each approximation graph is regular IF graph. Equivalently, G is said to be regular IF rough graph if in each approximation graph, each vertex has the same indegree and out degree. G is called a k-regular IF rough graph if each vertex in $\underline{G}$ and $\overline{G}$ has the same indegree k and out degree k.*

$$\sum_{\forall x_j \in X} T_{[\underline{W}(A_\sigma)]_G}\left(x_{ij}\right) = k_1 = \sum_{\forall x_j \in X} T_{[\underline{W}(A_\sigma)]_G}\left(x_{ji}\right),$$
$$\sum_{\forall x_j \in X} F_{[\underline{W}(A_\sigma)]_G}\left(x_{ij}\right) = k_2 = \sum_{\forall x_j \in X} F_{[\underline{W}(A_\sigma)]_G}\left(x_{ji}\right),$$
$$\sum_{\forall x_j \in X} T_{[\overline{W}(A_\sigma)]_G}\left(x_{ij}\right) = k_3 = \sum_{\forall x_j \in X} T_{[\overline{W}(A_\sigma)]_G}\left(x_{ji}\right),$$
$$\sum_{\forall x_j \in X} F_{[\overline{W}(A_\sigma)]_G}\left(x_{ij}\right) = k_4 = \sum_{\forall x_j \in X} F_{[\overline{W}(A_\sigma)]_G}\left(x_{ji}\right),$$

where k_i are constants $\forall i = 1, 2, 3, 4$.

Example 5.10. *Assume that $A = \{(x_1, 0.2, 0.2), (x_2, 0.2, 0.2), (x_3, 0.2, 0.2)\}$ is an IF set on $X = \{x_1, x_2, x_3\}$, and R is an IF relation on X as given in Table 18.*

Table 18. The IF relation R on X in Example 5.10.

R	x_1	x_2	x_3
x_1	$(0.5, 0.3)$	$(0.5, 0.3)$	$(0.5, 0.3)$
x_2	$(0.5, 0.3)$	$(0.5, 0.3)$	$(0.5, 0.3)$
x_3	$(0.5, 0.3)$	$(0.5, 0.3)$	$(0.5, 0.3)$

By computing, $< x_1 > R^+ =< x_2 > R^+ =< x_3 > R^+ = \{0.5, 0.5, 0.5\}$, $< x_1 > R^- =< x_2 > R^- =< x_3 > R^- = \{0.3, 0.3, 0.3\}$, *Similarly,* $R^+ < x_1 >= R^+ < x_2 >= R^+ < x_3 >= \{0.5, 0.5, 0.5\}$, $R^- < x_1 >= R^- < x_2 >= R^- < x_3 >= \{0.3, 0.3, 0.3\}$. *Therefore,* $R^+ < x_1 > R^+ = R^+ < x_2 > R^+ = R^+ < x_3 > R^+ = \{0.5, 0.5, 0.5\}$, $R^- < x_1 > R^- = R^- < x_2 > R^- = R^- < x_3 > R^- = \{0.3, 0.3, 0.3\}$. *This implies that* $[RA]_G = \left([\underline{R}A]_G, [\overline{R}A]_G\right)$ *is an IF rough set, where* $[\underline{R}A]_G$ *and* $[\overline{R}A]_G$ *are lower and upper approximations of A, respectively, given below:*

$$
\begin{aligned}
[\overline{R}A]_G = &\ \{(x_1, 0.2, 0.2), (x_2, 0.2, 0.2), (x_3, 0.2, 0.2)\},\\
[\underline{R}A]_G = &\ \{(x_1, 0.2, 0.2), (x_2, 0.2, 0.2), (x_3, 0.2, 0.2)\}.
\end{aligned}
$$

Assume that $Y = \{x_{12}, x_{23}, x_{31}\} \subseteq X \times X$. *Assume that W is an IF relation on Y given in Table 19.*

Table 19. The IF relation W on X in Example 5.10.

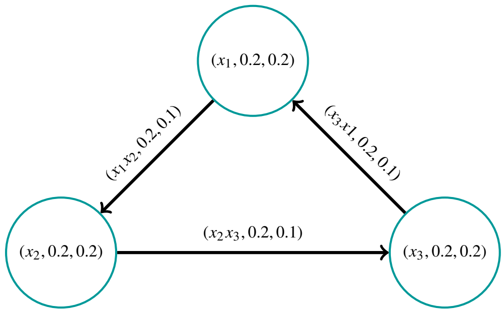
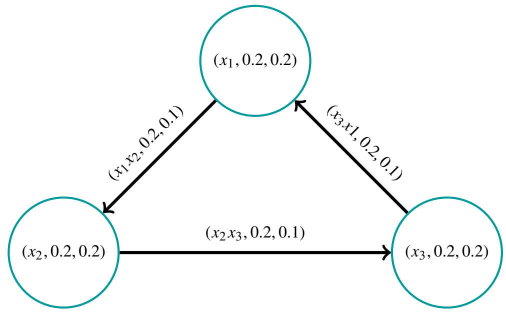
W	x_1x_2	x_2x_3	x_3x_1
x_1x_2	$(0.5, 0.3)$	$(0.5, 0.3)$	$(0.5, 0.3)$
x_2x_3	$(0.5, 0.3)$	$(0.5, 0.3)$	$(0.5, 0.3)$
x_3x_1	$(0.5, 0.3)$	$(0.5, 0.3)$	$(0.5, 0.3)$

By computing, $< x_1x_2 > R^+ =< x_2x_3 > R^+ =< x_3x_1 > R^+ = \{0.5, 0.5, 0.5\}$, $< x_1x_2 > R^- =< x_2x_3 > R^- =< x_3x_1 > R^- = \{0.3, 0.3, 0.3\}$, *Similarly,* $R^+ < x_1x_2 >= R^+ < x_2x_3 >= R^+ < x_3x_1 >= \{0.5, 0.5, 0.5\}$, $R^- < x_1x_2 >= R^- < x_2x_3 >= R^- < x_3x_1 >= \{0.3, 0.3, 0.3\}$. *Therefore,* $R^+ < x_1x_2 > R^+ = R^+ < x_2x_3 > R^+ = R^+ < x_3x_1 > R^+ = \{0.5, 0.5, 0.5\}$, $R^- < x_1x_2 > R^- = R^- < x_2x_3 > R^- = R^- < x_3x_1 > R^- = \{0.3, 0.3, 0.3\}$. *Assume that* $A_\sigma = \{(x_1x_2, 0.2, 0.1), (x_2x_3, 0.2, 0.1), (x_3x_1, 0.2, 0.1)\}$ *is an IF set on* Y. *Then, by definition, the upper and lower approximation relations are calculated as*

$$
\begin{aligned}
[\overline{W}A_\sigma]_G = \{(x_1x_2, 0.2, 0.1), (x_2x_3, 0.2, 0.1), (x_3x_1, 0.2, 0.1)\},\\
[\underline{W}A_\sigma]_G = \{(x_1x_2, 0.2, 0.1), (x_2x_3, 0.2, 0.1), (x_3x_1, 0.2, 0.1)\}.
\end{aligned}
$$

As a result, ($[WA_\sigma]_G, [\overline{W}A_\sigma]_G$) *is an IF rough relation on the set X. This implies that* $\underline{G} = \left([\underline{R}A]_G, \underline{Z}_i A_\sigma\right)$ *and* $\overline{G} = \left([\underline{R}A]_G, [\underline{W}A_\sigma]_G\right)$ *are IF graphs as shown in Table 20. It is a regular and total regular IF rough graph.*

Table 20. IF rough graph $G_1 = \left(\underline{G}, \overline{G}\right)$ in Example 5.10.

The IF lower approximation $\underline{G}$	The IF upper approximation $\overline{G}$
	

Definition 5.11. *An IF rough graph $G = (\underline{G}, \overline{G}) = ([RA]_G, [WA_\sigma]_G)$ is called totally regular if each approximation graph is an IF graph with the same total degree. Equivalently, if each approximation graph is a totally regular IF graph, then,*

$$\sum_{\forall x_j \in X} T_{[\underline{W}(A_\sigma)]_G}\left(x_{ij}\right) + T_{[\underline{R}(A)]_G}(x_i) = t_1 = \sum_{\forall x_j \in X} T_{[\underline{W}(A_\sigma)]_G}\left(x_{ji}\right) + T_{[\underline{R}(A)]_G}(x_i),$$
$$\sum_{\forall x_j \in X} F_{[\underline{W}(A_\sigma)]_G}\left(x_{ij}\right) + F_{[\underline{R}(A)]_G}(x_i) = t_2 = \sum_{\forall x_j \in X} F_{[\underline{W}(A_\sigma)]_G}\left(x_{ji}\right) + F_{[\underline{R}(A)]_G}(x_i),$$

$$\sum_{\forall x_j \in X} T_{[\overline{W}(A_\sigma)]_G}\left(x_{ij}\right) + T_{[\overline{R}(A)]_G}(x_i) = t_3 = \sum_{\forall x_j \in X} T_{[\overline{W}(A_\sigma)]_G}\left(x_{ji}\right) + T_{\overline{R}(A)}(x_i),$$
$$\sum_{\forall x_j \in X} F_{[\overline{W}(A_\sigma)]_G}\left(x_{ij}\right) + F_{[\overline{R}(A)]_G}(x_i) = t_4 = \sum_{\forall x_j \in X} F_{[\overline{W}(A_\sigma)]_G}\left(x_{ji}\right) + F_{\overline{R}(A)}(x_i),$$

where t_i are constants $\forall i = 1, 2, 3, 4$.

Theorem 5.12. *Assume that $G = (\underline{G}, \overline{G}) = ([RA]_G, [WA_\sigma]_G)$ is an IF rough graph. If G is a regular (totally regular) and $[RA]_G = \left([\underline{R}A]_G, [\overline{R}A]_G\right)$ is a constant function, then G is a totally regular (regular) IF rough graph.*

Proof. Assume that $G = (\underline{G}, \overline{G})$ is an IF rough graph, where $\underline{G} = \left([\underline{R}A]_G, [\underline{W}A_\sigma]_G\right)$ and $\overline{G} = \left([\overline{R}A]_G, [\overline{W}A_\sigma]_G\right)$. If G is regular and $[RA]_G$ is constant, then for any $x \in [RA]_G, \left(x_{ij}\right) \in [WA_\sigma]_G$,

$$\sum_{\forall x_j \in X} T_{[\underline{W}(A_\sigma)]_G}\left(x_{ij}\right) = k_1 = \sum_{\forall x_j \in X} T_{[\underline{W}(A_\sigma)]_G}\left(x_{ji}\right),$$
$$\sum_{\forall x_j \in X} F_{[\underline{W}(A_\sigma)]_G}\left(x_{ij}\right) = k_2 = \sum_{\forall x_j \in X} F_{[\underline{W}(A_\sigma)]_G}\left(x_{ji}\right),$$
$$\sum_{\forall x_j \in X} T_{[\overline{W}(A_\sigma)]_G}\left(x_{ij}\right) = k_3 = \sum_{\forall x_j \in X} T_{[\overline{W}(A_\sigma)]_G}\left(x_{ji}\right),$$
$$\sum_{\forall x_j \in X} F_{[\overline{W}(A_\sigma)]_G}\left(x_{ij}\right) = k_4 = \sum_{\forall x_j \in X} F_{[\overline{W}(A_\sigma)]_G}\left(x_{ji}\right),$$

and

$$T_{[\underline{R}(A)]_G}(x_i) = m_1, \quad F_{[\underline{R}(A)]_G}(x_i) = \quad m_2, \quad T_{[\overline{R}(A)]_G}(x_i) = m_3, \quad F_{[\overline{R}(A)]_G}(x_i) = m_4,$$

where k_i and m_i are constants $\forall i = 1, 2, 3, 4$. As a result,

$$\sum_{\forall x_j \in X} T_{[\underline{W}(A_\sigma)]_G}\left(x_{ij}\right) + T_{[\underline{R}(A)]_G}(x_i) = k_1 + m_1 = \sum_{\forall x_j \in X} T_{[\underline{W}(A_\sigma)]_G}\left(x_{ji}\right) + T_{[\underline{R}(A)]_G}(x_i),$$

$$\sum_{\forall x_j \in X} F_{[\underline{W}(A_\sigma)]_G}\left(x_{ij}\right) + F_{[\underline{R}(A)]_G}(x_i) = k_2 + m_2 = \sum_{\forall x_j \in X} F_{[\underline{W}(A_\sigma)]_G}\left(x_{ji}\right) + F_{[\underline{R}(A)]_G}(x_i),$$

$$\sum_{\forall x_j \in X} T_{[\overline{W}(A_\sigma)]_G}\left(x_{ij}\right) + T_{[\overline{R}(A)]_G}(x_i) = k_3 + m_3 = \sum_{\forall x_j \in X} T_{[\overline{W}(A_\sigma)]_G}\left(x_{ji}\right) + T_{[\overline{R}(A)]_G}(x_i),$$

$$\sum_{\forall x_j \in X} F_{[\overline{W}(A_\sigma)]_G}\left(x_{ij}\right) + F_{[\overline{R}(A)]_G}(x_i) = k_4 + m_4 = \sum_{\forall x_j \in X} F_{[\overline{W}(A_\sigma)]_G}\left(x_{ji}\right) + F_{[\overline{R}(A)]_G}(x_i),$$

Hence, G is totally regular. □

Theorem 5.13. *If an IF rough graph G is regular and $[RA]_G = \left([\underline{R}A]_G, [\overline{R}A]_G\right)$ is a constant function, then G' and G^μ are regular.*

Proof. Assume that $G = (\underline{G}, \overline{G})$ is an IF rough graph, where $\underline{G} = \left([\underline{R}A]_G, [\underline{W}A_\sigma]_G\right)$ and $\overline{G} = \left([\overline{R}A]_G, [\overline{W}A_\sigma]_G\right)$. If G is regular and $[RA]_G$ is $[RA]_G, x_{ij} \in [WA_\sigma]_G$,

$$\sum_{\forall x_j \in X} T_{[\underline{W}(A_\sigma)]_G}\left(x_{ij}\right) = k_1 = \sum_{\forall x_j \in X} T_{[\underline{W}(A_\sigma)]_G}\left(x_{ji}\right),$$

$$\sum_{\forall x_j \in X} F_{[\underline{W}(A_\sigma)]_G}\left(x_{ij}\right) = k_2 = \sum_{\forall x_j \in X} F_{[\underline{W}(A_\sigma)]_G}\left(x_{ji}\right),$$

$$\sum_{\forall x_j \in X} T_{[\overline{W}(A_\sigma)]_G}\left(x_{ij}\right) = k_3 = \sum_{\forall x_j \in X} T_{[\overline{W}(A_\sigma)]_G}\left(x_{ji}\right),$$

$$\sum_{\forall x_j \in X} F_{[\overline{W}(A_\sigma)]_G}\left(x_{ij}\right) = k_4 = \sum_{\forall x_j \in X} F_{[\overline{W}(A_\sigma)]_G}\left(x_{ji}\right),$$

and

$$T_{[\underline{R}(A)]_G}(x_i) = m_1, \quad F_{[\underline{R}(A)]_G}(x_i) = m_2, \quad T_{[\overline{R}(A)]_G}(x_i) = m_3, \quad F_{[\overline{R}(A)]_G}(x_i) = m_4,$$

Therefore,

$$\sum_{\forall x_j \in X} T_{[\underline{W}(A_\sigma)]_G}\left(x_{ij}\right) + T_{[\underline{R}(A)]_G}(x_i) = k_1 + m_1 = \sum_{\forall x_j \in X} T_{[\underline{W}(A_\sigma)]_G}\left(x_{ji}\right) + T_{[\underline{R}(A)]_G}(x_i),$$

$$\sum_{\forall x_j \in X} F_{[\underline{W}(A_\sigma)]_G}\left(x_{ij}\right) + F_{[\underline{R}(A)]_G}(x_i) = k_2 + m_2 = \sum_{\forall x_j \in X} F_{[\underline{W}(A_\sigma)]_G}\left(x_{ji}\right) + F_{[\underline{R}(A)]_G}(x_i),$$

$$\sum_{\forall x_j \in X} T_{[\overline{W}(A_\sigma)]_G}\left(x_{ij}\right) + T_{[\overline{R}(A)]_G}(x_i) = k_3 + m_3 = \sum_{\forall x_j \in X} T_{[\overline{W}(A_\sigma)]_G}\left(x_{ji}\right) + T_{[\overline{R}(A)]_G}(x_i),$$

$$\sum_{\forall x_j \in X} F_{[\overline{W}(A_\sigma)]_G}\left(x_{ij}\right) + F_{[\overline{R}(A)]_G}(x_i) = k_4 + m_4 = \sum_{\forall x_j \in X} F_{[\overline{W}(A_\sigma)]_G}\left(x_{ji}\right) + F_{[\overline{R}(A)]_G}(x_i),$$

where k_i and m_i are constants $\forall i = 1, 2, 3, 4$. According to Definitions 4.1 and 4.3, G^μ and G' are regular IF rough graphs, respectively. □

Theorem 5.14. *If an IF rough graph G is totally regular and $[RA]_G$ is a constant function, then $\dot{G}$ and G^{μ} are totally regular.*

Proof. Assume that $G = (\underline{G}, \overline{G})$ is an IF rough graph, where $\underline{G} = \left([\underline{R}A]_G, [\underline{W}A_\sigma]_G\right)$ and $\overline{G} = \left([\overline{R}A]_G, [\overline{W}A_\sigma]_G\right)$. If G is totally regular and $[RA]_G$ is constant, then, according to Theorems 5.12 and 5.13, we have G, G' and G^{μ} are regular IF rough graphs. Hence, G' and G^{μ} are totally regular IF rough graphs. □

Definition 5.15. *Assume that G is an IF rough graph. If there is a vertex that is adjacent to vertices with distinct indegree and out degree in each approximation space, then G is irregular.*

Definition 5.16. *Suppose that G is a connected rough IF graph. In any approximation graph, G is said to be a neighborly irregular IF rough graph if each of its two neighboring vertices has a unique degree pair.*

Example 5.17. *Assume that $X = \{x_1, x_2, x_3, x_4, x_5\}$ is a set. Assume that $G = (\underline{G}, \overline{G})$ is an IF rough graph on X, where $\underline{G} = \left([\underline{R}A]_G, [\underline{W}A_\sigma]_G\right)$ and $\overline{G} = \left([\overline{R}A]_G, [\overline{W}A_\sigma]_G\right)$ are IF graphs, as shown in Table 5. This graph is both an irregular IF rough graph and a neighboring irregular IF rough graph.*

6. Hybrid models applied to decision-making

Decision-making plays an important role in our daily life. Some choices are so significant that they have the power to alter our life. The decision-making process results in a selection of several options. Making decisions is seen to be highly helpful in obtaining as much information as possible from many sources and weighing all options available to address the issue or circumstance at hand. We come up with the finest remedy for the issue after going through this entire procedure utilizing left and right minimal fuzzy neighborhoods. Here, we showcase a real-world example of decision-making in action. Deep considerations of the problem can be developed by evaluating upper and lower approximations using the provided decision-making procedure. When working with several objects, the strategies that have been described can be used to avoid time-consuming calculations. This approach may be used for multi-criteria object selection in a variety of fields.

Example 6.1. ***Selection of suitable embroidery:*** *A magnificent assortment of bridal gowns has been exhibited at numerous fashion shows by leading fashion designers of our nation. Usually, brides show great interest in different wedding dresses; therefore, this collection also features stunning gowns. To create these ensembles with diverse design variations, several well-known fashion designers have participated in wedding-related fashion weeks. These bridal gowns differ significantly from city gowns, as they possess distinct value. The bridal attire aligns with customs, and this assortment includes mayoon, mehndi, baraat, and reception attire. Suppose a new fashion designer wants to create a bridal outfit. Brides tend to prefer more traditional looks through in lehengas and shararas produced with various types of embroidery, these such as Dabka, Beats, Naqshi, Mukesh, Pearls, Crystals, Studded Squins, Motifs, and Zari Work; these are available to all females. The designer intends to produce classic bridal outfits with a contemporary touch. He makes use of a collection X that includes x_1 = Dabka, x_2 = Mukesh, x_3 = Pearls, x_4 = Crystals, and x_5 = Work of Zari, applied to a traditional red with coffee color. $A = \{(x_1, 0.8, 0.1), (x_2, 0.7, 0.2), (x_3, 0.6, 0.1), (x_4, 0.8, 0.1), (x_5, 0.8, 0.1)\}$ is an IF* *set on X, representing the quality of the above embroideries. Let R be an IF relation on X, as given in Table 21.*

Table 21. The IF relation R on X in Subsection 6.1.

R	x_1	x_2	x_3	x_4	x_5
x_1	(0.9, 0.0)	(0.7, 0.1)	(0.6, 0.2)	(0.5, 0.1)	(0.3, 0.2)
x_2	(0.8, 0.1)	(0.4, 0.4)	(0.8, 0.1)	(0.7, 0.1)	(1.0, 0.0)
x_3	(0.7, 0.2)	(0.3, 0.1)	(0.0, 0.6)	(0.2, 0.2)	(0.6, 0.2)
x_4	(0.6, 0.1)	(0.5, 0.5)	(0.4, 0.4)	(0.7, 0.2)	(0.3, 0.4)
x_5	(0.9, 0.0)	(0.0, 1.0)	(0.1, 0.1)	(0.5, 0.5)	(0.3, 0.3)

By computing, $< x_1 > R^+ = \{0.6, 0.0, 0.0, 0.2, 0.3\}$, $< x_2 > R^+ = \{0.6, 0.3, 0.0, 0.2, 0.3\}$, $< x_3 > R^+ = \{0.6, 0.0, 0.1, 0.2, 0.3\}$, $< x_4 > R^+ = \{0.6, 0.0, 0.0, 0.2, 0.3\}$, $< x_5 > R^+ = \{0.6, 0.0, 0.0, 0.2, 0.3\}$, $< x_1 > R^- = \{0.2, 1.0, 0.6, 0.5, 0.4\}$, $< x_2 > R^- = \{0.2, 1.0, 0.6, 0.5, 0.4\}$, $< x_3 > R^- = \{0.2, 1.0, 0.6, 0.5, 0.4\}$, $< x_4 > R^- = \{0.2, 1.0, 0.6, 0.5, 0.4\}$, $< x_5 > R^- = \{0.2, 1.0, 0.6, 0.5, 0.4\}$. *Similarly,* $R^+ < x_1 >= \{0.3, 0.4, 0.0, 0.3, 0.0\}$, $R^+ < x_2 >= \{0.3, 0.4, 0.0, 0.3, 0.0\}$, $R^+ < x_3 >= \{0.3, 0.4, 0.2, 0.3, 0.0\}$, $R^+ < x_4 >= \{0.3, 0.4, 0.0, 0.3, 0.0\}$, $R^+ < x_5 >= \{0.3, 0.4, 0.0, 0.3, 0.1\}$, $R^- < x_1 >= \{0.2, 0.4, 0.6, 0.5, 1.0\}$, $R^- < x_2 >= \{0.2, 0.4, 0.6, 0.5, 1.0\}$, $R^- < x_3 >= \{0.2, 0.4, 0.6, 0.5, 1.0\}$, $R^- < x_4 >= \{0.2, 0.4, 0.6, 0.5, 1.0\}$, $R^- < x_5 >= \{0.2, 0.4, 0.6, 0.5, 1.0\}$. *Therefore,* $R^+ < x_1 > R^+ = \{0.3, 0.0, 0.0, 0.2, 0.0\}$, $R^+ < x_2 > R^+ = \{0.3, 0.3, 0.0, 0.2, 0.0\}$, $R^+ < x_3 > R^+ = \{0.3, 0.0, 0.1, 0.2, 0.0\}$, $R^+ < x_4 > R^+ = \{0.3, 0.0, 0.0, 0.2, 0.0\}$, $R^+ < x_5 > R^+ = \{0.3, 0.0, 0.0, 0.2, 0.1\}$, $R^- < x_1 > R^- = \{0.2, 1.0, 0.6, 0.5, 1.0\}$, $R^- < x_2 > R^- = \{0.2, 1.0, 0.6, 0.5, 1.0\}$, $R^- < x_3 > R^- = \{0.2, 1.0, 0.6, 0.5, 1.0\}$, $R^- < x_4 > R^- = \{0.2, 1.0, 0.6, 0.5, 1.0\}$, $R^- < x_5 > R^- = \{0.2, 1.0, 0.6, 0.5, 1.0\}$. *This implies that* $[RA]_G = \left([\underline{R}A]_G, [\overline{R}A]_G\right)$ *is an IF rough set, where* $[\underline{R}(A)]_G$ *and* $[\overline{R}A]_G$ *are lower and upper approximations of X, respectively, as given below:*

$$[\underline{R}A]_G = \{(x_1, 0.6, 0.1), (x_2, 0.6, 0.2), (x_3, 0.6, 0.1), (x_4, 0.6, 0.1), (x_5, 0.6, 0.1)\},$$
$$[\overline{R}A]_G = \{(x_1, 0.8, 0.1), (x_2, 0.7, 0.2), (x_3, 0.6, 0.1), (x_4, 0.8, 0.1), (x_5, 0.8, 0.1)\}.$$

It is clear that A is an IF rough set as $[\underline{R}A]_G \neq [\overline{R}A]_G$. *Let* $Y = \{x_{12}, x_{15}, x_{23}, x_{25}, x_{34}, x_{35}, x_{41}, x_{52}, x_{54}\} \subseteq X \times X$, *and W is an IF relation on Y, as given in Table 22.*

Table 22. The IF relation W on X in Subsection 6.1.

W	x_{12}	x_{15}	x_{23}	x_{25}	x_{34}	x_{35}	x_{41}	x_{52}	x_{54}
x_{12}	(0.4, 0.2)	(0.9, 0.0)	(0.6, 0.1)	(0.7, 0.1)	(0.4, 0.0)	(0.5, 0.2)	(0.4, 0.1)	(0.3, 0.4)	(0.1, 0.1)
x_{15}	(0.0, 0.0)	(0.3, 0.2)	(0.0, 0.0)	(0.3, 0.3)	(0.4, 0.5)	(0.3, 0.1)	(0.4, 0.0)	(0.0, 0.6)	(0.2, 0.4)
x_{23}	(0.3, 0.1)	(0.5, 0.0)	(0.0, 0.2)	(0.4, 0.3)	(0.2, 0.2)	(0.6, 0.1)	(0.7, 0.1)	(0.2, 0.0)	(0.2, 0.0)
x_{25}	(0.0, 0.3)	(0.3, 0.0)	(0.0, 0.4)	(0.0, 0.3)	(0.1, 0.1)	(0.1, 0.1)	(0.5, 0.1)	(0.0, 0.3)	(0.3, 0.3)
x_{34}	(0.4, 0.2)	(0.2, 0.0)	(0.2, 0.4)	(0.3, 0.1)	(0.0, 0.1)	(0.0, 0.1)	(0.2, 0.1)	(0.5, 0.1)	(0.4, 0.1)
x_{35}	(0.0, 0.0)	(0.2, 0.0)	(0.1, 0.1)	(0.1, 0.0)	(0.0, 0.4)	(0.0, 0.4)	(0.2, 0.1)	(0.0, 1.0)	(0.2, 0.2)
x_{41}	(0.6, 0.0)	(0.2, 0.0)	(0.5, 0.0)	(0.2, 0.0)	(0.4, 0.0)	(0.3, 0.0)	(0.7, 0.0)	(0.3, 0.0)	(0.3, 0.0)
x_{52}	(0.0, 0.3)	(0.3, 0.0)	(0.0, 0.4)	(0.0, 0.3)	(0.1, 0.1)	(0.1, 0.1)	(0.5, 0.1)	(0.0, 0.3)	(0.3, 0.3)
x_{54}	(0.5, 0.4)	(0.3, 0.3)	(0.0, 0.9)	(0.0, 0.8)	(0.1, 0.1)	(0.1, 0.3)	(0.5, 0.0)	(0.3, 0.4)	(0.3, 0.2)

Let $A_\sigma = \{(x_{12}, 0.6, 0.1), (x_{15}, 0.5, 0.0), (x_{23}, 0.7, 0.0), (x_{25}, 0.7, 0.0),$

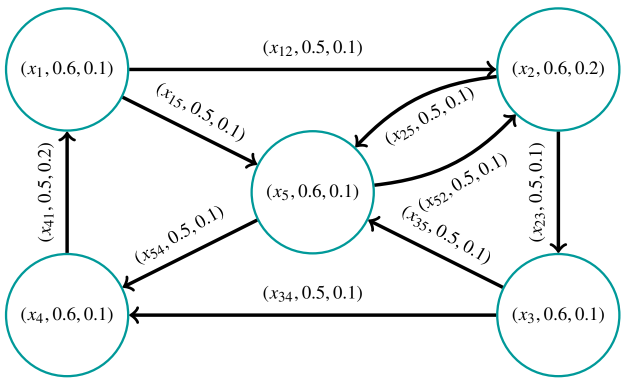
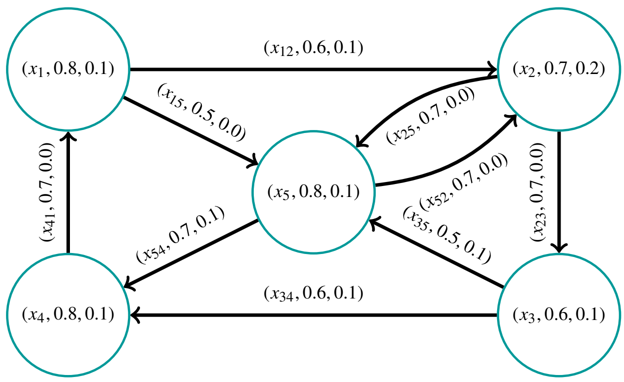
$(x_{34}, 0.6.0.1), (x_{35}, 0.5, 0.2), (x_{41}, 0.7, 0.0), (x_{52}, 0.7, 0.0), (x_{54}, 0.7, 0.1)\}$ *be an IF set on Y, representing the enhancement of beauty in the structure combining different types of different types of embroidery. By computing,* $< x_{12} > W^+ = \{0.3, 0.2, 0.0, 0.0, 0.0, 0.0, 0.2, 0.0, 0.1\}$, $< x_{15} > W^+ = \{0.0, 0.2, 0.0, 0.0, 0.0, 0.0, 0.2, 0.0, 0.1\}$, $< x_{23} > W^+ = \{0.0, 0.2, 0.1, 0.0, 0.0, 0.0, 0.2, 0.0, 0.1\}$, $< x_{25} > W^+ = \{0.0, 0.2, 0.0, 0.1, 0.0, 0.0, 0.2, 0.0, 0.1\}$, $< x_{34} > W^+ = \{0.0, 0.2, 0.0, 0.0, 0.1, 0.0, 0.2, 0.0, 0.1\}$, $< x_{35} > W^+ = \{0.0, 0.2, 0.0, 0.0, 0.0, 0.1, 0.2, 0.0, 0.1\}$, $< x_{41} > W^+ = \{0.0, 0.2, 0.0, 0.0, 0.0, 0.0, 0.2, 0.0, 0.1\}$, $< x_{52} > W^+ = \{0.0, 0.2, 0.0, 0.0, 0.0, 0.0, 0.2, 0.2, 0.1\}$, $< x_{54} > W^+ = \{0.0, 0.2, 0.0, 0.0, 0.0, 0.0, 0.2, 0.2, 0.1\}$, $< x_{12} > W^- = \{0.4, 0.3, 0.9, 0.8, 0.5, 0.4, 0.1, 0.6, 0.4\}$, $< x_{15} > W^- = \{0.4, 0.3, 0.9, 0.8, 0.5, 0.4, 0.1, 0.6, 0.4\}$, $< x_{23} > W^- = \{0.4, 0.3, 0.9, 0.8, 0.5, 0.4, 0.1, 0.6, 0.4\}$, $< x_{25} > W^- = \{0.4, 0.3, 0.9, 0.8, 0.5, 0.4, 0.1, 0.6, 0.4\}$, $< x_{34} > W^- = \{0.4, 0.3, 0.9, 0.8, 0.5, 0.4, 0.1, 0.6, 0.4\}$, $< x_{35} > W^- = \{0.4, 0.3, 0.9, 0.8, 0.5, 0.4, 0.1, 0.6, 0.4\}$, $< x_{41} > W^- = \{0.4, 0.3, 0.9, 0.8, 0.5, 0.4, 0.1, 0.6, 0.4\}$, $< x_{52} > W^- = \{0.4, 0.3, 0.9, 0.8, 0.5, 0.4, 0.1, 0.6, 0.4\}$, $< x_{54} > W^- = \{0.4, 0.3, 0.9, 0.8, 0.5, 0.4, 0.1, 0.6, 0.4\}$. $W^+ < x_{12} >= \{0.1, 0.0, 0.0, 0.0, 0.0, 0.0, 0.2, 0.0, 0.0\}$, $W^+ < x_{15} >= \{0.1, 0.2, 0.0, 0.0, 0.0, 0.0, 0.2, 0.0, 0.0\}$, $W^+ < x_{23} >= \{0.1, 0.0, 0.2, 0.0, 0.0, 0.0, 0.2, 0.0, 0.0\}$, $W^+ < x_{25} >= \{0.1, 0.0, 0.0, 0.1, 0.0, 0.0, 0.2, 0.0, 0.0\}$, $W^+ < x_{34} >= \{0.1, 0.0, 0.0, 0.0, 0.2, 0.0, 0.2, 0.0, 0.0\}$, $W^+ < x_{35} >= \{0.1, 0.0, 0.0, 0.0, 0.0, 0.1, 0.2, 0.0, 0.0\}$, $W^+ < x_{41} >= \{0.1, 0.0, 0.0, 0.0, 0.0, 0.0, 0.2, 0.0, 0.0\}$, $W^+ < x_{52} >= \{0.1, 0.0, 0.0, 0.0, 0.0, 0.0, 0.2, 0.1, 0.0\}$, $W^+ < x_{54} >= \{0.1, 0.0, 0.0, 0.0, 0.0, 0.0, 0.2, 0.0, 0.1\}$, $W^- < x_{12} >= \{0.4, 0.6, 0.3, 0.4, 0.4, 0.4, 0.0, 0.4, 0.9\}$, $W^- < x_{15} >= \{0.4, 0.6, 0.3, 0.4, 0.4, 0.4, 0.0, 0.4, 0.9\}$, $W^- < x_{23} >= \{0.4, 0.6, 0.3, 0.4, 0.4, 0.4, 0.0, 0.4, 0.9\}$, $W^- < x_{25} >= \{0.4, 0.6, 0.3, 0.4, 0.4, 0.4, 0.0, 0.4, 0.9\}$, $W^- < x_{34} >= \{0.4, 0.6, 0.3, 0.4, 0.4, 0.4, 0.0, 0.4, 0.9\}$, $W^- < x_{35} >= \{0.4, 0.6, 0.3, 0.4, 0.4, 0.4, 0.0, 0.4, 0.9\}$, $W^- < x_{41} >= \{0.4, 0.6, 0.3, 0.4, 0.4, 0.4, 0.0, 0.4, 0.9\}$, $W^- < x_{52} >= \{0.4, 0.6, 0.3, 0.4, 0.4, 0.4, 0.0, 0.4, 0.9\}$, $W^- < x_{54} >= \{0.4, 0.6, 0.3, 0.4, 0.4, 0.4, 0.0, 0.4, 0.9\}$. *Therefore,* $W^+ < x_{12} > W^+ = \{0.1, 0.0, 0.0, 0.0, 0.0, 0.0, 0.2, 0.0, 0.0\}$, $W^+ < x_{15} > W^+ = \{0.0, 0.2, 0.0, 0.0, 0.0, 0.0, 0.2, 0.0, 0.0\}$, $W^+ < x_{23} > W^+ = \{0.0, 0.0, 0.1, 0.0, 0.0, 0.0, 0.2, 0.0, 0.0\}$, $W^+ < x_{25} > W^+ = \{0.0, 0.0, 0.0, 0.1, 0.0, 0.0, 0.2, 0.0, 0.0\}$, $W^+ < x_{34} > W^+ = \{0.0, 0.0, 0.0, 0.0, 0.1, 0.0, 0.2, 0.0, 0.0\}$, $W^+ < x_{35} > W^+ = \{0.0, 0.0, 0.0, 0.0, 0.0, 0.1, 0.2, 0.0, 0.0\}$, $W^+ < x_{41} > W^+ = \{0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.2, 0.0, 0.0\}$, $W^+ < x_{52} > W^+ = \{0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.2, 0.1, 0.0\}$, $W^+ < x_{54} > W^+ = \{0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.2, 0.0, 0.1\}$, $W^- < x_{12} > W^- = \{0.4, 0.6, 0.9, 0.8, 0.5, 0.4, 0.1, 0.6, 0.9\}$, $W^- < x_{15} > W^- = \{0.4, 0.6, 0.9, 0.8, 0.5, 0.4, 0.1, 0.6, 0.9\}$, $W^- < x_{23} > W^- = \{0.4, 0.6, 0.9, 0.8, 0.5, 0.4, 0.1, 0.6, 0.9\}$, $W^- < x_{25} > W^- = \{0.4, 0.6, 0.9, 0.8, 0.5, 0.4, 0.1, 0.6, 0.9\}$, $W^- < x_{34} > W^- = \{0.4, 0.6, 0.9, 0.8, 0.5, 0.4, 0.1, 0.6, 0.9\}$, $W^- < x_{35} > W^- = \{0.4, 0.6, 0.9, 0.8, 0.5, 0.4, 0.1, 0.6, 0.9\}$, $W^- < x_{41} > W^- = \{0.4, 0.6, 0.9, 0.8, 0.5, 0.4, 0.1, 0.6, 0.9\}$, $W^- < x_{52} > W^- = \{0.4, 0.6, 0.9, 0.8, 0.5, 0.4, 0.1, 0.6, 0.9\}$, $W^- < x_{54} > W^- = \{0.4, 0.6, 0.9, 0.8, 0.5, 0.4, 0.1, 0.6, 0.9\}$. *Then,* $WA_\sigma = \left(\underline{W}A_\sigma, \overline{W}A_\sigma\right)$ *is an IF rough relation on X, where* $\underline{W}A_\sigma$ *and* $\overline{W}A_\sigma$ *are lower and upper approximations of* A_σ*, respectively, as shown in Table 23.*

Table 23. The IF relation edge set in Subsection 6.1.

WA_σ	x_{12}	x_{15}	x_{23}	x_{25}	x_{34}	x_{35}	x_{41}	x_{52}	x_{54}
$\underline{WA}_\sigma$	(0.5, 0.1)	(0.5, 0.1)	(0.5, 0.1)	(0.5, 0.1)	(0.5, 0.1)	(0.5, 0.1)	(0.5, 0.2)	(0.5, 0.1)	(0.5, 0.1)
$\overline{\overline{W}}A_\sigma$	(0.6, 0.1)	(0.5, 0.0)	(0.7, 0.0)	(0.7, 0.0)	(0.6, 0.1)	(0.5, 0.1)	(0.7, 0.0)	(0.7, 0.0)	(0.7, 0.1)

This implies that, $\underline{G} = \left([\underline{R}A]_G, \underline{WA}_\sigma\right)$ *and* $\overline{G} = \left([\overline{R}A]_G, [\overline{W}A_\sigma]_G\right)$ *are IF graphs, as shown in Table 24.*

Table 24. IF rough graph $G = \left(\underline{G}, \overline{G}\right)$ in Subsection 6.1.

$\underline{G} = \left([\underline{R}A]_G, \underline{Z}_i A_\sigma\right)$	$\overline{G} = \left([\overline{R}A]_G, [\overline{W}A_\sigma]_G\right)$
	

To find out the most suitable embroidery, we define the score values (S.V.), which acts as a single, interpretable score that combines lower and upper evidence to support embroidery design decisions under uncertainty. It enables ranking and thresholding of options, balances both approximation spaces for consistent judgments, and improves explainability by linking a scalar value to decision outcomes. For each embroidery x_i, $i \in \{1, 2, \ldots, 5\}$, $S.V(x_i)$ *is given by:*

$$S.V(x_i) = \sum_j \sqrt{\left(T_{[\overline{W}A_\sigma]_G}\left(x_{ij}\right) - T_{[\underline{WA}_\sigma]_G}\left(x_{ij}\right)\right)^2 + \left(F_{[\overline{W}A_\sigma]_G}\left(x_{ij}\right) - F_{[\underline{WA}_\sigma]_G}\left(x_{ij}\right)\right)^2}$$

and the decision is x_s *if* $S.V(x_s) = \max\{S.V(x_i), i = 1, 2 \ldots, 5\}$. *By calculation, we have*

- $S.V(x_1) = \sqrt{(0.5)^2 + (0.1)^2} + \sqrt{(0.4)^2 + (0.1)^2} = \sqrt{0.26} + \sqrt{0.17} \approx 0.922.$
- $S.V(x_2) = (0.6)^2 + (0.1)^2 + \sqrt{(0.6)^2 + (0.1)^2} = 2\sqrt{0.37} \approx 1.217.$
- $S.V(x_3) = \sqrt{(0.4)^2 + (0.1)^2} + \sqrt{(0.5)^2 + (0.1)^2} = \sqrt{0.17} + \sqrt{0.26} \approx 0.922.$
- $S.V(x_4) = \sqrt{(0.6)^2 + (0.2)^2} = \sqrt{0.40} \approx 0.632.$
- $S.V(x_5) = \sqrt{(0.6)^2 + (0.1)^2} + \sqrt{(0.6)^2 + (0.1)^2} = 2\sqrt{0.37} \approx 1.217.$

Clearly, x_2 *and* x_5 *have the maximum score. Then candidates* x_2 *and* x_5 *are the recommended decision. This indicates that any of them may be selected without loss of generality. We present our proposed method as an algorithm. This Algorithm 1 returns the optimal solution for the investment problem.*

Algorithm 1 The algorithm for determining a suitable embroidery

Input: A finite vertex set X of embroidery $x_1, x_2, \ldots, x_n$ and two finite IF relations R, W.

Output: The suitable embroidery x_i.

Input the vertex set $X = \{x_1, x_2, \ldots, x_n\}$.
Input an IF relation R on X. (2D matrix to link vertices to their relation values)
Input an IF set $A = (\rho^+, \rho^-)$ on X. (2D array or matrix where the first column is ρ^+ and the second is ρ^-)
Input the edge set Y of relations $y_1, y_2, \ldots, y_r$ where, $y_i = x_{jk}$, for some $j, k \in \{1, 2, \ldots, n\}$.
Input an IF relation W on $Y \subseteq X \times X$. (2D matrix to link edges to their relation values)
Input an IF set $A_\sigma = (\sigma^+, \sigma^-)$ on Y. (2D array or matrix where the first column is σ^+ and the second is σ^-)
for $i = 1 : n$ **do**
    read* $A(x_i)$ (The membership value $\rho^+(x_i)$ and the non-membership value $\rho^-(x_i)$ of x_i in the IF set A)
end for
for $i = 1 : n$ **do**
    $T_{[\overline{R}(A)]_G}(x_i) = 0$, $F_{[\overline{R}(A)]_G}(x_i) = 1$
    $T_{[\underline{R}(A)]_G}(x_i) = 1$, $F_{[\underline{R}(A)]_G}(x_i) = 0$
    for $j = 1 : n$ **do**
        $T_{[\underline{R}(A)]_G}(x_i) = \min\left\{T_A(x_i), \left(\max\left\{R^- < x_i > R^-(x_j), \rho^+\left(x_j\right)\right\}\right)\right\}$
        $F_{[\underline{R}(A)]_G}(x_i) = \max\left\{F_A(x_i), \left(\min\left\{R^+ < x_i > R^+(x_j), \rho^-\left(x_j\right)\right\}\right)\right\}$
        $T_{[\overline{R}(A)]_G}(x_i) = \max\left\{T_A(x_i), \left(\min\left\{R^+ < x_i > R^+(x_j), \rho^+\left(x_j\right)\right\}\right)\right\}$
        $F_{[\overline{R}(A)]_G}(x_i) = \min\left\{F_A(x_i), \left(\max\left\{R^- < x_i > R^-(x_j), \rho^-\left(x_j\right)\right\}\right)\right\}$
    end for
end for
for $i = 1 : r$ **do**
    read* $A_\sigma(y_i)$ (The membership value $\sigma^+(y_i)$ and non-membership value $\sigma^-(y_i)$ of x_i in the IF set A_σ)
end for
for $i = 1 : r$ **do**
    $T_{[\overline{W}A_\sigma]_G}(y_i) = 0$, $F_{[\overline{W}A_\sigma]_G}(y_i) = 1$
    $T_{[\underline{W}A_\sigma]_G}(y_i) = 1$, $F_{[\underline{W}A_\sigma]_G}(y_i) = 0$
    for $j = 1 : r$ **do**
        $T_{[\underline{W}A_\sigma]_G}(y_i) = \min\left\{T_{A_\sigma}(y_i), \left(\max\left\{W^- < y_i > W^-(y_j), \sigma^+\left(y_j\right)\right\}\right)\right\}$
        $F_{[\underline{W}A_\sigma]_G}(y_i) = \max\left\{F_{A_\sigma}(y_i), \left(\min\left\{W^+ < y_i > W^+(y_j), \sigma^-\left(y_j\right)\right\}\right)\right\}$
        $T_{[\overline{W}A_\sigma]_G}(y_i) = \max\left\{T_{A_\sigma}(y_i), \left(\min\left\{W^+ < y_i > W^+(y_j), \sigma^+\left(y_j\right)\right\}\right)\right\}$
        $F_{[\overline{W}A_\sigma]_G}(y_i) = \min\left\{F_{A_\sigma}(y_i), \left(\max\left\{W^- < y_i > W^-(y_j), \sigma^-\left(y_j\right)\right\}\right)\right\}$
    end for
end for
for $i = 1 : n$ **do**
    $S.V(x_i) = \sum_j \sqrt{\left(T_{[\overline{W}A_\sigma]_G}\left(x_{ij}\right) - T_{[\underline{W}A_\sigma]_G}\left(x_{ij}\right)\right)^2 + \left(F_{[\overline{W}A_\sigma]_G}\left(x_{ij}\right) - F_{[\underline{W}A_\sigma]_G}\left(x_{ij}\right)\right)^2}$
end for
if $x_s = \max_i x_i, i = 1, 2, \ldots, |X|$ **then**
    Print: The decision is x_s.
    if s has more than one value **then**
        Print: Any one of x_s.
    end if
end if

Comparative analysis

The proposed IF lower and upper approximations based on minimal IF neighborhoods effectively preserve the fundamental properties of classical rough set theory while enhancing its capabilities for uncertainty management. By constructing approximation operators through minimal IF neighborhoods derived from two IF binary relations on directed graphs, this framework maintains essential characteristics such as duality, inclusion properties, and boundary region definition that are central to Pawlak's original theory. Crucially, based on Lemmas 3.9 and 3.10, the proposed model preserves the monotonic property, which ensures that as input information becomes more refined or precise, the resulting approximations respond consistently. This monotonicity enables efficient evaluation of uncertainty in data by providing predictable behavior during computational processes. Furthermore, the approach demonstrably minimizes the boundary region more effectively than previous models [21, 23], where these new IF approximation spaces reduce the IF boundary region and increase the accuracy degree, as illustrated in Definitions 3.12, 3.13, and Example 3.14. This reduction in the boundary region directly translates to improved decision-making capabilities in hybrid models represented by directed graphs, since a smaller boundary region indicates less ambiguity between definable and indefinable sets, leading to more confident and accurate decisions in applications ranging from expert systems to complex network analysis. In addition, our model in Definition 3.7 is more general than previous models in [1, 11, 25, 26], as shown in Remark 3.20. We presented a single hybrid model of directed fuzzy rough graphs in Example 6.1 to demonstrate the practical application of our proposed IF approximations in decision-making contexts. Specifically, we illustrated our approach through a real-world example involving the selection of suitable embroidery for bridal fashion design, where the IF lower and upper approximations based on minimal IF neighborhoods were employed to evaluate alternatives and arrive at an optimal decision. These IF edges $A_\sigma = \{(x_{12}, 0.6, 0.1), (x_{15}, 0.5, 0.0), (x_{23}, 0.7, 0.0), (x_{25}, 0.7, 0.0), (x_{34}, 0.6.0.1), (x_{35}, 0.5, 0.2), (x_{41}, 0.7, 0.0), (x_{52}, 0.7, 0.0), (x_{54}, 0.7, 0.1)\}$, together with the IF vertex sets $A = \{(x_1, 0.8, 0.1), (x_2, 0.7, 0.2), (x_3, 0.6, 0.1), (x_4, 0.8, 0.1), (x_5, 0.8, 0.1)\}$, define the IF lower and upper approximations of the graph, which are used to create IF rough graphs. The fuzzy edges influence these approximations by affecting the minimal IF neighborhoods, which in turn determine the precision and accuracy of the boundary region that encapsulates uncertainty in the decision-making process. While this example serves as a concrete validation of our theoretical framework, it is important to emphasize that the underlying methodology is not limited to this particular scenario. The same principles utilizing minimal IF neighborhoods derived from two IF binary relations to construct approximation operators can be readily extended to address a wide range of similar real-life decision-making problems. Whether in healthcare diagnostics, financial risk assessment, supply chain optimization, or social network analysis, analogous IF rough graph models can be constructed by adapting the vertex and edge interpretations to the specific domain, while applying the same efficient algorithmic procedure as given in Algorithm 1 to compute approximations, reduce boundary regions, and enhance decision accuracy. Thus, the presented case functions as a representative template, showcasing a scalable and flexible approach that can be replicated across diverse application areas characterized by uncertainty and imprecise information.

7. Conclusions

Rough set theory and IF set theory are distinct mathematical frameworks used to model and address the challenges of comprehending and working with incomplete knowledge. When integrated, these two theories form the IF rough framework, which is recognized as a more adaptable and expressive system for modeling and processing incomplete data. The IF rough model is a broad framework constructed using the approximation space. This model is considered more compatible, accurate, and flexible compared to rough fuzzy models and fuzzy rough models, which can be viewed as specific instances of IF rough sets. To investigate various network models and pairwise interactions between objects, the research introduces a novel hybrid model. This model defines the IF lower and upper approximations using the minimal IF neighborhoods. The authors further create two new types of IF graph approximation spaces, based on any two IF binary relations formed on a non-empty universe. The purpose of creating these new spaces is to decrease the IF border region and increase the IF accuracy degree. Several important conclusions are obtained for both types of IF graph approximations, and the relationships between the current IF approximations are induced. Furthermore, it is demonstrated that the current IF graph approximations are more generic than the previous ones. Finally, the applications of IF rough graphs in decision-making problems are described, and an efficient algorithm is developed to solve these decision-making problems. A noted limitation of the paper is that the developed IF rough graph models and approximation spaces are confined to non-empty finite sets and corresponding finite graphs. Throughout the work, the universe of discourse is explicitly assumed to be finite, which restricts the applicability of the proposed IF rough frameworks and neighborhood constructions to finite structures only. Future work includes exploring IF topologies induced by rough IF graphs, leveraging the strong connection between graph theory and general topology to develop new topological structures based on the defined IF approximations. Additionally, the study suggests applying graph-theoretical indices such as the Wiener index and Laplacian energy within IF rough graphs to analyze complex systems like transport networks and decision-making utilities. Moreover, the efficient decision-making algorithm developed can be extended and applied to a broader range of real-world multi-criteria selection problems across various domains, enhancing the model's applicability and effectiveness in handling uncertainty and incomplete information.

Author contributions

Shi: Methodology, Data curation, Funding; Abbas: Validation, Formal analysis, Investigation; Khiamy: Software, Writing–original draft, Writing–review final form; Ibedou: Conceptualization, Supervision, Project administration. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Data availability

The datasets used and/or analyzed during the current study are available from the corresponding author on reasonable request.

Conflict of interest

The authors declare that they have no conflicts of interest.

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