Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(3): 8031–8064.
DOI: 10.3934/math.2026331
Received: 20 January 2026
Revised: 26 February 2026
Accepted: 03 March 2026
Published: 26 March 2026

Research article

Multicomponent stress–strength reliability inference under progressive Type-II censoring: A one-parameter model with applications to power systems and earthquake data

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Abstract: Stress–strength reliability models are essential for assessing the safety and performance of engineering systems operating under uncertainty and censored lifetime data. This paper develops a reliability framework for multicomponent stress–strength systems by modeling stress and strength using the Burr–Hatke distribution under progressive Type II censoring. Maximum likelihood estimators are derived for the model's parameters and the stress–strength reliability function, along with their asymptotic confidence intervals. To improve finite–sample inference, bootstrap confidence intervals are constructed. Bayesian estimation is performed under a generalized entropy loss function using gamma priors, using Lindley's approximation and Markov chain Monte Carlo techniques. The corresponding credible intervals and highest posterior density intervals are obtained for interval estimation. Extensive Monte Carlo simulations and applications to rear dump truck failure times and earthquake inter–event data demonstrate the effectiveness and robustness of the proposed approach under progressive censoring.

Keywords: multicomponent stress–strength model; reliability analysis; Lindley approximation; progressive Type II censoring; Burr–Hatke distribution; Bayesian inference; simulation analysis
Mathematics Subject Classification: 62F10, 62F15, 62N05, 62P30

Abbreviations

SS	Stress–Strength
MSSR	Multicomponent stress–strength reliability
PTIIC	Progressive Type II censoring
BuH	Burr–Hatke
MCMC	Markov chain Monte Carlo
HPD	Highest posterior density
MLE	Maximum likelihood estimator
ACI	Asymptotic confidence interval
Boot-p	Percentile bootstrap
Boot-t	Studentized bootstrap
CI	Confidence intervals
GELF	General entropy loss function
ELF	Entropy loss function
SELF	Squared error loss function
PELF	Precautionary loss function
MH	Metropolis–Hastings
CrI	Bayesian credible interval
MSE	Mean squared error
AL	Average length
CP	Coverage probability
AB	Average bias
C.S	Censoring scheme
KS	Kolmogorov–Smirnov
AIC	Akaike information criterion
BIC	Bayesian information criterion
ACF	Autocorrelation function
IP	Informative prior
NIP	Noninformative prior
ESS	Effective sample size

1. Introduction

Reliability analysis forms a core area of study in engineering, medicine, quality control, and risk assessment. It provides the mathematical basis for evaluating the performance and failure characteristics of systems and components. A fundamental concept in this field is the stress–strength model (SS), which provides a probabilistic framework for assessing the reliability of the system. In simple terms, a component fails when the external stress applied, represented by the random variable Y, exceeds its inherent strength X. Thus, the reliability δ of the system is expressed as the probability of survival $\delta = P(X > Y)$. Statistical inference for the SS model has been explored under different probability distributions. The early contribution of [1] established a general framework for reliability estimation, while [2] developed hypothesis testing procedures for the case where X and Y are normally distributed.

Subsequent research has extended these methods to other lifetime distributions, including the Weibull [3], the two-parameter exponential [4], the power Lindley [5], and recently, the exponentiated half-logistic Weibull distribution [6]. A detailed overview of the theory and applications of stress–strength models can be found in [7].

A natural extension of the classical SS model is the multicomponent stress–strength reliability (MSSR) system. This model reflects complex real-world systems where overall reliability depends on a set of k independent and identically distributed strength components, $X_1, X_2, \ldots, X_k$, that are simultaneously subjected to a specific random stress Y. The system functions successfully if at least s out of the k components ($1 \leq s \leq k$) survive the stress. Such an s-out-of-k configuration appears frequently in engineering applications. For instance, a bridge supported by k cables remains stable as long as at least s cables bear the load, and an internal combustion engine with multiple cylinders fails if fewer than a certain number are operational. The theoretical formulation for the MSSR model was established by [8], who defined the reliability parameter $\delta_{s,k}$ as follows:

$$\delta_{s,k} = P[\text{ At least } s \text{ of } (X_1, X_2, \ldots, X_k) \text{ exceed } Y]. \tag{1.1}$$

Using the theory of order statistics, this probability can be expressed as:

$$\delta_{s,k} = \sum_{p=s}^{k} \binom{k}{p} \int_{-\infty}^{\infty} [1 - F_X(y)]^p [F_X(y)]^{k-p} dF_Y(y), \tag{1.2}$$

where $F_X(\cdot)$ and $F_Y(\cdot)$ are the cumulative distribution functions (CDF) of the strength and stress variables, respectively.

In life-testing studies, censoring techniques are often introduced to minimize testing time and cost, particularly when studying highly reliable products that require long observation periods. In practice, different censoring plans are designed to improve efficiency and limit unnecessary expenses. Such methods are now standard in reliability engineering, machine performance assessments, radio signal analysis, and biomedical investigations. Among the most frequently used plans are Type I and Type II censoring. In Type I, the experiment is terminated at a fixed time, while in Type II testing, it stops after observing a certain number of failures; see [9]. However, these conventional schemes do not allow for the withdrawal of test units in the middle stages. Therefore, a progressive Type II censoring (PTIIC) scheme aims to address the limitations in the flexibility of conventional methods. Under the PTIIC scheme, N identical units are placed into the test, and at the time of the i^{th} failure, a predetermined number R_i of surviving units are randomly removed from the experiment. This process continues until the n^{th} failure occurs; at this point, all remaining $R_n = N - n - \sum_{i=1}^{n-1} R_i$ units are removed. The resulting ordered failure times, $X_{(1:n:N)}, X_{(2:n:N)}, \ldots, X_{(n:n:N)}$, constitute a progressively Type II censored sample with the scheme $(R_1, R_2, \ldots, R_n)$. This scheme generalizes conventional Type II censoring (when $R_1 = \ldots = R_{n-1} = 0, R_n = N - n$) and includes the complete sample case (when $R_1 = \ldots = R_n = 0$). The application of progressive censoring in reliability inference is demonstrated for distributions like the generalized Pareto [10] and, in the context of accelerated life testing, the power half-logistic distribution with an adaptive progressively censored framework [11].

The selection of an appropriate probability distribution is essential for reliable modeling because it must capture the actual failure behavior of the components. In this study, we use the one-parameter Burr–Hatke (BuH) distribution, first proposed by [12], as an alternative to the exponential model. For a positive random variable T with a scale parameter $\alpha > 0$, the BuH distribution is characterized by its CDF as follows:

$$F(t;\alpha) = 1 - \frac{e^{-\alpha t}}{t+1}, \quad t > 0, \tag{1.3}$$

and its corresponding probability density function (PDF):

$$f(t;\alpha) = \frac{e^{-\alpha t}[\alpha(t+1)+1]}{(t+1)^2}, \quad t > 0. \tag{1.4}$$

A key property of this model is that its PDF is strictly decreasing for all $\alpha > 0$, which makes it suitable for describing data with a higher risk of failure at the beginning of the system's life that gradually decreases over time. The hazard rate is given by

$$h(t;\alpha) = \alpha + \frac{1}{t+1},$$

which is strictly decreasing and converges to α as $t \to \infty$. This behavior reflects the infant mortality pattern observed in many electronic and mechanical systems, where failures are more likely early on and become less frequent as time progresses, which is a common characteristic in electronic and mechanical systems . Its mathematical tractability, owing to its simple one-parameter structure, facilitates straightforward implementation in standard inferential frameworks while maintaining interpretive simplicity for practitioners. The growing attention on BuH–based models in recent literature, evidenced by extensions such as the Burr–Hatke-G family [13], which attracted much attention from scholars such as [14] and [15]. The Burr-Hatke exponential distribution introduced by [16]. The Chen Burr–Hatke exponential distribution [17] further expands its flexibility to handle a wider variety of hazard shapes, including bathtub-type patterns.

The simplicity of the single-parameter formulation ensures interpretability, computational stability, and reduced estimation variability, especially under censored sampling schemes. At the same time, the BuH model retains sufficient flexibility while avoiding overparameterization. These characteristics suggest that the BuH distribution has strong potential for broader adoption in complex reliability systems, particularly in MSSR frameworks where analytical tractability and numerical stability are essential. Moreover, because the BuH distribution admits several natural multi-parameter extensions in relatively simple functional forms, it provides a convenient baseline model upon which more flexible families can be developed and studied within the MSSR context.

Considering the MSSR described in (1.2), the reliability function can be written as:

$$\delta_{s,k} = \sum_{p=s}^{k} \sum_{q=0}^{k-p} \binom{k}{p} \binom{k-p}{q} (-1)^q \int_{-\infty}^{\infty} [1 - F_X(y)]^{p+q} f_Y(y)\, dy. \tag{1.5}$$

Let $X_1, X_2, \ldots, X_k$ be independent random variables following the BuH distribution, BuH(α_1), and let Y (independent of Xs) follow the BuH distribution, BuH(α_2). Now, using Eqs (1.3) and (1.4) in Eq (1.5) we obtain $\delta_{s,k}$ as follows:

$$\delta_{s,k} = \sum_{p=s}^{k} \sum_{q=0}^{k-p} \binom{k}{p} \binom{k-p}{q} (-1)^q \int_{0}^{\infty} e^{-\xi y} (y+1)^{-(p+q+2)} (\alpha_2(y+1)+1)\, dy, \tag{1.6}$$

where $\xi = \alpha_1(p+q) + \alpha_2$. Setting $t = y + 1$, we get

$$\delta_{s,k} = \sum_{p=s}^{k} \sum_{q=0}^{k-p} \binom{k}{p} \binom{k-p}{q} (-1)^q e^{\xi} \int_1^{\infty} e^{-\xi t} t^{-(p+q+2)} (\alpha_2 t + 1)\, dt, \tag{1.7}$$

Finally, $\delta_{s,k}$ is given by

$$\delta_{s,k} = \sum_{p=s}^{k} \sum_{q=0}^{k-p} \binom{k}{p} \binom{k-p}{q} (-1)^q e^{\xi} \left(\alpha_2 E_{p+q+1}(\xi) + E_{p+q+2}(\xi) \right) \tag{1.8}$$

where $E_n(z) = \int_1^{\infty} e^{-zt} t^{-n} dt$ is the exponential integral function. The expression of $\delta_{s,k}$ in (1.8) is straightforward, since $E_n(z)$ is directly available in packages such as Mathematica via built-in routines, which enables efficient and accurate numerical evaluation.

The inference for the MSSR parameter $\delta_{s,k}$ has been an active area of research. Recent studies have developed classical and Bayesian estimation methods for various distributions under progressive censoring, including the Weibull [18], unit Gompertz [19], Topp–Leone [20], unit Kumaraswamy [21], Burr XII [22], and the unit Burr III distributions [23]. More recent work can be found in [24], who studied the reliability of a multicomponent system under the inverted exponential, see also the work of [25], in which progressive first-failure censoring was employed for multicomponent stress-strength systems. [26] used block adaptive Type II progressive hybrid censoring and k-records.

Despite these recent developments, the problem of estimating the MSSR for a system where both stress and strength follow the BuH distribution under PTIIC has not yet been explored in the literature. The BuH distribution offers a compelling balance between simplicity and flexibility. It is a single-parameter lifetime model capable of describing non-constant failure behavior while maintaining analytical simplicity. Unlike multiparameter models such as the Weibull, Burr XII, or generalized Pareto distributions, the BuH distribution avoids overparameterization and reduces numerical instability, which is an important advantage in complex settings like MSSR estimation under progressive censoring. The BuH distribution has shown competitive performance against classical alternatives like the exponential and Lindley distributions in modeling lifetime data. Moreover, its simple structure makes it suitable as a baseline model that can be extended to multi-parameter generalizations when additional flexibility is required.

By developing inference under progressive censoring and MSSR frameworks based on the BuH distribution, this work demonstrates that a well chosen single-parameter model can yield stable and competitive performance, specifically in complex reliability settings. The work aims to support statisticians and practitioners to consider this promising model in advanced reliability settings, while also encouraging further theoretical extensions built upon its simple yet expressive formulation. The maximum likelihood estimators (MLEs) of the model's parameters and the reliability measure are obtained, and the corresponding asymptotic confidence intervals are derived. In addition, percentile bootstrap and studentized bootstrap (Boot-p and Boot-t) confidence intervals are constructed. Bayesian inference is utilized with the generalized entropy loss function and gamma priors using Lindley's approximation and Markov chain Monte Carlo (MCMC) techniques. Furthermore, Bayesian intervals and the highest posterior density (HPD) intervals are constructed directly from the posterior samples generated via the MCMC algorithm. An intensive simulation study is carried out to assess the efficacy of the estimation methods under various censoring schemes. The practical relevance of the methodology is demonstrated through two applications: Generators' lifetime failure and earthquake inter-event time data.

The rest of this article is structured as follows: Section 2 develops the MLEs for the model's parameter and the reliability measure $\delta_{s,k}$, and outlines the construction of asymptotic bootstraps. Section 3 introduces the Bayesian estimation approach, using Lindley's approximation and the MCMC techniques, and describes the construction of Bayesian credible intervals as well as the highest posterior density intervals. Section 4 reports on the simulation study designed to assess the performance of the proposed estimators with various settings. Section 5 illustrates the practical applicability of the methods through a real data analysis. Concluding remarks and future work are presented in Section 6.

2. Maximum likelihood estimation of $\delta_{s,k}$

To determine the MLE of $\delta_{s,k}$, it is first necessary to obtain the MLEs of α_1 and α_2. Suppose that N systems are placed in a life test and each one is composed of K components. In the PTIIC scheme, only a portion of the data is observed. With this setup, full information is available for the n systems, each retaining k components of the original K. These censored observations are then used to formulate the likelihood function, which serves as the basis for estimating both α_1 and α_2 and, consequently, $\delta_{s,k}$. The samples can be represented in the following way:

$$\begin{array}{ccc} \text{Observed strength variables} & & \text{observed stress variables} \\ \begin{pmatrix} X_{11} & X_{12} & \cdots & X_{1k} \\ \vdots & \vdots & \ddots & \vdots \\ X_{n1} & X_{n2} & \cdots & X_{nk} \end{pmatrix} & \text{and} & \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix} \end{array}$$

Let $\{X_{i1}, X_{i2}, X_{i3}, \ldots, X_{ik}\},\ i = 1, \ldots, n$, represent k PTIIC observations from the BuH distribution with the parameter α_1 under the censoring scheme $(K, k, R_1, R_2, \ldots, R_k)$. On the other hand, assume that $(Y_1, Y_2, \ldots, Y_n)$ is a PTIIC sample obtained from BuH(α_2) according to the censoring plan $(N, n, S_1, S_2, \ldots, S_n)$. Given these samples, the likelihood function for the parameters α_1 and α_2 is

$$L(\alpha_1, \alpha_2) = c_1 \prod_{i=1}^{n} \left(c_2 \prod_{j=1}^{k} f(x_{ij})[1 - F(x_{ij})]^{R_j} \right) f(y_i)[1 - F(y_j)]^{S_i}, \tag{2.1}$$

where

$$c_1 = N(N - S_1 - 1)\ldots(N - S_1 - \cdots - S_{n-1} - n + 1),$$

$$c_2 = K(K - R_1 - 1)\ldots(K - R_1 - \cdots - R_{k-1} - k + 1).$$

By substituting the BuH PDF (1.4) and CDF (1.3) into the likelihood expression in (2.1), the likelihood function of the parameters (α_1, α_2) can be expressed as

$$\begin{aligned} L(\text{data} \mid \alpha_1, \alpha_2) =& c_1 \prod_{i=1}^{n} \left[c_2 \prod_{j=1}^{k} \left(\frac{e^{-\alpha_1 x_{ij}}(\alpha_1(x_{ij} + 1) + 1)}{(x_{ij} + 1)^2} \right) \left(\frac{e^{-\alpha_1 x_{ij}}}{x_{ij} + 1} \right)^{R_j} \right] \\ & \times \left[\frac{e^{-\alpha_2 y_i}(\alpha_2(y_i + 1) + 1)}{(y_i + 1)^2} \left(\frac{e^{-\alpha_2 y_i}}{y_i + 1} \right)^{S_i} \right]. \end{aligned} \tag{2.2}$$

The proposed likelihood formulation admits several important special cases. In particular, when $k = 1$, the model reduces to the conventional stress–strength setting, and the associated reliability measure coincides with $\delta = P(X > Y)$. Furthermore, when $R_j = 0$ and $S_i = 0$ for all j and i, the likelihood expression naturally simplifies to that of a complete sample.

From Eq (2.2), the log-likelihood function is considered as follows:

$$\begin{aligned}\ell \propto & -\alpha_1 \sum_{i=1}^{n}\sum_{j=1}^{k} x_{ij} + \sum_{i=1}^{n}\sum_{j=1}^{k} \log[\alpha_1(x_{ij}+1)+1] - 2\sum_{i=1}^{n}\sum_{j=1}^{k} \log(x_{ij}+1) \\ & -\alpha_1 \sum_{i=1}^{n}\sum_{j=1}^{k} x_{ij}R_j - \sum_{i=1}^{n}\sum_{j=1}^{k} R_j \log(x_{ij}+1) - \alpha_2 \sum_{i=1}^{n} y_i \\ & + \sum_{i=1}^{n} \log\left(\alpha_2(y_i+1)+1\right) - 2\sum_{i=1}^{n} \log(y_i+1) - \alpha_2 \sum_{i=1}^{n} S_i y_i - \sum_{i=1}^{n} S_i \log(y_i+1).\end{aligned} \tag{2.3}$$

The MLE $\hat{\eta} = (\hat{\alpha}_1, \hat{\alpha}_2)$ of the parameter vector $\eta = (\alpha_1, \alpha_2)$ is obtained by simultaneously solving the following non-linear equations:

$$\frac{\partial \ell}{d\alpha_1} = -\sum_{i=1}^{n}\sum_{j=1}^{k} x_{ij} + \sum_{i=1}^{n}\sum_{j=1}^{k} \frac{x_{ij}+1}{\alpha_1(x_{ij}+1)+1} - \sum_{i=1}^{n}\sum_{j=1}^{k} R_j x_{ij} = 0, \tag{2.4}$$

$$\frac{\partial \ell}{d\alpha_2} = -\sum_{i=1}^{n} y_i + \sum_{i=1}^{n} \frac{y_i+1}{\alpha_2(y_i+1)+1} - \sum_{i=1}^{n} S_i y_i = 0. \tag{2.5}$$

An analytical solution to Eqs (2.4) and (2.5) is not available, requiring a numerical solution via iterative methods such as the Newton-Raphson method, which is implemented using standard mathematical software. Alternatively, some software packages offer a direct approach to maximizing the log-likelihood function (2.3). Examples include Mathematica (NMaximize and FindMaximum functions) and R (optim and MaxLik functions).

The MLE of $\delta_{s,k}$ is obtained by using the invariance property of the MLE and is obtained as follows:

$$\hat{\delta}_{s,k} = \sum_{p=s}^{k}\sum_{q=0}^{k-p} \binom{k}{p}\binom{k-p}{q} (-1)^q e^{\hat{\xi}} [\hat{\alpha}_2 E_{p+q+1}(\hat{\xi}) + E_{p+q+2}(\hat{\xi})] \tag{2.6}$$

where $\hat{\xi} = \hat{\alpha}_1(p+q) + \hat{\alpha}_2$.

2.1. Asymptotic confidence interval

The asymptotic confidence interval (ACI) for $\delta_{s,k}$ is derived from the well-established asymptotic normality of the MLE. Specifically, $(\hat{\eta} - \eta)$ is approximated by a bivariate normal distribution with a zero mean vector and the variance-covariance matrix $I^{-1}(\hat{\eta})$, where $I(\hat{\eta})$ denotes the observed Fisher information matrix, given by

$$I(\hat{\eta}) = \begin{pmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{pmatrix} = \begin{pmatrix} -\frac{\partial^2 \ell}{\partial \alpha_1^2} & -\frac{\partial^2 \ell}{\partial \alpha_1 \partial \alpha_2} \\ -\frac{\partial^2 \ell}{\partial \alpha_2 \partial \alpha_1} & -\frac{\partial^2 \ell}{\partial \alpha_2^2} \end{pmatrix},$$

where

$$I_{11} = -\sum_{i=1}^{n}\sum_{j=1}^{k}\left(\frac{x_{ij}+1}{\alpha_1(x_{ij}+1)+1}\right)^2$$

$$I_{22} = -\sum_{i=1}^{n}\left(\frac{y_i+1}{\alpha_2(y_i+1)+1}\right)^2$$

$$I_{12} = I_{21} = 0.$$

The asymptotic variance of $\hat{\delta}_{s,k} = g(\hat{\eta})$ is obtained using the delta method, based on a first-order Taylor expansion of $g(\hat{\eta})$ around η. It follows that

$$var(\hat{\delta}_{s,k}) \approx \nabla g(\eta)^\top I^{-1}(\eta)\nabla g(\eta)$$

where $\nabla g(\eta) = \left(\frac{\partial g}{\partial \alpha_1}, \frac{\partial g}{\partial \alpha_2}\right)^\top$ denotes the gradient vector of partial derivatives. This approximation follows from the asymptotic normality of the MLE and standard delta method theory [27,28]. Since $I_{12} = I_{21} = 0$, the variance of $\hat{\delta}_{s,k}$ simplifies to

$$var(\hat{\delta}_{s,k}) = \left(\frac{\partial \delta_{s,k}}{\partial \alpha_1}\right)^2 [I^{-1}]_{11} + \left(\frac{\partial \delta_{s,k}}{\partial \alpha_2}\right)^2 [I^{-1}]_{22}.$$

The partial derivative $\frac{\partial \delta_{s,k}}{\partial \alpha_1}$ is obtained by differentiating Eq (1.7) with respect to α_1, which gives

$$\frac{\partial \delta_{s,k}}{\partial \alpha_1} = \sum_{p=s}^{k}\sum_{q=0}^{k-p}\binom{k}{p}\binom{k-p}{q}(-1)^q e^{\xi}(p+q)\int_1^\infty e^{-\xi t}t^{-(p+q+2)}\left(1-\alpha_2 t^2+(\alpha_2-1)t\right)dt.$$

Using the definition of the exponential integral function, this expression can be written as

$$\frac{\partial \delta_{s,k}}{\partial \alpha_1} = \sum_{p=s}^{k}\sum_{q=0}^{k-p}\binom{k}{p}\binom{k-p}{q}(-1)^q e^{\xi}(p+q)\left[E_{p+q+2}(\xi)-\alpha_2\, E_{p+q}(\xi)+(\alpha_2-1)\, E_{p+q+1}(\xi)\right]. \tag{2.7}$$

Similarly, we can obtain $\frac{\partial \delta_{s,k}}{\partial \alpha_2}$ as follows:

$$\frac{\partial \delta_{s,k}}{\partial \alpha_2} = \sum_{p=s}^{k}\sum_{q=0}^{k-p}\binom{k}{p}\binom{k-p}{q}(-1)^q e^{\xi}\left[\alpha_2\, E_{p+q+1}(\xi)-\alpha_2\, E_{p+q}(\xi)+E_{p+q+2}(\xi)\right]. \tag{2.8}$$

The variance $var(\hat{\delta}_{s,k})$ is obtained the MLEs $\hat{\alpha_1}$ and $\hat{\alpha_2}$. Thus, the $100(1-\gamma)\%$ ACI of $\delta_{s,k}$ is given by

$$\left(\hat{\delta}_{s,k} - z_{\gamma/2}\sqrt{var(\hat{\delta}_{s,k})}, \hat{\delta}_{s,k} + z_{\gamma/2}\sqrt{var(\hat{\delta}_{s,k})}\right),$$

where z_γ denotes the upper $\gamma/2$ percentile of $N(0,1)$.

Given that $\delta_{s,k}$ is bounded in the interval $(0,1)$, the standard normal approximation may not perform well near the boundaries. To address this, a logit transformation can be applied to improve the interval estimation, as recommended by [29]. Define the transformed parameter

$$\zeta = \log(\delta_{s,k}/(1-\delta_{s,k})).$$

Its MLE is

$$\hat{\zeta} = \log(\hat{\delta}_{s,k}/(1-\hat{\delta}_{s,k})).$$

Applying the delta method, the $100(1-\gamma)\%$ ACI for ζ can be written as (ζ_L, ζ_U), where

$$\zeta_L = \log\left(\frac{\hat{\delta}_{s,k}}{1-\hat{\delta}_{s,k}}\right) - z_{\gamma/2}\sqrt{\frac{var(\hat{\delta}_{s,k})}{\hat{\delta}^2_{s,k}(1-\hat{\delta}_{s,k})^2}}$$

and

$$\zeta_U = \log\left(\frac{\hat{\delta}_{s,k}}{1-\hat{\delta}_{s,k}}\right) + z_{\gamma/2}\sqrt{\frac{var(\hat{\delta}_{s,k})}{\hat{\delta}^2_{s,k}(1-\hat{\delta}_{s,k})^2}}.$$

Therefore, the $100(1-\gamma)\%$ logit-scale ACI for $\delta_{s,k}$ is given by: $\left(\frac{e^{\zeta_L}}{1+e^{\zeta_L}}, \frac{e^{\zeta_U}}{1+e^{\zeta_U}}\right)$.

2.2. *Bootstrap confidence intervals*

For small samples, the MLE may not follow its asymptotic normal distribution, which can affect the accuracy of confidence intervals. To address this issue, a parametric bootstrap procedure is used to construct confidence intervals for $\delta_{s,k}$. Bootstrap samples are generated from the fitted distributions using the MLEs. Specifically, we consider the percentile bootstrap (Boot-p) and studentized bootstrap (Boot-t) methods to obtain improved interval estimates. To ensure stable interval estimates, the number of bootstrap replications is set to $B = 1{,}000$; see [30–32].

2.2.1. Boot-p confidence interval

The following steps outline the construction of the Boot-p confidence interval:

(1) Compute the MLEs $\hat{\alpha}_1$ and $\hat{\alpha}_2$ from the observed PTIIC data $\tilde{x}$ and $\tilde{y}$.

(2) Given the PTIIC scheme $(R_1, R_2, \ldots, R_k)$ under which the original strength data were observed, generate PTIIC bootstrap samples $x^*_{i1}, x^*_{i2}, \ldots, x^*_{ik}$, $i = 1, 2, \ldots, n$, from the distribution $f_X(x; \hat{\alpha}_1)$ using the algorithm described in [33]. Similarly, under the PTIIC censoring scheme $(S_1, S_2, \ldots, S_n)$ corresponding to the original stress data, generate the PTIIC bootstrap sample $y^*_1, y^*_2, \ldots, y^*_n$ from the distribution $f_Y(y; \hat{\alpha}_2)$.

(3) Using the samples in Step 2, compute the bootstrap estimate $\hat{\delta}^*_{s,k}$ of the reliability parameter.

(4) Repeat Steps 2 and 3 $B = 1000$ times. This number of bootstrap replications is generally sufficient to ensure stable interval estimates. For percentile intervals, Efron and Tibshirani [30] recommended at least 1000 repetitions to obtain reliable quantiles. Andrews and Buchinsky [32] provided a formal framework for choosing B to achieve the desired accuracy levels. Sort these B estimates in ascending order as follows: $\hat{\delta}^*_{s,k}(1) \le \hat{\delta}^*_{s,k}(2) \le \cdots \le \hat{\delta}^*_{s,k}(B)$.

(5) The $100(1-\gamma)\%$ Boot-p confidence interval (CI) for $\delta_{s,k}$ is obtained by taking the $\gamma/2$ and $1-\gamma/2$ quantiles from the empirical distribution of the bootstrap estimators.

2.2.2. Boot-t confidence interval

Steps 1 to 4 of the Boot-p method are performed as described earlier, followed by two additional steps:

(5) For each $\hat{\delta}^*_{s,k}$, obtain $\hat{T}^* = \left(\hat{\delta}^*_{s,k} - \hat{\delta}_{s,k}\right) / \sqrt{var(\hat{\delta}^*_{s,k})}$, where, $var(\hat{\delta}^*_{s,k})$ is the variance of the bootstrap estimators of $\hat{\delta}^*_{s,k}$. Arrange these values as $\hat{T}^*_1 \leq \hat{T}^*_2 \leq \cdots \leq \hat{T}^*_B$.

(6) Use $\hat{T}^*_{[Bq]}$ to denote the $[Bq]^{\text{th}}$ ordered value of $\hat{T}^*$. The $100(1-\gamma)\%$ Boot-t confidence interval for $\delta_{s,k}$ is then given by

$$\left(\hat{\delta}_{s,k} + \hat{T}^*_{[B(\gamma/2)]}\sqrt{var(\hat{\delta}_{s,k})},\ \hat{\delta}_{s,k} + \hat{T}^*_{[B(1-\gamma/2)]}\sqrt{var(\hat{\delta}_{s,k})}\right),$$

where $[a]$ denotes the integer part of a.

3. Bayesian estimation

Within Bayesian inference, the unknown parameters are treated as random variables characterized by prior distributions. The choice of these priors reflects the analyst's beliefs about the parameters before observing the data. In this study, Bayesian estimation is used to obtain the reliability parameter $\delta_{s,k}$ under the general entropy loss function (GELF). The GELF is given by

$$L(\eta, \hat{\eta}) \propto \left[(\hat{\eta}/\eta)^c - c\ \log((\hat{\eta}/\eta) - 1\right],\ c{\neq}0$$

where η denotes the true parameter and $\hat{\eta}$ is its estimator. Under the GELF, the Bayes estimator is expressed as

$$\hat{\eta} = \left[E(\eta^{-c}|data)\right]^{-1/c}. \tag{3.1}$$

Special cases of (3.1) include the entropy loss function (ELF) when $c = 1$, the squared error loss function (SELF) when $c = -1$, and the precautionary loss function (PELF) when $c = -2$.

To obtain the Bayesian estimate for the MSSR parameter $\delta_{s,k}$, the parameters α_1, α_2 are assumed to have independent gamma priors with the PDFs:

$$\pi_1(\alpha_1) \propto \alpha_1^{a_1-1} e^{-b_1\alpha_1}, \quad \alpha_1 > 0; a_1, b_1 > 0, \tag{3.2}$$

$$\pi_2(\alpha_2) \propto \alpha_2^{a_2-1} e^{-b_2\alpha_2}, \quad \alpha_2 > 0; a_2, b_2 > 0. \tag{3.3}$$

The Gamma distribution was adopted as a prior for all model parameters due to its support on the positive real line, analytical flexibility, and widespread use in Bayesian reliability modeling. This choice ensures the parameters' admissibility and improves numerical stability under progressive Type II censoring. Moreover, Gamma priors can be tuned to represent informative, weakly informative, or noninformative beliefs.

In the absence of strong historical information, non-informative priors (NIPs) are commonly used in reliability and survival analysis to stabilize estimation without dominating the likelihood. For moderate to large sample sizes, Bayesian estimators under informative priors are known to yield results that are consistent with those obtained under NIPs. Consequently, the posterior inference in this study is expected to be robust with respect to reasonable variations in the priors' specification.

The joint PDF of α_1 and α_2 can be written as

$$\pi(\alpha_1, \alpha_2 \mid \text{data}) = \frac{L(\text{data} \mid \alpha_1, \alpha_2)\pi_1(\alpha_1)\pi_2(\alpha_2)}{\int_0^\infty \int_0^\infty L(\text{data} \mid \alpha_1, \alpha_2)\pi_1(\alpha_1)\pi_2(\alpha_2)\, d\alpha_1\, d\alpha_2}. \tag{3.4}$$

The integrals in (3.4) have no closed-form expression. Therefore, two different procedures are used to obtain the Bayesian estimates for $\delta_{s,k}$. These procedures are described in the following subsections.

3.1. Lindley's approximation

An approximate Bayesian estimator for $\delta_{s,k}$ is evaluated using Lindley's approximation technique [34]. Under the GELF, the Bayesian estimator of $\delta_{s,k}$ is written as

$$\hat{\delta}_{s,k} = \left[E\left(\delta_{s,k}^{-c} \mid \text{data}\right)\right]^{-1/c},$$

where the expectation term, $E\left(\delta_{s,k}^{-c} \mid \text{data}\right)$ is given by

$$E\left(\delta_{s,k}^{-c} \mid \text{data}\right) = \frac{\int_0^\infty \int_0^\infty \delta_{s,k}^{-c}\, e^{\ell(\eta)+\rho(\eta)}\, d\eta}{\int_0^\infty \int_0^\infty e^{\ell(\eta)+\rho(\eta)}\, d\eta}. \tag{3.5}$$

Here, $\ell(\eta)$ represents the log-likelihood function as defined in Eq (2.3) and $\rho(\eta)$ denotes the logarithm of the prior density function. The parameter vector η is defined as $\eta = (\alpha_1, \alpha_2)$.

When adopting Lindley's approximation by expanding around the MLE of the parameter vector η, the Bayesian estimator of $\delta_{s,k}$ under GELF is approximated as

$$\hat{\delta}_{s,k}^{\text{Lin}} = \left(\hat{\delta}_{s,k}^{-c} + \Phi + w_1\psi_1 + w_2\psi_2\right)^{-1/c}, \tag{3.6}$$

where

$$\Phi = 0.5(w_{11}\sigma_{11} + w_{22}\sigma_{22}) + w_{12}\sigma_{12},$$

$$\psi_r = \mu_1\sigma_{r1} + \mu_2\sigma_{r2}, \quad r = 1, 2,$$

$$\mu_r = \rho_r + 0.5A_r,$$

and

$$A_r = \sigma_{11}\ell_{11r} + \sigma_{22}\ell_{22r} + 2\sigma_{12}\ell_{12r}, \quad r = 1, 2.$$

On the basis of prior densities in Eqs (3.2) and (3.3), we have

$$\rho_1 = (a_1 - 1)\alpha_1^{-1} - b_1, \quad \rho_2 = (a_2 - 1)\alpha_2^{-1} - b_2.$$

The quantities w_i and w_{ij} are the first-and second- order partial derivatives of $\delta_{s,k}^{-c}$ with respect to α_i

$$w_i = \frac{\partial\delta_{s,k}^{-c}}{\partial\alpha_i}, \quad w_{ij} = \frac{\partial^2\delta_{s,k}^{-c}}{\partial\alpha_i\partial\alpha_j}, \quad i, j = 1, 2.$$

The matrix element σ_{ij} represents the (i, j)-th entry of the inverse Fisher information matrix, $I^{-1}(\eta)$.

The following derivatives are key to evaluating Eq (3.6):

$$\ell_{111} = \sum_{i=1}^{n}\sum_{j=1}^{k} \frac{2\left(1 + x_{ij}\right)^3}{\left(1 + \alpha_1\left(1 + x_{ij}\right)\right)^3},$$

$$\ell_{222} = \sum_{i=1}^{n} \frac{2\left(1 + y_i\right)^3}{\left(1 + \alpha_2\left(1 + y_i\right)\right)^3},$$

while the remaining third-order mixed derivatives ℓ_{ijr} are zero. Additionally,

$$w_i = -c\,\delta_{s,k}^{-c-1}\frac{\partial \delta_{s,k}}{\partial \alpha_i},$$

where $\frac{\partial \delta_{s,k}}{\partial \alpha_i}$, for $i = 1, 2$, are given by Eqs (2.7) and (2.8), respectively. The second derivatives, w_{ii}, are:

$$w_{ii} = c(c+1)\,\delta_{s,k}^{-c-2}\left(\frac{\partial \delta_{s,k}}{\partial \alpha_i}\right)^2 - c\,\delta_{s,k}^{-c-1}\frac{\partial^2 \delta_{s,k}}{\partial \alpha_i^2}.$$

The second derivatives $\frac{\partial^2 \delta_{s,k}}{\partial \alpha_i^2}$ are computed by differentiating Eq (1.7) twice with respect to α_i, yielding

$$\begin{aligned}\frac{\partial^2 \delta_{s,k}}{\partial \alpha_1^2} &= \sum_{p=s}^{k}\sum_{q=0}^{k-p}\binom{k}{p}\binom{k-p}{q}(-1)^q e^{\xi}(p+q)^2\int_1^{\infty} e^{-\xi t}t^{-(p+q+2)}\left(1 + (\alpha_2-1)t + (1-2\alpha_2)t^2 + \alpha_2 t^3\right)dt \\ &= \sum_{p=s}^{k}\sum_{q=0}^{k-p}\binom{k}{p}\binom{k-p}{q}(-1)^q e^{\xi}(p+q)^2\left[E_{p+q+2}(\xi) + (\alpha_2-2)E_{p+q+1}(\xi) + (1-2\alpha_2)E_{p+q}(\xi) + \alpha_2 E_{p+q-1}(\xi)\right].\end{aligned} \tag{3.7}$$

Similarly,

$$\frac{\partial^2 \delta_{s,k}}{\partial \alpha_2^2} = \sum_{p=s}^{k}\sum_{q=0}^{k-p}\binom{k}{p}\binom{k-p}{q}(-1)^q e^{\xi}\left[E_{p+q+2}(\xi) + \alpha_2 E_{p+q+1}(\xi) - (1+2\alpha_2)E_{p+q}(\xi) + \alpha_2 E_{p+q-1}(\xi)\right]. \tag{3.8}$$

Finally, all the expressions related to Eq (3.6) are evaluated at the MLE, $\hat{\eta} = (\hat{\alpha}_1, \hat{\alpha}_2)$.

3.2. MCMC Technique

The joint posterior density of the parameters α_1 and α_2 given in (3.4) does not admit a closed-form, which makes it difficult to derive the marginal posterior distributions analytically. Instead, the posterior densities of α_1 and α_2 can be expressed up to proportionality as follows:

$$\pi(\alpha_1 \mid \text{data}) \propto \alpha_1^{a_1-1}e^{-b_1\alpha_1}\prod_{i=1}^{n}\prod_{j=1}^{k}\left(\frac{e^{-\alpha_1 x_{ij}}(\alpha_1(x_{ij}+1)+1)}{(x_{ij}+1)^2}\right)\left(\frac{e^{-\alpha_1 x_{ij}}}{x_{ij}+1}\right)^{R_j}, \tag{3.9}$$

and

$$\pi(\alpha_2 \mid \text{data}) \propto \alpha_2^{a_2-1}e^{-b_2\alpha_2}\prod_{i=1}^{n}\left(\frac{e^{-\alpha_2 y_i}(\alpha_2(y_i+1)+1)}{(y_i+1)^2}\right)\left(\frac{e^{-\alpha_2 y_i}}{y_i+1}\right)^{S_i}. \tag{3.10}$$

The conditional posterior densities in (3.9) and (3.10) do not correspond to any standard probability distributions; therefore, direct sampling from these densities is not feasible. Consequently, posterior samples are generated using a Metropolis–Hastings (MH) algorithm [35, 36], in which normal distributions are used as the proposal kernels. The computational procedure is summarized below.

(1) Initialize with $(\alpha_1^{(0)}, \alpha_2^{(0)})$.

(2) Set $m = 1$.

(3) Update $\alpha_1^{(m)}$ from $\pi(\alpha_1 \mid \text{data})$ via the MH step with the proposal $N\left(\alpha_1^{(m-1)}, var(\alpha_1^{(m-1)})\right)$.

(4) Update $\alpha_2^{(m)}$ from $\pi(\alpha_2 \mid \text{data})$ via the MH step with the proposal $N(\alpha_2^{(m-1)}, 1)$.

(5) Compute the MSSR estimate $\delta_{s,k}^{(m)}$ using the expression in (1.8).

(6) Increase m by one and repeat Steps 3–5 until M samples are generated.

The Bayesian estimator of $\delta_{s,k}$ under the generalized entropy loss function (GELF) is then obtained as

$$\hat{\delta}_{s,k}^{MC} = \left[\frac{1}{M - M_0} \sum_{t=M_0+1}^{M} (\delta_{s,k}^{(t)})^{-c} \right]^{-1/c}, \tag{3.11}$$

where M_0 denotes the burn-in period. The choice of M and M_0 is specified in the simulation study. The $100(1-\gamma)\%$ Bayesian credible interval (CrI) is calculated by taking the $\gamma/2$ and $1-\gamma/2$ quantiles of the posterior distribution obtained from the MCMC sample.

In addition, the $100(1-\gamma)\%$ highest posterior density (HPD) interval, described by [37], is defined as the shortest interval covering the specified posterior probability. Let $\delta_{s,k}(1) \le \delta_{s,k}(2) \le \cdots \le \delta_{s,k}(M)$ denote the ordered posterior sample. The $100(1-\gamma)\%$ HPD interval of $\delta_{s,k}$ is

$$\left(\delta_{s,k}(j),\ \delta_{s,k}(j + [(1-\gamma)M])\right),$$

where the index j corresponds to the interval with the minimum width.

4. Simulation study

In this section, a comprehensive simulation study is carried out to examine the performance of the proposed estimators for the MSSR measure $\delta_{s,k}$ under the BuH distribution with PTIIC schemes. The study evaluates both point and interval estimators of $\delta_{s,k}$. For the point estimates, the MLEs and Bayesian estimators obtained via Lindley's approximation and MCMC methods are considered. For the Bayesian framework, estimation is performed under GELF for three values of the asymmetry parameter, $c = 1, -1$, and -2. The accuracy of the point estimators is evaluated using the mean squared error (MSE), the Monte Carlo standard error (MCSE) of the MSE, and the average bias (AB). For interval estimation, we construct 95% confidence and credible intervals and compare them using the average length (AL) and coverage probability (CP).

We consider several censoring schemes, summarized in Table 1. A total of N systems, each with K units, are placed into the test, and n systems of size k are observed under different PTIIC plans. For the strength variables, schemes $R_1, R_2, \ldots, R_9$ are adopted, while $S_1, S_2, \ldots, S_9$ are used for the stress variables. Compact notation is used; for example, a^{*r} means that 0 is repeated r times and $((0, 1)^{*3})$ denotes $(0, 1, 0, 1, 0, 1)$. In the censoring schemes R_1, R_2, and R_3, we take $s = 1$, while for R_4, R_5, and R_6 we set $s = 2$, meaning that reliability is evaluated when at least one or two components, respectively, remain operational at the specified stress level. In addition, for R_7, R_8, and R_9, the value $s = 3$ is considered.

Two parameter scenarios are considered: Case I with $(\alpha_1, \alpha_2) = (0.2, 1.8)$, where the corresponding reliability values are $\delta_{1,3} = 0.9416$, $\delta_{2,3} = 0.8651$, and $\delta_{3,6} = 0.8580$, and Case II with $(\alpha_1, \alpha_2) = (1.5, 0.5)$, for which the true values are $\delta_{1,3} = 0.5744$, $\delta_{2,4} = 0.4260$, and $\delta_{3,6} = 0.3973$. These two scenarios allow a comparative performance of the estimators to be examined under both high, moderate, and low reliability settings. Bayesian estimates are obtained under the GELF for three values of the asymmetry parameter: $c = 1, -1, -2$.

4.1. MCMC diagnostics

The total number of iterations and the burn-in period for the Bayesian MCMC procedure were determined using the Gelman–Rubin potential scale reduction factor ($\hat{R}$), see [38]. The statistic is defined as

$$\hat{R} = \sqrt{\frac{T-1}{T} + \frac{m+1}{mT}\frac{\mathscr{B}}{W}},$$

where W and $\mathscr{B}$ denote the within-chain and between-chain variances, respectively; T is the number of iterations per chain; and m is the number of parallel chains. In this study, $m = 5$ parallel chains were initialized from overdispersed starting values to ensure a reliable convergence assessment. For $\delta_{s,k}$, the values of $\hat{R}$ rapidly approached one and stabilized after approximately 2000 iterations under all considered PTIIC schemes; see Figures 1 and 2. Accordingly, for the simulation study, each chain was run for 12,000 iterations, with the first 2000 iterations discarded as burn-in. To reduce serial dependence between successive draws, every third draw was retained, yielding a total of 3333 posterior samples used for inference. Sampling efficiency was also evaluated using the effective sample size (ESS). The minimum ESS across all scenarios exceeded 1300, indicating low autocorrelation and ensuring accurate Monte Carlo estimation of the posterior summaries.

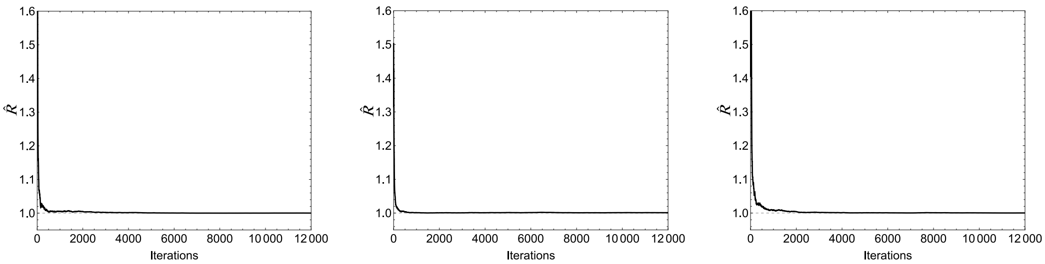

Figure 1. . Scale reduction factor for $\delta_{s,k}$ under Case I across different PTIIC schemes. From left to right: (R1,S1), (R6, S3), and (R9, S6).

Additional diagnostic plots for $\delta_{s,k}$ are presented in Figures 3 and 4 for different PTIIC schemes. For each censoring configuration, the left panel displays the posterior kernel density estimate, the middle panel shows the trace plot, and the right panel presents the running mean. The posterior densities are smooth and unimodal, indicating stable posterior behavior. The trace plots exhibit stable fluctuations around a constant level without visible trends, and the running means stabilize rapidly, providing further evidence of satisfactory mixing and convergence. Overall, these diagnostics confirm that the MCMC algorithm achieves reliable convergence and efficient sampling under all examined censoring schemes.

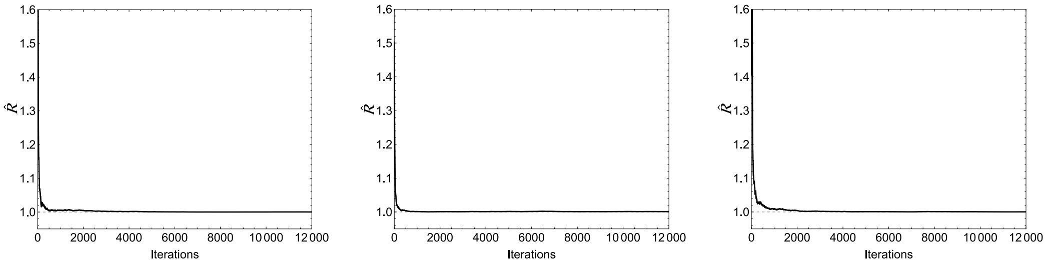

Figure 2. Scale reduction factor for $\delta_{s,k}$ under Case II across different PTIIC schemes. From left to right: (R2,S5), (R5, S7), and (R8, S2).

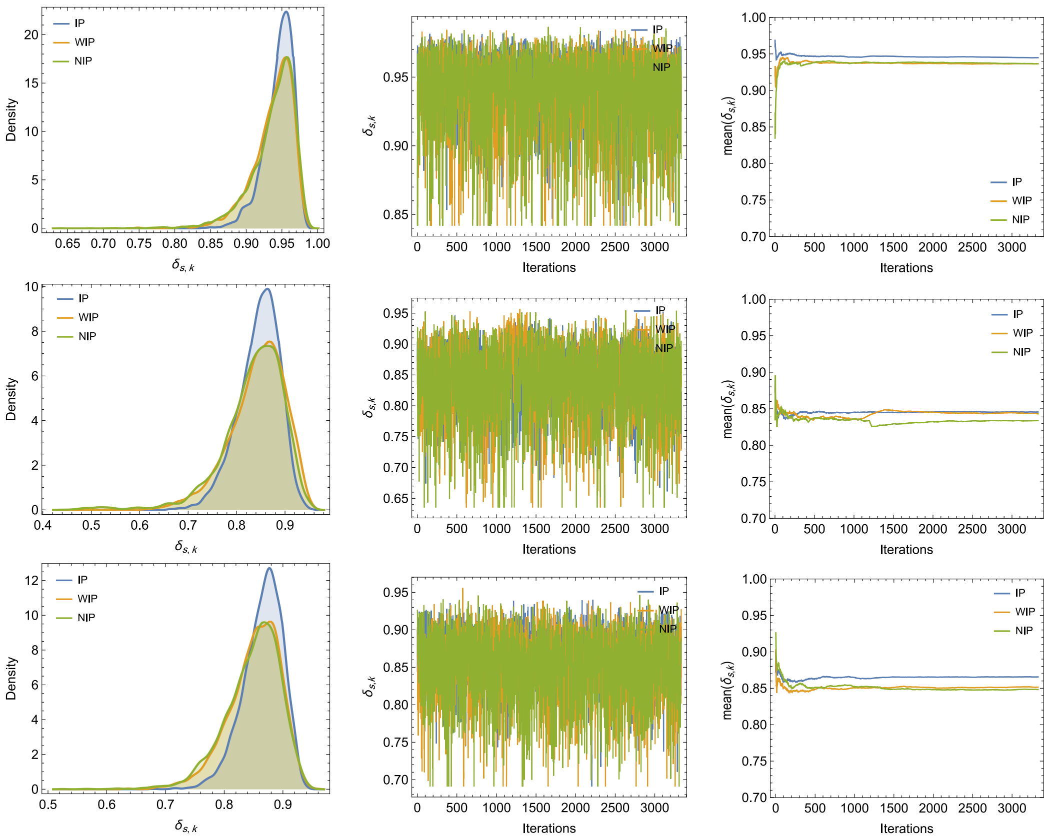

Figure 3. MCMC diagnostics under Case I across different PTIIC schemes. From top to bottom: (R1,S1), (R6, S3), and (R9, S6).

4.2. Prior sensitivity analysis

For Bayesian inference, independent Gamma priors $\Gamma(a_i, b_i)$ were assigned to α_i $(i = 1, 2)$ in the shape-rate parameterization, with a mean a_i/b_i and a variance a_i/b_i^2. In the main simulation study, we used informative priors (IPs) with the hyperparameters chosen such that the prior mean equals the true parameter value, while the prior variance was fixed at 0.5. To assess robustness with respect to prior specification, two alternative priors were considered: A weakly informative prior with the same prior mean and variance fixed at 4, and a NIP with $a_i = b_i = 0.0001$, $i = 1, 2$. Posterior summaries of $\delta_{s,k}$ were recomputed under these alternative priors for different PTIIC schemes. As illustrated in Figures 3 and 4, the posterior density curves corresponding to the informative prior, weakly informative prior, and NIP overlap in all scenarios. Although slight differences in dispersion are observed, particularly under the NIP, the central tendency of the posterior distribution remains essentially unchanged. The posterior means and credible intervals are highly consistent across prior choices.

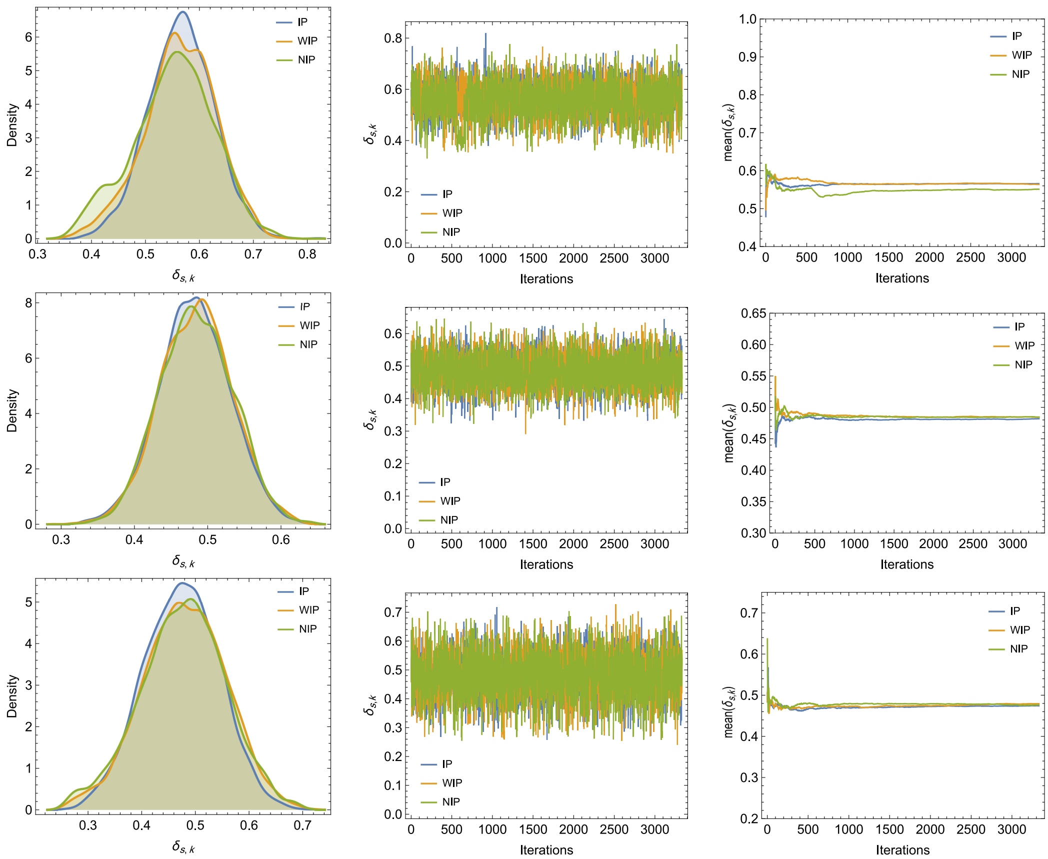

Figure 4. MCMC diagnostics under Case II across different PTIIC schemes. From top to bottom: (R2,S5), (R5, S7), and (R8, S2).

4.3. Results and discussions

According to the simulation results presented in Tables 2–9, the following conclusions are drawn.

- In general, all estimators of $\delta_{s,k}$ perform satisfactorily across the considered PTIIC schemes and cases, producing relatively small MSEs and ABs.
- As the effective sample size increases, both the MSEs and ABs decrease consistently for all estimation methods.
- In most scenarios, Bayesian estimators outperform the MLE in terms of MSE, particularly under informative priors and when using Lindley's approximation.

- Within the Bayesian framework, informative priors consistently yield smaller MSEs and average biases (ABs) than NIPs.

- Lindley's approximation provides a reliable and computationally efficient alternative to MCMC. Moreover, under both the Lindley and MCMC approaches, the PELF estimator typically achieves the smallest MSEs and ABs, followed by SELF, while ELF often exhibits slightly larger values.

- The MCSEs of the estimated MSEs are very small, confirming that the main simulation findings are numerically stable across censoring schemes and parameter settings. The MCSEs tend to be slightly larger under NIPs.

- For interval estimation, most methods achieve CPs reasonably close to the nominal level of 95%.

- Bayesian CrIs and HPD intervals generally produce shorter average lengths than classical intervals while maintaining high CPs.

- Under NIPs, the CrIs become wider, and CPs may slightly decrease in some censoring schemes.

- Boot-p and Boot-t intervals also achieve high CPs, although typically with wider average lengths.

- The ACI intervals are often shorter but may exhibit under coverage, particularly under the case of lower reliability. The logit-transformed intervals improve CPs relative to ACI, although occasionally with slightly longer intervals.

- Increasing the effective sample size reduces the average interval length and improves the stability of CPs across all methods.

- Overall, HPD intervals under informative priors frequently yield the shortest intervals with CPs close to or slightly exceeding the nominal level.

Table 1. PTIIC configurations used in the simulation study.

(k, K)		$C.S$	(n, N)		$C.S$
$(3, 4)$	R_1	$(0, 0, 1)$	$(20, 30)$	S_1	$(0^{*19}, 10)$
	R_2	$(1, 0, 0)$		S_2	$(10, 0^{*19})$
	R_3	$(0, 1, 0)$		S_3	$((0, 1)^{*10})$
$(4, 6)$	R_4	$(0, 0, 0, 2)$	$(30, 45)$	S_4	$(0^{*29}, 15)$
	R_5	$(2, 0, 0, 0)$		S_5	$(15, 0^{*29})$
	R_6	$(1, 0, 1, 0)$		S_6	$((0, 1)^{*15})$
$(6, 10)$	R_7	$(0, 0, 0, 0, 0, 4)$	$(50, 75)$	S_7	$(0^{*49}, 25)$
	R_8	$(4, 0, 0, 0, 0, 0)$		S_8	$(25, 0^{*49})$
	R_9	$(1, 0, 1, 0, 1, 1)$		S_9	$((0, 1)^{*25})$

Table 2. MSEs of the estimates of $\delta_{s,k}$ for Case I.

		Informative prior						Non-informative prior					
		Lindley			MCMC			Lindley			MCMC		
Scheme	MLE	SELF	ELF	PELF	SELF	ELF	PELF	SELF	ELF	PELF	SELF	ELF	PELF
R_1 S_1	0.00235	0.00069	0.00079	0.00065	0.00122	0.00129	0.00118	0.00290	0.00316	0.00277	0.00309	0.00362	0.00289
S_4	0.00170	0.00077	0.00083	0.00074	0.00080	0.00083	0.00079	0.00202	0.00213	0.00196	0.00245	0.00269	0.00235
S_7	0.00090	0.00058	0.00061	0.00057	0.00065	0.00068	0.00065	0.00096	0.00100	0.00095	0.00136	0.00143	0.00132
R_2 S_2	0.00204	0.00063	0.00072	0.00058	0.00108	0.00115	0.00104	0.00228	0.00245	0.00220	0.00343	0.00399	0.00320
S_5	0.00136	0.00063	0.00069	0.00060	0.00090	0.00095	0.00088	0.00172	0.00181	0.00168	0.00163	0.00175	0.00158
S_8	0.00079	0.00049	0.00051	0.00048	0.00061	0.00063	0.00060	0.00097	0.00099	0.00096	0.00116	0.00122	0.00113
R_3 S_3	0.00248	0.00081	0.00094	0.00076	0.00132	0.00140	0.00129	0.00284	0.00304	0.00274	0.00588	0.00690	0.00546
S_6	0.00149	0.00072	0.00079	0.00069	0.00113	0.00120	0.00110	0.00209	0.00220	0.00203	0.00190	0.00208	0.00183
S_9	0.00116	0.00076	0.00080	0.00074	0.00066	0.00068	0.00065	0.00106	0.00110	0.00104	0.00104	0.00108	0.00101
R_4 S_1	0.00209	0.00072	0.00080	0.00069	0.00109	0.00114	0.00106	0.00306	0.00328	0.00294	0.00364	0.00439	0.00334
S_4	0.00162	0.00082	0.00089	0.00080	0.00089	0.00093	0.00086	0.00148	0.00156	0.00144	0.00157	0.00168	0.00153
S_7	0.00074	0.00048	0.00051	0.00047	0.00082	0.00084	0.00081	0.00085	0.00087	0.00083	0.00101	0.00105	0.00099
R_5 S_2	0.00176	0.00059	0.00067	0.00055	0.00081	0.00086	0.00078	0.00191	0.00202	0.00185	0.00300	0.00341	0.00284
S_5	0.00116	0.00053	0.00058	0.00051	0.00071	0.00075	0.00070	0.00144	0.00150	0.00141	0.00192	0.00206	0.00186
S_8	0.00070	0.00047	0.00049	0.00046	0.00053	0.00055	0.00052	0.00091	0.00094	0.00089	0.00089	0.00093	0.00088
R_6 S_3	0.00169	0.00050	0.00057	0.00047	0.00088	0.00093	0.00086	0.00254	0.00272	0.00246	0.00368	0.00415	0.00348
S_6	0.00128	0.00065	0.00071	0.00062	0.00077	0.00081	0.00075	0.00155	0.00163	0.00151	0.00139	0.00149	0.00134
S_9	0.00080	0.00053	0.00056	0.00052	0.00059	0.00061	0.00058	0.00084	0.00086	0.00083	0.00094	0.00098	0.00092
R_7 S_1	0.00060	0.00034	0.00036	0.00033	0.00037	0.00038	0.00036	0.00102	0.00107	0.00099	0.00130	0.00148	0.00122
S_4	0.00051	0.00026	0.00027	0.00025	0.00031	0.00032	0.00030	0.00057	0.00059	0.00056	0.00057	0.00061	0.00055
S_7	0.00027	0.00018	0.00019	0.00018	0.00022	0.00022	0.00021	0.00028	0.00028	0.00027	0.00029	0.00030	0.00029
R_8 S_2	0.00068	0.00031	0.00034	0.00030	0.00034	0.00036	0.00033	0.00065	0.00068	0.00064	0.00088	0.00097	0.00084
S_5	0.00032	0.00018	0.00018	0.00017	0.00023	0.00024	0.00023	0.00057	0.00059	0.00056	0.00059	0.00062	0.00058
S_8	0.00021	0.00015	0.00016	0.00015	0.00022	0.00023	0.00022	0.00021	0.00022	0.00021	0.00023	0.00024	0.00023
R_9 S_3	0.00056	0.00023	0.00025	0.00023	0.00035	0.00036	0.00034	0.00119	0.00125	0.00116	0.00161	0.00183	0.00152
S_6	0.00041	0.00021	0.00023	0.00021	0.00029	0.00030	0.00029	0.00052	0.00054	0.00051	0.00059	0.00062	0.00057
S_9	0.00030	0.00022	0.00022	0.00021	0.00017	0.00017	0.00017	0.00032	0.00032	0.00031	0.00026	0.00026	0.00025

Table 3. MSEs of the estimates of $\delta_{s,k}$ for Case II.

		Informative prior						Non-informative prior					
		Lindley			MCMC			Lindley			MCMC		
Scheme	MLE	SELF	ELF	PELF	SELF	ELF	PELF	SELF	ELF	PELF	SELF	ELF	PELF
R_1 S_1	0.00573	0.00547	0.00641	0.00623	0.00446	0.00485	0.00433	0.00780	0.00801	0.00772	0.00743	0.00839	0.00703
S_4	0.00425	0.00390	0.00412	0.00381	0.00357	0.00385	0.00346	0.00473	0.00491	0.00464	0.00579	0.00649	0.00550
S_7	0.00232	0.00230	0.00243	0.00225	0.00273	0.00294	0.00264	0.00357	0.00361	0.00355	0.00367	0.00393	0.00356
R_2 S_2	0.00549	0.00421	0.00441	0.00416	0.00392	0.00429	0.00379	0.00509	0.00516	0.00509	0.00729	0.00801	0.00700
S_5	0.00329	0.00272	0.00278	0.00271	0.00291	0.00299	0.00290	0.00346	0.00349	0.00347	0.00425	0.00463	0.00410
S_8	0.00212	0.00186	0.00186	0.00186	0.00179	0.00184	0.00178	0.00232	0.00235	0.00231	0.00270	0.00276	0.00267
R_3 S_3	0.00589	0.00505	0.00547	0.00488	0.00468	0.00515	0.00450	0.00706	0.00720	0.00702	0.00636	0.00724	0.00601
S_6	0.00399	0.00350	0.00367	0.00343	0.00325	0.00358	0.00312	0.00472	0.00487	0.00466	0.00519	0.00583	0.00492
S_9	0.00218	0.00191	0.00192	0.00191	0.00215	0.00229	0.00209	0.00255	0.00258	0.00255	0.00284	0.00300	0.00277
R_4 S_1	0.00486	0.00456	0.00460	0.00455	0.00345	0.00343	0.00351	0.00633	0.00612	0.00667	0.00496	0.00546	0.00480
S_4	0.00360	0.00337	0.00351	0.00333	0.00218	0.00238	0.00211	0.00329	0.00343	0.00324	0.00437	0.00476	0.00422
S_7	0.00194	0.00199	0.00210	0.00194	0.00198	0.00209	0.00195	0.00199	0.00202	0.00199	0.00310	0.00335	0.00299
R_5 S_2	0.00490	0.00361	0.00362	0.00367	0.00277	0.00284	0.00278	0.00327	0.00327	0.00332	0.00507	0.00539	0.00497
S_5	0.00264	0.00214	0.00213	0.00218	0.00212	0.00215	0.00213	0.00266	0.00260	0.00271	0.00285	0.00301	0.00280
S_8	0.00119	0.00111	0.00113	0.00110	0.00159	0.00158	0.00160	0.00136	0.00134	0.00138	0.00133	0.00139	0.00132
R_6 S_3	0.00398	0.00323	0.00345	0.00318	0.00276	0.00292	0.00273	0.00540	0.00538	0.00545	0.00522	0.00549	0.00517
S_6	0.00322	0.00262	0.00262	0.00266	0.00278	0.00287	0.00276	0.00289	0.00285	0.00294	0.00415	0.00445	0.00405
S_9	0.00187	0.00171	0.00173	0.00171	0.00180	0.00183	0.00179	0.00156	0.00158	0.00156	0.00248	0.00264	0.00242
R_7 S_1	0.00437	0.00399	0.00422	0.00392	0.00254	0.00270	0.00252	0.00512	0.00480	0.00546	0.00474	0.00504	0.00466
S_4	0.00231	0.00212	0.00222	0.00209	0.00245	0.00253	0.00245	0.00312	0.00305	0.00318	0.00386	0.00412	0.00377
S_7	0.00171	0.00154	0.00154	0.00155	0.00170	0.00180	0.00166	0.00201	0.00202	0.00202	0.00208	0.00224	0.00201
R_8 S_2	0.00287	0.00213	0.00214	0.00218	0.00236	0.00243	0.00237	0.00362	0.00353	0.00371	0.00402	0.00426	0.00394
S_5	0.00238	0.00200	0.00198	0.00203	0.00191	0.00195	0.00191	0.00256	0.00248	0.00263	0.00262	0.00285	0.00253
S_8	0.00134	0.00124	0.00125	0.00124	0.00124	0.00124	0.00125	0.00137	0.00134	0.00139	0.00137	0.00139	0.00137
R_9 S_3	0.00402	0.00296	0.00299	0.00301	0.00269	0.00281	0.00267	0.00389	0.00386	0.00394	0.00448	0.00487	0.00434
S_6	0.00274	0.00226	0.00226	0.00230	0.00220	0.00227	0.00219	0.00267	0.00263	0.00271	0.00402	0.00421	0.00396
S_9	0.00157	0.00144	0.00145	0.00144	0.00142	0.00144	0.00142	0.00160	0.00156	0.00163	0.00214	0.00226	0.00209

Table 4. MCSEs of the estimated MSEs for Case I.

Scheme		MLE	Informative prior						Non-informative prior					
			Lindley			MCMC			Lindley			MCMC		
			SELF	ELF	PELF	SELF	ELF	PELF	SELF	ELF	PELF	SELF	ELF	PELF
R_1	S_1	0.00024	0.00007	0.00008	0.00007	0.00013	0.00014	0.00014	0.00034	0.00037	0.00033	0.00034	0.00040	0.00040
	S_4	0.00020	0.00009	0.00010	0.00009	0.00008	0.00008	0.00008	0.00022	0.00024	0.00021	0.00023	0.00026	0.00026
	S_7	0.00008	0.00005	0.00005	0.00005	0.00008	0.00008	0.00008	0.00010	0.00011	0.00010	0.00016	0.00017	0.00017
R_2	S_2	0.00018	0.00006	0.00007	0.00006	0.00011	0.00012	0.00012	0.00022	0.00024	0.00021	0.00037	0.00043	0.00043
	S_5	0.00013	0.00006	0.00006	0.00006	0.00011	0.00012	0.00012	0.00019	0.00020	0.00018	0.00015	0.00017	0.00017
	S_8	0.00008	0.00005	0.00005	0.00005	0.00006	0.00006	0.00006	0.00010	0.00011	0.00010	0.00013	0.00014	0.00014
R_3	S_3	0.00022	0.00007	0.00008	0.00007	0.00016	0.00017	0.00017	0.00027	0.00030	0.00026	0.00068	0.00079	0.00079
	S_6	0.00017	0.00008	0.00008	0.00007	0.00014	0.00015	0.00015	0.00021	0.00023	0.00021	0.00019	0.00021	0.00021
	S_9	0.00011	0.00007	0.00008	0.00007	0.00006	0.00007	0.00007	0.00011	0.00011	0.00011	0.00012	0.00013	0.00013
R_4	S_1	0.00027	0.00008	0.00008	0.00008	0.00014	0.00015	0.00015	0.00036	0.00039	0.00035	0.00048	0.00061	0.00061
	S_4	0.00016	0.00009	0.00010	0.00009	0.00010	0.00010	0.00010	0.00017	0.00018	0.00016	0.00016	0.00018	0.00018
	S_7	0.00007	0.00005	0.00005	0.00005	0.00010	0.00010	0.00010	0.00009	0.00010	0.00009	0.00010	0.00010	0.00010
R_5	S_2	0.00016	0.00006	0.00007	0.00006	0.00008	0.00009	0.00009	0.00019	0.00021	0.00019	0.00037	0.00043	0.00043
	S_5	0.00012	0.00005	0.00006	0.00005	0.00009	0.00009	0.00009	0.00019	0.00021	0.00019	0.00023	0.00025	0.00025
	S_8	0.00008	0.00006	0.00006	0.00006	0.00006	0.00006	0.00006	0.00013	0.00013	0.00013	0.00009	0.00009	0.00009
R_6	S_3	0.00017	0.00005	0.00006	0.00005	0.00008	0.00009	0.00009	0.00030	0.00033	0.00029	0.00072	0.00076	0.00076
	S_6	0.00012	0.00006	0.00007	0.00006	0.00007	0.00007	0.00007	0.00020	0.00022	0.00020	0.00013	0.00014	0.00014
	S_9	0.00008	0.00006	0.00006	0.00006	0.00008	0.00008	0.00008	0.00010	0.00010	0.00009	0.00009	0.00010	0.00010
R_7	S_1	0.00007	0.00005	0.00006	0.00005	0.00005	0.00005	0.00005	0.00016	0.00017	0.00016	0.00016	0.00018	0.00018
	S_4	0.00008	0.00003	0.00003	0.00003	0.00004	0.00004	0.00004	0.00007	0.00007	0.00007	0.00007	0.00008	0.00008
	S_7	0.00004	0.00003	0.00003	0.00003	0.00002	0.00003	0.00003	0.00003	0.00003	0.00003	0.00004	0.00004	0.00004
R_8	S_2	0.00007	0.00004	0.00005	0.00004	0.00004	0.00005	0.00005	0.00007	0.00007	0.00007	0.00013	0.00014	0.00014
	S_5	0.00004	0.00002	0.00002	0.00002	0.00003	0.00004	0.00004	0.00008	0.00008	0.00008	0.00006	0.00007	0.00007
	S_8	0.00002	0.00002	0.00002	0.00002	0.00004	0.00004	0.00004	0.00003	0.00003	0.00003	0.00002	0.00003	0.00003
R_9	S_3	0.00006	0.00003	0.00003	0.00003	0.00004	0.00004	0.00004	0.00022	0.00023	0.00021	0.00028	0.00032	0.00032
	S_6	0.00004	0.00002	0.00002	0.00002	0.00003	0.00004	0.00004	0.00008	0.00008	0.00008	0.00007	0.00007	0.00007
	S_9	0.00004	0.00003	0.00003	0.00003	0.00002	0.00002	0.00002	0.00004	0.00004	0.00004	0.00003	0.00003	0.00003

Table 5. MCSEs of the estimated MSEs for Case II.

Scheme		MLE	Informative prior						Non-informative prior					
			Lindley			MCMC			Lindley			MCMC		
			SELF	ELF	PELF	SELF	ELF	PELF	SELF	ELF	PELF	SELF	ELF	PELF
R_1	S_1	0.00053	0.00084	0.00169	0.00089	0.00041	0.00041	0.00041	0.00082	0.00080	0.00084	0.00056	0.00060	0.00060
	S_4	0.00044	0.00039	0.00041	0.00039	0.00032	0.00033	0.00033	0.00052	0.00052	0.00052	0.00046	0.00050	0.00050
	S_7	0.00027	0.00024	0.00025	0.00024	0.00024	0.00025	0.00025	0.00032	0.00032	0.00032	0.00032	0.00034	0.00034
R_2	S_2	0.00055	0.00040	0.00041	0.00040	0.00033	0.00035	0.00035	0.00057	0.00055	0.00057	0.00059	0.00061	0.00061
	S_5	0.00032	0.00026	0.00026	0.00026	0.00038	0.00037	0.00037	0.00032	0.00032	0.00032	0.00035	0.00037	0.00037
	S_8	0.00024	0.00020	0.00020	0.00021	0.00019	0.00019	0.00019	0.00023	0.00023	0.00023	0.00027	0.00027	0.00027
R_3	S_3	0.00061	0.00048	0.00050	0.00047	0.00044	0.00044	0.00044	0.00071	0.00070	0.00072	0.00051	0.00053	0.00053
	S_6	0.00040	0.00034	0.00036	0.00034	0.00033	0.00034	0.00034	0.00042	0.00042	0.00042	0.00042	0.00045	0.00045
	S_9	0.00024	0.00020	0.00020	0.00021	0.00020	0.00020	0.00020	0.00024	0.00024	0.00024	0.00027	0.00029	0.00029
R_4	S_1	0.00053	0.00062	0.00048	0.00107	0.00041	0.00037	0.00037	0.00074	0.00066	0.00088	0.00041	0.00041	0.00041
	S_4	0.00039	0.00033	0.00033	0.00033	0.00021	0.00021	0.00021	0.00032	0.00031	0.00032	0.00037	0.00037	0.00037
	S_7	0.00022	0.00020	0.00020	0.00020	0.00017	0.00017	0.00017	0.00020	0.00019	0.00020	0.00024	0.00025	0.00025
R_5	S_2	0.00064	0.00044	0.00043	0.00045	0.00029	0.00027	0.00027	0.00033	0.00032	0.00035	0.00043	0.00041	0.00041
	S_5	0.00034	0.00026	0.00025	0.00027	0.00018	0.00018	0.00018	0.00028	0.00027	0.00029	0.00025	0.00026	0.00026
	S_8	0.00011	0.00010	0.00010	0.00010	0.00019	0.00018	0.00018	0.00016	0.00015	0.00016	0.00013	0.00013	0.00013
R_6	S_3	0.00041	0.00030	0.00031	0.00030	0.00025	0.00024	0.00024	0.00057	0.00054	0.00059	0.00048	0.00047	0.00047
	S_6	0.00036	0.00026	0.00025	0.00027	0.00026	0.00025	0.00025	0.00027	0.00026	0.00028	0.00041	0.00040	0.00040
	S_9	0.00018	0.00015	0.00015	0.00016	0.00017	0.00017	0.00017	0.00016	0.00016	0.00016	0.00021	0.00022	0.00022
R_7	S_1	0.00052	0.00040	0.00039	0.00040	0.00024	0.00023	0.00023	0.00059	0.00047	0.00102	0.00036	0.00036	0.00036
	S_4	0.00030	0.00024	0.00024	0.00025	0.00023	0.00022	0.00022	0.00040	0.00038	0.00042	0.00031	0.00031	0.00031
	S_7	0.00021	0.00017	0.00017	0.00018	0.00015	0.00015	0.00015	0.00019	0.00019	0.00019	0.00017	0.00018	0.00018
R_8	S_2	0.00031	0.00021	0.00019	0.00022	0.00022	0.00021	0.00021	0.00042	0.00039	0.00044	0.00033	0.00032	0.00032
	S_5	0.00024	0.00018	0.00017	0.00019	0.00022	0.00020	0.00020	0.00026	0.00024	0.00027	0.00034	0.00034	0.00034
	S_8	0.00014	0.00012	0.00012	0.00012	0.00010	0.00010	0.00010	0.00018	0.00017	0.00018	0.00016	0.00015	0.00015
R_9	S_3	0.00049	0.00032	0.00031	0.00034	0.00026	0.00024	0.00024	0.00053	0.00051	0.00055	0.00034	0.00033	0.00033
	S_6	0.00034	0.00025	0.00024	0.00026	0.00021	0.00019	0.00019	0.00028	0.00026	0.00029	0.00030	0.00030	0.00030
	S_9	0.00018	0.00016	0.00015	0.00016	0.00016	0.00016	0.00016	0.00015	0.00015	0.00016	0.00019	0.00019	0.00019

Table 6. ABs of the estimates of $\delta_{s,k}$ for Case I.

			Informative prior						Non-informative prior					
			Lindley			MCMC			Lindley			MCMC		
Scheme		MLE	SELF	ELF	PELF	SELF	ELF	PELF	SELF	ELF	PELF	SELF	ELF	PELF
R_1	S_1	-0.00236	-0.00744	-0.00950	-0.00644	-0.00432	-0.00667	-0.00320	-0.01748	-0.02067	-0.01581	-0.01623	-0.02178	-0.01375
	S_4	0.00012	-0.00532	-0.00705	-0.00446	-0.00112	-0.00281	-0.00030	-0.00819	-0.01025	-0.00713	-0.01544	-0.01854	-0.01399
	S_7	-0.00014	-0.00396	-0.00513	-0.00337	-0.00318	-0.00430	-0.00263	-0.00623	-0.00749	-0.00559	-0.01030	-0.01188	-0.00954
R_2	S_2	0.00658	-0.00769	-0.00934	-0.00687	-0.00868	-0.01075	-0.00769	-0.01291	-0.01542	-0.01159	-0.02702	-0.03205	-0.02475
	S_5	0.00295	-0.00554	-0.00698	-0.00481	-0.00794	-0.00943	-0.00722	-0.00988	-0.01161	-0.00898	-0.01067	-0.01303	-0.00954
	S_8	0.00378	-0.00156	-0.00251	-0.00107	-0.00468	-0.00565	-0.00421	-0.00277	-0.00379	-0.00225	-0.01270	-0.01401	-0.01205
R_3	S_3	0.00024	-0.00916	-0.01118	-0.00816	-0.00548	-0.00768	-0.00442	-0.00997	-0.01275	-0.00852	-0.03973	-0.04614	-0.03685
	S_6	-0.00229	-0.00823	-0.00993	-0.00738	-0.01014	-0.01185	-0.00931	-0.01049	-0.01245	-0.00948	-0.01552	-0.01830	-0.01420
	S_9	-0.00281	-0.00627	-0.00741	-0.00570	-0.00593	-0.00700	-0.00540	-0.00505	-0.00622	-0.00446	-0.00665	-0.00803	-0.00597
R_4	S_1	-0.00110	-0.00744	-0.00917	-0.00660	-0.00155	-0.00344	-0.00064	-0.01223	-0.01486	-0.01087	-0.01571	-0.02109	-0.01329
	S_4	0.00012	-0.00448	-0.00591	-0.00377	-0.00696	-0.00846	-0.00624	-0.00828	-0.01002	-0.00739	-0.00184	-0.00405	-0.00080
	S_7	-0.00110	-0.00343	-0.00441	-0.00294	-0.00151	-0.00243	-0.00106	-0.00357	-0.00459	-0.00305	-0.00642	-0.00767	-0.00581
R_5	S_2	0.00230	-0.00852	-0.00997	-0.00781	-0.00808	-0.00981	-0.00724	-0.00679	-0.00883	-0.00573	-0.01838	-0.02230	-0.01659
	S_5	-0.00053	-0.00606	-0.00730	-0.00544	-0.00624	-0.00749	-0.00563	-0.00525	-0.00665	-0.00453	-0.01482	-0.01688	-0.01384
	S_8	0.00016	-0.00363	-0.00446	-0.00322	-0.00430	-0.00509	-0.00391	-0.00807	-0.00897	-0.00762	-0.00765	-0.00869	-0.00714
R_6	S_3	0.00287	-0.00690	-0.00839	-0.00617	-0.00327	-0.00507	-0.00241	-0.01168	-0.01398	-0.01049	-0.02415	-0.02865	-0.02211
	S_6	-0.00410	-0.00926	-0.01066	-0.00856	-0.00693	-0.00827	-0.00627	-0.00859	-0.01015	-0.00780	-0.00846	-0.01057	-0.00745
	S_9	-0.00161	-0.00512	-0.00602	-0.00467	-0.00502	-0.00589	-0.00460	-0.00179	-0.00269	-0.00133	-0.00794	-0.00909	-0.00737
R_7	S_1	-0.00280	-0.00646	-0.00710	-0.00615	-0.00234	-0.00293	-0.00205	-0.00771	-0.00849	-0.00731	-0.01425	-0.01628	-0.01334
	S_4	-0.00465	-0.00605	-0.00652	-0.00582	-0.00270	-0.00314	-0.00249	-0.00676	-0.00728	-0.00650	-0.00787	-0.00868	-0.00748
	S_7	-0.00047	-0.00254	-0.00281	-0.00241	-0.00431	-0.00460	-0.00417	-0.00306	-0.00335	-0.00291	-0.00370	-0.00406	-0.00352
R_8	S_2	-0.00399	-0.00757	-0.00810	-0.00730	-0.00398	-0.00453	-0.00371	-0.00779	-0.00843	-0.00746	-0.01068	-0.01203	-0.01006
	S_5	-0.00034	-0.00343	-0.00379	-0.00325	-0.00418	-0.00456	-0.00399	-0.00862	-0.00908	-0.00838	-0.01083	-0.01154	-0.01048
	S_8	-0.00201	-0.00378	-0.00403	-0.00366	-0.00318	-0.00341	-0.00306	-0.00182	-0.00205	-0.00170	-0.00545	-0.00577	-0.00530
R_9	S_3	-0.00279	-0.00649	-0.00703	-0.00623	-0.00407	-0.00465	-0.00379	-0.01226	-0.01304	-0.01186	-0.01689	-0.01885	-0.01600
	S_6	-0.00313	-0.00542	-0.00584	-0.00521	-0.00302	-0.00342	-0.00283	-0.00738	-0.00785	-0.00714	-0.00864	-0.00938	-0.00828
	S_9	-0.00302	-0.00485	-0.00512	-0.00471	-0.00260	-0.00285	-0.00247	-0.00461	-0.00489	-0.00447	-0.00338	-0.00370	-0.00321

Table 7. ABs of the estimates of $\delta_{s,k}$ for Case II.

			Informative prior						Non-informative prior					
			Lindley			MCMC			Lindley			MCMC		
Scheme		MLE	SELF	ELF	PELF	SELF	ELF	PELF	SELF	ELF	PELF	SELF	ELF	PELF
R_1	S_1	0.00831	-0.02543	-0.02919	-0.00927	-0.01886	-0.02892	-0.01395	-0.01944	-0.02731	-0.01480	-0.04343	-0.05595	-0.03716
	S_4	0.00959	-0.00906	-0.01519	-0.00565	-0.01706	-0.02457	-0.01339	-0.01306	-0.01913	-0.00967	-0.03515	-0.04464	-0.03051
	S_7	0.00043	-0.01032	-0.01441	-0.00815	-0.02023	-0.02497	-0.01791	-0.00152	-0.00550	0.00061	-0.01635	-0.02204	-0.01359
R_2	S_2	0.01230	-0.00375	-0.01105	0.00024	-0.02258	-0.03101	-0.01846	0.00128	-0.00609	0.00531	-0.03610	-0.04694	-0.03080
	S_5	0.00917	-0.00109	-0.00635	0.00169	-0.00349	-0.00941	-0.00061	0.00017	-0.00507	0.00294	-0.02059	-0.02804	-0.01698
	S_8	0.00880	0.00244	-0.00084	0.00415	-0.00392	-0.00749	-0.00216	-0.00304	-0.00635	-0.00132	-0.00404	-0.00799	-0.00211
R_3	S_3	0.00697	-0.01470	-0.02263	-0.01023	-0.02621	-0.03544	-0.02169	-0.00154	-0.00941	0.00292	-0.03915	-0.05134	-0.03315
	S_6	0.00694	-0.00764	-0.01339	-0.00452	-0.02303	-0.02972	-0.01975	-0.00955	-0.01524	-0.00645	-0.03266	-0.04136	-0.02841
	S_9	0.01095	0.00211	-0.00158	0.00405	-0.01415	-0.01834	-0.01209	-0.00054	-0.00423	0.00141	-0.01113	-0.01596	-0.00878
R_4	S_1	0.01128	-0.01213	-0.02016	-0.03706	-0.00440	-0.01412	0.00042	-0.00951	-0.01745	-0.00512	-0.03166	-0.04385	-0.02553
	S_4	0.00514	-0.00998	-0.01614	-0.00652	-0.01548	-0.02257	-0.01197	-0.01424	-0.02038	-0.01077	-0.02698	-0.03574	-0.02263
	S_7	-0.00470	-0.01311	-0.01713	-0.01095	-0.01208	-0.01657	-0.00987	-0.00398	-0.00805	-0.00180	-0.01873	-0.02431	-0.01598
R_5	S_2	0.01566	0.00150	-0.00579	0.00557	-0.00952	-0.01774	-0.00544	-0.00265	-0.01028	0.00154	-0.02375	-0.03376	-0.01878
	S_5	0.01046	0.00177	-0.00345	0.00457	-0.00553	-0.01126	-0.00270	0.00227	-0.00300	0.00508	-0.01319	-0.02019	-0.00977
	S_8	-0.00002	-0.00474	-0.00796	-0.00306	-0.00017	-0.00363	0.00155	0.00147	-0.00181	0.00318	-0.00712	-0.01095	-0.00523
R_6	S_3	0.00708	-0.01018	-0.01840	-0.00547	-0.01356	-0.02266	-0.00905	-0.00446	-0.01258	0.00018	-0.02308	-0.03479	-0.01728
	S_6	0.01361	0.00135	-0.00444	0.00454	-0.01002	-0.01651	-0.00680	0.00154	-0.00435	0.00476	-0.02016	-0.02830	-0.01615
	S_9	0.00379	-0.00297	-0.00666	-0.00102	-0.00546	-0.00952	-0.00345	-0.00423	-0.00794	-0.00228	-0.01634	-0.02100	-0.01406
R_7	S_1	0.00729	-0.01552	-0.02327	-0.01082	-0.01424	-0.02353	-0.00959	-0.00324	-0.01089	-0.00945	-0.02947	-0.04044	-0.02393
	S_4	0.00647	-0.00810	-0.01403	-0.00475	-0.00973	-0.01648	-0.00637	-0.00128	-0.00717	0.00201	-0.02017	-0.02862	-0.01598
	S_7	0.00883	0.00016	-0.00371	0.00224	-0.01371	-0.01799	-0.01160	-0.00347	-0.00729	-0.00142	-0.01363	-0.01890	-0.01105
R_8	S_2	0.01291	-0.00075	-0.00770	0.00315	-0.01152	-0.01927	-0.00768	-0.00058	-0.00763	0.00332	-0.02271	-0.03231	-0.01791
	S_5	0.00827	-0.00068	-0.00549	0.00192	-0.00770	-0.01310	-0.00503	0.00348	-0.00148	0.00614	-0.01988	-0.02631	-0.01671
	S_8	0.00143	-0.00349	-0.00650	-0.00191	-0.00162	-0.00489	0.00012	0.00151	-0.00154	0.00309	-0.00319	-0.00680	-0.00141
R_9	S_3	0.01782	-0.00090	-0.00852	0.00355	-0.01411	-0.02259	-0.00987	-0.00399	-0.01178	0.00050	-0.03049	-0.04094	-0.02519
	S_6	0.01223	0.00010	-0.00539	0.00314	-0.01001	-0.01625	-0.00692	-0.00083	-0.00637	0.00221	-0.01711	-0.02463	-0.01338
	S_9	0.00388	-0.00293	-0.00641	-0.00108	-0.00448	-0.00834	-0.00257	0.00360	0.00007	0.00547	-0.01449	-0.01889	-0.01233

Table 8. Average lengths and coverage probability of the estimates of $\delta_{s,k}$ for Case I.

										Informative prior				Non-informative prior			
		ACI		Logit		Boot-p		Boot-t		CrI		HPD		CrI		HPD	
Scheme		AL	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP
R_1	S_1	0.19934	0.915	0.20450	0.960	0.19179	0.940	0.19592	0.940	0.16766	0.985	0.16053	0.985	0.24194	0.955	0.22361	0.960
	S_4	0.16232	0.930	0.16541	0.945	0.15915	0.930	0.16076	0.950	0.14428	0.995	0.13964	0.995	0.18605	0.935	0.17666	0.930
	S_7	0.12711	0.960	0.12860	0.975	0.12298	0.935	0.12461	0.940	0.11898	0.985	0.11513	0.985	0.13767	0.930	0.13281	0.930
R_2	S_2	0.17790	0.896	0.18277	0.955	0.17082	0.930	0.17451	0.920	0.15880	0.980	0.15306	0.970	0.22971	0.965	0.21586	0.950
	S_5	0.14903	0.905	0.15166	0.930	0.14399	0.920	0.14710	0.915	0.13582	0.975	0.13103	0.975	0.16606	0.945	0.15886	0.950
	S_8	0.11593	0.955	0.11721	0.970	0.11294	0.965	0.11488	0.960	0.11073	0.980	0.10723	0.990	0.12695	0.945	0.12319	0.945
R_3	S_3	0.19115	0.910	0.19612	0.960	0.18395	0.935	0.18808	0.940	0.16297	0.985	0.15652	0.985	0.25332	0.915	0.23607	0.940
	S_6	0.15988	0.950	0.16265	0.950	0.15393	0.945	0.15797	0.955	0.14433	0.960	0.13979	0.950	0.17992	0.965	0.17142	0.965
	S_9	0.12466	0.925	0.12598	0.930	0.12241	0.920	0.12390	0.910	0.11593	0.975	0.11240	0.975	0.13002	0.955	0.12555	0.965
R_4	S_1	0.18030	0.915	0.18501	0.945	0.17470	0.925	0.17784	0.930	0.15114	0.985	0.14448	0.985	0.22411	0.955	0.20744	0.930
	S_4	0.14714	0.910	0.14986	0.945	0.14367	0.920	0.14562	0.935	0.13520	0.975	0.13036	0.965	0.15950	0.930	0.15129	0.920
	S_7	0.11586	0.975	0.11719	0.980	0.11396	0.985	0.11452	0.985	0.10749	0.945	0.10424	0.915	0.12346	0.945	0.11910	0.940
R_5	S_2	0.16503	0.905	0.16913	0.955	0.16090	0.920	0.16302	0.925	0.14599	0.990	0.14007	0.990	0.20285	0.920	0.19005	0.925
	S_5	0.13738	0.920	0.13963	0.950	0.13531	0.935	0.13687	0.935	0.12538	0.970	0.12095	0.975	0.15597	0.915	0.14893	0.905
	S_8	0.10689	0.950	0.10798	0.960	0.10554	0.965	0.10651	0.975	0.09992	0.980	0.09715	0.980	0.11373	0.935	0.11005	0.920
R_6	S_3	0.16974	0.945	0.17422	0.950	0.16768	0.960	0.16869	0.965	0.14827	0.995	0.14216	0.990	0.21864	0.950	0.20361	0.930
	S_6	0.14502	0.970	0.14737	0.970	0.14399	0.970	0.14389	0.970	0.12950	0.985	0.12519	0.985	0.15821	0.970	0.15042	0.970
	S_9	0.11140	0.940	0.11257	0.950	0.10994	0.950	0.11062	0.945	0.10467	0.960	0.10164	0.960	0.11979	0.955	0.11495	0.945
R_7	S_1	0.10160	0.920	0.10983	0.965	0.10626	0.965	0.10128	0.960	0.08603	0.980	0.08000	0.960	0.14200	0.955	0.12525	0.960
	S_4	0.08548	0.945	0.09012	0.955	0.08692	0.950	0.08479	0.935	0.07514	0.955	0.07129	0.960	0.09921	0.970	0.09084	0.940
	S_7	0.06283	0.925	0.06503	0.955	0.06398	0.935	0.06226	0.925	0.06227	0.965	0.05966	0.960	0.06915	0.970	0.06548	0.950
R_8	S_2	0.09621	0.910	0.10296	0.945	0.09751	0.940	0.09497	0.930	0.08377	0.975	0.07848	0.985	0.12156	0.960	0.10902	0.940
	S_5	0.07517	0.920	0.07895	0.945	0.07660	0.950	0.07435	0.945	0.07154	0.975	0.06800	0.975	0.09398	0.965	0.08750	0.975
	S_8	0.05998	0.960	0.06179	0.960	0.06093	0.960	0.05989	0.965	0.05678	0.955	0.05421	0.950	0.06497	0.980	0.06183	0.990
R_9	S_3	0.09801	0.945	0.10547	0.970	0.10010	0.955	0.09692	0.965	0.08608	0.990	0.08036	0.990	0.13880	0.950	0.12383	0.940
	S_6	0.08081	0.920	0.08498	0.925	0.08178	0.935	0.07998	0.925	0.07255	0.965	0.06870	0.975	0.09523	0.955	0.08814	0.965
	S_9	0.06285	0.930	0.06484	0.945	0.06350	0.950	0.06244	0.945	0.05804	0.980	0.05561	0.960	0.06596	0.960	0.06261	0.950

Table 9. Average lengths and coverage probability of the estimates of $\delta_{s,k}$ for Case II.

										Informative prior				Non-informative prior			
		ACI		Logit		Boot-p		Boot-t		CrI		HPD		CrI		HPD	
Scheme		AL	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP
R_1	S_1	0.30742	0.955	0.29865	0.970	0.28982	0.950	0.30059	0.970	0.28312	0.945	0.27751	0.940	0.30132	0.915	0.29248	0.871
	S_4	0.25384	0.945	0.24881	0.945	0.24584	0.965	0.25023	0.970	0.24710	0.950	0.24324	0.945	0.26536	0.930	0.25897	0.881
	S_7	0.19848	0.935	0.19600	0.940	0.19309	0.935	0.19695	0.945	0.19712	0.940	0.19407	0.915	0.21385	0.940	0.20967	0.910
R_2	S_2	0.27307	0.945	0.26690	0.960	0.26887	0.930	0.27096	0.945	0.26080	0.960	0.25618	0.955	0.28328	0.891	0.27673	0.881
	S_5	0.22572	0.945	0.22214	0.960	0.22280	0.930	0.22357	0.930	0.22274	0.955	0.21942	0.940	0.24363	0.930	0.23881	0.915
	S_8	0.17570	0.945	0.17399	0.945	0.17306	0.935	0.17458	0.935	0.17491	0.955	0.17211	0.940	0.18292	0.915	0.17986	0.886
R_3	S_3	0.29068	0.910	0.28324	0.930	0.28235	0.930	0.28674	0.935	0.27003	0.950	0.26499	0.935	0.29779	0.925	0.29049	0.891
	S_6	0.24014	0.920	0.23585	0.935	0.23206	0.920	0.23686	0.935	0.23234	0.925	0.22871	0.910	0.25709	0.925	0.25142	0.905
	S_9	0.18863	0.960	0.18652	0.970	0.18389	0.940	0.18755	0.940	0.18760	0.960	0.18484	0.950	0.20141	0.930	0.19764	0.925
R_4	S_1	0.26897	0.935	0.26267	0.940	0.26926	0.930	0.26661	0.940	0.24419	0.960	0.23922	0.945	0.25828	0.910	0.24972	0.866
	S_4	0.21923	0.920	0.21580	0.940	0.21589	0.925	0.21739	0.920	0.20795	0.970	0.20383	0.950	0.22221	0.896	0.21613	0.846
	S_7	0.16890	0.930	0.16733	0.930	0.16722	0.935	0.16673	0.930	0.16746	0.935	0.16488	0.920	0.18042	0.910	0.17697	0.876
R_5	S_2	0.23934	0.915	0.23477	0.925	0.24517	0.896	0.23871	0.905	0.22545	0.980	0.22121	0.945	0.24004	0.900	0.23332	0.866
	S_5	0.19536	0.935	0.19289	0.940	0.19584	0.925	0.19404	0.930	0.18969	0.965	0.18642	0.945	0.20454	0.925	0.20055	0.905
	S_8	0.14861	0.970	0.14753	0.975	0.14907	0.970	0.14748	0.965	0.14918	0.935	0.14683	0.935	0.15484	0.955	0.15255	0.945
R_6	S_3	0.25594	0.945	0.25045	0.960	0.25949	0.935	0.25365	0.935	0.23525	0.980	0.23021	0.955	0.25534	0.891	0.24763	0.871
	S_6	0.21066	0.930	0.20758	0.930	0.21173	0.900	0.20881	0.905	0.20067	0.950	0.19709	0.935	0.21616	0.876	0.21122	0.866
	S_9	0.16156	0.925	0.16017	0.935	0.16081	0.935	0.16054	0.940	0.16020	0.925	0.15775	0.905	0.16768	0.920	0.16472	0.896
R_7	S_1	0.25260	0.920	0.24738	0.945	0.25373	0.945	0.24909	0.945	0.22804	0.960	0.22183	0.935	0.23552	0.886	0.22682	0.861
	S_4	0.20874	0.960	0.20582	0.965	0.20683	0.955	0.20575	0.950	0.19699	0.930	0.19289	0.910	0.21183	0.900	0.20538	0.876
	S_7	0.16323	0.960	0.16183	0.960	0.16384	0.945	0.16250	0.955	0.15715	0.925	0.15425	0.900	0.17222	0.935	0.16820	0.925
R_8	S_2	0.22568	0.975	0.22186	0.975	0.23328	0.950	0.22339	0.950	0.20998	0.945	0.20556	0.930	0.22449	0.891	0.21792	0.881
	S_5	0.18181	0.940	0.17980	0.940	0.18750	0.920	0.18131	0.915	0.17778	0.950	0.17445	0.935	0.18675	0.915	0.18257	0.896
	S_8	0.13944	0.930	0.13856	0.935	0.14080	0.915	0.13961	0.940	0.14022	0.965	0.13779	0.955	0.14599	0.960	0.14346	0.950
R_9	S_3	0.24527	0.935	0.24042	0.940	0.24900	0.935	0.24228	0.930	0.21914	0.970	0.21377	0.950	0.23080	0.905	0.22205	0.861
	S_6	0.19890	0.965	0.19632	0.965	0.19833	0.940	0.19652	0.945	0.19005	0.945	0.18621	0.935	0.20234	0.886	0.19673	0.861
	S_9	0.15214	0.935	0.15100	0.940	0.15283	0.935	0.15110	0.930	0.15161	0.935	0.14893	0.920	0.15798	0.881	0.15522	0.871

5. Applications

In this section, we provide a real data analysis to illustrate the suitability of the proposed estimation methods for MSSR under the BuH distribution using PTIIC data.

5.1. Generator failure data

The proposed model is illustrated using a real dataset originally reported by [39], which comprises 36 failure times (measured in hours) of 500-MW generators recorded over a 6-year operational period in a local utility. After applying a scale transformation by dividing all observations by 10, [40] used these transformed data to construct artificial multicomponent systems under the assumptions $N = 6$ and $K = 5$. The resulting complete observed strength and stress samples are presented as follows:

$$X = \begin{pmatrix} 0.0618 & 0.2234 & 0.2433 & 0.2505 & 0.3551 \\ 0.0058 & 0.0421 & 0.1497 & 0.4378 & 0.4872 \\ 0.0224 & 0.057 & 0.2877 & 0.5085 & 1.1399 \\ 0.009 & 0.0113 & 0.0153 & 0.1104 & 0.3455 \\ 0.0834 & 0.2372 & 0.3166 & 0.5341 & 0.9188 \\ 0.0159 & 0.0596 & 0.1019 & 0.269 & 0.2879 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 0.2027 \\ 0.0121 \\ 0.0070 \\ 0.5272 \\ 0.0105 \\ 0.8952 \end{pmatrix}.$$

The BuH distribution is fitted separately to the strength and stress datasets and compared with the exponential and Lindley distributions, both of which are widely used single-parameter lifetime models. The fitting results are summarized in Table 10. This table reports the MLEs of the model parameters along with their standard errors, the Kolmogorov–Smirnov (KS) statistics and corresponding p-values, as well as the Akaike information criterion (AIC) and Bayesian information Criterion (BIC).

The results indicate that the BuH distribution provides an adequate fit for both the strength and stress data and performs competitively relative to the exponential and Lindley distributions in terms of goodness-of-fit measures and information criteria. Additional graphical diagnostics are presented in Figure 5, where the fitted probability density functions are overlaid on the empirical histograms, and the estimated survival functions are compared with the nonparametric Kaplan–Meier estimators along with their confidence intervals. Furthermore, the quantile-Quantile plots (QQ-plots) are displayed in Figure 6 compares the empirical quantiles with the corresponding theoretical quantiles for each fitted distribution. The QQ-plots show satisfactory agreement between the empirical and fitted distributions.

To demonstrate the applicability of the proposed methods, PTIIC samples were generated from the complete stress and strength data matrices X and Y under different censoring schemes. The censoring process was carried out as follows:

- The stress vector Y was censored using a predefined scheme S.
- The rows of the strength matrix X corresponding to the censored stress observations were removed.
- The progressive censoring scheme R was then applied to each row of the remaining strength matrix.

In addition to analyzing the MSSR based on the complete sample with $s = 3$, three progressively censored samples were considered. They are described as follows.

Scheme I: $R = (0, 0, 0, 1)$ and $S = (0, 0, 0, 2)$ with the parameters $(N = 6, K = 5, n = 4, k = 4, s = 2)$. The resulting censored samples are

$$X = \begin{pmatrix} 0.0618 & 0.2234 & 0.2433 & 0.2505 \\ 0.0058 & 0.0421 & 0.1497 & 0.4378 \\ 0.0224 & 0.057 & 0.2877 & 0.5085 \\ 0.0834 & 0.2372 & 0.3166 & 0.5341 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 0.2027 \\ 0.0121 \\ 0.0070 \\ 0.0105 \end{pmatrix}.$$

Scheme II: $R = (1, 0, 1)$ and $S = (1, 0, 1, 0)$ with the parameters $(N = 6, K = 5, n = 4, k = 3, s = 2)$. The corresponding censored data are

$$X = \begin{pmatrix} 0.0618 & 0.2433 & 0.2505 \\ 0.0058 & 0.1497 & 0.4378 \\ 0.0224 & 0.2877 & 0.5085 \\ 0.0159 & 0.1019 & 0.269 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 0.2027 \\ 0.0121 \\ 0.0070 \\ 0.8952 \end{pmatrix}.$$

Scheme III: $R = (2, 0, 0)$ and $S = (3, 0, 0)$ with the parameters $(N = 6, K = 5, n = 3, k = 3, s = 1)$. The progressively censored data are

$$X = \begin{pmatrix} 0.0618 & 0.2505 & 0.3551 \\ 0.009 & 0.1104 & 0.3455 \\ 0.0159 & 0.269 & 0.2879 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 0.2027 \\ 0.5272 \\ 0.8952 \end{pmatrix}.$$

Table 10. Fitting results of the BuH distribution compared with exponential and Lindley distributions to the strength and stress observations from generators' failure data.

Dataset	Model	MLEs Std. error	KS	p-value	AIC	BIC
X	BuH	3.1321 (0.7213)	0.1273	0.7157	-20.579	-19.177
	Exponential	3.9536 (0.7218)	0.1199	0.7817	-20.477	-19.076
	Lindley	4.6529 (0.7387)	0.1216	0.7669	-20.360	-18.958
Y	BuH	2.8047 (1.4784)	0.45508	0.1665	-1.6380	-1.8463
	Exponential	3.6260 (1.4803)	0.4571	0.1629	-1.4577	-1.6659
	Lindley	4.3090 (1.5190)	0.4585	0.1604	-1.3175	-1.5257

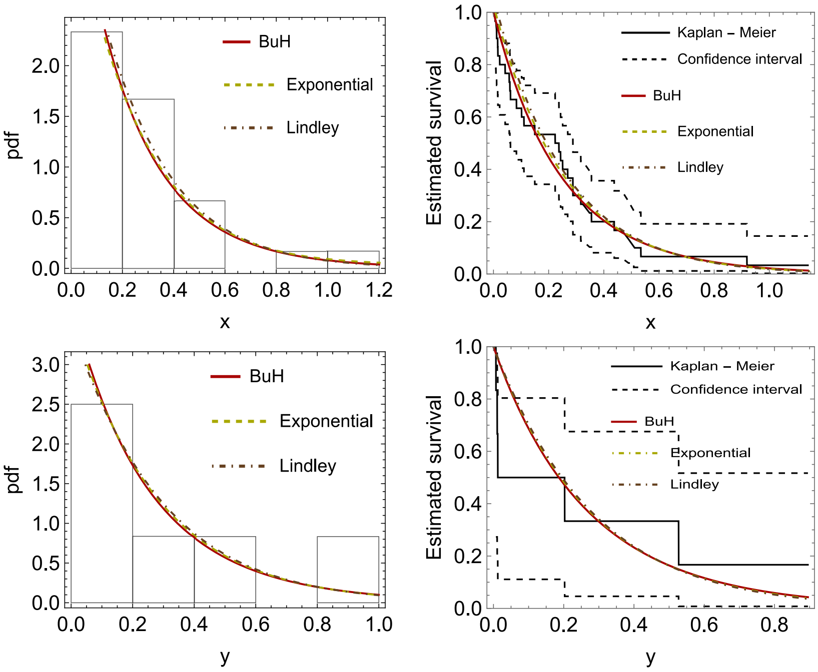

Figure 5. Fitted densities versus histograms and survival functions versus Kaplan–Meier estimators for the strength and stress observations from the generators' failure data.

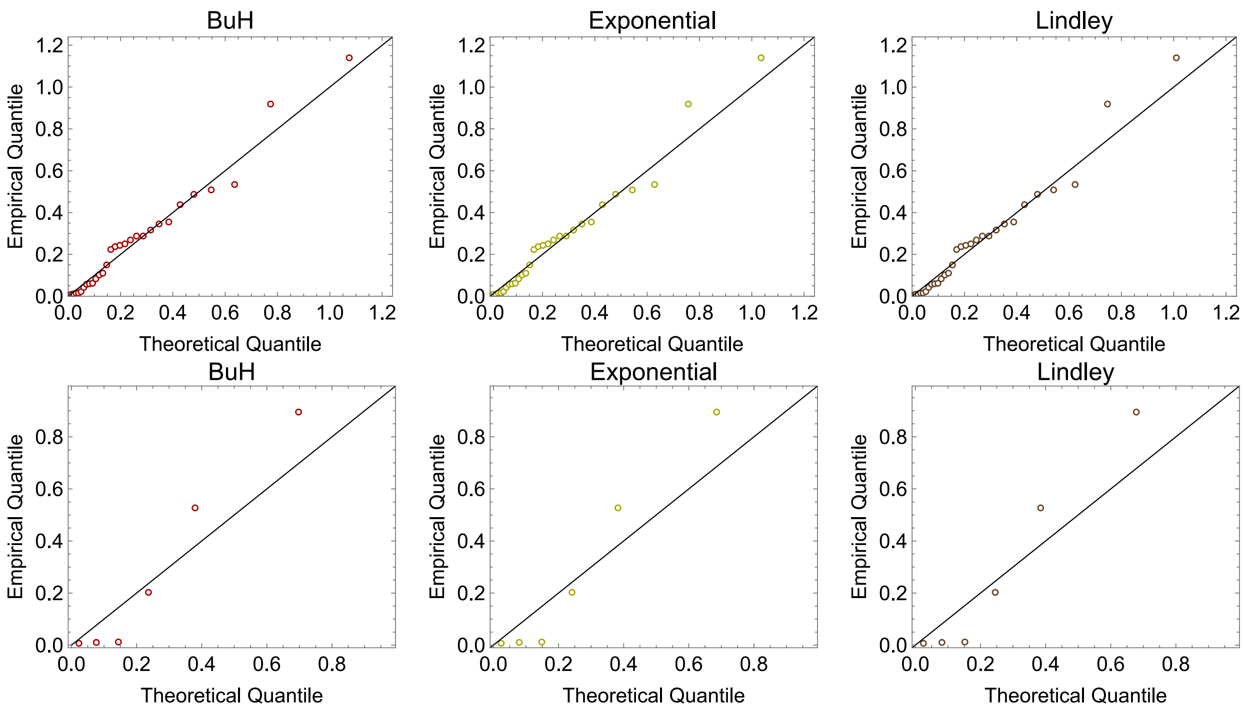

Figure 6. QQ-plots for the fitted distribution for the strength and stress observations from the generators' failure data.

According to the complete sample and the progressively censored samples, both the maximum likelihood and Bayesian estimates of $\delta_{s,k}$ were obtained. For the Bayesian estimation, three loss functions were considered: SELF, ELF, and PELF. Informative gamma priors were assigned to the parameters α_1 and α_2, with the hyperparameters set to produce a mean equal to the MLEs of the parameters and the prior variances are fixed at 0.5. The MCMC method was implemented with 100,000 iterations. The first 10,000 iterations were discarded as burn-in, and every tenth sample from the remaining iterations was retained for posterior inference. The resulting point estimates are reported in Table 11, while the 95% confidence and credible intervals are summarized in Table 12. The Boot-p and Boot-t intervals were obtained using 1000 bootstrap replications.

Convergence of the MCMC chains was assessed using several standard diagnostic tools. Marginal posterior distributions were examined through posterior histograms, trace plots were used to evaluate chain mixing, and autocorrelation function (ACF) plots up to 100 lags were inspected to assess sampling efficiency. As shown in Figures 7–10, the chains exhibit satisfactory mixing behavior and low autocorrelation across all censoring schemes. Moreover, the posterior histograms are smooth and consistent with the underlying posterior distributions, providing strong evidence of reliable convergence.

Table 11. The MLEs and Bayesian estimates of $\delta_{s,k}$ for the generators' failure data.

Scheme	MLE	Lindley			MCMC		
		SELF	ELF	PLF	SELF	ELF	PLF
Complete Sample	0.4742	0.4498	0.4199	0.4682	0.4776	0.4687	0.4819
Scheme I	0.7975	0.7428	0.7257	0.7537	0.8027	0.7997	0.8042
Scheme III	0.4983	0.4639	0.4228	0.4912	0.5042	0.4911	0.5104
Scheme III	0.4081	0.3309	0.2996	0.3638	0.3925	0.3730	0.4028

Table 12. The 95% interval estimates of $\delta_{s,k}$ for the generators' failure data.

Scheme	ACI	Logit	Boot-p	Boot-t	BCI	HPD
Complete sample	(0.2150,0.7334)	(0.2418, 0.7183)	(0.2804,0 .7538)	(0.2690, 0.7703)	(0.3547, 0.6076)	(0.3510, 0.6029)
Scheme I	(0.5253, 1.0697)	(0.42195, 0.9550)	(0.5116, 0.9417)	(0.4564, 0.9695)	(0.7007, 0.8906)	(0.7090, 0.8969)
Scheme II	(0.1749, 0.8218)	(0.2141, 0.7836)	(0.2535, 0.8093)	(0.2344, 0.8336)	(0.3511, 0.6642)	(0.35609, 0.6678)
Scheme III	(0.0754, 0.7408)	(0.14817, 0.7322)	(0.2161, 0.8549)	(0.21998, 0.8459)	(0.2530, 0.5968)	(0.2378, 0.5687)

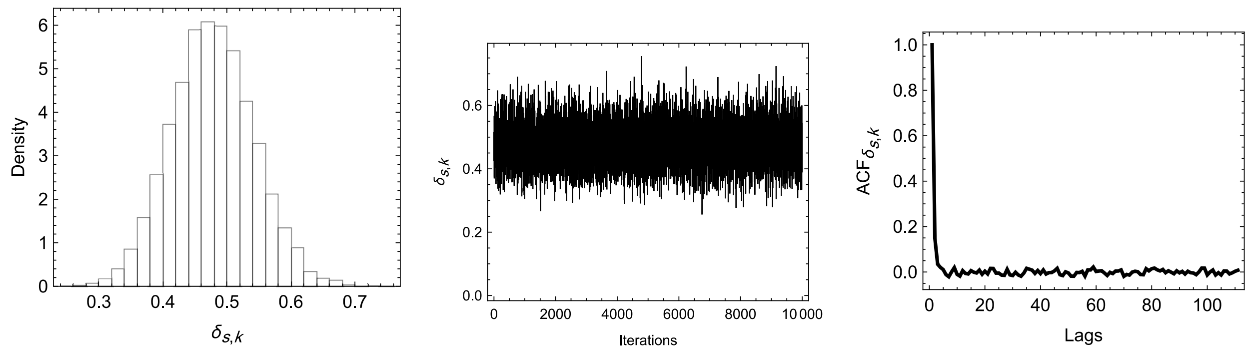

Figure 7. Posterior histogram, trace plot, and ACF plot for the generators' failure data based on the complete sample.

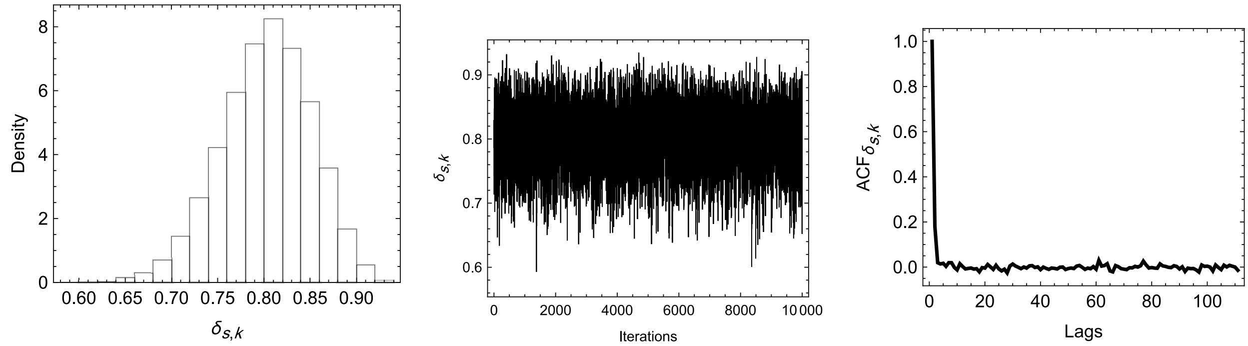

Figure 8. Posterior histogram, trace plot, and ACF plot for the generators' failure data based on Scheme I.

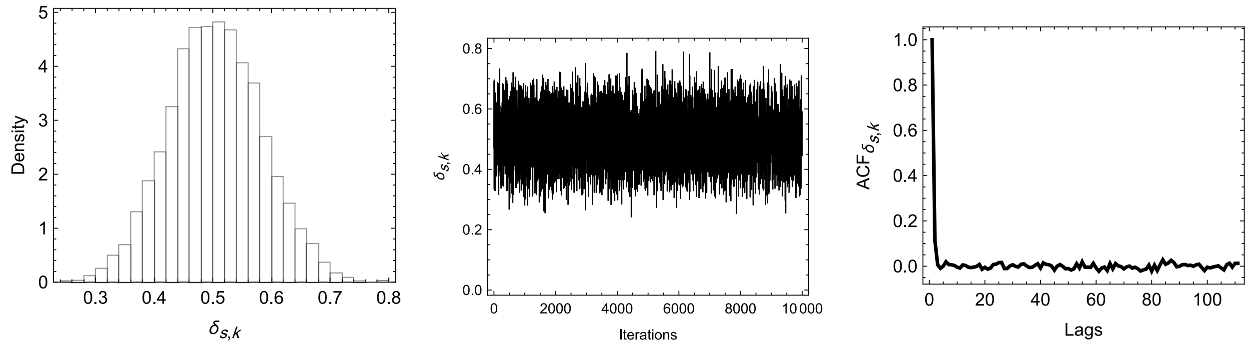

Figure 9. Posterior histogram, trace plot, and ACF plot for the generators' failure data based on Scheme II.

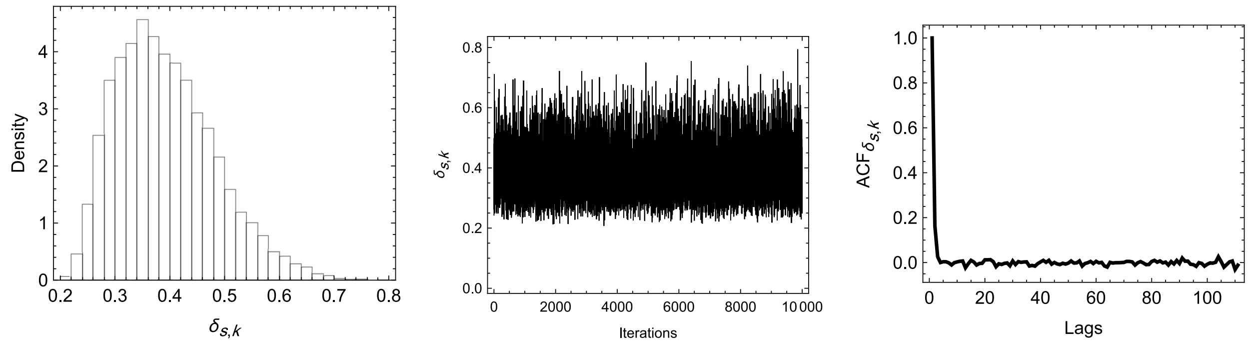

Figure 10. Posterior histogram, trace plot, and ACF plot for the generators' failure based on Scheme III.

5.2. *Earthquakes data*

The second dataset considered represents the periods between earthquakes, originally reported by [41]. It was previously analyzed for MSSR under the unit inverse Weibull distribution with Type II censoring by [42]. Following the MSSR framework, the s-out-of-k system is constructed, assuming $N = 10$ and $K = 5$. In the complete data setup, Y_1 represents the inter-event time of the first earthquake, while the associated strength variables $X_{11}, X_{12}, ..., X_{15}$ represent the subsequent five inter-event times.

Similarly, Y_2 corresponds to the next observed stress, with its five associated strengths, etc. In this way, the observed strength and stress matrices (X, Y) are constructed from the earthquake dataset and given below.

$$X = \begin{pmatrix} 0.0139 & 0.1336 & 0.0030 & 0.0365 & 0.0599 \\ 0.0038 & 0.0040 & 0.0937 & 0.0832 & 0.0887 \\ 0.0092 & 0.0294 & 0.0436 & 0.0082 & 0.0454 \\ 0.0584 & 0.0220 & 0.0840 & 0.0129 & 0.0246 \\ 0.0280 & 0.0402 & 0.0434 & 0.0735 & 0.0304 \\ 0.0567 & 0.0194 & 0.0780 & 0.0667 & 0.0121 \\ 0.0736 & 0.0328 & 0.0099 & 0.0256 & 0.0695 \\ 0.0460 & 0.0263 & 0.0375 & 0.0033 & 0.0145 \\ 0.0335 & 0.1901 & 0.0710 & 0.0556 & 0.0009 \\ 0.0562 & 0.0759 & 0.0203 & 0.0083 & 0.0036 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 0.0046 \\ 0.0076 \\ 0.0150 \\ 0.0157 \\ 0.0209 \\ 0.0319 \\ 0.0638 \\ 0.0721 \\ 0.1354 \\ 0.1617 \end{pmatrix}.$$

The BuH, exponential, and Lindley distributions were fitted to the strength and stress datasets, and the corresponding fitting results are summarized in Table 13. For both datasets, the three models exhibit comparable performance, with only marginal differences observed in the model selection criteria. Nevertheless, the BuH distribution provides a competitive fit for both the strength and stress observations. These findings indicate that the BuH distribution serves as a viable alternative to the exponential and Lindley distributions for modeling lifetime data. Figure 11 presents graphical assessments of the model fits, illustrating the estimated PDFs overlaid on the empirical histogram, as well as the estimated survival curves compared with the Kaplan–Meier estimator and its corresponding confidence intervals. The QQ-plots in Figure 12 show satisfactory agreement between the empirical and fitted distributions.

Table 13. Fitting results of the BuH distribution against exponential and Lindley distributions to the strength and stress observations from earthquake data.

Dataset	Model	MLEs Std. error	KS	p-value	AIC	BIC
X	BuH	22.1799 (3.2724)	0.0944	0.7640	-212.13	-210.22
	Exponential	23.1396 (3.2724)	0.0940	0.7685	-212.15	-210.24
	Lindley	24.0628 (3.2773)	0.0937	0.7725	-212.18	-210.27
Y	BuH	17.9624 (5.9812)	0.1729	0.9259	-36.800	-36.496
	Exponential	18.9143 (5.9812)	0.1735	0.9243	-36.798	-36.495
	Lindley	19.8227 (5.9938)	0.1740	0.9227	-36.798	-36.495

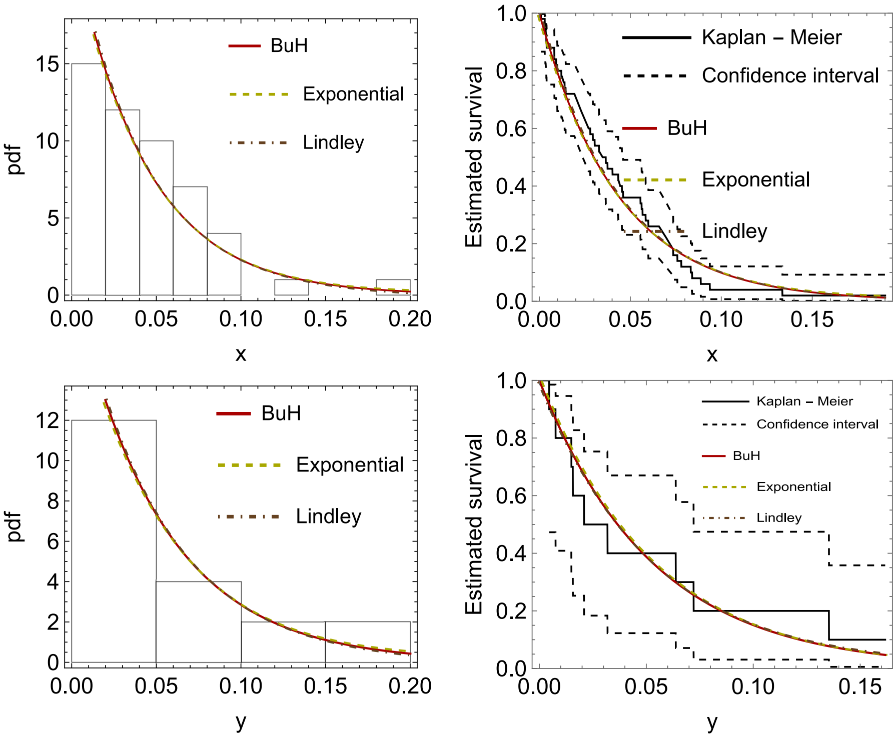

Figure 11. Fitted densities versus histograms, and survival functions versus Kaplan–Meier estimators for the strength and stress observations from earthquake data.

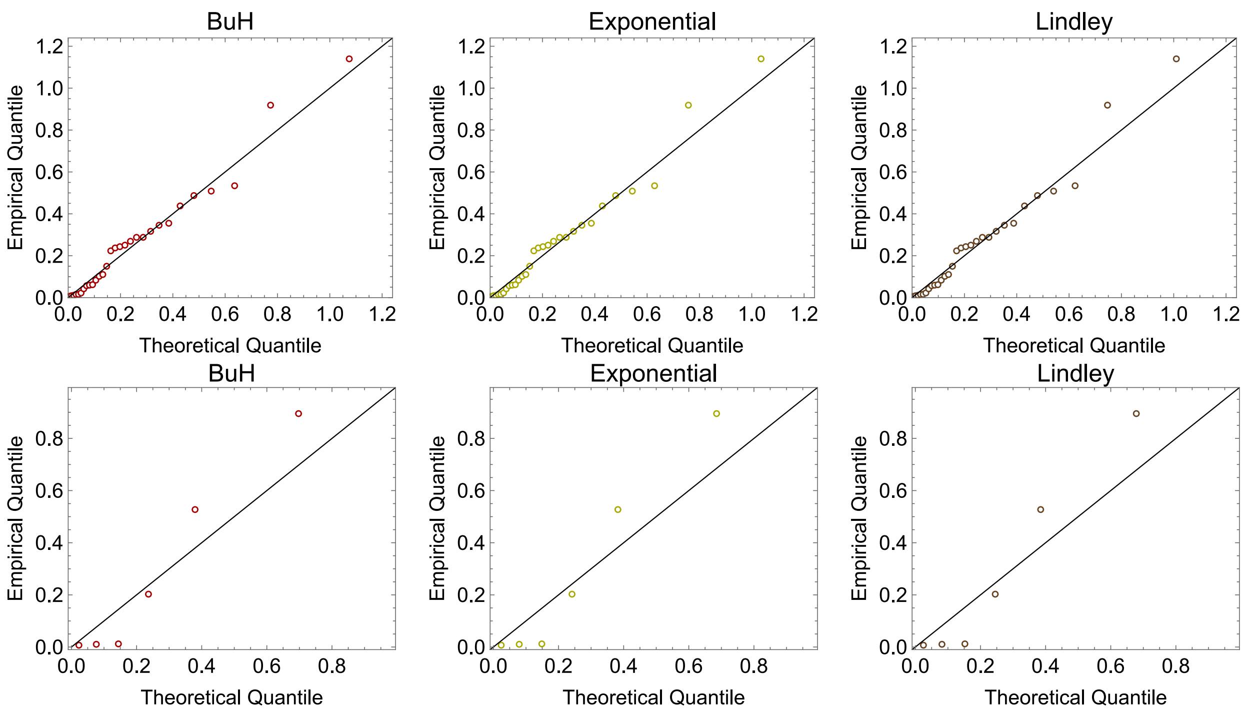

Figure 12. QQ-plots for the fitted distribution for the strength and stress observations from earthquake data.

In addition to analyzing the MSSR based on the complete sample with $s = 3$, three PTIIC samples were generated from the complete stress–strength data matrices X and Y under different censoring schemes.

Scheme I: $R = (1, 0, 0, 0)$ and $S = (0, 1, 0, 1, 0, 1, 0)$ with the parameters $(N = 10, K = 5, n = 7, k = 4, s = 2)$. The resulting censored samples are

$$X = \begin{pmatrix} 0.0030 & 0.0365 & 0.0599 & 0.1336 \\ 0.0038 & 0.0832 & 0.0887 & 0.0937 \\ 0.0129 & 0.0246 & 0.0584 & 0.0840 \\ 0.0280 & 0.0402 & 0.0434 & 0.0735 \\ 0.0099 & 0.0328 & 0.0695 & 0.0736 \\ 0.0033 & 0.0263 & 0.0375 & 0.0460 \\ 0.0036 & 0.0203 & 0.0562 & 0.0759 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 0.0046 \\ 0.0076 \\ 0.0157 \\ 0.0209 \\ 0.0638 \\ 0.0721 \\ 0.1617 \end{pmatrix}.$$

Scheme II: $R = (0, 1, 1)$ and $S = (1, 1, 1, 1, 1)$ with the parameters $(N = 10, K = 5, n = 5, k = 3, s = 1)$. The corresponding censored data are

$$X = \begin{pmatrix} 0.0030 & 0.0139 & 0.0365 & 0.0599 & 0.1336 \\ 0.0082 & 0.0092 & 0.0294 & 0.0436 & 0.0454 \\ 0.0280 & 0.0304 & 0.0402 & 0.0434 & 0.0735 \\ 0.0099 & 0.0256 & 0.0328 & 0.0695 & 0.0736 \\ 0.0009 & 0.0335 & 0.0556 & 0.0710 & 0.1901 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 0.0046 \\ 0.0150 \\ 0.0209 \\ 0.0638 \\ 0.1354 \end{pmatrix}.$$

Scheme III: $R = (0, 2, 0)$ and $S = (3, 2, 0, 0, 0)$ with the parameters $(N = 10, K = 5, n = 5, k = 3, s = 1)$. The progressively censored data are

$$X = \begin{pmatrix} 0.0030 & 0.0139 & 0.1336 \\ 0.0280 & 0.0304 & 0.0735 \\ 0.0033 & 0.0145 & 0.0460 \\ 0.0009 & 0.0335 & 0.1901 \\ 0.0036 & 0.0562 & 0.0759 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 0.0046 \\ 0.0209 \\ 0.0721 \\ 0.1354 \\ 0.1617 \end{pmatrix}.$$

Based on these progressively censored samples, the MLE and Bayesian estimates of $\delta_{s,k}$ were obtained. For Bayesian estimation, three different loss functions were considered: SELF, ELF, and PELF. Informative gamma priors were assigned to the parameters α_1 and α_2, with the hyperparameters set to produce a mean equal to the MLEs of the parameters, and the prior variances are fixed at 0.5. The MCMC method was implemented with 100,000 iterations, discarding the first 10,000 as burn-in and retaining every tenth sample for inference. The resulting point estimates are reported in Table 14, while the 95% confidence and credible intervals are summarized in Table 15. The Boot-p and Boot-t intervals were obtained using 1000 bootstrap replications.

Convergence of the MCMC chains was assessed through multiple diagnostic tools. Posterior histograms were examined to evaluate marginal distributions, trace plots were used to assess the mixing behavior of the chains, and ACF plots up to 100 lags were inspected to evaluate sampling efficiency. As illustrated in Figures 13–16, the chains displayed good mixing and low ACF across all schemes. The posterior histograms were smooth and consistent with the expected posterior distributions, confirming reliable convergence.

Table 14. The MLEs and Bayesian estimates of $\delta_{s,k}$ for the earthquake dataset.

Scheme	MLE	Lindley			MCMC		
		SELF	ELF	PELF	SELF	ELF	PELF
Complete Sample	0.4393	0.4276	0.4066	0.4393	0.4279	0.4038	0.4389
Scheme I	0.5205	0.5008	0.4726	0.5172	0.5083	0.4737	0.5232
Scheme III	0.5188	0.4967	0.4567	0.5209	0.5064	0.4469	0.5286
Scheme III	0.6653	0.6290	0.6001	0.6468	0.6392	0.5910	0.6566

Table 15. The 95% interval estimates of $\delta_{s,k}$ for the earthquake dataset.

Scheme	ACI	Logit	Boot-p	Boot-t	BCI	HPD
Complete sample	(0.2399,0.6387)	(0.2586,0.6377)	(0.2833,0.5673)	(0.2334,0.6082)	(0.2452,0.6242)	(0.2432,0.6215)
Scheme I	(0.2637,0.7774)	(0.2795,0.7524)	(0.3041,0.6859)	(0.2350,0.7387)	(0.2708,0.7453)	(0.2770,0.7500)
Scheme II	(0.2083,0.8292)	(0.2371,0.7890)	(0.2966,0.7756)	(0.2213,0.8627)	(0.2102,0.7905)	(0.2153,0.7938)
Scheme III	(0.3618,0.9688)	(0.3372,0.8859)	(0.3954,0.8621)	(0.3225,0.9152)	(0.3139,0.8896)	(0.3471,0.9076)

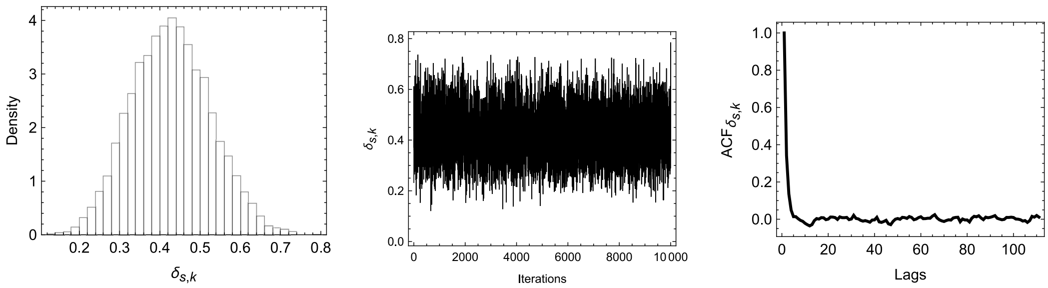

Figure 13. Posterior histogram, trace plot, and ACF plot based on the complete earthquake sample.

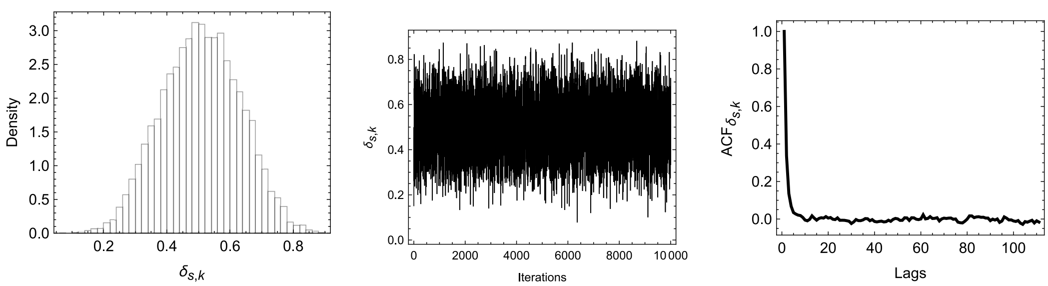

Figure 14. Posterior histogram, trace plot, and ACF plot for earthquake data based on Scheme I.

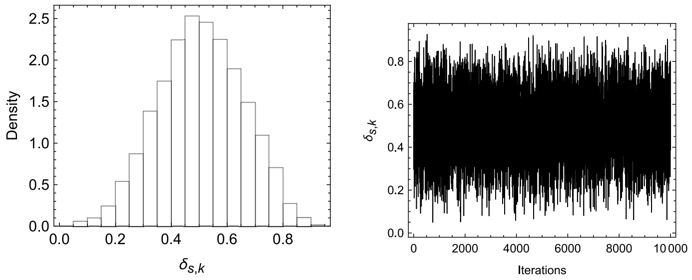

Figure 15. Posterior histogram, trace plot, and ACF plot for earthquake data based on Scheme II.

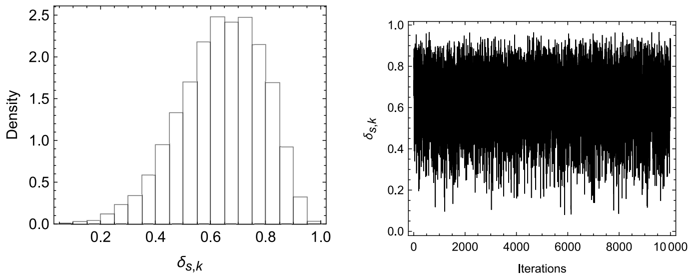

Figure 16. Posterior histogram, trace plot, and ACF plot for earthquake data based on Scheme III.

6. Conclusions

This study developed an inferential framework for MSSR when stress and strength follow the BuH distribution under PTIIC. MLE, bootstrap (Boot-p and Boot-t), and Bayesian procedures under GELF were considered. Bayesian estimation was implemented via Lindley's approximation and MCMC, and CrI and HPD intervals were constructed from posterior samples.

The simulation results show that all estimators improve as the effective sample size increases. Bayesian estimators under informative priors generally outperform MLE in terms of bias and MSE. In particular, Lindley's approximation under PELF consistently provides the smallest MSE across most PTIIC schemes while remaining computationally efficient. For interval estimation, HPD intervals maintain CPs close to the nominal 95% level and typically achieve a shorter length. Boot-p and Boot-t intervals improve upon ACI in small samples but tend to be wider. ACI is often shorter but may suffer from under-coverage in low-reliability settings, whereas the logit-transformed interval improves CP with only a slight increase in length. Overall, Bayesian estimation under informative priors, especially Lindley's approximation with PELF with HPD intervals, provides a reliable and efficient approach for MSSR under PTIIC.

The real data analyses involving generator failure times and earthquake inter-event times further demonstrated the flexibility and practical adequacy of the BuH distribution. In both applications, the model provided an excellent fit and yielded consistent reliability estimates across different progressive censoring schemes, confirming its applicability to real engineering and environmental reliability problems.

Future work may investigate MSSR under alternative censoring plans, particularly progressive schemes with random removals. Extending MSSR to generalized forms of the BuH model is another natural direction. The development of optimal testing plans under PTIIC to improve estimation efficiency is also of interest. Further research may consider multi-stress–strength systems and more complex system structures.

Author contributions

R. El-Desokey: Conceptualization, Methodology, Validation, Resources, Writing–original draft preparation; Mahmoud M. El-Awady: Conceptualization, Methodology, Software, Formal analysis, Data curation, Writing–review & editing, Supervision, Project administration; Hanan Haj Ahmad: Validation, Investigation, Writing–review & editing, Visualization, Funding acquisition; A. El-Gohary: Conceptualization, Methodology, Formal analysis, Investigation, Supervision. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Funding

This work was supported by the Deanship of Scientific Research, Vice Presidency for Graduate Studies and Scientific Research, King Faisal University, Saudi Arabia [GRANT No. KFU261131].

Conflict of interest

The authors declare that they have no conflicts of interest.

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