



Research article

Target set selection in plane graphs

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Abstract: The target set selection (TSS) problem was initially proposed to study the spread of information, ideas, or influence in social networks. It has been used to model many problems arising in various practical applications. In this paper, we study the TSS problem for plane graphs. We provide a characterization of plane graphs G for which $\min_3(G) = 3$. Consequently, we show that the minimum degree of such plane graphs is at most 4.

Keywords: target set selection; plane graph

Mathematics Subject Classification: 05C10, 05C82

1. Introduction

In this paper, we study the target set selection problem for plane graphs and characterize plane graphs G satisfying $\min_3(G) = 3$. All plane graphs considered in this paper are simple graphs embedded in the plane. Let v be a vertex of a graph G ; the degree of v in G is denoted by $d_G(v)$. In particular, we denote the minimum degree of G by $\delta(G)$. Let S be a vertex set of G and let $G[S]$ be the subgraph of G induced by S . In a plane graph, a face is a region of the plane bounded by edges. The outer face (also called the unbounded face) is the unique face that extends to infinity; all other faces are called inner faces (or internal faces). The degree of a face is the number of edges in its boundary, with cut edges counted twice.

The target set selection (TSS) problem was initially proposed by Kempe, Kleinberg, and Tardos [1] to study the spread of information, ideas, or influence through a social network [2, 3]. This model also captures many problems arising in economics, sociology, medicine, and computer science. Therefore, it has received much attention from both theoretical and practical perspectives. Examples include viral marketing [4], social networks [5], geometric networks [6], connectivity [7, 8], PTSS [9], design of the overall structure of the TSS framework [10], weighted Target Set Selection problem [11], monotone dynamic monopolies [12], dynamic monopolies [13, 14], the spread of

infectious diseases in a population or viruses in a network of computers [15, 16], and the diffusion of innovations or new products in a community [1, 4].

In the TSS problem, a network is modeled as an undirected graph whose vertices are equipped with a threshold function $\theta(v) : V(G) \rightarrow \mathbb{N}$, where $V(G)$ is the vertex set of G and \mathbb{N} is the set of positive integers.

A vertex u is called a neighbor of a vertex v if u is adjacent to v . A θ -activation (or simply an activation) of a vertex in TSS is defined via the course of a dynamic process, namely the θ -activation process: At step 0 of the process, we select an initial set S of vertices to be active while all other vertices are inactive. Starting from S , at every step $i > 0$, vertices become active according to the following rule:

Sequential updating rule [17]: Choose exactly one inactive vertex v that has at least $\theta(v)$ already-active neighbors to become active. At this phase, we also say that v is activated by $v_1, v_2, \dots, v_{\theta(v)}$ or, conversely, $v_1, v_2, \dots, v_{\theta(v)}$ activate v , where $v_1, v_2, \dots, v_{\theta(v)}$ are $\theta(v)$ active neighbours of v .

In an activation process, once a vertex becomes active, it remains active for the entire process and therefore, such a process is also called *irreversible* [15, 18]. The process ends when no more vertices can get activated. If S can activate all the vertices of G in an activation process (which may go through several activation rounds), then we call S a θ -target set of G . In particular, if $\theta(v) = k$ for any vertex v of G then we call S a k -target set of G , where k is a positive integer. It can be seen that, if $\theta(v) > d(v)$ for some vertex v , then v must be selected into every θ -target set.

Target set selection (TSS) asks for a θ -target set of minimum size. Such a set is called an optimal θ -target set, and its cardinality, denoted by $\min_{\theta}(G)$ [18, 19], is called the θ -target number. We write $\min_k(G) = t$ if the minimum size of a k -target set is t . The original version of the problem considered in [1] asks for a set of a given size that activates as many vertices as possible.

Notice that, at each step $i > 0$ in an activation process, we may have one candidate inactive to be selected. In contrast to a ‘sequential’ updating rule, if we select all candidate inactive vertices at the same time to become active, then we call such an updating rule the *parallel updating rule* [17]. It was shown that, for any graph, an optimal target set under the sequential updating rule is also optimal under the parallel updating rule, and vice versa [17]. For this reason, throughout the following, we only consider the sequential updating rule.

In a TSS problem, various types of thresholds were introduced to meet some specific requirements. The two typical and well-studied types are the *majority threshold*, denoted by θ_{\geq} , and *strict majority threshold*, denoted by $\theta_{>}$. In a majority threshold of a graph G , we have $\theta_{\geq}(v) = \lceil d(v)/2 \rceil$ for every vertex v of G while in a strict majority threshold, we have $\theta_{>}(v) = \lceil (d(v) + 1)/2 \rceil$ for every vertex v of G [16, 20, 21].

Unsurprisingly, the TSS problem is nondeterministic polynomial-time (NP)-hard in general. In fact, Dreyer and Roberts [15] showed that computing $\min_k(G)$ is NP-hard for any constant $k \geq 3$. Moreover, Chen [3] proved that the problem remains NP-hard even for $k \leq 2$ on bounded bipartite graphs. Consequently, much research has focused on specific graph classes, such as block-cactus graphs, chordal graphs, planar graphs [22], Hamming graphs [23], chordal rings [12], tori [24], hexagonal grids [17], sparse graphs, ‘cliquish’ graphs [21] as well as trees, multipartite graphs, and grids [15].

In this paper, we study the TSS problem for plane graphs. We characterize plane graphs G for which $\min_3(G) = 3$. As a consequence, we show that the minimum degree of such graphs is at most 4.

2. Bounds on a plane graph

Definition 2.1. Let S be a θ -target set of G . If the vertices of $V(G) \setminus S$ can be sequentially activated by $\Pi_0 : v_{|S|+1}, \dots, v_n$, then Π_0 is called a \min_θ -dynamic sequence. For such a sequence, define

$$d_{\Pi_0}(v_i) = d_{G[v_1, v_2, \dots, v_i]}(v_i).$$

$d_{\Pi_0}(v_i)$ is the number of neighbors of v_i in the subgraph induced by $G[v_1, v_2, \dots, v_i]$. These neighbors are called Π_0 -neighbors.

Property 2.2. For any plane graph G ,

$$\min_\theta(G) \geq \max\left\{\theta, \frac{1}{\theta}((\theta - 3)n + 6)\right\}.$$

Proof. Let $S = \{v_1, v_2, \dots, v_{|S|}\}$ be a θ -target set of G and let $v_{|S|+1}, \dots, v_n$ be a $\min_\theta(G)$ -dynamic sequence of S . Then for each $v_i \in \{v_{|S|+1}, \dots, v_n\}$, we have $d_{\Pi_0}(v_i) \geq \theta$.

When a vertex becomes active, the edges connecting it to already active vertices are counted as newly activated edges.

From $v_{|S|+1}$ to v_n , the total number of such edges accumulated during the process is $\sum_{i=|S|+1}^n d_{\Pi_0}(v_i)$, and each such edge is counted exactly once. Since all these edges belong to G , this sum cannot exceed the total number m of edges in G .

By Euler formula,

$$\theta(n - |S|) \leq \sum_{i=|S|+1}^n d_{\Pi_0}(v_i) \leq m \leq 3n - 6.$$

Therefore,

$$|S| \geq \max\left\{\theta, \frac{1}{\theta}((\theta - 3)n + 6)\right\}.$$

By the arbitrariness of S , this completes our proof. \square

In particular, for $\theta = 3$, we have $\min_3(G) \geq \max\{3, 2\} = 3$.

Moreover, if $\min_3(G) = 3$, then from the proof of Property 2.2, we obtain a bound on the number of edges m of G . Specifically, since

$$\sum_{i=|S|+1}^n d_{\Pi_0}(v_i) \leq 3n - 6$$

and $|S| = 3$, we have

$$3(n - 3) \leq \sum_{i=4}^n d_{\Pi_0}(v_i) \leq 3n - 6,$$

which implies $3n - 9 \leq m \leq 3n - 6$. The upper bound $m \leq 3n - 6$ is the well-known bound for planar graphs from Euler's formula, while the lower bound $m \geq 3n - 9$ indicates that any plane graph with $\min_3(G) = 3$ is at most three edges short of being triangulated, i.e., it can miss at most three edges compared to a maximal planar graph. This bound holds for all such graphs, regardless of whether they are themselves maximal planar. This structural restriction plays a crucial role in the characterization of such graphs in Theorem 2.5 and the degree bound in Theorem 2.6.

Definition 2.3. Let G be a plane graph and let F be a face of G with a boundary consisting of l vertices, where $l \geq 3$. For a positive integer k with $3 \leq k \leq l$, a k -embedding into F is an operation that produces a new plane graph G' as follows:

(1) Embed a new vertex v in the interior of F (if F is an inner face) or in the outer region (if F is the outer face).

(2) Connect v by k new edges to k distinct vertices on the boundary of F .

The resulting graph is denoted by $G' = G \sqcup v$ (see Figure 1). The vertex v is called an embedded vertex, and the k edges are called embedding edges.

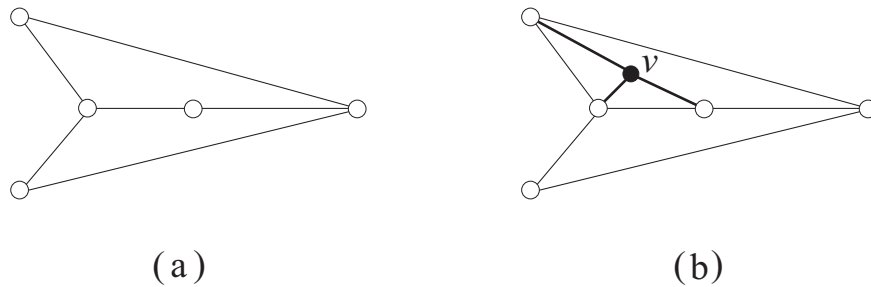


Figure 1. (a). Plane graph G ; (b). one 3-embedding $G \sqcup v$ of G (black vertex and bold edge).

More generally, let $\Pi = (v_1, v_2, \dots, v_t)$ be a sequence of vertices. We denote by

$$G \sqcup \{v_1, v_2, \dots, v_t\}$$

the plane graph obtained by successively applying k_i -embeddings for v_1, v_2, \dots, v_t into the current graph, where k_i is the number of edges added when embedding v_i . The sequence Π is called an embedding sequence of G . For each vertex v_i , we define

$$d_{\Pi}(v_i) = d_{G \sqcup \{v_1, v_2, \dots, v_i\}}(v_i),$$

i.e., the degree of v_i in the graph after embedding v_1, \dots, v_i . This value equals the number of embedding edges added for v_i .

Let $S = \{v_1, v_2, \dots, v_{|S|}\}$ be a θ -target set of plane graph G and $v_{|S|+1}, \dots, v_n$ be a \min_{θ} -dynamic sequence of S .

The activation process of $v_{|S|+1}, \dots, v_n$ can be viewed as a sequential embedding from plane graph $G[S]$:

$$\begin{aligned} G[S] &\rightarrow G[S] \sqcup v_{|S|+1} \rightarrow G[S] \sqcup \{v_{|S|+1}, v_{|S|+2}\} \rightarrow \\ &\dots \rightarrow G[S] \sqcup \{v_{|S|+1}, v_{|S|+2}, \dots, v_n\} = G. \end{aligned}$$

Moreover, we notice that

$$d_{G \sqcup \{v_1, v_2, \dots, v_i\}}(v_i) = d_{G \sqcup \{v_1, v_2, \dots, v_i\}}(v_i).$$

Then, $d_{\Pi}(v_i)$ under the \min_{θ} -dynamic sequence is equivalent to sequential embedding from $G[S]$.

Property 2.4. Let F be a face of degree k in a plane graph, and let $\Pi : v_1, v_2, \dots, v_t$ be a sequence of vertices embedded successively into the interior of F . Suppose that for each $i \in \{1, 2, \dots, t\}$, we have

$d_{\Pi}(v_i) \geq 3$, where $d_{\Pi}(v_i)$ denotes the number of edges incident to v_i at the time of its embedding. Then the following hold:

(1) For each $i \in \{1, 2, \dots, t\}$, we have $d_{\Pi}(v_i) \leq k$.

(2) The minimum degree of the resulting plane graph $F \cup \{v_1, v_2, \dots, v_t\}$ satisfies

$$\delta(F \cup \{v_1, v_2, \dots, v_t\}) \leq k.$$

(3) Moreover, $\sum_{i=1}^t (d_{\Pi}(v_i) - 3) \leq k - 3$, where the equality holds if and only if every face in the interior of $F \cup \{v_1, v_2, \dots, v_t\}$ is a triangle.

Note. The same conclusion holds if the vertices are embedded into the outer face of F .

Proof. (1) Consider the embedding of vertex v_1 . Initially, it can only connect to vertices on the boundary of F , so $d_{\Pi}(v_1) \leq k$. When v_1 is embedded, the interior of F is divided into $d_{\Pi}(v_1)$ faces. Since $d_{\Pi}(v_1) \geq 3$, and each resulting face has degree at most k , we have $d_{\Pi}(v_2) \leq k$. Repeat this argument. When v_i is embedded, the interior of F is divided into

$$1 + \sum_{j=1}^i (d_{\Pi}(v_j) - 1) \tag{2.1}$$

faces, each of degree at most k . Hence $d_{\Pi}(v_i) \leq k$ for all i .

(2) Note that $d_{F \sqcup \{v_1, v_2, \dots, v_t\}}(v_t) = d_{\Pi}(v_t)$. By (1),

$$\delta(F \sqcup \{v_1, v_2, \dots, v_t\}) \leq d_{F \sqcup \{v_1, v_2, \dots, v_t\}}(v_t) = d_{\Pi}(v_t) \leq k.$$

(3) After embedding all vertices, the number of inner faces of $F \sqcup \{v_1, v_2, \dots, v_t\}$ is

$$1 + \sum_{j=1}^t (d_{\Pi}(v_j) - 1).$$

Now add a new vertex to the outer face of this graph and connect it to all k vertices on the boundary of F . The resulting planar graph, denoted by G , has

$$f = k + 1 + \sum_{j=1}^t (d_{\Pi}(v_j) - 1)$$

faces and

$$e = 2k + \sum_{j=1}^t d_{\Pi}(v_j)$$

edges.

If every inner face of $F \sqcup \{v_1, v_2, \dots, v_t\}$ is a triangle, then G is a maximal planar graph, so by Euler's formula, $f = \frac{2}{3}e$. Substituting and simplifying yields

$$\sum_{j=1}^t (d_{\Pi}(v_j) - 3) = k - 3.$$

Conversely, if equality holds, then G is maximal planar, implying that all inner faces are triangles. \square

Theorem 2.5. A maximal planar graph G satisfies $\min_3(G) = 3$ if and only if G can be obtained by a series of k -embeddings from one of the graphs in Figure 2, where $3 \leq k \leq 6$.

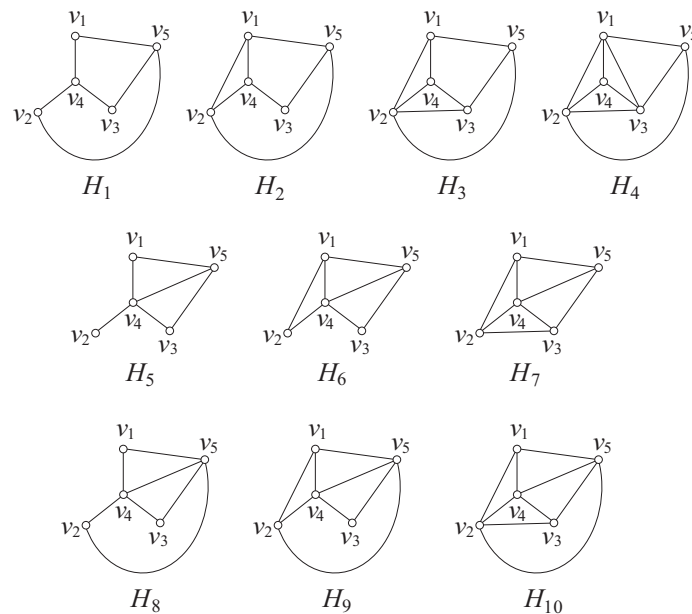


Figure 2. All labeled maximal planar graphs on five vertices that satisfy the activation conditions yields exactly ten distinct graphs.

Proof. We prove the theorem in two directions.

(\Leftarrow) First, we verify that each of the ten graphs H_1, H_2, \dots, H_{10} shown in Figure 2 satisfies $\min_3(H_i) = 3$. This can be verified directly by examining the structure of these graphs: Each contains a set of three vertices that can activate the entire graph under the threshold $\theta = 3$. The verification is straightforward and is omitted here for brevity.

Next, we show that if a graph G satisfies $\min_3(G) = 3$, then performing a k -embedding (with $3 \leq k \leq 6$) into any face of G preserves the property $\min_3(G') = 3$ for the resulting graph G' . By Property 2.4, any such embedding adds a vertex v with $d_{\Pi}(v) = k$ and divides the face into k triangular faces. The new vertex can be activated by its k neighbors, and the activation process can be extended to the original vertices. Hence $\min_3(G') \leq 3$. Since $\min_3(G') \geq 3$ by Property 2.2, we have $\min_3(G') = 3$. By induction, any graph obtained by a series of such embeddings from any H_i satisfies $\min_3(G) = 3$.

(\Rightarrow) Necessity. Conversely, let G be a maximal planar graph with $\min_3(G) = 3$. Let $S = \{v_1, v_2, v_3\}$ be an optimal 3-target set of G , and let $\Pi_0 : v_4, v_5, \dots, v_n$ be a min-dynamic sequence.

Consider the subgraph induced by the first five vertices $G[v_1, v_2, \dots, v_5]$. Since v_1, v_2, v_3 form a triangle (as G is maximal planar), the vertex v_4 must be activated by v_1, v_2, v_3 . Hence v_4 lies inside this triangle and is adjacent to all three vertices, forming a K_4 .

The activation of v_5 can occur in several ways:

- v_5 may be activated by three vertices among v_1, v_2, v_3 (in which case v_5 lies inside a triangular face of the K_4 and connects to all three vertices of that face);
- v_5 may be activated by v_4 together with two of v_1, v_2, v_3 (in which case v_5 lies inside the original triangle $v_1v_2v_3$ but not inside any triangular face of the K_4);
- Other combinations are also possible.

By systematically enumerating all possible activation orders and embedding patterns—considering all possible choices of the three activating neighbors for v_5 —we obtain a complete list of all possible 5-vertex subgraphs that can arise from a valid 3-dynamic activation sequence starting from a triangle. The ten graphs H_1, H_2, \dots, H_{10} shown in Figure 2 are exactly all such configurations. This enumeration is straightforward and can be verified by exhaustive checking; no other configurations satisfy the activation conditions.

Now, for each subsequent vertex v_i ($i \geq 6$), its embedding into the current graph $G[v_1, \dots, v_{i-1}]$ must satisfy the conditions of Property 2.4. In particular, by Property 2.4(3), the sum of $(d_{\Pi}(v_i) - 3)$ over all embedded vertices is bounded by the degree of the face into which they are embedded.

Since G is maximal planar, all faces are triangles, and the only faces available for embedding are those created by previous embeddings. A detailed case analysis shows that the only way to maintain the condition $\min_3(G) = 3$ throughout the embedding process is to ensure that the maximum face degree never exceeds 6. This is precisely the condition captured in Figure 3.

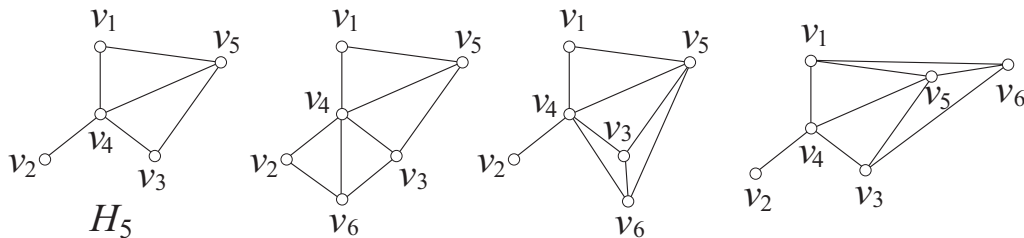


Figure 3. The embedding process is to ensure that the maximum face degree never exceeds 6.

Finally, by repeatedly applying Property 2.4 and the fact that G is maximal planar, we conclude that the entire graph G must be obtainable by a series of k -embeddings from one of the graphs in Figure 2, with each embedding satisfying $3 \leq k \leq 6$. This completes the proof. \square

Remark on the edge bound. From the proof of Property 2.2, we have

$$(3(n - 3) \leq \sum_{i=4}^n d_{\Pi}(v_i) \leq m),$$

and hence $m \geq 3n - 9$. This lower bound holds for any plane graph with $\min_3(G) = 3$, regardless of whether it is maximal planar. It indicates that such graphs are at most three edges short of being triangulated, but does not require the graph itself to be triangulated.

Note. Why are there exactly ten such graphs?

The triangle $v_1v_2v_3$ is fixed. The embedding of v_4 inside this triangle yields a K_4 , but different choices of which face becomes the outer face affect subsequent placements of v_5 . The vertex v_5 can be placed in any of the four triangular faces of K_4 , and different choices may yield isomorphic or non-isomorphic labeled graphs. Enumerating all labeled maximal planar graphs on five vertices that satisfy the activation conditions yields exactly ten distinct graphs, shown as H_1, \dots, H_{10} in Figure 2.

It is a classic result in planar graph theory that every planar graph has minimum degree at most five, i.e., $\delta(G) \leq 5$ for any planar graph G . This follows directly from Euler's formula and the fact that the average degree is less than six. However, for plane graphs satisfying $\min_3(G) = 3$, we obtain a stronger bound as stated in the following theorem.

Theorem 2.6. *If a plane graph G satisfies $\min_3(G) = 3$, then*

$$\delta(G) \leq 4.$$

In other words, such graphs cannot contain a vertex of degree five, and consequently, they are sparser in terms of minimum degree than general planar graphs. This result further refines the structural characterization of graphs with $\min_3(G) = 3$.

Proof. It is a classic result that $\delta(G) \leq 5$ for any planar graph G . We will show that under the condition $\min_3(G) = 3$, the case $\delta(G) = 5$ is impossible, thus establishing the stronger bound $\delta(G) \leq 4$.

Let G be a plane graph with $\min_3(G) = 3$. By Theorem 2.5, G can be obtained from one of the graphs $H_i (i \in \{1, 2, \dots, 10\})$ shown in Figure 2.2 through a series of k -embeddings, i.e.,

$$G = H_i \cup \{v_6, v_7, \dots, v_n\},$$

where v_6, \dots, v_n are the vertices embedded after the initial five vertices of H_i .

Assume for contradiction that $\delta(G) \geq 5$. Then by Property 2.4(2), no face of degree 4 can appear in any intermediate graph during the embedding process; otherwise, a vertex embedded into such a face would have degree at most 4 in the final graph. Consequently, the initial graph H_i cannot contain any face of degree 4. Examining Figure 2, the only graphs among H_1, \dots, H_{10} with no face of degree 4 are H_5 and H_6 . Thus $i \in \{5, 6\}$.

Case 1: G is obtained from H_5 .

H_5 has one face of degree 6 (the outer face) and two inner faces of degree 3. By Property 2.4, all vertices v_6, \dots, v_n must be embedded in the outer face F ; otherwise, embedding into a degree-3 face would create a vertex of degree 3, contradicting $\delta(G) \geq 5$. Then by Property 2.4(3),

$$\sum_{i=6}^n (d_{\Pi}(v_i) - 3) \leq 6 - 3 = 3.$$

Since $d_{\Pi}(v_i) \geq 3$ for all i , one of the following subcases must hold:

Subcase 1.1. Among the vertices v_6, v_7, \dots, v_n , there is exactly one vertex and we assume that v_s satisfies $d_{\Pi}(v_s) = 6$, while all other vertices v_i satisfy $d_{\Pi}(v_i) = 3$.

If $s < n$, $d_G(v_n) = d_{\Pi}(v_n) = 3$. This contradicts $\delta(G) > 4$. Therefore, $s = n$ and $d_{\Pi}(v_{n-1}) = 3$. Then

$$d_G(v_{n-1}) \leq d_{\Pi}(v_{n-1}) + 1 = 4.$$

This leads to a contradiction.

Subcase 1.2. Exactly one vertex v_s satisfies $d_{\Pi}(v_s) = 4$, one vertex v_t satisfies $d_{\Pi}(v_t) = 5$, and all other vertices satisfy $d_{\Pi}(v_i) = 3$.

The total excess is $(4 - 3) + (5 - 3) = 3$, so this sequence is allowed by the inequality. To avoid creating vertices of degree less than 5 in the final graph, we must have $s = n - 1$ and $t = n$. However, a detailed analysis of the embedding geometry (see Figures 3 and 4) shows that in this configuration, at least one of the vertices on the boundary of the outer face will have degree at most 4 in the final graph. This contradicts $\delta(G) \geq 5$.

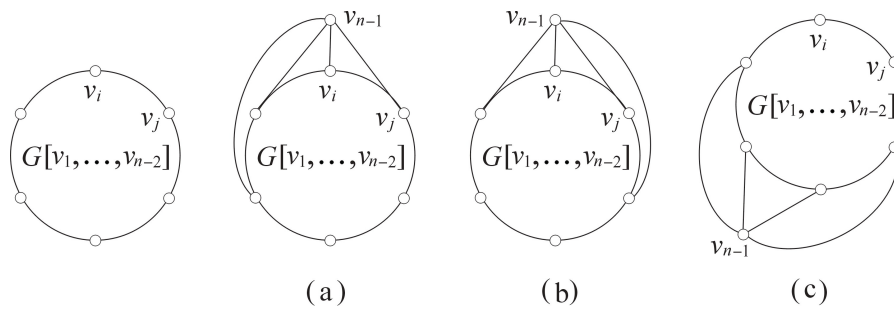


Figure 4. The three embedding methods.

Subcase 1.3. Among v_6, v_7, \dots, v_n , there is exactly one vertex v_s satisfying $d_{\Pi}(v_s) = 5$, while the remaining vertices satisfy $d_{\Pi}(v_i) = 3$.

At this vertex, it must be that $s = n$ and $d_{\Pi}(v_{n-1}) = 3$. Therefore,

$$d_G(v_{n-1}) \leq d_{\Pi}(v_{n-1}) + 1 = 4,$$

which is a contradiction.

Subcase 1.4. Among v_6, v_7, \dots, v_n , there are at most three vertices satisfying $d_{\Pi}(v_i) = 4$, while the remaining vertices satisfy $d_{\Pi}(v_i) = 3$. At this vertex, $d_G(v_n) \leq 4$, which is a contradiction.

Case 2: G is obtained from H_6 .

Similar to the discussion of H_5 , v_6, v_7, \dots, v_n are all embedded in the 5-degree face (outer face) of H_6 . Then, from (1), we have

$$\sum_{i=6}^n (d_{\Pi}(v_i) - 3) = 5 - 3 \leq 2.$$

One of the following cases is true:

Subcase 2.1. Among v_6, v_7, \dots, v_n , there is exactly one vertex; we let it be v_s , satisfying $d_{\Pi}(v_s) = 5$, while the all remaining vertices v_i satisfy $d_{\Pi}(v_i) = 3$. At this vertex, we have $s = n$. Therefore, $d_{\Pi}(v_{n-1}) = 3$. Then,

$$d_G(v_{n-1}) \leq d_{\Pi}(v_{n-1}) + 1 = 4.$$

This is a Contradiction.

Subcase 2.2. In v_6, v_7, \dots, v_n , there are at most two vertices satisfying $d_{\Pi}(v_i) = 4$, while the remaining vertices satisfy $d_{\Pi}(v_i) = 3$. At this vertex, $d_G(v_n) \leq 4$. This is a Contradiction.

Both subcases lead to a contradiction. Hence, no graph obtained from H_6 can satisfy $\delta(G) \geq 5$.

Since all cases lead to contradictions, our assumption that $\delta(G) \geq 5$ is false. Therefore, $\delta(G) \leq 4$. \square

The characterization of plane graphs with $\min_3(G) = 3$ reveals that such graphs are at most three edges short of being triangulated, with an edge count satisfying $3n-9 \leq m \leq 3n-6$. This bound follows directly from the proof of Property 2.2 and holds for any plane graph with $\min_3(G) = 3$, regardless of whether it is maximal planar. Moreover, we improve the classical planar graph degree bound from $\delta(G) \leq 5$ to $\delta(G) \leq 4$ under this condition, showing that vertices of degree five are excluded. These results highlight the restrictive nature of the target set selection problem with threshold three in the planar setting.

Author contributions

Haining Jiang and Haiyun Wan: Conceptualization, Writing-original draft, Writing-review & editing. All authors read and approved the final manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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