



Research article

Finite-time synchronization for fractional-order delayed quaternion valued neural networks by using the negative definition of matrix

Zhimin Wan^{1,*}, Zhengqiu Zhang^{2,*} and Zhenbo Cheng¹

¹ Clinical Laboratory of Hunnan Provincial People's Hospital, The First Hospital of Hunan Normal University, Changsha, 410000, China

² College of Mathematics, Hunan University, Changsha, 410006, China

* **Correspondence:** Email: 18570686560@163.com; zhangzhengqiu@hnu.edu.cn.

Abstract: In the study, the finite-time synchronization (FTSN) for a kind of master-slave quaternion-valued fractional-order neural networks (MSQVFONNS) is discussed. By using the negative definition of matrix and the properties of the determinant, two novel criteria on the FTSN for the considered MSFOQVNNS are established. The negative definition of the matrix and the properties of the determinant are introduced to study the FTSN for neural networks (NNs) in our article. Since until, studies about the FTSN for the NNs are rare and researchers have only used the linear matrix inequality (LMI), finite time stability theorems (FTSTs) of fractional order and Lyapunov direct method to study the FTSN for the MSFOQVNNS, so far, our method to study the FTSN for the MSFOQVNNS is of definite significance.

Keywords: fractional-order quaternion-valued neural networks; finite-time synchronization; the negative definition of matrix; the properties of the determinant

Mathematics Subject Classification: 34K24

1. Introduction

Fractional calculus can better describe the dynamic behaviors of neurons because of the memory and hereditary behaviors involved. Some researchers have applied the fractional calculus to neural networks (NNs) and created fractional-order NNs (FONNs). So far, the dynamic properties of FONNs have been extensively investigated [1–3]. Some scholars have found that quaternions are an effective tool to handle multidimensional data; they have created quaternion valued NNs. Compared with real-valued and complex-valued NNs, the quaternion valued NNs have remarkable advantages in extensive key fields including color image compression, attitude control and the like. Synchronization among master-slave NNs has been widely applied in safety communication, image encryption, and some other fields. Some researchers have done wide research on the infinite time synchronization for

the master-slave FONNs (MSFONNs). For example, In [4], by utilizing the matrix measure approach, the synchronization of the MSFONNs was discussed. By employing a novel approach based on fractional-order Dini-like derivatives within the matrix measure framework, the synchronization between the MSFONNs was obtained. In [5], the global Mittag-Leffler synchronization (GMLS) problem of Caputo fractional-order inertial memristive neural networks (FOIMNNs) was studied. Using the Filippov discontinuous theory, Lyapunov stability theory, and Mittag-Leffler convergence, several sufficient conditions were derived to ensure the GMLS of FOIMNNs. In [6], the quasi-synchronization (QS) for a class of master-slave fractional-order complex-valued bidirectional associative memory (BAM) neural networks (MSFOCVBAMNNs) was investigated. To begin, by utilizing Laplace transform and the Mittag-Leffler function property, a novel fractional differential inequality was obtained. Then, sufficient conditions were obtained to assure the QS for the considered MSFOCVBAMNNs by utilizing the nondecomposable method. In [7], the synchronization problem of the master-slave fractional variable-order delayed neural networks (MSFVODNNs) was explored under the sampled-data control scheme. Novel synchronization criteria were gained in the form of linear matrix inequality (LMI) to guarantee the asymptotic stability of the error states of MSFVODNNs. The paper [8] mainly used intermittent quantized controller to examine asymptotic synchronization between MSFONNs. First, by using the advantages of intermittent properties, a new lemma with asymptotic stability inequalities was put forward. Second, combining intermittent properties with quantization skill, two different categories of intermittent quantized controllers were constructed to ensure the synchronization of the MSFONNs. In [9], the function matrix projective synchronization (FMPS) for the MSFONNs was discussed. By designing the adaptive control strategy with the controlling strength updated rules and the unknown parameter adaptive rules, the FMPS was realized by establishing a Lyapunov function for the considered MSFONNs. The paper [10] investigated the QS and complete synchronization (CS) for master-slave fractional-order fuzzy bidirectional associative memory neural networks (MSFOBAMNNs). Some synchronization criteria were derived for the explored systems by utilizing the Lyapunov function approach, fractional calculation and some inequality skills.

However, the studies on the finite-time synchronization (FTSN) for MSFONNs are rare. So far, we only find a few articles which investigated the FTSN of the MSFONNs. In [11], a hybrid controller with a sampled data control was investigated to achieve finite-time master-slave synchronization for the MSFONNs. A Lyapunov-Krasovskii functional was constructed to obtain the sufficient conditions. The obtained conditions were expressed via LMI. The finite-time synchronization (FTSN) was studied in [12] for delayed fractional-order quaternion valued Cohen-Grossberg neural networks (FQVCGNNs). To begin, a fractional-order finite-time stability theorem (FTST) was established by utilizing the definition of a fractional-order integral. Then, a novel quaternion-valued feedback controller and a quaternion-valued adaptive controller were designed, respectively, to realize the FTSN of the FQVCGNNs. In [13], the FTSN of the MSFONNs was studied. Using the inequality on the fractional-order derivative of the composite function, a novel fractional-order finite-time inequality was obtained. Using this novel inequality, the designed feedback controllers and the fractional-order power law inequality, two novel criteria were obtained to assure the FTSN of the MSFONNs. In [14], this synchronization problem over a finite-time domain for a class of fractional-order stochastic NNS was discussed. Some criteria on the synchronization for the networks were obtained systematically by implementing MSFOQVNNs indirect method and applying the FTST. In [15], the FTSN of the MSFOQVNNs was considered without applying decomposition. First, two inequalities about quaternion were gained to broaden the current achievements in quaternion field. Secondly, a fractional differential inequality was established by using Laplace transform and applying the definition of the Mittag-Leffler function. Then, by employing the

presented inequalities and two different quaternion control strategies, some new conditions were derived to assure the FTSN of delayed FOQVNNs. In [16], the FTSN of FOQVNNs with time delay was studied. Without separating the quaternion-valued system into two complex valued or four real-valued systems, the FTSN criteria were derived by using Lyapunov direct method.

Up to present, researchers often have applied LMI ([11]), FTST ([12,14]), a fractional inequality ([13]), Laplace transform ([15]), and the Lyapunov direct method ([16]) to studying the FTSN for the considered MSFONNs.

Studies concerning the FTSN for MSFONNs, considering only LMI, FTST, fractional inequality, and Laplace transform have been used to study the FTSN for MSFONNs. Inspired by past study contributions on the FTSN for MSFONNs, in this article, we introduce a new method to study this topic for the MSFONNs. Namely, we will study the FTSN for a kind of master slave quaternion-valued fractional order NNS (MSQVFONNs) by utilizing the negative definition of matrix and the properties of the determinant.

Our proof is divided into the four steps as follows:

Step 1. Transform the given MSFOQVNNs into a real-valued fractional order system of four differential equations.

Step 2. By constructing a V function and applying the real-valued differential equation system, we get

$$D_t^\mu \leq [L_q(t)]^T AL_q(t) - \eta,$$

where, D_t^μ denotes the fractional derivative, μ is the fractional order number. $[L_q(t)]^T AL_q(t)$ is a quadratic type, $\eta > 0$ is a constant.

Step 3. Using the properties of the determinant and the negative definition of matrix, we prove $A < 0$.

Step 4. Via Step 3, we have

$$\begin{aligned} & D_t^\mu V(t) \\ & \leq [L_q(t)]^T AL_q(t) - \eta \\ & < -\eta. \end{aligned}$$

Step 5. Integrating on both sides of the inequality in Step 4, we have

$$\lim_{t \rightarrow T} V(t) = 0, V(t) = 0, t \geq T.$$

This completes the proof of the FTSN.

Usually, there are two methods to study the synchronization of quaternion-valued NNs: (a) the decomposition method; (b) nondecomposition method.

In this paper, the quaternion-valued NNs are broken down into a real-valued system of four differential equations by decomposition method. One of the advantages of the decomposition method is that we can get a differential equation group of four differential equations, and then from this differential equation group, the inequality in Step 2 can be obtained. Furthermore, by Step 3-Step 5, the FTSN for the considered MSFONNs can be achieved. Another advantage of the decomposition method is that the study on the FTSN for the MSFONNs is transformed into the computation of the determinant. By applying an LMI to study the FTSN, the conditions obtained are so complicated that the results cannot easily tested, and by applying FTST to studying the FTSN, it is difficult to obtain the fractional order inequalities. By applying our method to study the FTSN, however, we only need to compute the determinant, which is easier and more computationally efficient. Although the proof process is cumbersome, and many complicated formulas are needed in the computation, but the idea is simple, in that, we only need to compute the discussed determinant. However, the limitation of our

method is that when the dimension of the determinant increases, the computation will become more difficult. Therefore, the determinant computed cannot exceed an order of eight. In order to obtain four differential equations of the same dimension number, it is indispensable to transfer the quaternion-valued NNs into a real-valued NNs of several equations. The main contributions of this article are therefore: (a) A new method is introduced to investigate the FTSN for the MSFONNs: the negative definition of matrix is introduced; (b) The properties of the determinant are cited to study the FTSN for MSFONNs; (c) Two novel FTSN criteria are obtained for the considered MSFOQVNNs.

The arrangement of this paper is scheduled as follows: In Section 2, some definitions and lemmas are introduced. In Section 3, two criteria on the FTSN for the considered MSFONNs are given. In Section 4, two examples are given to certify the validity of our main results. In Section 5, we present the conclusion of this paper and some outlooks for future work.

2. Preliminaries

Based on the model in [15], we consider a kind of FOQVNNs with the Caputo fractional operator of the order μ expressed by (2.1) for $q, p = 1, 2, \dots, w$,

$$D_t^\mu[\alpha_q(t)] = -a_q\alpha_q(t) + \sum_{p=1}^w b_{qp}F_p(\alpha_p(t)) + \sum_{p=1}^w c_{qp}F_p(\alpha_p(t-\tau)) + r_q, \quad (2.1)$$

where, w denotes the number of neurons, $t \geq 0$, $\alpha_q(t)$ represents the q th neuron's state variable, $0 < \mu < 1$ is a constant, $a_q > 0$ denotes the self-feedback weight; $b_{qp}, c_{qp} \in \mathcal{Q}$ are the interconnection weights, $F_q(\alpha_q(t)) : \mathcal{R}^+ \rightarrow \mathcal{Q}$ is the activation function, r_q is the constant external input, the integer τ is the time delay.

The initial values of System (2.1) are constructed as

$$\alpha_q(\gamma) = \hat{\phi}_q(\gamma), \gamma \in [-\tau, 0].$$

Let

$$\alpha_q(n) = \alpha_q^R + i\alpha_q^I + j\alpha_q^J + k\alpha_q^K;$$

$$b_{pq} = b_{pq}^R + ib_{pq}^I + jb_{pq}^J + kb_{pq}^K;$$

$$F_p(\alpha_p(t)) = F_p^R(\alpha_p^R(t)) + iF_p^I(\alpha_p^I(t)) + jF_p^J(\alpha_p^J(t)) + kF_p^K(\alpha_p^K(t))(\text{Assumption1});$$

$$c_{pq} = c_{pq}^R + ic_{pq}^I + jc_{pq}^J + kc_{pq}^K;$$

$$F_p(\alpha_p(t-\tau)) = F_p^R(\alpha_p^R(t-\tau)) + iF_p^I(\alpha_p^I(t-\tau)) + jF_p^J(\alpha_p^J(t-\tau)) + kF_p^K(\alpha_p^K(t-\tau))(\text{Assumption1});$$

$$r_q = r_q^R + ir_q^I + jr_q^J + kr_q^K.$$

Then Eq (2.1) can be articulated as:

$$\left\{ \begin{aligned}
 & D_t^\mu [\alpha_q^R(t)] \\
 & = -a_q \alpha_q^R(t) + \sum_{p=1}^w \left\{ b_{pq}^R F_p^R(\alpha_p^R(t)) - b_{pq}^I F_p^I(\alpha_p^I(t)) - b_{pq}^J F_p^J(\alpha_p^J(t)) - b_{pq}^K F_p^K(\alpha_p^K(t)) \right. \\
 & \quad \left. + c_{pq}^R F_p^R(\alpha_p^R(t-\tau)) - c_{pq}^I F_p^I(\alpha_p^I(t-\tau)) - c_{pq}^J F_p^J(\alpha_p^J(t-\tau)) - c_{pq}^K F_p^K(\alpha_p^K(t-\tau)) \right\} + r_q^R \\
 & D_t^\mu [\alpha_q^I(t)] \\
 & = -a_q \alpha_q^I(t) + \sum_{p=1}^w \left\{ b_{pq}^I F_p^R(\alpha_p^R(t)) + b_{pq}^R F_p^I(\alpha_p^I(t)) + b_{pq}^J F_p^K(\alpha_p^K(t)) - b_{pq}^K F_p^J(\alpha_p^J(t)) \right. \\
 & \quad \left. + c_{pq}^I F_p^R(\alpha_p^R(t-\tau)) + c_{pq}^R F_p^I(\alpha_p^I(t-\tau)) - c_{pq}^K F_p^J(\alpha_p^J(t-\tau)) + c_{pq}^J F_p^K(\alpha_p^K(t-\tau)) \right\} + r_q^I \\
 & D_t^\mu [\alpha_q^J(t)] \\
 & = -a_q \alpha_q^J(t) + \sum_{p=1}^w \left\{ b_{pq}^J F_p^R(\alpha_p^R(t)) + b_{pq}^R F_p^J(\alpha_p^J(t)) - b_{pq}^I F_p^K(\alpha_p^K(t)) + b_{pq}^K F_p^I(\alpha_p^I(t)) \right. \\
 & \quad \left. + c_{pq}^J F_p^R(\alpha_p^R(t-\tau)) + c_{pq}^R F_p^J(\alpha_p^J(t-\tau)) - c_{pq}^I F_p^K(\alpha_p^K(t-\tau)) + c_{pq}^K F_p^I(\alpha_p^I(t-\tau)) \right\} + r_q^J \\
 & D_t^\mu [\alpha_q^K(t)] \\
 & = -a_q \alpha_q^K(t) + \sum_{p=1}^w \left\{ b_{pq}^R F_p^K(\alpha_p^K(t)) + b_{pq}^K F_p^R(\alpha_p^R(t)) + b_{pq}^I F_p^J(\alpha_p^J(t)) - b_{pq}^J F_p^I(\alpha_p^I(t)) \right. \\
 & \quad \left. + c_{pq}^R F_p^K(\alpha_p^K(t-\tau)) + c_{pq}^K F_p^R(\alpha_p^R(t-\tau)) + c_{pq}^I F_p^J(\alpha_p^J(t-\tau)) - c_{pq}^J F_p^I(\alpha_p^I(t-\tau)) \right\} + r_q^K.
 \end{aligned} \right. \quad (2.2)$$

The initial conditions of System (2.2) are as follows:

$$\alpha_q^\lambda(s) = \phi_q^\lambda(s), \quad s \in [-\tau, 0],$$

$\lambda = R, I, J, K, \phi_q(s)$ is a continuous function.

Take System (2.2) as the master system, and the slave system is depicted as for $q, p = 1, 2, \dots, w$

$$\begin{aligned}
 & \left. \begin{aligned}
 & D_t^\mu[\beta_q^R(t)] \\
 & = -a_q \beta_q^R(t) + \sum_{p=1}^w \left\{ b_{pq}^R F_p^R(\beta_p^R(t)) - b_{pq}^I F_p^I(\beta_p^I(t)) - b_{pq}^J F_p^J(\beta_p^J(t)) - b_{pq}^K F_p^K(\beta_p^K(t)) \right. \\
 & + c_{pq}^R F_p^R(\beta_p^R(t-\tau)) - c_{pq}^I F_p^I(\beta_p^I(t-\tau)) - c_{pq}^J F_p^J(\beta_p^J(t-\tau)) - c_{pq}^K F_p^K(\beta_p^K(t-\tau)) \left. \right\} + r_q \\
 & + l_{1q}(t) \\
 & D_t^\mu[\beta_q^I(t)] \\
 & = -a_q \beta_q^I(t) + \sum_{p=1}^w \left\{ b_{pq}^I F_p^R(\beta_p^R(t)) + b_{pq}^R F_p^I(\beta_p^I(t)) + b_{pq}^J F_p^K(\beta_p^K(t)) - b_{pq}^K F_p^J(\beta_p^J(t)) \right. \\
 & + c_{pq}^I F_p^R(\beta_p^R(t-\tau)) + c_{pq}^R F_p^I(\beta_p^I(t-\tau)) - c_{pq}^K F_p^J(\beta_p^J(t-\tau)) + c_{pq}^J F_p^K(\beta_p^K(t-\tau)) \left. \right\} + r_q \\
 & + l_{2q}(t) \\
 & D_t^\mu[\beta_q^J(t)] \\
 & = -a_q \beta_q^J(t) + \sum_{p=1}^w \left\{ b_{pq}^J F_p^R(\beta_p^R(t)) + b_{pq}^R F_p^J(\beta_p^J(t)) - b_{pq}^I F_p^K(\beta_p^K(t)) + b_{pq}^K F_p^I(\beta_p^I(t)) \right. \\
 & + c_{pq}^J F_p^R(\beta_p^R(t-\tau)) + c_{pq}^R F_p^J(\beta_p^J(t-\tau)) - c_{pq}^I F_p^K(\beta_p^K(t-\tau)) + c_{pq}^K F_p^I(\beta_p^I(t-\tau)) \left. \right\} + r_q \\
 & + l_{3q}(t) \\
 & D_t^\mu[\beta_q^K(t)] \\
 & = -a_q \beta_q^K(t) + \sum_{p=1}^w \left\{ b_{pq}^R F_p^K(\beta_p^K(t)) + b_{pq}^K F_p^R(\beta_p^R(t)) + b_{pq}^I F_p^J(\beta_p^J(t)) - b_{pq}^J F_p^I(\beta_p^I(t)) \right. \\
 & + c_{pq}^R F_p^K(\beta_p^K(t-\tau)) + c_{pq}^K F_p^R(\beta_p^R(t-\tau)) + c_{pq}^I F_p^J(\beta_p^J(t-\tau)) - c_{pq}^J F_p^I(\beta_p^I(t-\tau)) \left. \right\} + r_q \\
 & + l_{4q}(t).
 \end{aligned}
 \right\} \tag{2.3}
 \end{aligned}$$

where, the parameters in System (2.3) are the same as ones as in System (2.2); $l_{1q}(t), l_{2q}(t), l_{3q}(t), l_{4q}(t)$ are the controllers, and $\beta_q^I(t), \beta_q^J(t), \beta_q^R(t), \beta_q^K(t)$ are the status variables. The initial conditions of System (2.3) are planed as follows:

$$\beta_q^\lambda(s) = \psi_q^\lambda(s), s \in [-k, 0], \lambda = R, I, J, K.$$

Setting $X_q^\lambda(t) = \beta_q^\lambda(t) - \alpha_q^\lambda(t), \lambda = R, I, J, K.$ denote

$$F_p^R(X_p^R(t)) = F_p^R(\beta_p^R(t)) - F_p^R(\alpha_p^R(t)), F_p^I(X_p^I(t)) = F_p^I(\beta_p^I(t)) - F_p^I(\alpha_p^I(t)),$$

$$F_p^J(X_p^J(t)) = F_p^J(\beta_p^J(t)) - F_p^J(\alpha_p^J(t)), F_p^K(X_p^K(t)) = F_p^K(\beta_p^K(t)) - F_p^K(\alpha_p^K(t)).$$

Then, the error system between (2.2) and (2.3) can be depicted as

$$\left\{ \begin{aligned}
 & D_t^\mu [X_q^R(t)] \\
 &= -a_q X_q^R(t) + \sum_{p=1}^w \left\{ b_{pq}^R F_p^R(X_p^R(t)) + b_{pq}^I F_p^I(X_p^I(t)) + b_{pq}^J F_p^K(X_p^K(t)) - b_{pq}^J F_p^J(X_p^J(t)) + c_{pq}^I \times \right. \\
 &\quad \left. F_p^R(X_p^R(t-\tau)) + c_{pq}^R F_p^I(X_p^I(t-\tau)) - c_{pq}^K F_p^J(X_p^J(t-\tau)) + c_{pq}^J F_p^K(X_p^K(t-\tau)) + l_{1q}(t) \right\}; \\
 & D_t^\mu [X_q^I(t)] \\
 &= -a_q X_q^I(t) + \sum_{p=1}^w \left\{ b_{pq}^I F_p^R(X_p^R(t)) + b_{pq}^R F_p^I(X_p^I(t)) + b_{pq}^J F_p^K(X_p^K(t)) - b_{pq}^K F_p^J(X_p^J(t)) + c_{pq}^I \times \right. \\
 &\quad \left. F_p^R(X_p^R(t-\tau)) + c_{pq}^R F_p^I(X_p^I(t-\tau)) - c_{pq}^K F_p^J(X_p^J(t)) + c_{pq}^J F_p^K(X_p^K(t-\tau)) + l_{2q}(t) \right\}; \\
 & D_t^\mu [X_q^J(t)] \\
 &= -a_q X_q^J(t) + \sum_{p=1}^w \left\{ b_{pq}^J F_p^R(X_p^R(t)) + b_{pq}^R F_p^J(X_p^J(t)) - b_{pq}^I F_p^K(X_p^K(t)) + b_{pq}^K F_p^I(X_p^I(t)) + c_{pq}^J \times \right. \\
 &\quad \left. F_p^R(X_p^R(t-\tau)) + c_{pq}^R F_p^J(X_p^J(t-\tau)) - c_{pq}^I F_p^K(X_p^K(t-\tau)) - c_{pq}^K F_p^I(X_p^I(t-\tau)) + l_{3q}(t) \right\}; \\
 & D_t^\mu [X_q^K(t)] \\
 &= -a_q X_q^K(t) + \sum_{p=1}^w \left\{ b_{pq}^R F_p^K(X_p^K(t)) + b_{pq}^K F_p^R(X_p^R(t)) + b_{pq}^I F_p^J(X_p^J(t)) - b_{pq}^J F_p^I(X_p^I(t)) + c_{pq}^R \times \right. \\
 &\quad \left. F_p^K(X_p^K(t-\tau)) + c_{pq}^K F_p^R(X_p^R(t-\tau)) + c_{pq}^I F_p^J(X_p^J(t-\tau)) - c_{pq}^J F_p^I(X_p^I(t-\tau)) + l_{4q}(t) \right\}
 \end{aligned} \right. \quad (2.4)$$

Definition 2.1 (A. Kilbas et al. [19]). The fractional integral of function $h(t)$ is defined as

$$I^\mu[h(t)] = \frac{1}{\Gamma^*(\mu)} \int_{t_0}^t (t-u)^{\mu-1} h(u) du, \quad \mu > 0,$$

where $\Gamma^*(\cdot)$ is the well-known gamma function given by

$$\gamma^*(\mu) = \int_0^\infty t^{\mu-1} e^{-t} dt.$$

Definition 2.2 (A. Kilbas et al. [19]). The Caputo fractional derivative of function $h(t)$ is defined by

$$D_t^\mu[h(t)] = \frac{1}{\gamma^*(n-\mu)} \int_{t_0}^t \frac{h^n(s)}{(t-s)^{\mu-n+1}} ds,$$

where $0 \leq n-1 < \mu < n$, $\gamma^*(\cdot)$ is the gamma function. Particularly, when $0 < \mu < 1$,

$$D_t^\mu[h(t)] = \frac{1}{\Gamma^*(1-\mu)} \int_{t_0}^t \frac{(h)'(s)}{(t-s)^\mu} ds.$$

Definition 2.3. The drive system (2.2) and the response system (2.3) are said to achieve the FTSN if, for arbitrary solutions of System (2.2) and System (2.3) denoted by $[\alpha_1(t), \alpha_2(t), \dots, \alpha_w(t)]^T$ and

$[\beta_1(t), \beta_2(t), \dots, \beta_w(t)]$, there is a positive function T which is related to the initial values of the error system such that

$$\lim_{t \rightarrow T} |\alpha_i^\lambda(t) - \beta_i^\lambda(t)| = 0, \quad |\alpha_i^\lambda(t) - \beta_i^\lambda(t)| = 0, \lambda = R, I, J, K, t > T.$$

Definition 2.4. The real quadratic form $f(x_1, x_2, \dots, x_n)$ is said to be the positive definite quadratic form if for not being all zero real numbers (c_1, c_2, \dots, c_n) , one has $f(c_1, c_2, \dots, c_n) > 0$.

Definition 2.5. The real symmetric matrix A is said to be positive-definite if the real quadratic form $X^T A X$ is a positive quadratic form.

Lemma 2.1 (A. Kilbas et al [19]). If the CFD $D_t^\mu[h(t)]$ is integrable, then the μ integration of $D_t^\mu[h(t)]$ is defined as

$$I^\mu[D_t^\mu(h(t))] = h(t) - \sum_{k=0}^{\hat{n}-1} \frac{(h)^{(k)}(t_0)}{k!} (t - t_0)^k.$$

Especially, for $0 < \mu \leq 1$, one has

$$I^\mu[D_t^\mu(h(t))] = h(t) - h(t_0).$$

Lemma 2.2 (All members of Algebra [17]) Consider the real symmetrical matrix $K = (k_{i,j})_{n \times n}$. Then, $K < 0$ if and only if all of the following requirements are satisfied: For $i = \text{odd}, i \leq n$,

$$K_i = \begin{vmatrix} k_{11} & k_{12} & \cdots & k_{1i} \\ k_{21} & k_{22} & \cdots & k_{2i} \\ \cdots & \cdots & \cdots & \cdots \\ k_{i1} & k_{i2} & \cdots & k_{ii} \end{vmatrix} < 0;$$

for $i = \text{even}, i \leq n$,

$$K_i = \begin{vmatrix} k_{11} & k_{12} & \cdots & k_{1i} \\ k_{21} & k_{22} & \cdots & k_{2i} \\ \cdots & \cdots & \cdots & \cdots \\ k_{i1} & k_{i2} & \cdots & k_{ii} \end{vmatrix} > 0.$$

Lemma 2.3 (Q.Peng et al [18]). Assume that $0 < \mu < 1$. Then,

$$D_t^\mu[g^2(x)] \leq 2g(x)D_t^\mu g(x);$$

$$D_t^\mu|g(x)| \leq \text{sign}[g(x)]D_t^\mu g(x).$$

Lemma 2.4. If μ is the Caputo fractional-order, then

$$D_t^\mu[f_1(x) + f_2(x) + \cdots + f_n(x)] = D_t^\mu f_1(x) + D_t^\mu f_2(x) + \cdots + D_t^\mu f_n(x).$$

Proof. From the Definition 2.2, the proof of Lemma 2.4 can be finished.

In this work, the following assumptions hold:

Assumption 1:

$$F_p(\alpha_p(t)) = F_p^R(\alpha_p^R(t)) + iF_p^I(\alpha_p^I(t)) + jF_p^J(\alpha_p^J(t)) + kF_p^K(\alpha_p^K(t - \tau));$$

Assumption 2: There exist positive constants L^R, L^I, L^J, L^K such that

$$|F_p^R(y_1) - F_p^R(y_2)| \leq L^R|y_1 - y_2|, |F_p^I(y_1) - F_p^I(y_2)| \leq L^I|y_1 - y_2|,$$

$$|F_p^J(y_1) - F_p^J(y_2)| \leq L^J|y_1 - y_2|, |F_p^K(y_1) - F_p^K(y_2)| \leq L^K|y_1 - y_2|;$$

for all $y_1, y_2 \in R, p = 1, 2, \dots, w, |\cdot|$ is the norm of the Euclidean space R .

Some notations are cited as follows:

$$\begin{aligned} r_1 &= \max_{1 \leq q \leq w} \left\{ \sum_{p=1}^w |b_{pq}^R| L^R \right\}, r_2 = \max_{1 \leq q \leq w} \left\{ \sum_{p=1}^w |b_{qp}^R| L^R \right\}, r_3 = \max_{1 \leq q \leq w} \left\{ \sum_{p=1}^w |b_{pq}^R| L^I \right\}; \\ r_4 &= \max_{1 \leq q \leq w} \left\{ \sum_{p=1}^w |b_{qp}^I| L^I \right\}, r_5 = \max_{1 \leq q \leq w} \left\{ \sum_{p=1}^w |b_{pq}^K| L^K \right\}, r_6 = \max_{1 \leq q \leq w} \left\{ \sum_{p=1}^w |b_{qp}^J| L^K \right\}; \\ r_7 &= \max_{1 \leq q \leq w} \left\{ \sum_{p=1}^w |b_{qp}^K| L^J \right\}, r_8 = \max_{1 \leq q \leq w} \left\{ \sum_{p=1}^w |b_{pq}^K| L^J \right\}, r_9 = \max_{1 \leq q \leq w} \left\{ \sum_{p=1}^w |c_{pq}^I| L^R \right\}; \\ r_{10} &= \max_{1 \leq q \leq w} \left\{ \sum_{p=1}^w |c_{qp}^I| L^R \right\}, r_{11} = \max_{1 \leq q \leq w} \left\{ \sum_{p=1}^w |c_{pq}^R| L^I \right\}, r_{12} = \max_{1 \leq q \leq w} \left\{ \sum_{p=1}^w |c_{qp}^R| L^I \right\}; \\ r_{13} &= \max_{1 \leq q \leq w} \left\{ \sum_{p=1}^w |c_{pq}^K| L^J \right\}, r_{14} = \max_{1 \leq q \leq w} \left\{ \sum_{p=1}^w |c_{qp}^K| L^J \right\}, r_{15} = \max_{1 \leq q \leq w} \left\{ \sum_{p=1}^w |c_{pq}^J| L^K \right\}; \\ r_{16} &= \max_{1 \leq q \leq w} \left\{ \sum_{p=1}^w |c_{qp}^J| L^K \right\}, r_{17} = \max_{1 \leq q \leq w} \left\{ \sum_{p=1}^w |b_{pq}^I| L^R \right\}, r_{18} = \max_{1 \leq q \leq w} \left\{ \sum_{p=1}^w |b_{qp}^I| L^R \right\}; \\ r_{19} &= \max_{1 \leq q \leq w} \left\{ \sum_{p=1}^w |b_{pq}^R| L^I \right\}, r_{20} = \max_{1 \leq q \leq w} \left\{ \sum_{p=1}^w |b_{qp}^R| L^I \right\}, r_{21} = \max_{1 \leq q \leq w} \left\{ \sum_{p=1}^w |b_{pq}^J| L^K \right\}; \\ r_{22} &= \max_{1 \leq q \leq w} \left\{ \sum_{p=1}^w |b_{qp}^J| L^K \right\}, r_{23} = \max_{1 \leq q \leq w} \left\{ \sum_{p=1}^w |b_{pq}^K| L^J \right\}, r_{24} = \max_{1 \leq q \leq w} \left\{ \sum_{p=1}^w |b_{qp}^K| L^J \right\}; \\ r_{25} &= \max_{1 \leq q \leq w} \left\{ \sum_{p=1}^w |c_{pq}^I| L^R \right\}, r_{26} = \max_{1 \leq q \leq w} \left\{ \sum_{p=1}^w |c_{qp}^I| L^R \right\}, r_{27} = \max_{1 \leq q \leq w} \left\{ \sum_{p=1}^w |c_{pq}^R| L^I \right\}; \\ r_{28} &= \max_{1 \leq q \leq w} \left\{ \sum_{p=1}^w |c_{qp}^R| L^I \right\}, r_{29} = \max_{1 \leq q \leq w} \left\{ \sum_{p=1}^w |c_{pq}^K| L^J \right\}, r_{30} = \max_{1 \leq q \leq w} \left\{ \sum_{p=1}^w |c_{qp}^K| L^J \right\}; \\ r_{31} &= \max_{1 \leq q \leq w} \left\{ \sum_{p=1}^w |c_{pq}^J| L^K \right\}, r_{32} = \max_{1 \leq q \leq w} \left\{ \sum_{p=1}^w |c_{qp}^J| L^K \right\}, r_{33} = \max_{1 \leq q \leq w} \left\{ |b_{pq}^J| L^R \right\}; \\ r_{34} &= \max_{1 \leq q \leq w} \left\{ \sum_{p=1}^w |b_{qp}^J| \right\}, r_{35} = \max_{1 \leq q \leq w} \left\{ \sum_{p=1}^w |b_{pq}^J| \right\}, r_{36} = \max_{1 \leq q \leq w} \left\{ \sum_{p=1}^w |b_{qp}^J| L^J \right\}; \\ r_{37} &= \max_{1 \leq q \leq w} \left\{ \sum_{p=1}^w |b_{pq}^I| L^K \right\}, r_{38} = \max_{1 \leq q \leq w} \left\{ \sum_{p=1}^w |b_{qp}^I| L^K \right\}, r_{39} = \max_{1 \leq q \leq w} \left\{ \sum_{p=1}^w |b_{pq}^K| L^I \right\}; \end{aligned}$$

$$\begin{aligned}
r_{40} &= \max_{1 \leq q \leq w} \left\{ \sum_{p=1}^w |b_{qp}^K| L^I \right\}, r_{41} = \max_{1 \leq q \leq w} \left\{ \sum_{p=1}^w |c_{pq}^J| L^R \right\}, r_{42} = \max_{1 \leq q \leq w} \left\{ \sum_{p=1}^w |c_{qp}^J| L^R \right\}; \\
r_{43} &= \max_{1 \leq q \leq w} \left\{ \sum_{p=1}^w |c_{pq}^R| L^J \right\}, r_{44} = \max_{1 \leq q \leq w} \left\{ \sum_{p=1}^w |c_{qp}^R| L^J \right\}, r_{45} = \max_{1 \leq q \leq w} \left\{ \sum_{p=1}^w |c_{pq}^I| L^K \right\}; \\
r_{46} &= \max_{1 \leq q \leq w} \left\{ \sum_{p=1}^w |c_{qp}^I| L^K \right\}, r_{47} = \max_{1 \leq q \leq w} \left\{ \sum_{p=1}^w |c_{pq}^K| L^I \right\}, r_{48} = \max_{1 \leq q \leq w} \left\{ \sum_{p=1}^w |c_{qp}^K| L^I \right\}; \\
r_{49} &= \max_{1 \leq q \leq w} \left\{ \sum_{p=1}^w |b_{pq}^R| L^K \right\}, r_{50} = \max_{1 \leq q \leq w} \left\{ \sum_{p=1}^w |b_{qp}^R| L^K \right\}, r_{51} = \max_{1 \leq q \leq w} \left\{ \sum_{p=1}^w |b_{pq}^K| L^R \right\}; \\
r_{52} &= \max_{1 \leq q \leq w} \left\{ \sum_{p=1}^w |b_{qp}^K| L^R \right\}, r_{53} = \max_{1 \leq q \leq w} \left\{ \sum_{p=1}^w |b_{pq}^I| L^J \right\}, r_{54} = \max_{1 \leq q \leq w} \left\{ \sum_{p=1}^w |b_{qp}^I| L^J \right\}; \\
r_{55} &= \max_{1 \leq q \leq w} \left\{ \sum_{p=1}^w |b_{pq}^J| L^I \right\}, r_{56} = \max_{1 \leq q \leq w} \left\{ \sum_{p=1}^w |b_{qp}^J| L^I \right\}, r_{57} = \max_{1 \leq q \leq w} \left\{ \sum_{p=1}^w |c_{pq}^R| L^K \right\}; \\
r_{58} &= \max_{1 \leq q \leq w} \left\{ \sum_{p=1}^w |c_{qp}^R| L^K \right\}, r_{59} = \max_{1 \leq q \leq w} \left\{ \sum_{p=1}^w |c_{pq}^K| L^R \right\}, r_{60} = \max_{1 \leq q \leq w} \left\{ \sum_{p=1}^w |c_{qp}^K| L^R \right\}; \\
r_{61} &= \max_{1 \leq q \leq w} \left\{ \sum_{p=1}^w |c_{pq}^I| L^J \right\}, r_{62} = \max_{1 \leq q \leq w} \left\{ \sum_{p=1}^w |c_{qp}^I| L^J \right\}, r_{63} = \max_{1 \leq q \leq w} \left\{ \sum_{p=1}^w |c_{pq}^J| L^I \right\}; \\
r_{64} &= \max_{1 \leq q \leq w} \left\{ \sum_{p=1}^w |c_{qp}^J| L^I \right\},
\end{aligned}$$

$$k_{11} = -2a_q + \sum_{i=1}^3 r_i + r_5 + r_8 + r_9 + r_{11} + r_{13} + r_{15} + r_{18} + r_{34} + r_{52},$$

$$k_{22} = -2a_q + r_4 + r_{17} + r_{19} + r_{21} + r_{23} + r_{25} + r_{27} + r_{29} + r_{31} + r_{40} + r_{56},$$

$$k_{33} = -2a_q + r_7 + r_{24} + r_{33} + \sum_{i=35}^{37} r_i + r_{39} + r_{41} + r_{43} + r_{45} + r_{47} + r_{54},$$

$$k_{44} = -2a_q + r_6 + r_{22} + r_{38} + \sum_{i=49}^{51} r_i + r_{53} + r_{55} + r_{57} + r_{59} + r_{61} + r_{63},$$

$$k_{55} = -2k_1 + r_{10} + r_{26} + r_{42} + r_{60},$$

$$k_{66} = -2k_2 + r_{12} + r_{28} + r_{30} + r_{48} + r_{64},$$

$$k_{77} = -2k_3 + r_{14} + r_{30} + r_{44} + r_{62}, k_{88} = -2k_4 + r_{16} + r_{32} + r_{46} + r_{58}.$$

The controllers in (2.3) are constructed as follows:

$$l_{1q}(t) = -\frac{\eta_1 + k_1 [X_q^R(t - \tau)]^2}{X_q^R(t)}, \quad (2.5)$$

$$l_{2q} = -\frac{k_2[X_q^I(t-\tau)]^2}{X_q^I(t)} - u_2 X_q^J(t), \quad (2.6)$$

$$l_{3q} = -\frac{k_3[X_q^J(t-\tau)]^2}{X_q^J(t)} - u_3 X_q^K(t), \quad (2.7)$$

$$l_{4q} = -\frac{k_4[X_q^K(t-\tau)]^2}{X_q^K(t)} - u_4 X_q^R(t-\tau) - u_1 X_q^I(t-\tau) - u_5 X_q^J(t-\tau) \quad (2.8)$$

or

$$l_{1q} = \text{sign}[X_q^R(t)] \{ -\gamma_1 |X_q^R(t-\tau)| - \gamma_2 |X_q^I(t-\tau)| - \gamma_3 |X_q^J(t-\tau)| - \gamma_4 |X_q^K(t-\tau)| \}; \quad (2.9)$$

$$l_{2q} = -\eta_2 \text{sign}[X_q^I(t)]; \quad (2.10)$$

$$l_{3q} = \text{sign}[X_q^J(t)] \{ -\gamma_6 |X_q^I(t)|^{0.5} |X_q^J(t)|^{0.5} - \gamma_7 |X_q^J(t)|^{0.5} \times |X_q^R(t-\tau)|^{0.5} \}; \quad (2.11)$$

$$l_{4q} = \text{sign}[X_q^K(t)] \{ -\gamma_9 |X_q^J(t-\tau)|^{0.5} |X_q^K(t-\tau)|^{0.5} \}, \quad (2.12)$$

where, $\eta_1 > 0, \eta_2 > 0, \gamma_m > 0, k_m > 0 (m = 1, 2, 3, 4), \gamma_6, \gamma_7, u_i (i = 1, 2, \dots, 5)$ are constants.

3. Main results

Theorem 3.1. Under Assumption 1, the System (2.2) and System (2.3) can realize the FTSN by employing the controllers in (2.5) if the following conditions hold:

(h_1)

$$k_{11} < 0, k_{66} < 0, k_{77} < 0;$$

(h_2)

$$k_{22} < 0, k_{88} < 0;$$

(h_3)

$$k_{33} k_{22} > 0.25 u_2^2;$$

(h_4)

$$k_{22} [k_{33} k_{44} - 0.25 u_3^2] < 0.25 u_2^2 k_{44};$$

(h_5)

$$k_{33} [k_{44} k_{55} - 0.25 u_4^2] < 0.25 u_3^2 k_{55};$$

with the finite-time

$$t_1 = \left[\frac{Z(0)\mu\Gamma^*(\mu)}{2w\eta_1} \right]^{\frac{1}{\mu}}, Z(0) = \sum_{q=1}^w \left[(X_p^R(0))^2 + (X_p^I(0))^2 + (X_p^J(0))^2 + (X_p^K(0))^2 \right].$$

Statement 1. $k_{22} < 0$ and (h_3) imply $k_{33} < 0$. (h_5) and $k_{33} < 0$ imply $k_{44}k_{55} - 0.25u_4^2 > 0$. $k_{66} < 0$, implies (h_6)

$$k_{66}k_{33}[k_{44}k_{55} - 0.25u_4^2] > 0.$$

Proof of *Theorem 3.1*. If $X_q^R(t) = X_q^I(t) = X_q^J(t) = X_q^K(t) = X_q^R(t - \tau) = X_q^I(t - \tau) = X_q^J(t - \tau) = X_q^K(t - \tau) = 0$, then the proof of *Theorem 3.1* has been finished. So we assume that $X_q^R(t), X_q^I(t), X_q^J(t), X_q^K(t), X_q^R(t - \tau), X_q^I(t - \tau), X_q^J(t - \tau), X_q^K(t - \tau)$ are not all zero. Construct a Lyapunov function:

$$Z(t) = \sum_{q=1}^w \left[(X_p^R(t))^2 + (X_p^I(t))^2 + (X_p^J(t))^2 + (X_p^K(t))^2 \right].$$

Via System (2.4), we obtain by using Assumption 1

$$\begin{aligned} & D_t^\mu[Z(t)] \\ & \leq 2 \sum_{q=1}^w \left\{ X_q^R(t) D_t^\mu[X_q^R(t)] + X_q^I(t) D_t^\mu[X_q^I(t)] + X_q^J(t) D_t^\mu[X_q^J(t)] + X_q^K(t) D_t^\mu[X_q^K(t)] \right\} \\ & = 2 \sum_{q=1}^w \left\{ X_q^R(t) \left[-a_q X_q^R(t) + \sum_{p=1}^w (b_{pq}^R F_p^R(X_p^R(t)) + b_{pq}^I F_p^I(X_p^I(t)) + b_{pq}^K F_p^K(X_p^K(t)) \right. \right. \\ & \quad \left. \left. - b_{pq}^K F_p^J(X_p^J(t)) + c_{pq}^I F_p^R(X_p^R(t - \tau)) + c_{pq}^R F_p^I(X_p^I(t - \tau)) - c_{pq}^K F_p^J(X_p^J(t - \tau)) + c_{pq}^J \times \right. \right. \\ & \quad \left. \left. F_p^K(X_p^K(t - \tau)) + l_{1q}(t) \right] + X_q^I(t) \left[-a_q X_q^I(t) + \sum_{p=1}^w (b_{pq}^I F_p^R(X_p^R(t)) + F_p^I(X_p^I(t)) b_{pq}^R + \right. \right. \\ & \quad \left. \left. b_{pq}^J F_p^K(X_p^K(t)) - b_{pq}^K F_p^J(X_p^J(t - \tau)) + c_{pq}^I F_p^R(X_p^R(t - \tau)) + c_{pq}^R F_p^I(X_p^I(t - \tau)) - c_{pq}^K \times \right. \right. \\ & \quad \left. \left. F_p^J(X_p^J(t - \tau)) + c_{pq}^J F_p^K(X_p^K(t - \tau)) \right) + l_{2q}(t) \right] + X_q^J(t) \left[-a_q X_q^J(t) + \sum_{p=1}^w (F_p^R(X_p^R(t)) \right. \\ & \quad \left. \times b_{pq}^J + b_{pq}^R F_p^J(X_p^J(t)) - b_{pq}^I F_p^K(X_p^K(t)) + b_{pq}^K F_p^I(X_p^I(t)) + c_{pq}^J F_p^R(X_p^R(t - \tau)) + c_{pq}^R \times \right. \\ & \quad \left. F_p^J(X_p^J(t - \tau)) - c_{pq}^I F_p^K(X_p^K(t - \tau)) - c_{pq}^K F_p^I(X_p^I(t - \tau)) \right) + l_{3q}(t) \right] + X_q^K(t) \left[-X_q^K(t) \right. \\ & \quad \left. \times a_q + \sum_{p=1}^w (b_{pq}^R F_p^K(X_p^K(t)) + b_{pq}^K F_p^R(X_p^R(t)) + b_{pq}^I F_p^J(X_p^J(t)) - b_{pq}^J F_p^I(X_p^I(t)) + c_{pq}^R \times \right. \\ & \quad \left. F_p^K(X_p^K(t - \tau)) + c_{pq}^K F_p^R(X_p^R(t - \tau)) + c_{pq}^I F_p^J(X_p^J(t - \tau)) - c_{pq}^J F_p^I(X_p^I(t - \tau)) \right) + l_{4q}(t) \left. \right\} \\ & \leq 2 \sum_{q=1}^w \left\{ X_q^R(t) \left[-a_q X_q^R(t) + \sum_{p=1}^w (|b_{pq}^R| L^R |X_p^R(t)| + |b_{pq}^I| L^I |X_p^I(t)| + |b_{pq}^K| L^K |X_p^K(t)| + |b_{pq}^J| \right. \right. \end{aligned}$$

$$\begin{aligned}
& |L_J|X_p^J(t)| + |c_{pq}^I|L^R|X_p^R(t-\tau)| + |c_{pq}^R|L^I|X_p^I(t-\tau)| + |c_{pq}^K|L^J|X_p^J(t-\tau)| + |c_{pq}^J|L^K \times \\
& |X_p^K(t-\tau)| - \frac{\eta_1 + k_1[X_q^R(t-\tau)]^2}{X_q^R(t)} \Big] + X_q^I(t) \Big[-a_q X_q^I(t) + \sum_{p=1}^w (|b_{pq}^I|L^R|X_p^R(t)| + |X_p^I(t)| \\
& \times |L_I|b_{pq}^R| + |b_{pq}^J|L^K|X_p^K(t)| + |b_{pq}^K|L^J|X_p^J(t)| + |c_{pq}^I|L^R|X_p^R(t-\tau)| + |c_{pq}^R|L^I|X_p^I(t-\tau)| + \\
& |c_{pq}^K|L^J|X_p^J(t)| + |c_{pq}^J|L^K|X_p^K(t-\tau)|) - u_2 X_q^J(t) - \frac{k_2[X_q^I(t-\tau)]^2}{X_q^I(t)} \Big] + X_q^J(t) \Big[-a_q X_p^J(t) \\
& + \sum_{p=1}^w (|b_{pq}^J|L^R|X_p^R(t)| + |b_{pq}^R|L^J|X_p^J(t)| + |b_{pq}^I|L^K|X_p^K(t)| + |b_{pq}^K|L^I|X_p^I(t)| + |X_p^R(t-\tau)| \\
& \times |c_{pq}^J|L^R + |c_{pq}^R|L^J|X_p^J(t-\tau)| + |c_{pq}^I|L^K|X_p^K(t-\tau)| + |c_{pq}^K|L^I|X_p^I(t-\tau)|) - u_3 X_q^K(t) \\
& - k_3 \frac{[X_q^J(t-\tau)]^2}{X_q^J(t)} \Big] + X_q^K(t) \Big[-a_q X_q^K(t) + \sum_{p=1}^w (|b_{pq}^R||X_p^K(t)|L^K + |b_{pq}^K||X_p^R(t)|L^R + |b_{pq}^I|L^J \\
& \times |X_p^J(t)| + |b_{pq}^J|L^I|X_p^I(t)| + |c_{pq}^R|L^K|X_p^K(t-\tau)| + |c_{pq}^K|L^R|X_p^R(t-\tau)| + |c_{pq}^I|L^J|X_p^J(t-\tau)| \\
& + |c_{pq}^J|L^I|X_p^I(t-\tau)|) - \frac{k_4[X_q^K(t-\tau)]^2}{X_q^K(t)} - u_4 X_q^R(t-\tau) - u_1 X_q^I(t-\tau) - u_5 X_q^J(t-\tau) \Big].
\end{aligned} \tag{3.1}$$

Applying $2ab \leq a^2 + b^2$, one has

$$\begin{aligned}
2 \sum_{q=1}^w X_q^R(t) \sum_{p=1}^w |b_{pq}^R|L^R|X_p^R(t)| & \leq \sum_{q=1}^w \sum_{p=1}^w |b_{pq}^R|L^R \left([X_q^R(t)]^2 + [X_p^R(t)]^2 \right) \\
& \leq \sum_{q=1}^w [r_1 + r_2][X_q^R(t)]^2,
\end{aligned} \tag{3.2}$$

$$\begin{aligned}
2 \sum_{q=1}^w \sum_{p=1}^w X_q^R(t) |b_{pq}^I|L^I|X_p^I(t)| & \leq \sum_{q=1}^w \sum_{p=1}^w |b_{pq}^I|L^I \left([X_q^R(t)]^2 + [X_p^I(t)]^2 \right) \\
& \leq \sum_{q=1}^w \left(\sum_{p=1}^w |b_{pq}^I|L^I [X_q^R(t)]^2 + \sum_{q=1}^w \left(\sum_{p=1}^w |b_{qp}^I|L^I [X_q^I(t)]^2 \right) \right) \\
& \leq \sum_{q=1}^w \left(r_3 [X_q^R(t)]^2 + r_4 [X_q^I(t)]^2 \right);
\end{aligned} \tag{3.3}$$

$$\begin{aligned}
2 \sum_{q=1}^w \sum_{p=1}^w X_q^R(t) |b_{pq}^K|L^K|X_p^K(t)| & \leq \sum_{q=1}^w \sum_{p=1}^w |b_{pq}^K|L^K \left([X_q^R(t)]^2 + [X_p^K(t)]^2 \right) \\
& \leq \sum_{q=1}^w \left(r_5 [X_q^R(t)]^2 + r_6 [X_q^K(t)]^2 \right),
\end{aligned} \tag{3.4}$$

$$\begin{aligned}
2 \sum_{q=1}^w X_q^R(t) \sum_{p=1}^w |b_{pq}^K| L^J |X_p^J(t)| &\leq \sum_{q=1}^w \sum_{p=1}^w |b_{pq}^K| L^J ([X_p^J(t)]^2 + [X_q^R(t)]^2) \\
&\leq \sum_{q=1}^w (r_7 [X_q^J(t)]^2 + r_8 [X_q^R(t)]^2),
\end{aligned} \tag{3.5}$$

$$\begin{aligned}
2 \sum_{q=1}^w \sum_{p=1}^w X_q^R(t) |c_{pq}^J| L^R |X_p^R(t - \tau)| &\leq \sum_{q=1}^w \sum_{p=1}^w |c_{pq}^J| L^R ([X_q^R(t)]^2 + [X_p^R(t - \tau)]^2) \\
&\leq \sum_{q=1}^w (r_9 [X_q^R(t)]^2 + r_{10} [X_q^R(t - \tau)]^2),
\end{aligned} \tag{3.6}$$

$$\begin{aligned}
2 \sum_{q=1}^w X_q^R(t) \sum_{p=1}^w |c_{pq}^R| L^I |X_p^I(t - \tau)| &\leq \sum_{q=1}^w \sum_{p=1}^w |c_{pq}^R| L^I ([X_q^R(t)]^2 + [X_p^I(t - \tau)]^2) \\
&\leq \sum_{q=1}^w (r_{11} [X_q^R(t)]^2 + r_{12} [X_q^I(t - \tau)]^2),
\end{aligned} \tag{3.7}$$

$$\begin{aligned}
2 \sum_{q=1}^w X_q^R(t) \sum_{p=1}^w |c_{pq}^K| L^J |X_p^J(t - \tau)| &\leq \sum_{q=1}^w \sum_{p=1}^w |c_{pq}^K| L^J ([X_q^R(t)]^2 + [X_p^J(t - \tau)]^2) \\
&\leq \sum_{q=1}^w (r_{13} [X_q^R(t)]^2 + r_{14} [X_q^J(t - \tau)]^2),
\end{aligned} \tag{3.8}$$

$$\begin{aligned}
2 \sum_{q=1}^w X_q^R(t) \sum_{p=1}^w |c_{pq}^J| L^K |X_p^K(t - \tau)| &\leq \sum_{q=1}^w \sum_{p=1}^w |c_{pq}^J| L^K ([X_q^R(t)]^2 + [X_p^K(t - \tau)]^2) \\
&\leq \sum_{q=1}^w (r_{15} [X_q^R(t)]^2 + r_{16} [X_q^K(t - \tau)]^2),
\end{aligned} \tag{3.9}$$

$$\begin{aligned}
2 \sum_{q=1}^w X_q^I(t) \sum_{p=1}^w |b_{pq}^I| L^R |X_p^R(t)| &\leq \sum_{q=1}^w \sum_{p=1}^w |b_{pq}^I| L^R ([X_q^I(t)]^2 + [X_p^R(t)]^2) \\
&\leq \sum_{q=1}^w (r_{17} [X_q^I(t)]^2 + r_{18} [X_q^R(t)]^2),
\end{aligned} \tag{3.10}$$

$$\begin{aligned}
2 \sum_{q=1}^w X_q^I(t) \sum_{p=1}^w |b_{pq}^R| L^I |X_p^I(t)| &\leq \sum_{q=1}^w \sum_{p=1}^w |b_{pq}^R| L^I ([X_q^I(t)]^2 + [X_p^I(t)]^2) \\
&\leq \sum_{q=1}^w (r_{19} [X_q^I(t)]^2 + r_{20} [X_q^I(t)]^2),
\end{aligned} \tag{3.11}$$

$$\begin{aligned}
2 \sum_{q=1}^w X_q^I(t) \sum_{p=1}^w |b_{pq}^J| L^K |X_p^K(t)| &\leq \sum_{q=1}^w \sum_{p=1}^w |b_{pq}^J| L^K \left([X_q^I(t)]^2 + [X_p^K(t)]^2 \right) \\
&\leq \sum_{q=1}^w \left(r_{21} [X_q^I(t)]^2 + r_{22} [X_q^K(t)]^2 \right), \tag{3.12}
\end{aligned}$$

$$\begin{aligned}
2 \sum_{q=1}^w X_q^I(t) \sum_{p=1}^w |b_{pq}^K| L^J |X_p^J(t)| &\leq \sum_{q=1}^w \sum_{p=1}^w |b_{pq}^K| L^J \left([X_q^I(t)]^2 + [X_p^J(t)]^2 \right) \\
&\leq \sum_{q=1}^w \left(r_{23} [X_q^I(t)]^2 + r_{24} [X_q^J(t)]^2 \right), \tag{3.13}
\end{aligned}$$

$$\begin{aligned}
2 \sum_{q=1}^w X_q^I(t) \sum_{p=1}^w |c_{pq}^I| L^R |X_p^R(t - \tau)| &\leq \sum_{p=1}^w |c_{pq}^I| L^R \left([X_q^I(t)]^2 + [X_p^R(t - \tau)]^2 \right) \\
&\leq \sum_{q=1}^w \left(r_{25} [X_q^I(t)]^2 + r_{26} [X_q^R(t - \tau)]^2 \right), \tag{3.14}
\end{aligned}$$

$$\begin{aligned}
2 \sum_{q=1}^w X_q^I(t) \sum_{p=1}^w |c_{pq}^R| L^I |X_p^I(t - \tau)| &\leq \sum_{q=1}^w \sum_{p=1}^w |c_{pq}^R| L^I \left([X_q^I(t)]^2 + [X_p^I(t - \tau)]^2 \right) \\
&\leq \sum_{q=1}^w \left(r_{27} [X_q^I(t)]^2 + r_{28} [X_q^I(t - \tau)]^2 \right), \tag{3.15}
\end{aligned}$$

$$\begin{aligned}
2 \sum_{q=1}^w X_q^I(t) \sum_{p=1}^w |c_{pq}^K| L^J |X_p^J(t - \tau)| &\leq \sum_{q=1}^w \sum_{p=1}^w |c_{pq}^K| L^J \left([X_q^I(t)]^2 + [X_p^J(t - \tau)]^2 \right) \\
&\leq \sum_{q=1}^w \left(r_{29} [X_q^I(t)]^2 + r_{30} [X_q^J(t - \tau)]^2 \right), \tag{3.16}
\end{aligned}$$

$$\begin{aligned}
\sum_{q=1}^w 2X_q^I(t) \sum_{p=1}^w |c_{pq}^J| L^K |X_p^K(t - \tau)| &\leq \sum_{p=1}^w |c_{pq}^J| L^K \left([X_q^I(t)]^2 + [X_p^K(t - \tau)]^2 \right) \\
&\leq \sum_{q=1}^w \left(r_{31} [X_q^I(t)]^2 + r_{32} [X_q^K(t - \tau)]^2 \right), \tag{3.17}
\end{aligned}$$

$$\begin{aligned}
2 \sum_{q=1}^w X_q^J(t) \sum_{p=1}^w |b_{pq}^J| L^R |X_p^R(t)| &\leq \sum_{p=1}^w |b_{pq}^J| L^R \left([X_q^J(t)]^2 + [X_p^R(t)]^2 \right) \\
&\leq \sum_{q=1}^w \left(r_{33} [X_q^J(t)]^2 + r_{34} [X_q^R(t)]^2 \right), \tag{3.18}
\end{aligned}$$

$$\begin{aligned}
2 \sum_{q=1}^w X_q^J(t) \sum_{p=1}^w |b_{pq}^J| L^J |X_p^J(t)| &\leq \sum_{q=1}^w \sum_{p=1}^w |b_{pq}^J| L^J ([X_q^J(t)]^2 + [X_p^J(t)]^2) \\
&\leq \sum_{q=1}^w (r_{35} [X_q^J(t)]^2 + r_{36} [X_q^J(t)]^2), \tag{3.19}
\end{aligned}$$

$$\begin{aligned}
2 \sum_{q=1}^w X_q^J(t) \sum_{p=1}^w |b_{pq}^I| L^K |X_p^K(t)| &\leq \sum_{q=1}^w \sum_{p=1}^w |b_{pq}^I| L^K ([X_q^J(t)]^2 + [X_p^K(t)]^2) \\
&\leq \sum_{q=1}^w (r_{37} [X_q^J(t)]^2 + r_{38} [X_q^K(t)]^2), \tag{3.20}
\end{aligned}$$

$$\begin{aligned}
2 \sum_{q=1}^w X_q^J(t) \sum_{p=1}^w |b_{pq}^K| L^I |X_p^I(t)| &\leq \sum_{q=1}^w \sum_{p=1}^w |b_{pq}^K| L^I ([X_q^J(t)]^2 + [X_p^I(t)]^2) \\
&\leq \sum_{q=1}^w (r_{39} [X_q^J(t)]^2 + r_{40} [X_q^I(t)]^2), \tag{3.21}
\end{aligned}$$

$$\begin{aligned}
2 \sum_{q=1}^w X_q^J(t) \sum_{p=1}^w |c_{pq}^J| L^R |X_p^R(t - \tau)| &\leq \sum_{q=1}^w \sum_{p=1}^w |c_{pq}^J| L^R ([X_q^J(t)]^2 + [X_p^R(t - \tau)]^2) \\
&\leq \sum_{q=1}^w (r_{41} [X_q^J(t)]^2 + r_{42} [X_q^R(t - \tau)]^2), \tag{3.22}
\end{aligned}$$

$$\begin{aligned}
2 \sum_{q=1}^w X_q^J(t) \sum_{p=1}^w |c_{pq}^R| L^J |X_p^J(t - \tau)| &\leq \sum_{q=1}^w \sum_{p=1}^w |c_{pq}^R| L^J ([X_q^J(t)]^2 + [X_p^J(t - \tau)]^2) \\
&\leq \sum_{q=1}^w (r_{43} [X_q^J(t)]^2 + r_{44} [X_q^J(t - \tau)]^2), \tag{3.23}
\end{aligned}$$

$$\begin{aligned}
2 \sum_{q=1}^w X_q^J(t) \sum_{p=1}^w |c_{pq}^I| L^K |X_p^K(t - \tau)| &\leq \sum_{p=1}^w |c_{pq}^I| L^K ([X_q^J(t)]^2 + [X_p^K(t - \tau)]^2) \\
&\leq \sum_{q=1}^w (r_{45} [X_q^J(t)]^2 + r_{46} [X_q^K(t - \tau)]^2), \tag{3.24}
\end{aligned}$$

$$\begin{aligned}
2 \sum_{q=1}^w X_q^J(t) \sum_{p=1}^w |c_{pq}^K| L^I |X_p^I(t - \tau)| &\leq \sum_{q=1}^w \sum_{p=1}^w |c_{pq}^K| L^I ([X_q^J(t)]^2 + [X_p^I(t - \tau)]^2) \\
&\leq \sum_{q=1}^w (r_{47} [X_q^J(t)]^2 + r_{48} [X_q^I(t - \tau)]^2), \tag{3.25}
\end{aligned}$$

$$\begin{aligned}
2 \sum_{q=1}^w X_q^K(t) \sum_{p=1}^w |b_{pq}^R| L^K |X_p^K(t)| &\leq \sum_{q=1}^w \sum_{p=1}^w |b_{pq}^R| L^K ([X_q^K(t)]^2 + [X_p^K(t)]^2) \\
&\leq \sum_{q=1}^w (r_{49} [X_q^K(t)]^2 + r_{50} [X_q^K(t)]^2),
\end{aligned} \tag{3.26}$$

$$\begin{aligned}
2 \sum_{q=1}^w X_q^K(t) \sum_{p=1}^w |b_{pq}^K| L^R |X_p^R(t)| &\leq \sum_{p=1}^w |b_{pq}^K| L^R ([X_q^K(t)]^2 + [X_p^R(t)]^2) \\
&\leq \sum_{q=1}^w (r_{51} [X_q^K(t)]^2 + r_{52} [X_q^R(t)]^2),
\end{aligned} \tag{3.27}$$

$$\begin{aligned}
2 \sum_{q=1}^w X_q^K(t) \sum_{p=1}^w |b_{pq}^I| L^J |X_p^J(t)| &\leq \sum_{q=1}^w \sum_{p=1}^w |b_{pq}^I| L^J ([X_q^K(t)]^2 + [X_p^J(t)]^2) \\
&\leq \sum_{q=1}^w (r_{53} [X_q^K(t)]^2 + r_{54} [X_q^J(t)]^2),
\end{aligned} \tag{3.28}$$

$$\begin{aligned}
2 \sum_{q=1}^w X_q^K(t) \sum_{p=1}^w |b_{pq}^J| L^I |X_p^I(t)| &\leq \sum_{q=1}^w \sum_{p=1}^w |b_{pq}^J| L^I ([X_q^K(t)]^2 + [X_p^I(t)]^2) \\
&\leq \sum_{q=1}^w (r_{55} [X_q^K(t)]^2 + r_{56} [X_q^I(t)]^2),
\end{aligned} \tag{3.29}$$

$$\begin{aligned}
2 \sum_{q=1}^w X_q^K(t) \sum_{p=1}^w |c_{pq}^R| L^K |X_p^K(t - \tau)| &\leq \sum_{p=1}^w |c_{pq}^R| L^K ([X_q^K(t)]^2 + [X_p^K(t - \tau)]^2) \\
&\leq \sum_{q=1}^w (r_{57} [X_q^K(t)]^2 + r_{58} [X_q^K(t - \tau)]^2),
\end{aligned} \tag{3.30}$$

$$\begin{aligned}
2 \sum_{q=1}^w X_q^K(t) \sum_{p=1}^w |c_{pq}^K| L^R |X_p^R(t - \tau)| &\leq \sum_{q=1}^w \sum_{p=1}^w |c_{pq}^K| L^R ([X_q^K(t)]^2 + [X_p^R(t - \tau)]^2) \\
&\leq \sum_{q=1}^w (r_{59} [X_q^K(t)]^2 + r_{60} [X_q^R(t - \tau)]^2),
\end{aligned} \tag{3.31}$$

$$\begin{aligned}
2 \sum_{q=1}^w X_q^K(t) \sum_{p=1}^w |c_{pq}^I| L^J |X_p^J(t - \tau)| &\leq \sum_{q=1}^w \sum_{p=1}^w |c_{pq}^I| L^J ([X_q^K(t)]^2 + [X_p^J(t - \tau)]^2) \\
&\leq \sum_{q=1}^w (r_{61} [X_q^K(t)]^2 + r_{62} [X_q^J(t - \tau)]^2)
\end{aligned} \tag{3.32}$$

and

$$\begin{aligned} 2 \sum_{q=1}^w X_q^K(t) \sum_{p=1}^w |c_{pq}^J| L^I |X_p^I(t-\tau)| &\leq \sum_{q=1}^w \sum_{p=1}^w |c_{pq}^J| L^I ([X_q^K(t)]^2 + [X_p^I(t-\tau)]^2) \\ &\leq \sum_{q=1}^w (r_{63} [X_q^K(t)]^2 + r_{64} [X_q^I(t-\tau)]^2). \end{aligned} \quad (3.33)$$

Substituting (3.2)-(3.33) into (3.1) gives

$$D_t^\mu Z(t) \leq L_q^T(t) P L_q(t) - 2w\eta_1, \quad (3.34)$$

where

$$L_q(t) = \sum_{q=1}^w (X_q^R(t), X_q^I(t), X_q^J(t), X_q^K(t), X_q^R(t-\tau), X_q^I(t-\tau), X_q^J(t-\tau), X_q^K(t-\tau))^T,$$

$$P = (p_{ig})_{8 \times 8}, P_i = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1i} \\ p_{21} & p_{22} & \cdots & p_{2i} \\ \cdots & \cdots & \cdots & \cdots \\ p_{i1} & p_{i2} & \cdots & p_{ii} \end{pmatrix},$$

where

$$\begin{aligned} p_{11} &= k_{11}, p_{12} = p_{13} = p_{14} = p_{15} = p_{16} = p_{17} = p_{18} = 0, \\ p_{22} &= k_{22}, p_{23} = -0.5u_2, p_{24} = p_{25} = p_{26} = p_{27} = p_{28} = 0, \\ p_{33} &= k_{33}, p_{34} = -0.5u_3, p_{35} = p_{36} = p_{37} = p_{38} = 0, \\ p_{44} &= k_{44}, p_{45} = -0.5u_4, p_{46} = -0.5u_1, p_{47} = -0.5u_5, p_{48} = 0; \\ p_{55} &= k_{55}, p_{56} = p_{57} = p_{58} = 0, \\ p_{66} &= k_{66}, p_{67} = p_{68} = 0, p_{77} = k_{77}, p_{78} = 0, p_{88} = k_{88}. \end{aligned}$$

Because of (h_1) , we have

$$P_1 = p_{11} = k_{11} < 0. \quad (3.35)$$

Due to (h_2) , we have $k_{22} < 0$, so

$$\begin{aligned} &\begin{vmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{vmatrix} \\ &= k_{11}k_{22} \\ &> 0. \end{aligned} \quad (3.36)$$

Because of $k_{11} < 0$ and (h_3) , one has

$$= \begin{vmatrix} P_3 & & \\ k_{11} & 0 & 0 \\ 0 & k_{22} & -0.5u_2 \\ 0 & -0.5u_2 & k_{33} \end{vmatrix}$$

$$\begin{aligned}
&= k_{11}[k_{22}k_{33} - 0.25u_2^2] \\
&< 0.
\end{aligned} \tag{3.37}$$

Because of (h_4) , we have

$$\begin{aligned}
&P_4 \\
&= \begin{vmatrix} k_{11} & 0 & 0 & 0 \\ 0 & k_{22} & -0.5u_2 & 0 \\ 0 & -0.5u_2 & k_{33} & -0.5u_3 \\ 0 & 0 & -0.5u_3 & k_{44} \end{vmatrix} \\
&= k_{11} \begin{vmatrix} k_{22} & -0.5u_2 & 0 \\ -0.5u_2 & k_{33} & -0.5u_3 \\ 0 & -0.5u_3 & k_{44} \end{vmatrix} \\
&= k_{11}(k_{22}[k_{33}k_{44} - 0.25u_3^2] - 0.25u_2^2k_{44}) \\
&> 0.
\end{aligned} \tag{3.38}$$

Due to (h_5) ,

$$\begin{aligned}
&P_5 \\
&= \begin{vmatrix} k_{11} & 0 & 0 & 0 & 0 \\ 0 & k_{22} & -0.5u_2 & 0 & 0 \\ 0 & 0 & k_{33} & -0.5u_3 & 0 \\ 0 & 0 & -0.5u_3 & k_{44} & -0.5u_4 \\ 0 & 0 & 0 & -0.5u_4 & k_{55} \end{vmatrix} \\
&= k_{11}k_{22} \begin{vmatrix} k_{33} & -0.5u_3 & 0 \\ -0.5u_3 & k_{44} & -0.5u_4 \\ 0 & -0.5u_4 & k_{55} \end{vmatrix} \\
&= k_{11}k_{22}\{k_{33}[k_{44}k_{55} - 0.25u_4^2] - 0.25u_3^2k_{55}\} \\
&< 0.
\end{aligned} \tag{3.39}$$

Due to $k_{11}k_{22} > 0$ and (h_6) , we have

$$\begin{aligned}
&P_6 \\
&= \begin{vmatrix} k_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & k_{22} & -0.5u_2 & 0 & 0 & 0 \\ 0 & 0 & k_{33} & -0.5u_3 & 0 & 0 \\ 0 & 0 & 0 & k_{44} & -0.5u_4 & 0 \\ 0 & 0 & 0 & -0.5u_4 & k_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & k_{66} \end{vmatrix} \\
&= k_{11}k_{22}k_{66} \begin{vmatrix} k_{33} & -0.5u_3 & 0 \\ 0 & k_{44} & -0.5u_4 \\ 0 & -0.5u_4 & k_{55} \end{vmatrix} \\
&= k_{11}k_{22}k_{66}k_{33}[k_{44}k_{55} - 0.25u_4^2]
\end{aligned}$$

$$> 0. \quad (3.40)$$

Because $k_{77} < 0, P_6 > 0$, then

$$P_7 = k_{77}P_6 < 0. \quad (3.41)$$

Because $k_{88} < 0, P_7 < 0$, then

$$P_8 = k_{88}P_7 > 0. \quad (3.42)$$

By using Lemma 2.2, from (3.35)-(3.42), we get

$$P < 0.$$

Via Definition 2.4 and Definition 2.5, because $X_q^R(t), X_q^I(t), X_q^J(t), X_q^K(t), X_q^R(t - \tau), X_q^I(t - \tau), X_q^J(t - \tau), X_q^K(t - \tau)$ are not all zero, then

$$[L_q(t)]^T P L_q(t) < 0. \quad (3.43)$$

Substituting (3.43) into (3.34) gives

$$D_t^\mu Z(t) < -2w\eta_1. \quad (3.44)$$

Integrating (3.44) over $[0, t]$, we get

$$\begin{aligned} Z(t) &\leq Z(0) - \frac{\eta_1}{\Gamma^*(\mu)} \int_0^t (t-s)^{\mu-1} ds \\ &= Z(0) - \frac{2w\eta_1 t^\mu}{\mu\Gamma^*(\mu)}. \end{aligned} \quad (3.45)$$

Let

$$Z(0) - \frac{2w\eta_1 t^\mu}{\mu\Gamma^*(\mu)} \leq 0,$$

then

$$t \geq t_1 = \left(\frac{Z(0)\mu\Gamma^*(\mu)}{2w\eta_1} \right)^{\frac{1}{\mu}}.$$

So from (3.45), it follows that when $t \geq t_1$,

$$\lim_{t \rightarrow t_1} Z(t) = 0, Z(t) = 0, t \geq t_1.$$

Namely

$$\begin{aligned} \lim_{t \rightarrow t_1} |\alpha_q^\lambda(t) - \beta_q^\lambda(t)| &= 0, \lambda = R, I, J, K; \\ |\alpha_q^\lambda(t) - \beta_q^\lambda(t)| &= 0, \lambda = R, I, J, K, t \geq t_1. \end{aligned}$$

This completes the arguments of Theorem 3.1.

Theorem 3.2. Under Assumption 1, then System (2.2) and the System (2.3) can realize the FTSN by employing the controllers in (2.6) if the following conditions hold:

(h_7)

$$k_{11}^* = -a_q + \sum_{p=1}^w \left[|b_{qp}^R| + |b_{qp}^I| + |b_{qp}^J| + |b_{qp}^K| \right] L^R < 0;$$

(h₈)

$$k_{22}^* = -a_q + \sum_{p=1}^w [|b_{qp}^I| + |b_{qp}^R| + |b_{qp}^J| + |b_{qp}^K|] L^I < 0;$$

(h₉)

$$k_{22}^* k_{33}^* > 0.25 \gamma_6^2;$$

(h₁₀)

$$k_{44}^* = -a_q + \sum_{p=1}^w (|b_{qp}^K| + |b_{qp}^J| + |b_{qp}^R| + |b_{qp}^I|) L^K < 0;$$

(h₁₁)

$$\frac{1}{8} \gamma_6 \gamma_7^2 + 0.25 \gamma_1 \gamma_6^2 - \gamma_1 n_{22} n_{33} < 0;$$

(h₁₂)

$$0.25 \gamma_9^2 < \gamma_3 \gamma_4,$$

with $k_{33}^* = -a_q + \sum_{p=1}^w (|b_{qp}^K| + |b_{qp}^R| + |b_{qp}^I| + |b_{qp}^J|) L^J$, and the finite-time

$$t_2 = \left[\frac{Z(0) \mu \Gamma^*(\mu)}{2w\eta_2} \right]^{\frac{1}{\mu}}, Z_1(0) = \sum_{q=1}^w [|X_p^R(0)| + |X_p^I(0)| + |X_p^J(0)| + |X_p^K(0)|].$$

Proof. Without loss of generalization, we assume that $X_q^R(t), X_q^I(t), X_q^J(t), X_q^K(t), |X_q^R(t - \tau)|, |X_q^I(t - \tau)|, |X_q^J(t - \tau)|, |X_q^K(t - \tau)|$ are not all zero. Construct a Lyapunov function:

$$Z_1(t) = \sum_{q=1}^w [|X_q^R(t)| + |X_q^I(t)| + |X_q^J(t)| + |X_q^K(t)|].$$

Via System (2.4), we obtain the following inequality by using Assumption 1:

$$\begin{aligned} & D_t^\mu [Z_1(t)] \\ & \leq \sum_{q=1}^w \left\{ \text{sign}[X_q^R(t)] \left[-a_q X_q^R(t) + \sum_{p=1}^w (b_{pq}^R F_p^R(X_p^R(t)) + b_{pq}^I F_p^I(X_p^I(t)) + b_{pq}^K F_p^K(X_p^K(t)) \right. \right. \\ & \quad - b_{pq}^J F_p^J(X_p^J(t)) + c_{pq}^I F_p^R(X_p^R(t - \tau)) + c_{pq}^R F_p^I(X_p^I(t - \tau)) - c_{pq}^K F_p^J(X_p^J(t - \tau)) + c_{pq}^J \times \\ & \quad \left. \left. F_p^K(X_p^K(t - \tau)) + l_{1q}(t) \right] + \text{sign}[X_q^I(t)] \left[-a_q X_q^I(t) + \sum_{p=1}^w (b_{pq}^I F_p^R(X_p^R(t)) + F_p^I(X_p^I(t)) \right. \right. \\ & \quad \times b_{pq}^R - b_{pq}^K F_p^J(X_p^J(t)) + b_{pq}^J F_p^K(X_p^K(t))) - b_{pq}^K F_p^J(X_p^J(t - \tau)) + c_{pq}^I F_p^R(X_p^R(t - \tau)) + c_{pq}^R \\ & \quad \times F_p^I(X_p^I(t - \tau)) - c_{pq}^K F_p^J(X_p^J(t - \tau)) + c_{pq}^J F_p^K(X_p^K(t - \tau))) + l_{2q}(t) \left] + \text{sign}[X_q^J(t)] \right. \\ & \quad \times \left[-a_q X_q^J(t) + \sum_{p=1}^w (F_p^R(X_p^R(t)) b_{pq}^J + b_{pq}^R F_p^J(X_p^J(t)) - b_{pq}^I F_p^K(X_p^K(t)) + b_{pq}^K F_p^I(X_p^I(t)) \right. \\ & \quad \left. \left. + c_{pq}^J F_p^R(X_p^R(t - \tau)) + c_{pq}^R F_p^J(X_p^J(t - \tau)) - c_{pq}^I F_p^K(X_p^K(t - \tau)) - c_{pq}^K F_p^I(X_p^I(t - \tau)) \right) + \right. \end{aligned}$$

$$\begin{aligned}
& l_{3q}(t) + \text{sign}[X_q^K(t)] \left[-a_q X_q^K(t) + \sum_{p=1}^w \left(b_{pq}^R F_p^K(X_p^K(t)) + b_{pq}^K F_p^R(X_p^R(t)) + b_{pq}^I F_p^J(X_p^J(t)) \right. \right. \\
& \quad \left. \left. - b_{pq}^J F_p^I(X_p^I(t)) + c_{pq}^R F_p^K(X_p^K(t-\tau)) + c_{pq}^K F_p^R(X_p^R(t-\tau)) + c_{pq}^I F_p^J(X_p^J(t-\tau)) - c_{pq}^J \times \right. \right. \\
& \quad \left. \left. F_p^I(X_p^I(t-\tau)) + l_{4q}(t) \right) \right] \\
\leq & \sum_{q=1}^w \left\{ -a_q |X_q^R(t)| + \sum_{p=1}^w \left(|b_{pq}^R| L^R |X_p^R(t)| + |b_{pq}^I| L^I |X_p^I(t)| + |b_{pq}^K| L^K |X_p^K(t)| + |b_{pq}^K| L^J \times \right. \right. \\
& |X_p^J(t)| + |c_{pq}^I| L^R |X_p^R(t-\tau)| + |c_{pq}^R| L^I |X_p^I(t-\tau)| + |c_{pq}^K| L^J |X_p^J(t-\tau)| + |c_{pq}^J| L^K \times \\
& |X_p^K(t-\tau)| \left. \right) - \gamma_1 |X_q^R(t-\tau)| - \gamma_2 |X_q^I(t-\tau)| - \gamma_3 |X_q^J(t-\tau)| - \gamma_4 |X_q^K(t-\tau)| - a_q |X_q^I(t)| \\
& + \sum_{p=1}^w \left(|b_{pq}^I| L^R |X_p^R(t)| + |b_{pq}^R| L^I |X_p^I(t)| + |b_{pq}^J| L^K |X_p^K(t)| + |b_{pq}^K| L^J |X_p^J(t)| + |b_{pq}^K| L^J |X_p^J(t-\tau)| \right. \\
& \left. + |c_{pq}^I| L^R |X_p^R(t-\tau)| + |c_{pq}^R| L^I |X_p^I(t-\tau)| + |c_{pq}^K| L^J |X_p^J(t-\tau)| + |c_{pq}^J| L^K |X_p^K(t-\tau)| \right) - \eta_2 - \\
& \times a_q |X_q^J(t)| + \sum_{p=1}^w \left(|b_{pq}^I| L^R |X_p^R(t)| + |b_{pq}^R| L^J |X_p^J(t)| + |b_{pq}^I| L^K |X_p^K(t)| + |b_{pq}^K| L^I |X_p^I(t)| + |c_{pq}^I| \right. \\
& L^R |X_p^R(t-\tau)| + |c_{pq}^R| L^J |X_p^J(t-\tau)| + |c_{pq}^I| L^K |X_p^K(t-\tau)| + |c_{pq}^K| L^I |X_p^I(t-\tau)| \left. \right) - \gamma_6 \times \\
& [X_q^I(t) X_q^J(t)]^{0.5} - \gamma_7 [X_q^J(t) X_q^R(t-\tau)]^{0.5} - a_q |X_q^K(t)| + \sum_{p=1}^w \left(|b_{pq}^R| L^K |X_p^K(t)| + |b_{pq}^K| L^R |X_p^R(t)| \right. \\
& \left. + |b_{pq}^I| L^J |X_p^J(t)| + |b_{pq}^J| L^I |X_p^I(t)| + |c_{pq}^K| L^K |X_p^K(t-\tau)| + |c_{pq}^K| L^R |X_p^R(t-\tau)| + |c_{pq}^I| L^J \times \right. \\
& |X_p^J(t-\tau)| + |c_{pq}^J| L^I |X_p^I(t-\tau)| \left. \right) - \gamma_8 |X_q^R(t-\tau)|^{0.5} |X_q^I(t-\tau)|^{0.5} |X_q^J(t-\tau)|^{0.5} \\
& \times |X_q^K(t-\tau)|^{0.5} \left. \right\} \\
\leq & \sum_{q=1}^w \left\{ -a_q |X_q^R(t)| + \sum_{p=1}^w \left(|b_{qp}^R| L^R |X_q^R(t)| + |b_{qp}^I| L^I |X_q^I(t)| + |b_{qp}^K| L^K |X_q^K(t)| + |b_{qp}^K| L^J |X_q^J(t)| \right. \right. \\
& \left. \left. + |c_{qp}^I| L^R |X_q^R(t-\tau)| + |c_{qp}^R| L^I |X_q^I(t-\tau)| + |c_{qp}^K| L^J |X_q^J(t-\tau)| + |c_{qp}^J| L^K |X_q^K(t-\tau)| \right) - \gamma_1 \times \right. \\
& |X_q^R(t-\tau)| - \gamma_2 |X_q^I(t-\tau)| - \gamma_3 |X_q^J(t-\tau)| - \gamma_4 |X_q^K(t-\tau)| - a_q |X_q^I(t)| + \sum_{p=1}^w \left(|b_{qp}^I| L^R |X_q^R(t)| \right. \\
& \left. + |b_{qp}^R| L^I |X_q^I(t)| + |b_{qp}^J| L^K |X_q^K(t)| + |b_{qp}^K| L^J |X_q^J(t)| + |b_{qp}^K| L^J |X_q^J(t-\tau)| + |c_{qp}^I| L^R |X_q^R(t-\tau)| + \right. \\
& \left. |c_{qp}^R| L^I |X_q^I(t-\tau)| + |c_{qp}^K| L^J |X_q^J(t-\tau)| + |c_{qp}^J| L^K |X_q^K(t-\tau)| \right) - \eta_2 - a_q |X_q^J(t)| + \sum_{p=1}^w \left(|b_{qp}^J| L^R \right. \\
& \times |X_q^R(t)| + |b_{qp}^R| L^I |X_q^I(t)| + |b_{qp}^I| L^K |X_q^K(t)| + |b_{qp}^K| L^I |X_q^I(t)| + |c_{qp}^J| L^R |X_q^R(t-\tau)| + |c_{qp}^R| L^I \times \\
& |X_q^I(t-\tau)| + |c_{qp}^I| L^K |X_q^K(t-\tau)| + |c_{qp}^K| L^I |X_q^I(t-\tau)| \left. \right) - \gamma_6 |X_q^I(t) X_q^J(t)|^{0.5} - \gamma_7 |X_q^J(t)|^{0.5} \times \\
& |X_q^R(t-\tau)|^{0.5} - a_q |X_q^K(t)| + \sum_{p=1}^w \left(|b_{qp}^R| L^K |X_q^K(t)| + |b_{qp}^K| L^R |X_q^R(t)| + |b_{qp}^I| L^J |X_q^J(t)| + |b_{qp}^J| L^I \times \right.
\end{aligned}$$

$$\begin{aligned}
& |X_q^I(t)| + |c_{qp}^K|L^K|X_q^K(t-\tau)| + |c_{qp}^R|L^R|X_q^R(t-\tau)| + |c_{qp}^J|L^J|X_q^J(t-\tau)| + |c_{qp}^I|L^I|X_q^I(t-\tau)| \\
& - \gamma_9|X_q^J(t-\tau)|^{0.5}|X_q^K(t-\tau)|^{0.5} \} \\
= & H_q(t)NH_q^T(t) - w\eta_2,
\end{aligned} \tag{3.46}$$

with $H_q(t) = [\sqrt{|X_q^R(t)|}, \sqrt{|X_q^I(t)|}, \sqrt{|X_q^J(t)|}, \sqrt{|X_q^K(t)|}, \sqrt{|X_q^R(t-\tau)|}, \sqrt{|X_q^I(t-\tau)|}, \sqrt{|X_q^J(t-\tau)|}, \sqrt{|X_q^K(t-\tau)|}]$, $N = (n_{ij})_{8 \times 8}$,

$$N_i = \begin{bmatrix} n_{11} & n_{12} & \cdots & n_{1i} \\ n_{21} & n_{22} & \cdots & n_{2i} \\ \cdots & \cdots & \cdots & \cdots \\ n_{i1} & n_{i2} & \cdots & n_{ii} \end{bmatrix},$$

and

$$\begin{aligned}
n_{11} &= k_{11}^*, \\
n_{12} &= n_{13} = n_{14} = n_{15} = n_{16} = n_{17} = n_{18} = 0; \\
n_{22} &= k_{22}^*, \\
n_{23} &= -0.5\gamma_6, n_{24} = n_{25} = n_{26} = n_{27} = n_{28} = 0, n_{33} = k_{33}^*; \\
n_{34} &= 0, n_{35} = -0.5\gamma_7, n_{36} = n_{37} = n_{38} = 0; \\
n_{44} &= k_{44}^*, n_{45} = n_{46} = n_{47} = n_{48} = 0; \\
n_{55} &= -\gamma_1, n_{56} = 0, n_{57} = n_{58} = 0, n_{66} = -\gamma_2; \\
n_{67} &= n_{68} = 0, n_{77} = -\gamma_3, n_{78} = -0.5\gamma_9, n_{88} = -\gamma_4.
\end{aligned}$$

From (h_7) , one has

$$N_1 = n_{11} < 0. \tag{3.47}$$

Because of (h_8) , one has

$$N_2 = \begin{vmatrix} n_{11} & 0 \\ 0 & n_{22} \end{vmatrix} = n_{11}n_{22} > 0. \tag{3.48}$$

From (h_9) , we obtain

$$N_3 = \begin{vmatrix} n_{11} & 0 & 0 \\ 0 & n_{22} & -0.5\gamma_6 \\ 0 & -0.5\gamma_6 & n_{33} \end{vmatrix} = n_{11}[n_{22}n_{33} - 0.25\gamma_6^2] < 0 \tag{3.49}$$

and

$$N_4 = \begin{vmatrix} n_{11} & 0 & 0 & 0 \\ 0 & n_{22} & -0.5\gamma_6 & 0 \\ 0 & -0.5\gamma_6 & n_{33} & 0 \\ 0 & 0 & 0 & n_{44} \end{vmatrix} = n_{11}n_{44}(n_{22}n_{33} - 0.25\gamma_6^2) > 0. \tag{3.50}$$

Because of (h_{11}) , then

$$\begin{aligned}
 N_5 &= \begin{vmatrix} n_{11} & 0 & 0 & 0 & 0 \\ 0 & n_{22} & -0.5\gamma_6 & 0 & 0 \\ 0 & -0.5\gamma_6 & n_{33} & 0 & -0.5\gamma_7 \\ 0 & 0 & 0 & n_{44} & 0 \\ 0 & 0 & -0.5\gamma_7 & 0 & -\gamma_1 \end{vmatrix} \\
 &= n_{11} \begin{vmatrix} n_{22} & -0.5\gamma_6 & 0 & 0 \\ -0.5\gamma_6 & n_{33} & 0 & -0.5\gamma_7 \\ 0 & 0 & n_{44} & 0 \\ 0 & -0.5\gamma_7 & 0 & n_{55} \end{vmatrix} \\
 &= n_{11}n_{44} \begin{vmatrix} n_{22} & -0.5\gamma_6 & 0 \\ -0.5\gamma_6 & n_{33} & -0.5\gamma_7 \\ 0 & -0.5\gamma_7 & -\gamma_1 \end{vmatrix} \\
 &= n_{11}n_{44} \left[\frac{\gamma_6\gamma_7^2}{8} + 0.25\gamma_1\gamma_6^2 - \gamma_1n_{22}n_{33} \right] \\
 &< 0
 \end{aligned} \tag{3.51}$$

and

$$N_6 = -\gamma_2N_5 > 0; \tag{3.52}$$

$$N_7 = \gamma_2\gamma_3N_5 < 0. \tag{3.53}$$

Due to (h_{10}) , (h_{11}) , then

$$\begin{aligned}
 N_8 &= \begin{vmatrix} n_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & n_{22} & -0.5\gamma_6 & 0 & 0 & /0 & 0 & 0 \\ 0 & -0.5\gamma_6 & n_{33} & 0 & -0.5\gamma_7 & 0 & 0 & 0 \\ 0 & 0 & 0 & n_{44} & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.5\gamma_7 & 0 & -\gamma_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\gamma_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\gamma_3 & -0.5\gamma_9 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.5\gamma_9 & -\gamma_4 \end{vmatrix} \\
 &= -\gamma_4N_7 - 0.25\gamma_9^2N_6 \\
 &= N_5[0.25\gamma_2\gamma_9^2 - \gamma_2\gamma_3\gamma_4] \\
 &= n_{11}n_{44} \left[\frac{\gamma_6\gamma_7^2}{8} + 0.25\gamma_1\gamma_6^2 - \gamma_1n_{22}n_{33} \right] [0.25\gamma_2\gamma_9^2 - \gamma_2\gamma_3\gamma_4] \\
 &> 0.
 \end{aligned} \tag{3.54}$$

By using Lemma 2.2 from (3.47)–(3.54), we obtain

$$N < 0.$$

Therefore

$$[H_q(t)]^T N H_q(t) < 0. \quad (3.55)$$

Substituting (3.55) into (3.46) gives

$$D_t^\mu Z_1(t) < -w\eta_2. \quad (3.56)$$

Integrating (3.56) over $[0, t]$, we get

$$\begin{aligned} Z_1(t) &\leq Z_1(0) - \frac{\eta_2}{\Gamma^*(\mu)} \int_0^t (t-s)^{\mu-1} ds \\ &= Z_1(0) - \frac{2w\eta_2 t^\mu}{\mu\Gamma^*(\mu)}. \end{aligned} \quad (3.57)$$

Let

$$Z_1(0) - \frac{2w\eta_2 t^\mu}{\mu\Gamma^*(\mu)} \leq 0,$$

then

$$t \geq t_2 = \left(\frac{Z_1(0)\mu\Gamma^*(\mu)}{2w\eta_2} \right)^{\frac{1}{\mu}}.$$

So from (3.57), it follows that when $t \geq t_2$,

$$\lim_{t \rightarrow t_2} Z_1(t) = 0, Z_1(t) = 0, t \geq t_2.$$

So

$$\lim_{\rightarrow t_2} |\alpha_q^\lambda(t) - \beta_q^\lambda(t)| = 0, \lambda = R, I, J, K;$$

$$|\alpha_q^\lambda(t) - \beta_q^\lambda(t)| = 0, \lambda = R, I, J, K, t \geq t_2.$$

This completes the arguments of Theorem 3.2.

Remark 1. Up to now, researchers have always applied LMI [11], FTST [12,14], fractional inequalities [13], Laplace transforms [15] and Lyapunov direct method [16] to studying the FTSN for the considered MSFONNS. In this study, a new method is introduced to study the FTSN for MSQVFONNS.

Remark 2. So far, the method (the negative definition of a matrix combined with the properties of the determinant) has not been applied by scholars in integer-order neural network synchronization problems.

4. Examples

Two examples are given for verifying the correctness of our results.

Example 4.1. Consider System (2.2), System (2.3) via the controllers (2.5–2.8) for $q, w = 1, 2$, where $\mu = 0.4, a_1 = a_2 = 7, L^R = 1, L^I = 0.5, L^J = 1.2, L^K = 0.7, b_{11} = 0.3 - 0.2i - 0.1j - 0.4k, b_{12} = -0.1 + 0.1i + 0.5j + 0.2k, b_{21} = -0.5 + 0.3i - 0.2j - 0.1k, b_{22} = 0.1 - 0.1i + 0.3j - 0.2k, c_{11} = -0.3 + 0.1i + 0.5j + 0.2k, c_{12} = 0.3 - 0.2i - 0.4j - 0.1k, c_{21} = -0.2 + 0.3i + 0.2j - 0.4k, c_{22} = 0.4 + 0.1i - 0.3j + 0.2k, r_1 = 1 - 0.1i + 0.5j - 0.3k, r_2 = -0.1 + 0.2i - 0.5j + 0.4k, \eta_1 = 4, u_1 = 1, u_2 = -1, u_3 = 2.5, u_4 = -0.5, u_5 =$

1.5, $k_1 = 2, k_2 = 3, k_3 = 2.5, k_4 = 4, F_p(\alpha_p) = |\alpha_p - 1| + 0.5|\alpha_p + 1| + 1.2|\alpha_p - 2| + 0.7|\alpha_p + 1.5|k, \tau = 1$.

The finite-time of synchronization $t_1 = \left[\frac{Z(0)\mu\Gamma^*(\mu)}{2w\eta_1} \right]^{\frac{1}{\mu}} = 16.76$. Then,

(h_1)

$$k_{11} = -7.69 < 0, k_{66} = -3.39 < 0, k_{77} = -2.36 < 0;$$

(h_2)

$$k_{22} = -9.18 < 0, k_{88} = -6.04 < 0;$$

(h_3)

$$64.6272 = k_{33}k_{22} > 0.25u_2^2 = 0.25;$$

(h_4)

$$-533.6949 = k_{22}[k_{33}k_{44} - 0.25u_3^2] < 0.25u_2^2k_{44} = -2.12;$$

(h_5)

$$-101.0486 = k_{33}[k_{44}k_{55} - 0.25u_4^2] < 0.25u_3^2k_{55} = -2.6563;$$

(h_6)

$$k_{66}k_{33}[k_{44}k_{55} - 0.25u_4^2] = 397.1212 > 0.$$

Thus, by Theorem 3.1, system (2.2) and System (2.3) can realize the FTSN via the controller (2.5–2.8). The methods and controllers designed in our example are very different from those in past articles [11, 12, 14–16]. Thus, the result in the example cannot be tested the methods therein.

The curves of master system and slave system are shown in Figure 1, the curves of the error system are shown in Figure 2.

Example 4.2. Consider System (2.2), System (2.3) applying the controllers (2.9–2.12) for $q, w = 1, 2$, where $\mu = 0.7, a_1 = a_2 = 10, L^R = 0.8, L^I = 1.5, L^J = 0.5, L^K = 1, b_{11} = b_{21} = -0.2 + 0.5i + 0.3j - k, b_{12} = b_{22} = 0.3 - 0.2i - 0.1j - 3k, c_{11} = 0.4 - 2i + 0.2j - 1.5k, c_{12} = 0.2 + i - 0.1j + 2k, c_{21} = -0.1 + 3i + 0.2j - k, c_{22} = 0.3 - i + 0.3j + k, r_1 = 0.6 - 0.3i + 0.2j - 0.1k, r_2 = -0.5 + 0.1i - 0.4j + 0.3k, \eta_2 = 1, \gamma_1 = 3, \gamma_2 = 1.5, \gamma_3 = 2, \gamma_4 = 2.5, \gamma_6 = 1, \gamma_7 = 0.5, \gamma_9 = 3.5, F_p(\alpha_p) = 2|\alpha_p + 0.5| + 1.5|\alpha_p - 1| + 0.5|\alpha_p - 1.5|j + |\alpha_p + 2|k, \tau = 2$. The finite-time of synchronization: $t_2 = \left[\frac{Z_1(0)\mu\Gamma^*(\mu)}{2w\eta_2} \right]^{\frac{1}{\mu}} = 26.07$. Then,

(h_7)

$$k_{11}^* = -5.52 < 0;$$

(h_8)

$$n_{22} = k_{22}^* = -1.6 < 0;$$

(h_9)

$$11.52 = k_{22}^*k_{33}^* > 0.25\gamma_6^2 = 0.25;$$

(h_{10})

$$k_{44}^* = -4.4 < 0;$$

(h_{11})

$$n_{33} = k_{33}^* = -7.2, \frac{1}{8}\gamma_6\gamma_7^2 + 0.25\gamma_1\gamma_6^2 - \gamma_1n_{22}n_{33} = -33.7788 < 0;$$

(h_{12})

$$3.0625 = 0.25\gamma_9^2 < \gamma_3\gamma_4 = 5.$$

Thus, by Theorem 3.2, System (2.2) and System (2.3) can realize the FTSN via the controller (2.9–2.12). Because the argument method and the controllers in our example all are very different, of course, the synchronization result in this example cannot be verified by methods in past papers [11, 12, 14,

15]. The curves of the corresponding master system and slave system are shown in Figure 3 and the curves of the corresponding error system are shown in Figure 4.

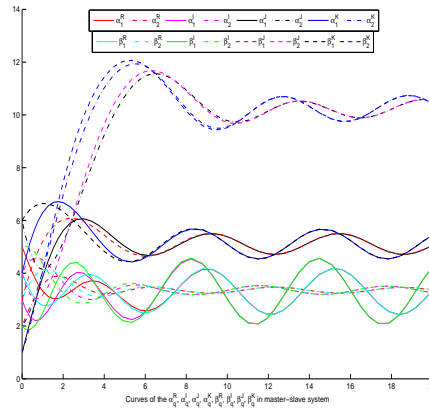


Figure 1. The figure of the status variables of the master-slave system in Example.4.1.

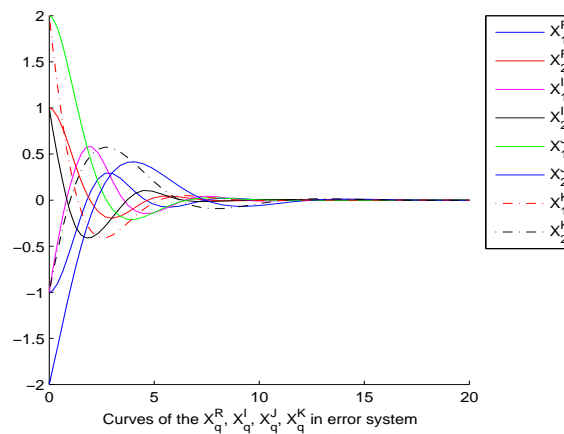


Figure 2. The figure of the status variables of the error system in Example.4.1.

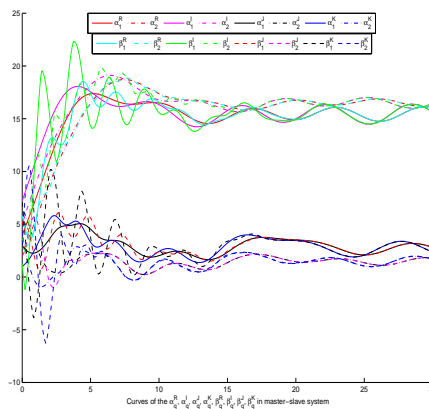


Figure 3. The figure of the status variables of the master-slave system in Example.4.2.

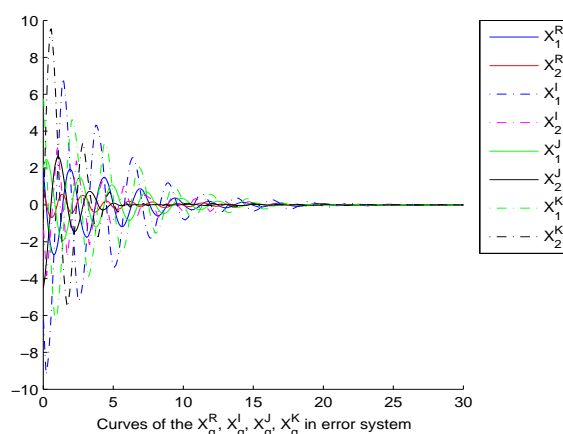


Figure 4. The figure of the status variables of the error system in Example.4.2.

5. Conclusions

In the study, the FTSN for the considered MSQVFONNs has been investigated. Without using the past study methods of the FTSN such as LMI and FTST, by using the negative definition of matrix and the properties of the determinant, two novel criteria on the FTSN for the considered MSFOQVNNs have been obtained. In our article, new methods are introduced to investigate the FTSN for the MSFONNs: the negative definition of a matrix is introduced to study the FTSN for the MSFONNs, and the properties of the determinant are cited to study the FTSN for the MSFONNs. In the near future, we will study the FTSN for master-slave discrete-time MSFONNs.

Use of Generative-AI tools declaration

The author(s) declare(s) they have used Artificial Intelligence (AI) tools by transforming Chinese into English.

Author contributions

Zhimin Wan writes the paper, Zhengqiu Zhang check wole paper and dram figures in the paper, Zhenbo Cheng provide the fund support.

Conflict of interest

No conflict of interest exits in the submission of this paper, which is approved by all authors for publication.

Acknowledgments

This article was supported by the Hunan Provincial Natural Science Foundation (2025JJ50583) and the Natural Science Foundation of Fujian Province (2022 J011240).

References

1. X. Y. Li, Z. S. Cheng, Y. M. Xin, Y. Shang, Dynamic behavior of three-layer fractional-order neural networks with multiple delays, *Cogn. Comput.*, **17** (2025), <https://doi.org/10.1007/s12559-025-10411-7>
2. M. S. Ali, G. Narayanan, V. Shekher, A. Alsaedi, B. Ahmad, Global Mittag-Leffler stability analysis of impulsive fractional order complex-valued BAM neural networks with time varying delays, *Commun. Nonlinear Sci.*, **83** (2020), 105088. <https://doi.org/10.1016/j.cnsns.2019.105088>
3. Z. H. Deng, J. Zhang, Z. Y. Yang, Quasi-uniform stability for fractional-order fuzzy neural networks with uncertain proportional delay, *J. Appl. Math. Comput.*, **70** (2024), 639–656. <https://doi.org/10.1007/s12190-023-01978-1>
4. S. Jose, V. Parthiban, Synchronization of fractional order time delayed neural networks using matrix measure approach, *Eur. Phys. J. Spec. Top.*, **234** (2025), 1635–1644. <https://doi.org/10.1140/epjs/s11734-024-01300-5>
5. Y. Wang, J. M. Li, The global Mittag-Leffler synchronization problem of Caputo fractional-order inertial memristive neural networks with time-varying delays, *Soft Comput.*, **28** (2024), 8247–8257. <https://doi.org/10.1007/s00500-024-09627-w>
6. L. B. Wang, G. G. Xu, Quasi-synchronization of fractional-order complex-value BAM neural networks with time delays and discontinuous activations, *Comput. Appl. Math.*, **43** (2024), 391. <https://doi.org/10.1007/s40314-024-02903-3>
7. R. Kiruthika, A. Manivannan, Robust sampled-data synchronization of chaotic fractional variable order neural networks with time delays, *Eur. Phys. J. Spec. Top.*, **234** (2025), 1645–1658. <https://doi.org/10.1140/epjs/s11734-024-01242-y>
8. T. Y. Jing, T. Y. He, Synchronization of fractional-order neural networks via intermittent quantized control: Optimal Algorithm, *Int. J. Theor. Phys.*, **63** (2024), 164. <https://doi.org/10.1007/s10773-024-05701-z>
9. J. M. He, L. J. Pei, Function matrix projective synchronization for unknown and delayed fractional-order neural network, *Neural Comput. Appl.*, **70** (2024), 639–656.
10. J. P. Yang, H. L. Li, J. K. Yang, L. Zhang, H. J. Jiang, Quasi-synchronization and complete synchronization of fractional-order fuzzy BAM neural networks via nonlinear control, *Neural Process. Lett.*, **54** (2022), 3303–3319. <https://doi.org/10.1007/s11063-022-10769-x>
11. R. Kiruthika, A. Manivannan, Master-slave finite-time synchronization of chaotic fractional-order neural networks under hybrid sampled-data control: An LMI Approach, *Neural Process. Lett.*, **57** (2025), 15. <https://doi.org/10.1007/s11063-025-11733-1>
12. Z. W. Wu, X. B. Nie, Finite-time synchronization of fractional-order quaternion-valued delayed Cohen-Grossberg neural networks, *Neural Process. Lett.*, **55** (2023), 12255–12271.
13. F. F. Du, J. G. Lu, Novel methods of finite-time synchronization of fractional-order delayed memristor-based Cohen CGrossberg neural networks, *Nonlinear Dynam.*, **111** (2023), 18985–19001. <https://doi.org/10.1007/s11071-023-08880-2>
14. S. Kanakalakshmi, R. Sakthivel, S. A. Karthick, C. Wang, A. Leelamani, Finite-time non-fragile control for synchronization of fractional-order stochastic neural networks, *Soft Comput.*, **27** (2023), 2453–2463. <https://doi.org/10.1007/s00500-022-07692-7>

15. H. Y. Yan, Y. H. Qiao, L. J. Duan, J. Miao, New inequalities to finite-time synchronization analysis of delayed fractional-order quaternion-valued neural networks, *Neural Comput. Appl.*, **34** (2022), 9919–9930. <https://doi.org/10.1007/s00521-022-06976-1>
16. W. W. Zhang, H. Y. Zhao, C. L. Sha, Y. Wang, Finite time synchronization of delayed quaternion valued neural networks with fractional order, *Neural Process. Lett.*, **53** (2021), 3607–3618. <https://doi.org/10.1007/s11063-021-10551-5>
17. Advanced Algebra, a trial textbook for higher education, compiled by the the all members of Algebra Group of the teaching and research section of Geometry and Algebra, Department of Mathematics Mechanics, Beijing University, published by Higher Education Press, 1978.3.
18. Q. Peng, J. G. Jian, Asymptotic synchronization of second fractional order fuzzy neural networks with impulsive effects, *Chaos, Soliton. Fract.*, **168** (2023), 113150. <https://doi.org/10.1016/j.chaos.2023.113150>
19. A. Kilbas, H. Srivastava, J. Trujillo, Theory and application of fractional differential equations, 2026, Elsevier, New York.



©2026 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (<http://creativecommons.org/licenses/by/4.0>)