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Research article

Laplacian controllability analysis of chain graph based on minimal perfect critical vertex set

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Abstract: The aim is to find all of the minimum leader set for chain graphs. By analyzing the subgraph structure rather than the eigenvectors, minimal perfect critical vertex sets (MPCS) of chain graph are found. It is proved that there is one and only one MPCS in chain graph (CG) (1, 1, ⋯, 1; 1, 1, ⋯, 1), and it is a 4-MPCS. Based on this, all the minimum leader sets of the chain graph are given.

Keywords: minimal perfect critical vertex set; chain graph; Laplacian controllability; minimum leader set; subgraph structure
Mathematics Subject Classification: 05C50, 15A18, 93A16

1. Introduction

1.1. Motivation

Controllability is the primary and fundamental issue in the field of network control, and selecting as few control vertices as possible is of great significance for improving control efficiency and reducing costs. Various kinds of controllability have been proposed, For example, the leader vertices may be virtual vertices or specific individuals in the swarms. The matrix used may be an adjacency matrix or a Laplacian matrix. In this paper, Laplacian controllability means that the matrix used in Kalman controllability is the Laplacian matrix of an undirected graph, which was first proposed by Tanner in 2004 [1]. Let $\boldsymbol{L}$ be the Laplacian matrix of undirected graph $G=(V,E)$, $F\subset V$ be the followers and $\bar{F}=V\backslash F$ be the leaders and the Laplacian matrix $\boldsymbol{L}=\begin{bmatrix}\boldsymbol{L}_{FF} & \boldsymbol{L}_{F\bar{F}}\\ \boldsymbol{L}_{\bar{F}F} & \boldsymbol{L}_{\bar{F}\bar{F}}\end{bmatrix}$. Let $\boldsymbol{A}=\boldsymbol{L}_{FF}$ and $\boldsymbol{B}=\boldsymbol{L}_{F\bar{F}}$. If $\boldsymbol{C}=[\boldsymbol{B},\boldsymbol{A}\boldsymbol{B},\boldsymbol{A}^2\boldsymbol{B},\cdots,\boldsymbol{A}^{n-1}\boldsymbol{B}]$, where $n=|F|$ has full row rank, then $\bar{F}$ is a leader set which makes graph G Laplacian controllable, referred to as controllable leaders. The controllable leader set with the fewest number of vertices is called the minimum leader set. Two important issues in the study of Laplacian controllability of networks are as follows: First, how many vertices are there in the minimum leader set? Second, how to find the minimum leader set and how many types of the minimum leader set are there?

In the research results of Laplace controllability, Proposition 2.1 (2) in Reference [2] and Proposition 1 (2) in Reference [3] both show that the key to influencing the leading set does not lie in the rank of the Laplacian matrix, in the magnitude of the eigenvalues, nor in whether the eigenvectors are linearly dependent or not. Instead, it lies in the positions of the zero components among all the eigenvectors. For multiple eigenvalues, the situation regarding the zero components of their eigenvectors is complex. This is because after taking linear combinations of different eigenvectors corresponding to the same eigenvalue, new zero components may emerge, and both the number and positions of these new zero components are uncertain. In 2023, we proposed the concept of the MPCS [4] and addressed the problem of selecting the minimum leader set from the subgraph structure prospect.

This paper will study the minimum leader set of chain graphs based on the MPCS. A chain graph is a graph in which no vertex-induced subgraph contains C_3, C_5, or $2K_2$. The chain graph in Figure 1 is denoted as $CG\ (m_1, m_2, \cdots, m_h; n_1, n_2, \cdots, n_h)$. A chain graph is also known as a difference graph or a double nested graph. In a connected bipartite graph with a given number of vertices and edges, the graph with the largest spectral radius of the adjacency matrix is the chain graph [5]. The chain graph is also closely related to the threshold graph. In 2019, Shun-Pin Hsu studied the controllability of the threshold [6]. Due to these important properties, chain graphs occupy an important position in graph spectrum theory. In 2023, Alazemi et al. [7] studied the necessary and sufficient conditions for the divisor matrix (quotient matrix) of a chain graph to have no repeated eigenvalues. They proved that all the vertices in the minimum leader sets of two types of chain graphs,

$$\mathrm{CG}(1,2,\ldots,2;2,2,\cdots,2)\text{、}\mathrm{CG}(1,1,\cdots,1,2,2,1,\cdots,1;1,1,\cdots,1),$$

where the two cells with vertex number 2 are located in the “middlemost” positions, that is, at $\left\lceil\frac{h-1}{2}\right\rceil$ and $\left\lceil\frac{h+1}{2}\right\rceil$, and where h is the number of cells in U come from twin vertex pairs. The quotient matrices of these two types of graphs have no repeated eigenvalues and contain pairs of twins. Therefore, Alazemi et al. conjectured that for a chain graph whose quotient matrix has only simple eigenvalues and each cell contains at most 2 vertices, the vertices in its minimum leader set all come from twin vertex pairs. The fact that the quotient matrix has no repeated eigenvalues does not imply that the Laplacian matrix of the graph also has no repeated eigenvalues. For a graph without repeated eigenvalues, the minimum leader vertex set can be obtained by using an algebraic method to check the eigenvectors of each eigenvalue one by one. In 2024, Alshamary et al. [8] studied the following chain graph:

$$CG(k,1,1,\cdots,1;1,1,\cdots,1)\ (k\geq 2).$$

The quotient matrix of this chain graph also has no repeated eigenvalues. The chain graphs studied in References [7,8] all contain twin vertices, and the vertices in the minimum leader sets all come from twin vertices. This verifies Alazemiq’s conjecture from one side. This paper will further study chain graphs without twins. In such graphs, because of the absence of twins, the leader set cannot be solved using the methods in references [7,8].

1.2. The main contributions

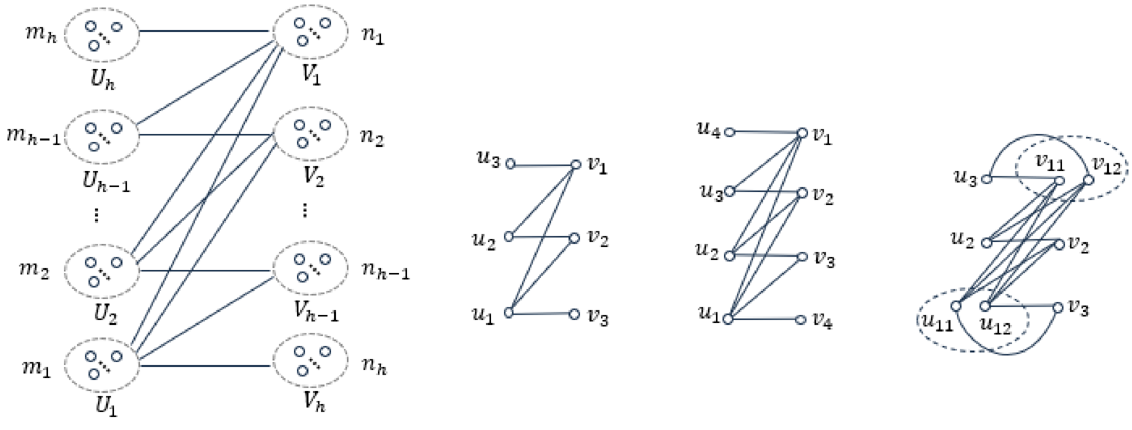
Different from existing studies that focus on eigenvectors, this paper is concerned with the graphical structure of MPCS, which is closely related to the minimum leader sets.

- First, all the MPCSs of the chain graph $CG1$ are given. It is proved that it has one and only one 4-MPCS, and the graphical structure of this MPCS is given.
- Second, it is proved that CGt $(t \geq 2)$ has only 2-MPCS.
- Third, the methods for obtaining all the minimum leader sets of $CG1$ and CGt, the number of minimum leader sets, and the number of the least leader vertices are given. Thus, the minimum Laplace controllability problems of these two types of chain graphs are completely solved.
- Fourth, the methodology used in this paper provides a reference framework for addressing the problem of selecting the minimum leader set in other chain graphs. With the help of the MPCS, we only focus on studying $\boldsymbol{L}_{SS}$ and $\boldsymbol{L}_{\bar{S}S}$, or even $\boldsymbol{L}_{uS}$, which enables us to solve the controllability problem from the local subgraph structure.

2. Basic concepts and preparatory theories

2.1. Concepts and symbols of a chain graph

The basic graph theory terms that appear in this paper can be found in the classical literature [9], and will not be elaborated here. A chain graph is a bipartite graph $G = (X, Y; E)$. Let h be a positive integer, let $X = \bigcup_{i=1}^{h} U_i$ and $Y = \bigcup_{i=1}^{h} V_i$, where both U_i and V_i are isolated vertex sets with m_i and n_i vertices, respectively. Each vertex in U_i is adjacent with the vertices in $V_1, V_2, \cdots, V_{h+1-i}$. As shown in Figure 1 below, such a chain graph is denoted as $CG(m_1, m_2, \cdots, m_h; n_1, n_2, \cdots, n_h)$.



(a) $CG(m_1, m_2, \cdots, m_h; n_1, n_2, \cdots, n_h)$ (b) CG1(3) (c) CG1(4) (d) CG(2,1,1;2,1,1)

Figure 1. Chain graphs.

For chain graph $CG\left(\underbrace{1,1,\ldots,1}_{h};\underbrace{1,1,\ldots,1}_{h}\right)$. That is, there is only one vertex in both U_i and V_i.Without loss of generality, let$U_i = \{u_i\}$ and $V_i = \{v_i\}$. For the purpose of simplicity, we denote $CG\left(\underbrace{1,1,\ldots,1}_{h};\underbrace{1,1,\ldots,1}_{h}\right) \triangleq CG1(\mathrm{h})$. Then, from the graphical structure of $CG1(\mathrm{h})$, it is easy to see

that the degrees of its vertices have the following properties.
Property 1. $\mathrm{d}(u_i) = \mathrm{d}(v_i) = h + 1 - i(i = 1, 2, \cdots, h)$;
Property 2. The degrees of all vertices in X are mutually distinct; the degrees of all vertices in Y are mutually distinct, too.

2.2. The definition of MPCS and preparatory theory

In order to study the MPCS in $CG1$, we first present the concept of the minimal perfect critical vertex set proposed in Reference [4,11]. Let S be a vertex set, if there exists an eigenvector $\boldsymbol{y}$ of Laplacian matrix $\boldsymbol{L}$ such that $\boldsymbol{y}_{\bar{S}} = 0$, then S is called a critical vertex set (CS); If S is a critical vertex set, and there exists an eigenvector $\boldsymbol{y}$ such that $\boldsymbol{y}_{\bar{S}} = \mathbf{0}$ and $\boldsymbol{y}_{v_i} \neq 0(\forall v_i \in S)$, then S is called a perfect critical vertex set (PCS). If S is a perfect critical vertex set, and no proper subset of it is a perfect critical vertex set, then S is called a minimal perfect critical vertex set (MPCS). When $|S| = k$, then S is called a k-MPCS.

For example, see CG1(2, 1, 1; 2, 1, 1) in Figure 1(d), $\lambda = 4$ is it's triple eigenvalue. The corresponding eigenvectors are (the value is rounded to four decimal places).

$$\mathbf{y}_1 = (-0.1912, 0.1912, 0, 0, 0.6808, -0.6808, 0, 0),$$

$$\mathbf{y}_2 = 0.3536(-1, -1, 1, 1, -1, -1, 1, 1),$$

$$\mathbf{y}_3 = (-0.6808, 0.6808, 0, 0, -0.1912, 0.1912, 0, 0).$$

See $\boldsymbol{y}_1$, we know that $S_1 = \{u_{11}, u_{12}, u_3, u_4, v_{11}, v_{12}\}$ is a CS, but S_1 is not a PCS. $S_2 = \{u_{11}, u_{12}, v_{11}, v_{12}\}$ is a PCS, but S_2 is not an MPCS because there exists another eigenvector $\boldsymbol{y}_4$ and

$$\boldsymbol{y}_4 = 0.6808\boldsymbol{y}_1 - 0.1912\boldsymbol{y}_3 = (0, 0, 0, 0, 0.5, -0.5, 0, 0).$$

Hence, $S_3 = \{v_{11}, v_{12}\}$ is a 2-MPCS.

In terms of eigenvalues and eigenvectors of a Laplacian matrix, [2,10] presented a necessary and sufficient algebraic condition on controllability.
Property 3. [2,10] The undirected graph G is controllable under the leader set $\bar{F}$ if and only $\boldsymbol{y}_{\bar{F}} \neq \mathbf{0}$ ($\forall \boldsymbol{y}$ is an eigenvector of the Laplacian matrix of G), where $F \subset V(G)$ is a follower vertex set and $\bar{F} = V(G) \backslash F$ is leader vertex set.

By the concept of MPCS, Property3 can be restated as the following Property 4 (see Remark 2 in [4]).
Property 4. [4] The undirected graph G is controllable under the leader set $\bar{F}$ if and only if, for each MPCS S, $S \cap \bar{F} \neq \emptyset$.

There is a close relationship between MPCS and the minimum leader set. In other words, when we find all MPCS of G, we find the minimum leader set and hence the minimum number of leader vertices.

For example, for chain graph $CG(2,1,1; 2,1,1)$, all its MPCS are

$$\{u_{11}, u_{12}\}, \{v_{11}, v_{12}\}, \{u_2, u_3, v_2, v_3\},$$

Noticing that these three sets are mutually disjoint, take one vertex from each set to form $\bar{F}$, namely $\bar{F} = \{w_1, w_2, w_3\}$, where $w_1 \in \{u_{11}, u_{12}\}, w_2 \in \{v_{11}, v_{12}\}$ and $w_3 \in \{u_2, u_3, v_2, v_3\}$. By Property 1, $CG(2,1,1; 2,1,1)$ is controllable under the leader set $\bar{F}$. Hence, the minimum number of leaders is 3 and all together there are 12 minimum leader sets.

It can be seen that MPCS plays an important role in finding the minimum leader sets. But the difficulty in finding the minimum leader sets based on Property 4 lies in the need to identify all the MPCSs in the graph. It is for this reason that all MPCSs of the chain graphs will be studied in the following parts.

3. MPCS and Laplacian controllability of chain graph $CG1$

Let S be a subset of $V(G)$. Whether S can become an MPCS is only related to the graphical structure of $G[S]$ and $[S,\bar{S}]$. Therefore, MPCS has a clear graphical structure. Next, we will prove that the chain graph CG1 has only one MPCS.

3.1. 4-MPCS of chain graph $CG1$

Although the minimal perfect critical set is defined by eigenvectors, it has special graphical structure. We have proved that ([4] Theorem 1). 2-MPCSs are twins vertices, and there is no 1-MPCS or 3-MPCS in any graph ([4] Theorem 2). According to the definition of the chain graph and the graphical structure of 2-MPCS, we know that there is no 2-MPCS in $CG1$.

Theorem 1. Let $h \geq 3$, then there exists a 4-MPCS S in $CG1$, and the following are true:
(1) If $h = 2l + 1$ is odd, then $S = \{u_{l+1}, u_{l+2}, v_{l+1}, v_{l+2}\}$;
(2) If $h = 2l$ is even, then $S = \{u_l, u_{l+1}, v_l, v_{l+1}\}$.
Proof: By the definition of MPCS, we only need to construct an eigenvector such that its components on S are all nonzero, while its components on $\bar{S}$ are exactly zero.
(1) Let $h = 2l + 1$ be odd, $S = \{u_{l+1}, u_{l+2}, v_{l+1}, v_{l+2}\}$. Set the components of the vector $\boldsymbol{x}$ as

$$x_{u_{l+1}} = x_{v_{l+1}} = -x_{u_{l+2}} = -x_{v_{l+2}} = 1,$$

0 for the others. We will prove that $\boldsymbol{x}$ is an eigenvector of the Laplacian matrix $\boldsymbol{L} = \begin{bmatrix} \boldsymbol{L}_{SS} & \boldsymbol{L}_{S\bar{S}} \\ \boldsymbol{L}_{\bar{S}S} & \boldsymbol{L}_{\bar{S}\bar{S}} \end{bmatrix}$.

It is easy to see that

$$\boldsymbol{L}_{SS}\boldsymbol{x}_S = \begin{bmatrix} l+1 & 0 & -1 & 0 \\ 0 & l & 0 & 0 \\ -1 & 0 & l+1 & 0 \\ 0 & 0 & 0 & l \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = l\boldsymbol{x}_S.$$

However, in the row of $\boldsymbol{L}_{\bar{S}S}$, the row $\boldsymbol{L}_{u_iS}$ corresponding to the vertex $u_i (1 \leq i \leq l)$, there are exactly two components that are -1, corresponding to the vertex v_{l+1} and v_{l+2}, the others are zero, thus $\boldsymbol{L}_{u_iS}\boldsymbol{x}_S = \boldsymbol{0}$. For the vertex $u_j (j \geq l + 3)$, the row $\boldsymbol{L}_{u_jS}$ is a zero row, then we also have $\boldsymbol{L}_{u_jS}\boldsymbol{x}_S = \boldsymbol{0}$. Similarly, for $v_j (j \neq l + 1, l + 2)$, $\boldsymbol{L}_{v_jS}\boldsymbol{x}_S = \boldsymbol{0}$. Thus, we know

$$\boldsymbol{L}_{\bar{S}S}\boldsymbol{x}_S = \boldsymbol{0}.$$

That means $\boldsymbol{x}$ is an eigenvector of the Laplacian matrix, S is a CS. Since any proper subset of S is not a critical set, it is a minimal perfect critical set, that is, S is an MPCS.
(2) Similarly, it can be proved.

According to Theorem 1, $\{u_2, u_3, v_2, v_3\}$ in Figure 1(b) is an MPCS of $CG1(3)$. And $\{u_2, u_3, v_2, v_3\}$ in Figure 1(c) is an MPCS of $CG1(4)$. From the perspective of the overall graph topology, the vertices in these two MPCSs exhibit obvious symmetry in their positions. And whether h is odd or even, the vertex at the center with subscript $\left\lceil \frac{h}{2} \right\rceil$ is in a 4-MPCS. However, the set $\{u_1, u_2, v_1, v_2\}$ in Figure 1(b), which possesses the same symmetry, is not an MPCS. This shows that symmetry alone does not determine an MPCS.

3.2. No k-MPCS（$k \geq 5$）in chain graph $CG1$

In this part, we will prove that there is no k-MPCS ($k \geq 5$) in $CG1$, For this purpose, we first give the following lemmas.

Lemma 1. [11] Let S be a k-MPCS, $\forall v \in \bar{S}$, we have $|[\{v\}, S]| \neq 1$ and $|[\{v\}, S]| \neq k-1$.

Lemma 2. Let S be a k-MPCS in $CG1, S = \left\{u_{i_1}, u_{i_2}, \ldots, u_{i_p}; v_{j_1}, v_{j_2}, \ldots, v_{j_q}\right\}$, where $i_1 < i_2 < \cdots < i_p, j_1 < j_2 < \cdots < j_q$, then (1) there exists at most one vertex u in $\{u_{i_1}, u_{i_2}, \ldots, u_{i_p}\}$, such that u is not adjacent with any vertex in S; (2) there exists at most one vertex v in $\{v_{j_1}, v_{j_2}, \ldots, v_{j_q}\}$, such that v is not adjacent with any vertex in S.

Proof: (1) Assume that there exist two vertices, say u', u, in $\{u_{i_1}, u_{i_2}, \ldots, u_{i_p}\}$, they are not adjacent with $v(\forall v \in S)$. Consider matrix $\boldsymbol{L}_{SS}$, two rows corresponding to u', u, these two corresponding rows have only the main diagonal elements as the degrees of the corresponding vertices, the other elements are 0. Therefore, from $\boldsymbol{L}_{SS}\boldsymbol{x} = \lambda \boldsymbol{x}$, we have $\lambda = d_u = d'_u$, thus $d_u = d_{u'}$, this contradicts Property 2.

(2) Similarly to the proof in (1), the conclusion holds.

The eigenvalues corresponding to MPCS are, in many cases, integers. From the proof of Lemma 2, if a vertex in MPCS is isolated in $G[S]$, then its corresponding eigenvalue is the degree of that isolated vertex, hence an integer eigenvalue.

Note 1. Each vertex with subscript greater than $h + 2 - i_1$ definitely does not belong to S, that is $i_p \leq h + 2 - i_1$. In fact, suppose $v_{h+2-i_1} \in S$, notice that v_{h+2-i_1} is not adjacent to $v(\forall v \in S)$, Therefore, according to Lemma 2, the conclusion holds. Let $v_{h+2-i_1} \notin S$, suppose $v_{h+l-i_1} \in S(l \geq 3)$, Without loss of generality, let $v_{h+l_0-i_1}$ be such a vertex in S with the smallest subscript, i.e.,

$$x_{v_{h+2-i_1}} = x_{v_{h+3-i_1}} = \cdots = x_{v_{h+l_0-1-i_1}} = 0. \tag{1}$$

Taking into account again $u_{i_1+1-l_0} \notin S$, then, we have

$$\left(x_{v_{i_1}} + x_{v_{i_2}} + \cdots + x_{v_{h+2-i_1}}\right) + \left(x_{v_{h+3-i_1}} + x_{v_{h+4-i_1}} + \cdots + x_{v_{h+l_0-1-i_1}}\right) + x_{v_{h+l_0-i_1}} = 0.$$

Consider u_{i_1-1}, and notice that $u_{i_1-1} \notin S$, therefore,

$$x_{v_{i_1}} + x_{v_{i_2}} + \cdots + x_{v_{h+1-i_1}} + x_{v_{h+2-i_1}} = 0. \tag{2}$$

Combined with (1) and (2), it is easy to see that $x_{v_{h+l_0-i_1}} = 0$, this contradicts $v_{h+l_0-i_1} \in S$.

Lemma 3. Let S be a k-MPCS in $CG1, S = \left\{u_{i_1}, u_{i_2}, \ldots, u_{i_p}; v_{j_1}, v_{j_2}, \ldots, v_{j_q}\right\}$, where $i_1 < i_2 < \cdots < i_p, j_1 < j_2 < \cdots < j_q$, then we have $p = q$ and $i_1 = j_1, i_2 = j_2, \cdots, i_p = j_p$.

Proof: Notice the chain graph $CG1$, swap the positions of vertices u_i and v_i ($i = 1, 2, \cdots, h$), the resulting graph is still $CG1$, that is, the Laplacian matrix of $CG1$ remains invariant under congruence transformation. For each eigenvector $\boldsymbol{x}$, there is the following relationship among its components:

$$\begin{cases} x_{u_i} = kx_{v_i} \\ x_{v_i} = kx_{u_i} \end{cases},$$

where $k \in \mathcal{R}$ is a nonzero real number. From the above equation, we know that $k^2 = 1$, that is either $k = 1$ or $k = -1$. Therefore, $x_{u_i} = x_{v_i} (i = 1, 2, \ldots, h)$ or $x_{u_i} = -x_{v_i} (i = 1, 2, \cdots, h)$. This means x_{u_i} and x_{v_i} be zero or not be zero simultaneously, thus, from the definition of MPCS, we have u_i

and v_i either belong to S or not belong to S simultaneously.

It is known from Lemma 3 that if S is a k-MPCS of $CG1(\mathrm{h})$, then S must have the following form.

$$S = \left\{u_{i_1}, u_{i_2}, \dots, u_{i_p}; v_{i_1}, v_{i_2}, \dots, v_{i_p}\right\}.$$

Lemma 4. Let S be a k-MPCS of $CG1, S = \left\{u_{i_1}, u_{i_2}, \dots, u_{i_p}; v_{i_1}, v_{i_2}, \dots, v_{i_p}\right\}$, where $i_1 < i_2 < \cdots < i_p$, then $v_{h+1-i_1} \in S$.

Proof: Notice that i_1 is the smallest subscript of the vertices in $S \cap X$, according to the graphical structure of $CG1$, v_{h+1-i_1}only adjacent with u_{i_1}in $S \cap X$, and there is no edge between v_{h+1-i_1} and $v(\forall v \in Y)$. By Lemma 1, we have $v_{h+1-i_1} \notin \bar{S}$, that is $v_{h+1-i_1} \in S$.

By Lemma 4, $v_{h+1-i_1} \in S$, thus, we have $i_p \geq h + 1 - i_1 \geq i_1$, combined with Note , it can be seen that Note 2 holds.

Note 2. If $v_{h+2-i_1} \in S$, then $i_p = h + 2 - i_1$. If $v_{h+2-i_1} \notin S$, then $i_p = h + 1 - i_1$.

Lemma 5. Let S be a k-MPCS of $CG1, S = \left\{u_{i_1}, u_{i_2}, \dots, u_{i_p}; v_{i_1}, v_{i_2}, \dots, v_{i_p}\right\}$, where $i_1 < i_2 < \cdots < i_p$. Let λ_S be the corresponding eigenvalue of S. If $v_{h+2-i_1} \in S$, then $\lambda_S = \mathrm{d}(v_{h+2-i_1})$; If $v_{h+2-i_1} \notin S$, then $\lambda_S = \mathrm{d}(v_{i_1})$.

The detailed proof of Lemma is deferred to Appendix A. By Lemma 5, for MPCS in the chain graph $CG1$, the corresponding eigenvalue is an integer. Moreover, when $v_{h+2-i_1} \notin S, \lambda_S = h + 1 - i_1$, which is the degree of the vertex in S with the smallest subscript; when $v_{h+2-i_1} \in S, \lambda_S = i_1 - 1$, which is the degree of the vertex in S with the largest subscript.

Lemma 6. Let S be a k-MPCS of $CG1$, $S = \left\{u_{i_1}, u_{i_2}, \dots, u_{i_p}; v_{i_1}, v_{i_2}, \dots, v_{i_p}\right\}$, where $i_1 < i_2 < \cdots < i_p$, then

(1) $i_1, i_2, \cdots, i_p$ is a sequence of consecutive positive integers, namely $i_2 = i_1 + 1, i_3 = i_1 + 2, \cdots, i_p = i_1 + (p - 1)$.

(2) If $v_{h+2-i_1} \in S$, then $i_p = h + 2 - i_1$; if $v_{h+2-i_1} \notin S$, then $i_p = h + 1 - i_1$.

The long proof of Lemma can be found in Appendix B. By Lemma 6 and there is no 2-MPCS or 3-MPCS in $CG1$, we know that

$$\left\{u_{i_1+1}, v_{i_1+1}\right\} \subseteq S.$$

Drawing on the preceding lemmas, it can be proved that $CG1$ admits no k-MPCS with $k \geq 5$. This is shown in the following Theorem. The proof of Theorem is provided in Appendix C.

Theorem 2. There is no k-MPCS$(k \geq 5)$ in $CG1$.

Thus far, the process of identifying all MPCSs for CG1 is now complete. From Theorems 1 and 2, when $h \geq 3$, there is one and only one MPCS in $CG1$ and it is a 4-MPCS, its graphical structure is shown in Figure. For example, $\{u_2, u_3, v_2, v_3\}$ in Figure 1(b) is only one MPCS of $CG1(3)$. And $\{u_2, u_3, v_2, v_3\}$ in Figure 1(c) is the only MPCS of $CG1(4)$.

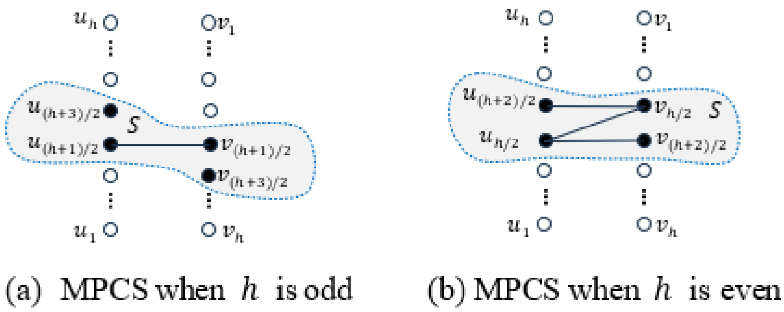


Figure 2. MPCS in chain graph $CG1$ (Omit all edges that do not belong to $G[S]$).

3.3. All of the minimum leader sets in chain graph $CG1$

By Theorems 1 and 2, there is at most one MPCS in $CG1$, therefore, from Proterty 4, it is known that the following Theorem 3 hold.

Theorem 3. The minimum number of leader is 1 for chain graph $CG1$; when $h \geq 3$, the leader can be any one of vertex in 4-MPCS as shown in Figure 2, when $h = 1, 2$, then any one vertex in $CG1$ can be selected as leader.

Figure 2 provide the graphical structure of all MPCS of $CG1$, Theorem 3 gives all the methods for selecting leader vertices that make $CG1$ controllable. In this way, the controllability problem of $CG1$ has been completely solved.

4. Analysis of MPCS and controllability of the chain graph CGt

Next, we denote CGt as chain graph $CG(t, t, \cdots, t; t, t, \cdots, t)(t \geq 2)$, namely there are exactly $t(t \geq 2)$ vertices in each U_i, V_i. Let $U_i = \{u_i^{(1)}, u_i^{(2)}, \cdots, u_i^{(t)}\}, V_i = \{v_i^{(1)}, v_i^{(2)}, \cdots, v_i^{(t)}\}$.

For chain graph CGt, Lemmas 1 to 5 in the previous text and Notes 1 to 3 all hold true. Next, we will prove that there is no other MPCS in CGt except 2-MPCS.

Theorem 4. Let $S = \left\{u_{i_1}, u_{i_2}, \dots, u_{i_p}; v_{i_1}, v_{i_2}, \dots, v_{i_p}\right\}$, where $i_1 < i_2 < \cdots < i_p, u_{i_j} \in U_{i_j}, v_{i_j} \in V_{i_j}$, then S must not be an MPCS of CGt, that is, there are no other MPCSs in CGt except 2-MPCS.

Proof: When $p = 1$, it is obvious that S is not composed of twin vertices, therefore, it is not 2-MPCS. The following assumptions $p \geq 2$.

When $v_{h+2-i_1} \in S$, $i_p = h + 2 - i_1$. Noticing that the t vertices in U_i are indistinguishable in CGt, their degrees are equal to $t(h + 1 - i)$, we denote any vertex in U_i as u_i. Let the Laplacian matrix $\boldsymbol{L}_{SS}$, as shown in Figure 3 from Lemma, we know that the eigenvalue corresponding to S is $\lambda_S = t(i_1 - 1)$. Let $\boldsymbol{E}$ be the identity matrix, then the element on the main diagonal of j-th row and the $(p + j)$-th column in matrix $\boldsymbol{L}_{SS} - \lambda_S \boldsymbol{E}$ $(1 \leq j \leq p - 1)$ is

$$\mathrm{d}\left(u_{i_j}\right) - t(i_1 - 1) = t(h + 2 - i_j - i_1).$$

On the other hand, noticing that $h + 1 - i_j$ is the maximum subscript of the vertices which is adjacent with u_{i_j} in V, there are at most $h + 2 - i_1 - i_j$ vertices in $v_{i_1}, v_{i_2}, \cdots, v_{i_p}$ that are adjacent with u_{i_s}. Therefore, for $\boldsymbol{L}_{SS} - \lambda_S \boldsymbol{E}$, the maximum number of other non-zero elements in this row, that is, the number of -1s, is

$$h + 2 - i_j - i_1.$$

$$\begin{array}{c} \quad u_{i_1} \quad u_{i_2} \quad \cdots \quad u_{i_{p-1}} \quad u_{i_p} \quad v_{i_1} \quad v_{i_2} \quad \cdots \quad v_{i_{p-1}} \quad v_{i_p} \\ \left[\begin{array}{ccccc|ccccc} \mathrm{d}(u_{i_1}) & & & & & -1 & -1 & \cdots & -1 & 0 \\ & \mathrm{d}(u_{i_2}) & & & & -1 & -1 & \cdots & 0 & 0 \\ & & \ddots & & & \vdots & \vdots & & \vdots & \vdots \\ & & & \mathrm{d}(u_{i_{p-1}}) & & -1 & 0 & \cdots & 0 & 0 \\ & & & & \mathrm{d}(u_{i_p}) & 0 & 0 & \cdots & 0 & 0 \\ -1 & -1 & \cdots & -1 & 0 & \mathrm{d}(v_{i_1}) & & & & \\ -1 & -1 & \cdots & 0 & 0 & & \mathrm{d}(v_{i_2}) & & & \\ \vdots & \vdots & & \vdots & \vdots & & & \ddots & & \\ -1 & 0 & \cdots & 0 & 0 & & & & \mathrm{d}(v_{i_{p-1}}) & \\ 0 & 0 & \cdots & 0 & 0 & & & & & \mathrm{d}(v_{i_p}) \end{array}\right] \begin{array}{c} u_{i_1} \\ u_{i_2} \\ \vdots \\ u_{i_{p-1}} \\ u_{i_p} \\ v_{i_1} \\ v_{i_2} \\ \vdots \\ v_{i_{p-1}} \\ v_{i_p} \end{array} \end{array}$$

Figure 3. The Laplace submatrix $\boldsymbol{L}_{SS}$ of chain graph CGt where $S = \{u_{i_1}, u_{i_2}, \dots, u_{i_p}; v_{i_1}, v_{i_2}, \dots, v_{i_p}\}$.

In view of the above two aspects, we known that when $t \geq 2$, we can delete the rows and columns of $\boldsymbol{L}_{SS} - \lambda_S \boldsymbol{E}$ where the vertices u_{i_p}, v_{i_p} are located. The obtained matrix is a strictly diagonally dominant matrix. Thus $\boldsymbol{y}_{S\setminus\{u_{i_p}, v_{i_p}\}} = \boldsymbol{0}$, Combined with Eq (2), we have $\boldsymbol{y}_S = \boldsymbol{0}, S$ is not an MPCS.

If $\mathrm{v}_{\mathrm{h}+2-\mathrm{i}_1} \notin \mathrm{S}$, namely $\mathrm{i}_\mathrm{p} = \mathrm{h} + 1 - \mathrm{i}_1, \lambda_\mathrm{S} = \mathrm{t}(\mathrm{h} + 1 - \mathrm{i}_1)$ is the corresponding eigenvalue of S. In $\mathbf{L}_{\mathrm{SS}} - \lambda_\mathrm{S}\mathbf{E}$, take row $\mathrm{u}_{\mathrm{i}_\mathrm{j}} (2 \leq \mathrm{j} \leq \mathrm{p})$ minus the row $\mathrm{u}_{\mathrm{i}_1}$, and row $\mathrm{v}_{\mathrm{i}_\mathrm{j}} (2 \leq \mathrm{j} \leq \mathrm{p})$ minus the row $\mathrm{v}_{\mathrm{i}_1}$, both of the elements in j-th row and the $(\mathrm{p} + \mathrm{j})$-th row are

$$\mathrm{t}(\mathrm{i}_\mathrm{j} - \mathrm{i}_1).$$

And the sum of the other elements of this row is $i_j - i_1$, then, in this matrix, delete the row and column where the vertices u_{i_1} and v_{i_1} are located, the obtained matrix is a strictly diagonally dominant matrix$(t \geq 2)$, thus S is not an MPCS. This completes the proof.

Note that if the number of vertices in each U_i and V_i in the chain graph is not less than 2, similarly to the proof process of Theorem 4, it can be known that the following Corollary 1 holds.

Corollary 1. Let chain graph $G = CG(m_1, m_2, \cdots, m_h; n_1, n_2, \cdots, n_h)$ and

$$\min_{1 \leq i \leq h} \{|m_i|, |n_i|\} \geq 2.$$

Let $S = \left\{u_{i_1}, u_{i_2}, \dots, u_{i_p}; v_{i_1}, v_{i_2}, \dots, v_{i_p}\right\}$, where $i_1 < i_2 < \cdots < i_p, u_{i_j} \in U_{i_j}, v_{i_j} \in V_{i_j}$, then S is not an MPCS of G, That is, in G, with the exception of the 2-MPCS, no other MPCSs exist.

Thus far, we have proved that $CG1$ admits a unique 4-MPCS while CGt admits only 2-MPCS. The size of U_i or V_i is of great importance. When a set U_i or V_i has more than one vertex, these vertices are considered replicas of one another and are identical in status. The pairwise relationship between them results in twin pairs, which gives rise to a 2-MPCS. The absence of such vertices in $CG1$ explains why it possesses only a 4-MPCS and no 2-MPCS.

5. Conclusions

This paper completely solves the controllability problems of two types of chain graphs, $CG1$ and

CGt, all of their MPCS given. Based on MPCS, the selection method for all the minimum leader sets of these graphs and the minimum number of leaders are given. These two types of chain graphs have their particularities in structure, how to select the minimum leader sets of a general chain graph $CG(m_1, m_2, \cdots, m_h; n_1, n_2, \cdots, n_h)$ is an interesting problem worthy of further study. The characterization of MPCS for $CG1$ and CGt naturally motivates the enumeration problem for arbitrary chain graphs. Recent years have seen considerable progress in counting substructures such as subtrees and subpaths in structured graphs [12]. MPCS is distinguished as a special substructure by its relevance both to eigenvalues and, more importantly, to the zero entries of eigenvectors. This relevance, together with the regularity of chain graphs, suggests that systematic techniques may be adapted to enumerate MPCS. Is it certain that there are no MPCS with more than 5 vertices in the chain graph? In what kind of chain graph must there be a 4-MPCS? If a chain graph has no repeated eigenvalues, does it necessarily have a 4-MPCS? Issues such as these are all worthy of further research. In particular, for chain graph $CG(t, 1, \cdots, 1; 1, 1, \cdots, 1)$ $(t \geq 2)$, References [8] proved that when $t \geq h, G$ is controllable by $t - 1$ leaders. The vertices in the minimum leader set all come from U_1. This indicates that in this situation, G has no other MPCS except for 2-MPCS. However, no conclusion is given for the case of $t < h$. Here, we put forward a conjecture:

Conjecture 1. For chain graph $G = CG(\underbrace{t, 1, \ldots, 1}_{h}; \underbrace{1, 1, \ldots, 1}_{h})$, where $t \geq 2$:

(1) If $t \geq h$, there is no other MPCS in G except for the 2-MPCS.

(2) If $t < h$, in addition to the 2-MPCS in G, there exists a unique vertex subset S that is an MPCS and S is a 4-MPCS.

Use of Generative-AI tools declaration

The author declares that she has not used any artificial intelligence (AI) tools in creating this article.

Conflict of interest

The author declares that she has no interests in this paper.

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Appendix A (Proof of Lemma 5)

Proof: $v \notin S$ if and only if $x_v = 0$, therefore, vertices not in S can be added to (2) to make it an equation with consecutive subscripts, that is,

$$x_{v_{i_1}} + x_{v_{i_1+1}} + \cdots + x_{v_{h+1-i_1}} + x_{v_{h+2-i_1}} = 0.$$

If $v_{h+2-i_1} \in S$, then $x_{v_{h+2-i_1}} \neq 0$. Notice that v_{h+2-i_1} is not adjacent with any other vertices in S, therefore,

$$\boldsymbol{L}_{v_{h+2-i_1}S}\boldsymbol{x}_S = d\left(v_{h+2-i_1}\right) \cdot x_{v_{h+2-i_1}} = (i_1 - 1) \cdot x_{v_{h+2-i_1}}.$$

On the other hand, because S is an MPCS, we know $\boldsymbol{L}_{v_{h+2-i_1}S}\boldsymbol{x}_S = \lambda_S x_{v_{h+2-i_1}}$. Therefore, $\lambda_S = i_1 - 1$.

If $v_{h+2-i_1} \notin S$, then we have $x_{v_{h+2-i_1}} = 0$. Consider the vertex u_{i_1} and $\boldsymbol{L}_{u_{i_1}S}\boldsymbol{x}_S = \lambda_S x_{u_{i_1}}$,

$$\boldsymbol{L}_{u_{i_1}S}\boldsymbol{x}_S = d\left(u_{i_1}\right) \cdot x_{u_{i_1}} - \left(x_{v_{i_1}} + x_{v_{i_2}} + \cdots + x_{v_{h+1-i_1}}\right),$$

Therefore, by (2), the eigenvalue is equal to the degree of vertex u_{i_1}, namely $\lambda_S = h + 1 - i_1$.

Appendix B (Proof of Lemma 6)

Proof: (1) If $v_{h+2-i_1} \in S$, then $x_{v_{h+2-i_1}} \neq 0$, and from Lemma 5, we have $\lambda_S = i_1 - 1$. take u_{i_1} into account. By

$$\boldsymbol{L}_{u_{i_1}S}\boldsymbol{x}_S = \lambda_S x_{u_{i_1}},$$

we know that $(h + 1 - i_1)x_{u_{i_1}} - \left(x_{v_{i_1}} + x_{v_{i_1+1}} + \cdots + x_{v_{h+1-i_1}}\right) = (i_1 - 1)x_{u_{i_1}}$. By (2),

$$x_{v_{h+2-i_1}} = (2i_1 - h - 2)x_{u_{i_1}}. \tag{3}$$

In the same way, consider v_{h+1-i_1}. From Lemma 4, $v_{h+1-i_1} \in S$, So by

$$\boldsymbol{L}_{v_{h+1-i_1}S}\boldsymbol{x}_S = \lambda_S x_{v_{h+1-i_1}},$$

we get

$$i_1 x_{v_{h+1-i_1}} - x_{u_{i_1}} = (i_1 - 1)x_{v_{h+1-i_1}}, \tag{4}$$

Thus, we obtain $x_{v_{h+1-i_1}} = x_{u_{i_1}}$.

Next, we will prove it by induction: When $0 \le l \le h+1-2i_1$, there is always the formula $x_{v_{h+1-i_1-l}} = x_{u_{i_1+l}} = x_{u_{i_1}} \neq 0$ holding true. Consequently $\{v_{h+1-i_1-l}, u_{i_1+l}\} \subseteq S$, and this means Lemma 6 (1) holds. From Eq (4), we know that when $l = 0$, the conclusion holds. Suppose $x_{v_{h-i_1}} = x_{u_{i_1+1}} = x_{u_{i_1}}, x_{v_{h-i_1-1}} = x_{u_{i_1+2}} = x_{u_{i_1}}, \cdots, \quad x_{v_{h-i_1-l+2}} = x_{u_{i_1+l-1}} = x_{u_{i_1}}$. From Eq (3) and the inductive hypothesis, we have

$$(x_{v_{h+1-i_1-(l-1)}} + x_{v_{h+1-i_1-(l-2)}} + \cdots + x_{v_{h+1-i_1}}) + x_{v_{h+2-i_1}} = l x_{u_{i_1}} + (2i_1 - h - 2)x_{u_{i_1}} \neq 0,$$

Consider the vertex u_{i_1+l}, from Eq (2), we get $u_{i_1+l} \in S$. Thus by $\boldsymbol{L}_{u_{i_1+l}S}\boldsymbol{x}_S = \lambda_S x_{u_{i_1+l}}$, we have

$$x_{u_{i_1+l}} = x_{u_{i_1}}.$$

Similarly, since $x_{u_{i_1}} + x_{u_{i_2}} + \cdots + x_{u_{i_1+l}} = (l+1)x_{u_{i_1}} \neq 0$, we have $v_{h+1-i_1-l} \in S$. Thus by $\boldsymbol{L}_{v_{h+1-i_1-l} \to S}\boldsymbol{x}_S = \lambda_S x_{v_{h+1-i_1-l}}$, we get

$$x_{v_{h+1-i_1-l}} = x_{u_{i_1}}.$$

Therefore, $x_{v_{h+1-i_1-l}} = x_{u_{i_1+l}} = x_{u_{i_1}} (0 \le l \le h+1-2i_1)$.

If $v_{h+2-i_1} \notin S$, it can be proved similarly:

$$x_{u_{i_1}} = (2i_1 - h - 1)x_{v_{h+1-i_1}} \neq 0,$$

$$x_{v_{h+1-i_1-l}} = x_{u_{i_1+l}} = x_{v_{h+1-i_1}} \neq 0 (0 \le l \le h - 2i_1).$$

(3) From (1) and Note 1, the conclusion holds.

Appendix C (Proof of Theorem 2)

Proof: Suppose S is a k-MPCS$(k \ge 5)$ in $CG1$, let $S = \{u_{i_1}, u_{i_2}, \dots, u_{i_p}; v_{i_1}, v_{i_2}, \dots, v_{i_p}\}$, where $i_1 < i_2 < \cdots < i_p$, By Lemma 4, we know that the vertices with subscript $i_1, i_1+1, h+1-i_1$ must belong to S and $2i_1 \le h+1$.

Case 1. When $v_{h+2-i_1} \in S$ and $2i_1 = h+1$.

At this point, h is odd. Let $h = 2l+1$, then $i_1 = l+1, h+2-i_1 = l+2$. Therefore, S must contain $u_{l+1}, u_{l+2}, v_{l+1}, v_{l+2}$, from Theorem 1 and the minimality of MPCS, we know that S is a 4-MPCS, that is $k = 4$, which contradicts $k \ge 5$.

Case 2. When $v_{h+2-i_1} \in S$ and $2i_1 < h+1$.

At this time, there is the following inequality:

$$i_1 \le \frac{h}{2} < \frac{h+1}{2} < \frac{h+2}{2} \le h+1-i_1. \tag{5}$$

By this inequality, if $h = 2l+1$ is odd, that means $i_1 < l+1 < l+2 < h+2-i_1$, from Lemma

6 (1), we have $\{u_{l+1}, u_{l+2}, v_{l+1}, v_{l+2}\} \subseteq S$. Similarly to Case 1, a contradiction can be obtained. If $h = 2l$ is even, by the inequality (5) and Lemma 6 (1), we have $\{u_l, u_{l+1}, v_l, v_{l+1}\} \subseteq S$, Similarly, a contradiction can be obtained in the same way as in Case 1.

Case 3. When $v_{h+2-i_1} \notin S$ and $2i_1 = h + 1$.

At this point, h is an odd number. Let $h = 2l + 1$, then $i_1 = l + 1$. By Lemma 6 (1), we have $i_p = h + 1 - i_1 = l + 1$, hence, $S = \{u_{l+1}, v_{l+1}\}$. But $\{u_{l+1}, v_{l+1}\}$ are not twins, thus $S = \{u_{l+1}, v_{l+1}\}$ is not MPCS.

Case 4. When $v_{h+2-i_1} \notin S$ and $2i_1 < h + 1$.

Similarly to Case 2, the inequality (5) also holds. If $h = 2l + 1$ is odd, then $i_1 \leq l < l + 1 < l + 2 \leq h + 1 - i_1$. Similarly to Case 2, there is a contradiction; If $h = 2l$ is even, then $i_1 \leq l < l + 1 \leq h + 1 - i_1$, also similarly to Case 2, there is a contradiction.