Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(3): 6699–6719.
DOI:10.3934/math.2026277
Received: 09 January 2026
Revised: 12 February 2026
Accepted: 27 February 2026
Published: 16 March 2026

Research article

A class of solitary and periodic wave structures in (3+1)-dimensional space-time fractional KdV-BBM equation with Jacobi elliptic functions

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Abstract: This paper considers the (3+1)-dimensional space-time fractional variable Korteweg-de Vries-Benjamin-Bona-Mahoney (KdV-BBM) equation. This equation is of critical importance in studying long-wave phenomena in nonlinear dispersive media. It is commonly accepted that classical equations fail to capture memory and fractal properties of physical phenomena. In fluid dynamics and plasma physics, fractional equations accurately capture the memory effect in nonlinear one-way wave propagation. Thus, we employ a fractional complex transformation that reduces the controlling nonlinear partial differential equations (NPDE) to a nonlinear ordinary differential equation (NODE). The newly introduced Jacobi elliptic function expansion method is systematically implemented to capture various types of exact or analytical wave solutions. Our findings explicitly reveal that by tuning the β parameter in the fractional variable KdV-BBM equation, one can capture solitary and double periodic nonlinear structures in terms of Jacobi sine and cosine functions. The major outcome is that it enormously regulates the topology of various solitary and double periodic nonlinear waves. Therefore, our investigation explicitly reveals that conformable fractional derivatives serve to be highly efficient operators in exploring higher-dimensional nonlinear or solitary and double periodic waves. The anticipated research could be considered a foundation that could be used in future experiments in nonlinear optics and water flows.

Keywords: solitary waves; periodic waves; shock waves; KdV-BBM equation; fractional derivative
Mathematics Subject Classification: 35Q51, 35Q53, 37K40, 90C33

1. Introduction

Nonlinear partial differential equations (NPDEs) are a highly important and interesting area of study. NPDEs are significant in nature, and their mathematical solutions enhance our comprehension of it. NPDEs are utilized in meteorology, fluid dynamics, solid-state physics, hydrodynamics, thermal conduction studies, fiber optics, plasma physics, and various other scientific and engineering disciplines. The concept of traveling waves represents a significant advancement in deriving mathematical solutions to nonlinear evolution equations (NLEEs). Extensive study has been conducted on resolving NLEE wave issues with the concept of moving waves. These waves frequently resemble bells or curves. To evaluate the efficacy of numerical approaches, accurate responses are also required. They verify the accuracy and reliability of the computer models employed to analyze complex systems [1, 2].

Soliton theory examines unique wave phenomena. Waves keep their form. These are found in optical fibers, plasma, and fluids [3, 4]. Behavior is defined by nonlinear equations. These equations elucidate wave interactions and stability. This information is needed for contemporary technology. Advancements in quantum physics and communications occur. These solitary waves stimulate scholars' interest. Examining them reveals intricate system functionality. The velocity of a soliton is constrained. Properties provide comprehensive physical information. The theory of solitary waves permits the propagation of solitons. Soliton wave packets preserve their form and speed in nonlinear dispersive media. These waves arise from the equilibrium of nonlinearity and dispersion effects under specific limitations.

The authors investigated the existence and Hyers-Ulam stability for random impulsive stochastic functional differential equations with finite delays [5]. Solitary wave solutions bear similar characteristics to the real-life phenomena of a tsunami, internal waves in the ocean, and optical fiber pulses. Because of the existence of these solitary wave solutions to the momentum equation, we are able to understand the propagation of similar phenomena in real life. This is because solitary wave solutions are more stable and do not break during propagation over large distances.

In the last few decades, a number of authors have proposed simple techniques for finding exact solutions to NPDEs. These include the separation's variables method [6, 7], Hirota's bilinear method [8], inverse scattering method [9], Backlund transformation [10], the homogeneous balance method [11, 12], the discrete tanh method [13], the tanh method [14], and the Jacobi elliptic function expansion method [15]. Recently, researchers have been looking at exact solutions for nonlinear dynamics studies, and much more so for fractional nonlinear partial differential equations (FNPDEs) [13, 16, 17].

Finding periodic and solitary wave solutions using Jacobi elliptic functions is the main novelty. The range of wave shapes permitted by Jacobi functions (*sn*, *cn*, and *dn*) is far greater than that of normal trigonometric functions (*sin* and *cos*). Nonlinear equations in four variables (x, y, z, t) are notoriously challenging to solve. We make use of a sophisticated fractional transformation. This "shortcut" in mathematics transforms the partial fractional differential equation into an ordinary differential equation (ODE) that is easier to understand. As a result, high-dimensional phenomena can be solved for without the math breaking under its own weight. This work maps out a "zoo" of waveforms rather than identifying a single solution: Solitary waves: individual pulses that hold their shape and reinforce themselves. Periodic waves: recurring patterns that simulate regular fluid or plasma oscillations. Particularly originating from the Burgers part of the equation, shock waves signify sudden variations in density or pressure.

The shock-shaped solutions indicate wave patterns with abrupt changes or discontinuity in the shape of a wave function, similar to a shock wave. The general features of these solutions include sharp changes or discontinuities; these kinds of changes, in a medium, often illustrate abrupt changes in pressure, density, or velocity. In real life, shock-shaped waves have significant importance in different areas, like supersonic aerodynamics, where it is called a shock wave for airplanes, explosions, traffic flow, plasma physics, etc. It is very significant to study these kinds of wave patterns to identify their movements in the presence of rapid energy transfer among them.

Also, the novelty of the paper is that for the well-known model, the (3+1)-dimensional time-fractional Korteweg-de Vries-Benjamin-Bona-Mahoney (KdV-BBM) equation of complex nonlinear waves in fluid dynamics and plasma physics, we have been able to obtain a rich class of traveling periodic wave solutions. Since most of the previous works have dealt with integer-order models in one space and one time dimensions [18], there appears to be a large gap in combining three-dimensional spatial evolution and multifractional temporal dimensions. By virtue of the Jacobi elliptic function method, a generalized framework for solutions such as solitary waves, periodic waves, and even singular waves may be given. More explicitly, the novelty of the current study is defined by the dimensional extension, fractional parameter influence and solution versatility.

Fractional calculus has enabled mathematical modeling of complex processes like anomalous diffusion and viscoelasticity. We model various physical phenomena using nonlinear fractional partial differential equations (FNPDEs). Many fields of engineering, biology, chemistry, and physics use fractional calculus theoretically and practically. Fractional derivatives and integrals help scientists describe complex processes, improving several fields. Fractional operators like Atangana-Baleanu, Riemann-Liouville, Caputo-Fabrizio, and Beta-time derivatives solve complex fractional nonlinear equations. Conformal derivatives are useful for fractional model analysis in engineering and applied science due to their simplicity and practicality in understanding dynamic systems [19–21].

Definition: *The conformable fractional derivative to the function $f : (0, \infty) \to R$ of order β, as defined in [22], can be rewritten as:*

$$T_\beta(f)(t) = \lim_{\beta \to 0} \frac{f(t + \beta t^{1-\beta}) - f(t)}{\beta} \tag{1.1}$$

for all $t > 0$ and $\beta \in (0, 1]$.

Rules governing conformable fractional derivatives can be restated as follows [21, 22]: The variables f and g can be changed and are differentiable for all $t > 0$, and a_1 and a_2 are real numbers, thus we can say the following:

- $T_\beta(a_1 f + a_2 g) = a_1 T_\beta(f) + a_2 T_\beta(g)$.
- $T_\beta(t^p) = pt^{(p-\beta)}$ for all $p \in R$.
- $T_\beta(fg) = gT_\beta(f) + fT_\beta(g)$.
- $T_\beta \frac{f}{g} = \frac{gT_\beta(f) - fT_\beta(g)}{g^2}$.
- If function f is differentiable, then $T_\beta(f)(t) = t^{1-\beta} \frac{df}{dt}$.

From the physical point of view, it can be considered as a local measure of instantaneous rate of change, taking care of fractional-order dynamics and hence capable of modeling memory effects and hereditary properties of the systems under consideration. The presence of the scaling factor t1-mean in the definition easily distinguishes the conformable derivative from classical ones and reflects the fractional scaling of the time or space variables in the underlying system. The introduction of such a scaling within the framework of a local derivative is important to model phenomena like anomalous diffusion, viscoelasticity, etc. In essence, the conformable fractional derivative yields a mathematically sound definition of fractional-order differentiation, which is physically meaningful, interpretable as the fractional rate of change, and non-necessarily local. This makes it particularly suitable for describing systems where fractional effects are significant but a local interpretation of the derivative is desired.

While traditional methods of symbolic computation play an important role in finding exact solutions to nonlinear wave equations, they often involve tedious algebraic complexity. On the other hand, non-symbolic methods such as the Hamilton approach and different variational techniques provide a simpler platform to investigate the overall complexity of the problem. Interestingly, recent investigations have shown the effectiveness of these non-symbolic methods in simplifying the computation process for fractional equations and are seen to be an alternative to avoid the common problems associated with the complexity of the computation itself [23–25].

2. Exploring Jacobi elliptic functions under conformable fractional analysis

The following gives further details about the FNPDEs:

$$G(U, \frac{\partial U}{\partial t}, \frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z}, \frac{\partial^2 U}{\partial x^2}, \frac{\partial^2 U}{\partial y^2}, \frac{\partial^2 U}{\partial z^2}, \frac{\partial^3 U}{\partial x^3}, \frac{\partial^3 U}{\partial y^3}, \frac{\partial^3 U}{\partial z^3}, ...) = 0 \tag{2.1}$$

if $U(x, y, z, t) = U(\xi), \xi = ax + by + cz - kt$ is a solution for the NPDEs Eq (2.1). Then (2.1) transformed into

$$G(U, -k\frac{\partial U}{\partial \xi}, a\frac{\partial U}{\partial \xi}, b\frac{\partial U}{\partial \xi}, c\frac{\partial U}{\partial \xi}, a^2\frac{\partial^2 U}{\partial \xi^2}, b^2\frac{\partial^2 U}{\partial \xi^2}, c^2\frac{\partial^2 U}{\partial \xi^2}, a^3\frac{\partial^3 U}{\partial \xi^3}, b^3\frac{\partial^3 U}{\partial \xi^3}, c^3\frac{\partial^3 U}{\partial \xi^3}, ...) = 0, \tag{2.2}$$

and the following function $U(x, y, z, t) = u(\xi), \xi = a\frac{x^\beta}{\beta} + b\frac{y^\beta}{\beta} + c\frac{z^\beta}{\beta} - k\frac{t^\beta}{\beta}$ is a solution of the associated FNPDE:

$$G(U, D_t^\beta U, D_x^\beta U, D_y^\beta U, D_z^\beta U, (D_x^\beta)^2 U, (D_y^\beta)^2 U, (D_z^\beta)^2 U, ...) = 0 \tag{2.3}$$

such that the conformable fractional differential operators are $D_t^\beta, D_x^\beta, D_y^\beta, and D_z^\beta$. Using the fifth property of conformable fractional derivatives, we obtain

$$\begin{aligned} &D_t^\beta u = -k\frac{du}{d\xi}, \quad D_x^\beta u = a\frac{du}{d\xi}, \quad D_y^\beta u = b\frac{du}{d\xi}, \quad D_z^\beta u = c\frac{du}{d\xi}, \\ &(D_t^\beta)^2 u = k^2\frac{d^2u}{d\xi^2}, \quad (D_x^\beta)^2 u = a^2\frac{d^2u}{d\xi^2}, \quad (D_y^\beta)^2 u = b^2\frac{d^2u}{d\xi^2}, \quad (D_z^\beta)^2 u = c^2\frac{d^2u}{d\xi^2}, ... \end{aligned} \tag{2.4}$$

Subsequently, with additional details, the transformation of (2.3) is as follows:

$$G(U, -k\frac{\partial U}{\partial \xi}, a\frac{\partial U}{\partial \xi}, b\frac{\partial U}{\partial \xi}, c\frac{\partial U}{\partial \xi}, a^2\frac{\partial^2 U}{\partial \xi^2}, b^2\frac{\partial^2 U}{\partial \xi^2}, c^2\frac{\partial^2 U}{\partial \xi^2}, a^3\frac{\partial^3 U}{\partial \xi^3}, b^3\frac{\partial^3 U}{\partial \xi^3}, c^3\frac{\partial^3 U}{\partial \xi^3}, ...) = 0. \tag{2.5}$$

We use the transformation below:

$$U(\xi) = \sum_{i=0}^{n} a_i H^i(\xi). \tag{2.6}$$

Let constants a_i (for $i = 0, 1, 2, ..., n$)) be defined for resolution, with n established by equating the orders of the greatest derivative and the nonlinear term in (2.5). If $U(\xi)$ represents the solutions for the given new ansatz [26], then:

$$H'(\xi) = \sqrt{d_0 + d_1 H(\eta) + d_2 H^2(\eta) + d_3 H^3(\eta) + d_4 H^4(\eta)}. \tag{2.7}$$

The real parameters d_0, d_1, d_2, d_3, and d_4 are used, whereas the prime symbol represents differentiation with regard to ξ. By inserting the ansatz (2.7) and (2.6) into (2.5), a set of algebraic equations comprising the variables d_0, d_1, d_2, d_3, d_4, k, λ, and a_i may be developed. Resolving Eq (2.7) is not a systematic technique. Nevertheless, the incorporation of the five arbitrary parameters d_0, d_1, d_2, d_3, and d_4 in this ansatz is justified, as its solution may be expressed using functions like Jacobi elliptic functions, commonly employed in nonlinear problems. Moreover, the incorporation of distinctive solutions like $sech\xi$ and $\tanh\xi$ within these functions facilitates the comprehension of solitary and shock wave dynamics. The coefficients a_i (for $i = 0, 1, 2, ..., n$) in Eq (2.6) are mostly determined by the characteristics of the associated PDE system. Solutions are categorized according to parameters d_0, d_1, d_2, d_3, and d_4, resulting in three principal situations.

By inserting the solved algebraic system solutions into (2.6) and performing integration on (2.7)'s solutions, we can express ξ as $a\frac{x^\beta}{\beta} + b\frac{y^\beta}{\beta} + c\frac{z^\beta}{\beta} - k\frac{t^\beta}{\beta}$. We can figure out the fractional moving wave solutions for Eq (2.1) using this method. The tanh-function method [27] is what you get when you assume $H = \tanh\xi$ in Eq (2.6). For the same reason, this method is called the sech-function method when $H = sech\xi$. There is also the Jacobi elliptic function method when $H \in \operatorname{sn}\xi, \operatorname{cn}\xi, \operatorname{cs}\xi$. Eq (2.7) is used because it can handle solitary waves ($H = \operatorname{sech}\xi$), shock waves ($H = \tanh\xi$), and periodic waves (via Jacobi elliptic functions) when $H \in \operatorname{sn}\xi$, $\operatorname{cn}\xi$, $\operatorname{cs}\xi$. The right choices of parameters (d_0, d_1, d_2, and d_4) reveal these solutions. The Jacobi elliptic functions ($\operatorname{cn}\xi \equiv \operatorname{cn}(\xi|m)$), ($\operatorname{sn}\xi \equiv \operatorname{sn}(\xi|m)$), and ($\operatorname{dn}\xi \equiv \operatorname{dn}(\xi|m)$) (for modulus ($m \in (0, 1)$)) exhibit double periodicity and trigonometric-like properties, specifically

$$\begin{aligned} &sn^2(\xi) + cn^2(\xi) = 1, \quad dn^2(\xi) + m^2 sn^2(\xi) = 1, \quad (cn(\xi))' = -sn(\xi)dn(\xi), \\ &(sn(\xi))' = cn(\xi)dn(\xi), \quad (dn(\xi))' = -m^2 sn(\xi)cn(\xi). \end{aligned} \tag{2.8}$$

The Jacobi elliptic functions undergo a transformation into trigonometric functions, which can be represented by the following examples, as the value of the parameter m approaches zero:

$$\begin{aligned} &sn(\xi) \to \sin(\xi), \quad cn(\xi) \to \cos(\xi), \quad dn(\xi) \to 1, \\ &cs(\xi) \to \cot(\xi), \quad ds(\xi) \to csc(\xi). \end{aligned} \tag{2.9}$$

As the parameter m tends toward 1, the Jacobi elliptic functions evolve into hyperbolic functions.

$$\begin{aligned} &sn(\xi) \to \tanh(\xi), \quad cn(\xi) \to sech(\xi), \quad dn(\xi) \to sech(\xi), \\ &cs(\xi) \to csch(\xi), \quad ds(\xi) \to csch(\xi). \end{aligned} \tag{2.10}$$

In recent years, symbolic calculation methods have shown significant progress in finding solutions to high-dimensional nonlinear evolution equations. In particular, multi-soliton solutions can be obtained, and complicated wave interactions are discussed based on Hirota's bilinear method in a (3+1)-dimensional system [28,29]. Even though Hirota's method is powerful in finding exact N-soliton solutions, the Jacobi elliptic function method is indispensable in the study of the (3+1)-dimensional space-time fractional KdV-BBM equation. The main reason for choosing the Jacobi elliptic function method is its unique character in providing doubly periodic solutions. This kind of solution allows a parameter-driven smooth transition between the periodic wave trains and solitary pulses in terms of the elliptic modulus m, and this feature complements the soliton-focused results usually obtained via bilinear forms.

2.1. Solitary and periodic waves in the fractional (3+1)-dimensional KdV-BBM equation

The (3+1)-dimensional fractional KdV-BBM equation [30] is a considerable step up from the original nonlinear dispersive wave equations. It has both higher-dimensional spatial effects and fractional-order derivatives to show how complex physical systems remember things and act in ways that aren't local. The fractional KdV-BBM equation takes the normal BBM and KdV frameworks and adds their nonlinear and dispersive features together. It is easier to represent odd wave propagation events that happen in viscoelastic mediums, porous flows, and other non-classical transport conditions.

The newly derived soliton solutions have the potential to impart necessary insights into the characterization of nonlinear wave behavior in various physical configurations. In particular, the derived solutions can find application in characterizing wave dynamics in shallow water systems, plasma physics, as well as nonlinear optical fibers, where the nonlinear-dispersive effects can have a significant impact on wave transmission. As a result, the derived solutions characterizing solitary as well as periodic wave behavior in (3+1)-dimensional fractional systems can have potential applications in predicting and controlling nonlinear wave behavior in such complex physical systems.

Though fractional KdV-BBM has been investigated in (1+1) and (2+1) dimensions with various operators, there is an absence in addressing (3+1)-dimensional space-time fractional forms with conformable operators. The gap is addressed in this paper, which is not only restricted to solitary wave solutions but to the whole array of periodic Jacobi elliptic solutions. Not to mention, in fractional equations, new insight is provided to explain the significance of the parameter β to differentiate periodic oscillations from solitary peaks, which is sometimes disregarded in lower-dimensional equations.

$$\frac{\partial^{2\beta}}{\partial x^{\beta}\partial t^{\beta}}U+\mu_1\frac{\partial^{\beta}}{\partial x^{\beta}}(U\frac{\partial^{\beta}}{\partial x^{\beta}}U)+\mu_2\frac{\partial^{4\beta}}{\partial x^{4\beta}}U-\mu_3\frac{\partial^{4\beta}}{\partial x^{3\beta}\partial t^{\beta}}U+\mu_4\frac{\partial^{2\beta}}{\partial y^{2\beta}}U-\mu_5\frac{\partial^{2\beta}}{\partial z^{2\beta}}U=0. \tag{2.11}$$

Solitons and nonlinear waves in fluid dynamics, plasma physics, and weak dispersion environments can be studied using this equation, which is linked to many well-known nonlinear evolution equations for real-valued variables $U = U(x, y, z, t)$. If $\mu_3 = \mu_4 = \mu_5 = 0$, Eq (2.11) simplifies to the fractional KdV equation.

$$\frac{\partial^{\beta}}{\partial t^{\beta}}U+\mu_1(U\frac{\partial^{\beta}}{\partial x^{\beta}}U)+\mu_2\frac{\partial^{3\beta}}{\partial x^{3\beta}}U=0. \tag{2.12}$$

Setting μ_3 to zero makes Eq (2.11) simpler and turns it into the fractional 3D KdV-BBM equation

$$\frac{\partial^{2\beta}}{\partial x^{\beta}\partial t^{\beta}}U+\mu_1\frac{\partial^{\beta}}{\partial x^{\beta}}(U\frac{\partial^{\beta}}{\partial x^{\beta}}U)+\mu_2\frac{\partial^{4\beta}}{\partial x^{4\beta}}U+\mu_4\frac{\partial^{2\beta}}{\partial y^{2\beta}}U-\mu_5\frac{\partial^{2\beta}}{\partial z^{2\beta}}U=0. \tag{2.13}$$

When $\mu_3 = \mu_5 = 0$, the Eq (2.11) reduces to the fractional 2D KdV-BBM equation

$$\frac{\partial^{2\beta}}{\partial x^{\beta}\partial t^{\beta}}U+\mu_1\frac{\partial^{\beta}}{\partial x^{\beta}}(U\frac{\partial^{\beta}}{\partial x^{\beta}}U)+\mu_2\frac{\partial^{4\beta}}{\partial x^{4\beta}}U+\mu_4\frac{\partial^{2\beta}}{\partial y^{2\beta}}U=0. \tag{2.14}$$

Using the Jacobi elliptic function approach, the primary objective of this research is to analyze solitary and periodic wave solutions to NPDEs. This analytical method simplifies the derivation of correct periodic and solitary wave solutions, which in turn provides physicists with valuable insights into the modeling and investigation of complex physical occurrences. As a result, this method increases the comprehension of NPDEs. The following is the process by which the solutions to Eq (2.11) that are represented as traveling waves are derived:

$$U(x,y,z,t) = U(\xi), \qquad \xi = a\frac{x^\beta}{\beta} + b\frac{y^\beta}{\beta} + c\frac{z^\beta}{\beta} - k\frac{t^\beta}{\beta}, \tag{2.15}$$

with a, b, c and k being real constants. When Eq (2.11) turns into Eq (2.15), Eq (2.11) is simplified into the following NODE:

$$\frac{1}{2}\left(2\,a^4\mu_2 + 2\,a^3k\mu_3\right)U_{\xi\xi} + \frac{1}{2}\,U\left(a^2U\mu_1 - 2\,\mu_5\,c^2 + 2\,\mu_4\,b^2 - 2\,ka\right) = 0. \tag{2.16}$$

To achieve accurate solutions, we start by balancing the nonlinear term U^2 and the highest-order derivative term U''. Based on this equilibrium, the leading-order contribution occurs at n=2. This suggests that the nonlinear and dispersive effects occur in the same order. Eq (2.6) is appropriately defined based on the provided information. Therefore, we can express Eq (2.6) as follows:

$$U(\xi) = \sum_{i=0}^{2} e_i H^i(\xi). \tag{2.17}$$

By substituting (2.17) with (2.7) into (2.16) and equating the coefficients of $H^i(\xi)$ to zero, where $H^i = \sqrt{d_0 + d_1H + d_2H^2 + d_3H^3 + d_4H^4}$, we can derive a system of equations with respect to the unknowns $e_0, e_1, e_2, a, b, c, k, d_0, d_1, d_2, d_3$, and d_4.

$$\begin{aligned}
&10\,a^2\left(\mu_2\left(d_3e_2 + \frac{2}{5}\,d_4e_1\right)a^2 + k\mu_3\left(d_3e_2 + \frac{2}{5}\,d_4e_1\right)a + \frac{1}{5}\,e_1e_2\mu_1\right) = 0,\\
&12\,a^2e_2\left(a^2d_4\mu_2 + ad_4k\mu_3 + 1/12\,e_2\mu_1\right) = 0,\\
&(4\,\mu_2\,d_0e_2 + \mu_2\,d_1e_1)\,a^4 + 4\,\mu_3\,k\left(d_0e_2 + \frac{1}{4}\,d_1e_1\right)a^3 + a^2\mu_1\,{e_0}^2 - 2\,ake_0 - 2\,e_0\left(-\mu_4\,b^2 + \mu_5\,c^2\right) = 0,\\
&(6\,\mu_2\,d_1e_2 + 2\,\mu_2\,d_2e_1)\,a^4 + 6\,\mu_3\,k\left(d_1e_2 + \frac{1}{3}\,d_2e_1\right)a^3 + 2\,a^2e_0e_1\mu_1 - 2\,ae_1k - 2\,e_1\left(-\mu_4\,b^2 + \mu_5\,c^2\right) = 0,\\
&(8\,\mu_2\,d_2e_2 + 3\,\mu_2\,d_3e_1)\,a^4 + 8\,\mu_3\,k\left(d_2e_2 + \frac{3}{8}\,d_3e_1\right)a^3 + \left(2\,\mu_1\,e_0e_2 + \mu_1\,{e_1}^2\right)a^2 - 2\,ae_2k\\
&-2\,e_2\left(-\mu_4\,b^2 + \mu_5\,c^2\right) = 0.
\end{aligned} \tag{2.18}$$

Upon solving the system of equations given in (2.18), we obtain

$$k = \frac{\mu_4\,b^2 - \mu_5\,c^2}{a}, \mu_1 = 0, \mu_2 = -\frac{\left(\mu_4\,b^2 - \mu_5\,c^2\right)\mu_3}{a^2}, \tag{2.19}$$

where $a, b, c, m, \mu_3, \mu_4, \mu_5, e_0, e_1$, and e_2 are arbitrary constants. Consequently, the resolution of Eq (2.16) can be articulated as follows:

$$U(x,y,z,t) = e_0 + e_1H(a\frac{x^\beta}{\beta} + b\frac{y^\beta}{\beta} + c\frac{z^\beta}{\beta} - k\frac{t^\beta}{\beta}) + e_2H^2(a\frac{x^\beta}{\beta} + b\frac{y^\beta}{\beta} + c\frac{z^\beta}{\beta} - k\frac{t^\beta}{\beta}). \tag{2.20}$$

We find many moving wave solutions for Eq (2.16) by changing the values of d_0, d_1, d_2, d_3, and d_4 in Eq (2.7).

Case 1. If $d_0 = 1$, then $d_1 = 0$. The values of d_2, d_3, and d_4 are m^2, $-(1 + m^2)$, 0, and 0, respectively. Thus, the periodic wave solution for Eq (2.16) is given by the equations $H = sn(\xi, m)$ and $H = cd(\xi, m)$, which are the solutions of Eq (2.7).

$$U(x,y,z,t) = 1 + \frac{1}{2}sn(a\frac{x^\beta}{\beta} + b\frac{y^\beta}{\beta} + c\frac{z^\beta}{\beta} - k\frac{t^\beta}{\beta}) + \frac{1}{20}sn^2(a\frac{x^\beta}{\beta} + b\frac{y^\beta}{\beta} + c\frac{z^\beta}{\beta} - k\frac{t^\beta}{\beta}), \quad (2.21)$$

$$U(x,y,z,t) = 1 + \frac{1}{2}cd(a\frac{x^\beta}{\beta} + b\frac{y^\beta}{\beta} + c\frac{z^\beta}{\beta} - k\frac{t^\beta}{\beta}) + \frac{1}{20}cd^2(a\frac{x^\beta}{\beta} + b\frac{y^\beta}{\beta} + c\frac{z^\beta}{\beta} - k\frac{t^\beta}{\beta}). \quad (2.22)$$

Both Figures 1 and 2 provide an illustration of the structure that is typical for this particular subject.

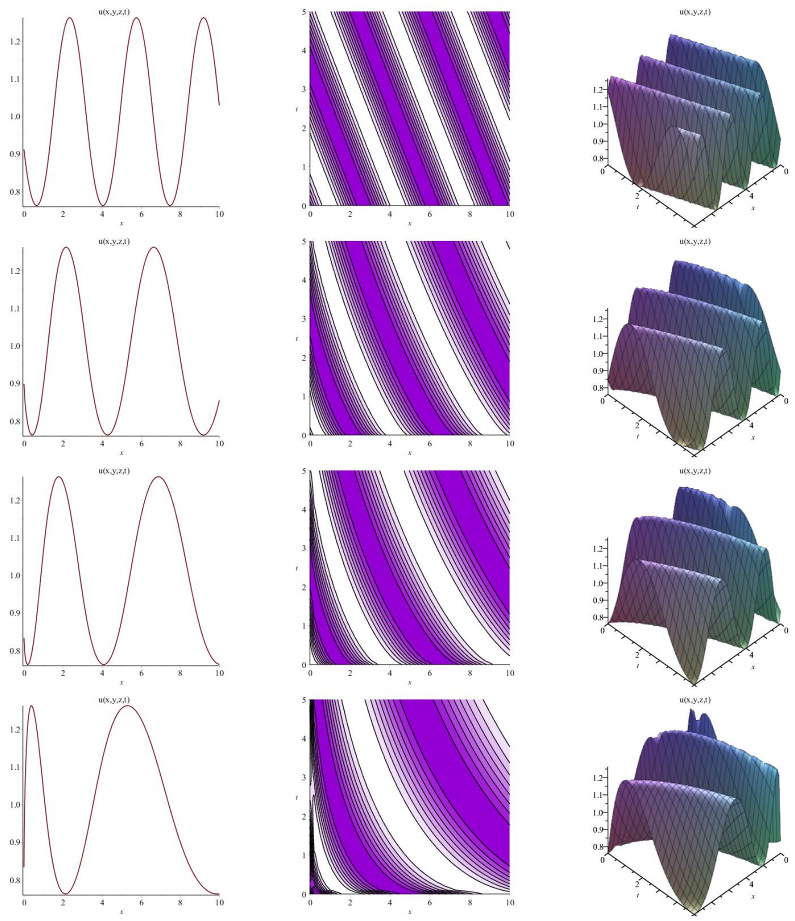

Figure 1. 2D and 3D plots of the periodic wave solutions for U, based on the Jacobi elliptic sine function solutions of Eq (2.21), and shown for various values of $\beta = .99, .8, .7, .5$, $\mu_3 = \mu_4 = \mu_5 = c = a = e_0 = 1, b = e_1 = .5, e_2 = .05$, and $m = 2$.

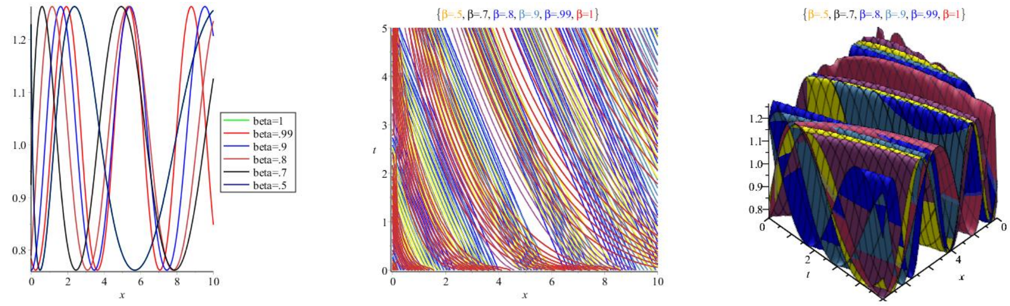

Figure 2. The periodic wave solutions for U, which are obtained from the Jacobi elliptic sine function solutions of Eq (2.21), are shown in 2D and 3D for various values of $\beta = 1, .99, .9, .8, .7, .5$.

Equation (2.21) describes a complicated superposition of Jacobi elliptic sine functions, and physically it corresponds to a periodic wave train that propagates through a nonlinear dispersive medium. Due to the presence of both sn and sn^2 terms in the expression, it would appear to be asymmetrical, as happens for shallow water waves, or in plasma physics where wave crests and troughs are not exactly symmetric due to higher-order nonlinear effects.

A degenerate periodic wave solution is produced when the parameter m approaches zero, leading to a simplification of Eq (2.21):

$$U(x,y,z,t) = 1 + \frac{1}{2}\sin(a\frac{x^\beta}{\beta} + b\frac{y^\beta}{\beta} + c\frac{z^\beta}{\beta} - k\frac{t^\beta}{\beta}) + 2\sin^2(a\frac{x^\beta}{\beta} + b\frac{y^\beta}{\beta} + c\frac{z^\beta}{\beta} - k\frac{t^\beta}{\beta}), \quad (2.23)$$

where $k = \frac{\mu_4 b^2 - \mu_5 c^2}{a}$.

Figures 3 and 4 clearly show this tendency by showing the periodic wave solutions of U in both 2D and 3D plots. The examples illustrate how the Jacobi elliptic sine function solutions shift in response to a change in the parameter β. The way in which this parameter influences the structure and shape of the waveforms that are created is demonstrated below.

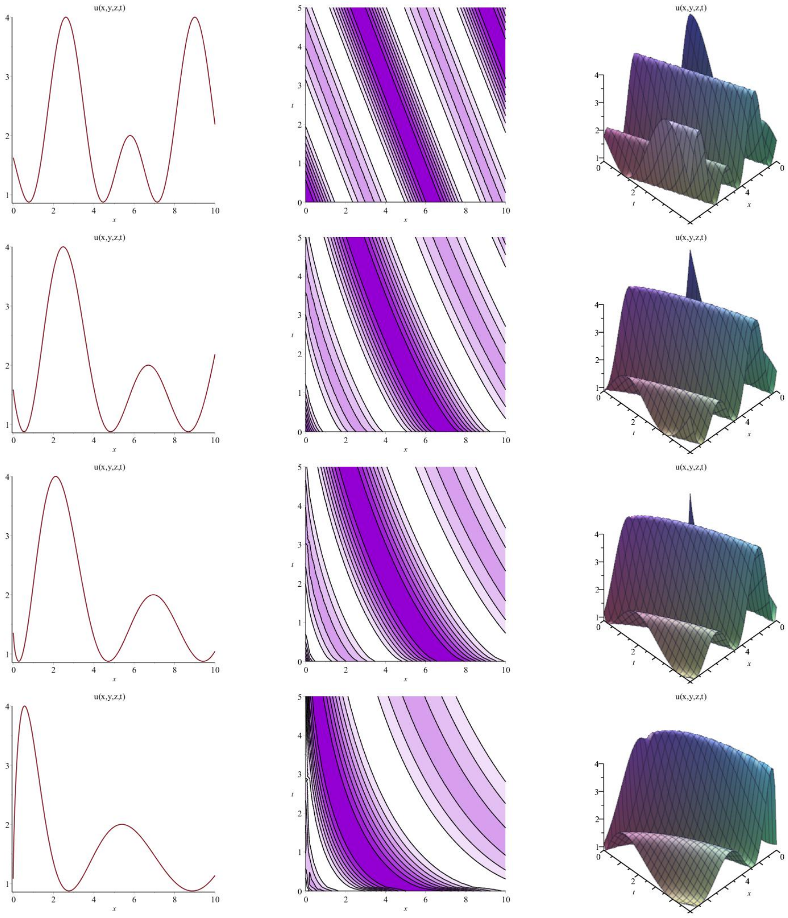

Figure 3. For several values of the parameter $\beta = .99, .8, .7, .5$, $\mu_3 = \mu_4 = \mu_5 = c = a = e_0 = e_1 = 1, e_2 = 2, b = .5$, and $m = 0$, 2D contour plots and 3D surface representations of the periodic wave solutions for U, obtained from Eq (2.23), are displayed.

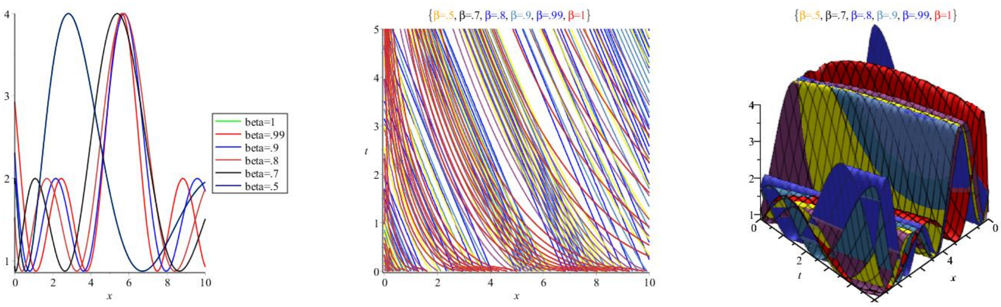

Figure 4. The results of the periodic wave equation U in Eq (2.23) for $beta = 1, 0.99, 0.9, 0.8, 0.7, 0.5$ are shown in a 2D contour map and a 3D simulation.

These visualizations highlight the effect of β on the structure and behavior of the wave patterns.

As far as Eq (2.23) is concerned, we have clarified the role of the combination $\sin\sin^2\ldots\sin^2$ in terms of an asymmetric periodic wave train characteristic of nonlinear dispersive media in (3+1) space.

Case 2. The values of d_0, d_1, d_2, d_3, and d_4 are $1-m^2, 0, 2m^2-1, 0$, and $-m^2$, respectively. By replacing the supposed solution of the form $H = cn(\xi, m)$, which represents the Jacobi elliptic cosine function with modulus m, into Eq (2.7), we obtain a category of periodic wave solutions governed by elliptic functions. This substitution changes the original equation into an identity that uses features of elliptic functions. This lets us figure out what the parameters need to be for the solution to work. As a result, we find a Jacobi elliptic wave solution to Eq (2.16), which is a periodic traveling wave whose properties are set by the modulus m and the wave parameters. These answers are especially helpful because they connect completely sinusoidal behavior (when $m = 0$) with solitary wave behavior (when $m = 1$). This gives us a wider range of exact solutions in the context of nonlinear wave theory. Following the substitution of the solution $H = cn(\xi, m)$ into Eq (2.7), we are able to obtain the Jacobi elliptic wave solution for Eq (2.16):

$$U(x,y,z,t) = \frac{1}{2} + \frac{1}{2}cn(a\frac{x^\beta}{\beta} + b\frac{y^\beta}{\beta} + c\frac{z^\beta}{\beta} - k\frac{t^\beta}{\beta}) + \frac{1}{2}cn^2(a\frac{x^\beta}{\beta} + b\frac{y^\beta}{\beta} + c\frac{z^\beta}{\beta} - k\frac{t^\beta}{\beta}). \quad (2.24)$$

The solution $U(x, y, z, t)$, as given in Eq (2.24), corresponds to a periodically wave-like profile in multiple dimensions, where there is an inherent bias or inhomogeneity in the crest-to-trough ratio of the wave profile. The inclusion of the $\frac{1}{2}$ constant term implies the propagation of the wave in a non-zero equilibrium position. The physical implication of the transition as the parameter β varies from 0.99 to 0.5 is an important step in understanding the role of the 'memory effect' of the medium on the phase velocities of the wave as they propagate in space.

Figures 5 and 6 show the typical forms of these solutions using both 2D and 3D plots to illustrate their shape and behavior. Figures 5 and 6 exhibit cosine function waveforms with their essential features.

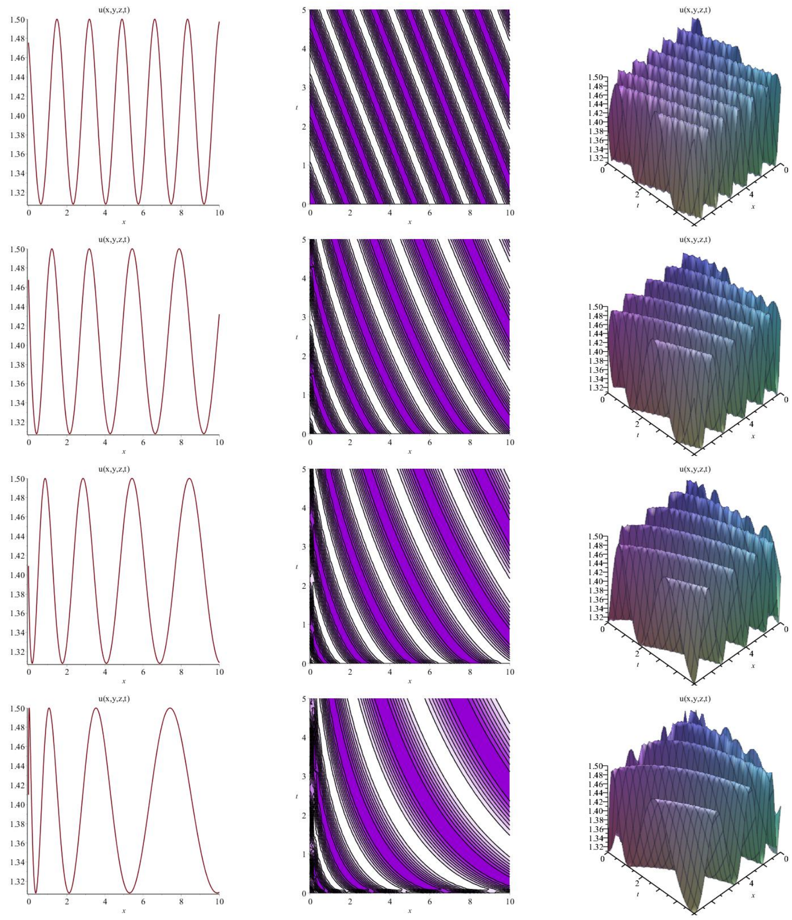

Figure 5. The 2D contour plot and 3D graph of the Jacobi elliptic cosine function solutions for Eq (2.24) are shown for various values of $\beta = 0.99, 0.8, 0.7, 0.5$, $\mu_3 = \mu_4 = \mu_5 = c = a = 1, b = e_0 = e_1 = e_2 = .5$, and $m = 2$.

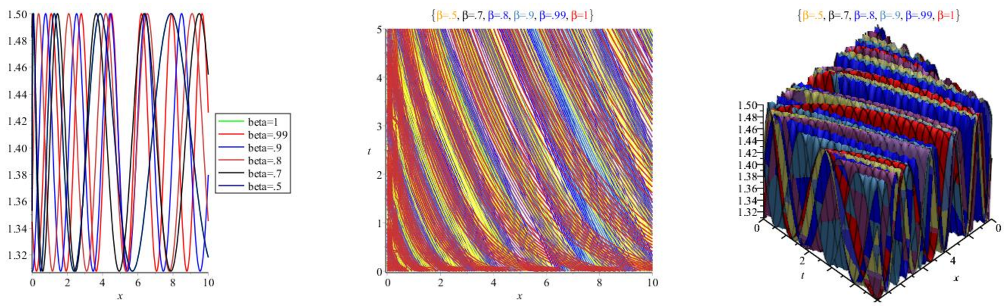

Figure 6. 2D and 3D visualizations of the cosine function solutions of the Jacobi elliptic functions for Eq (2.24) are illustrated for various values of $\beta = 1, 0.99, 0.9, 0.8, 0.7, 0.5$.

As the parameter nears 1, Eq (2.24) undergoes a mathematical degeneration, resulting in the periodic Jacobi elliptic function solutions ultimately transforming into localized waveforms. When $m = 1$, the elliptic functions transform into hyperbolic functions, resulting in soliton-like solutions. These solitary waves are unique because they stay in one place and don't spread out. They keep their shape and speed over time, which is a classic example of how solitons behave in nonlinear systems.

$$U(x,y,z,t) = 1 + sech(a\frac{x^\beta}{\beta} + b\frac{y^\beta}{\beta} + c\frac{z^\beta}{\beta} - k\frac{t^\beta}{\beta}) + sech^2(a\frac{x^\beta}{\beta} + b\frac{y^\beta}{\beta} + c\frac{z^\beta}{\beta} - k\frac{t^\beta}{\beta}), \quad (2.25)$$

where $k = \frac{\mu_4 b^2 - \mu_5 c^2}{a}$, the usual form of which can be seen in Figures 7 and 8.

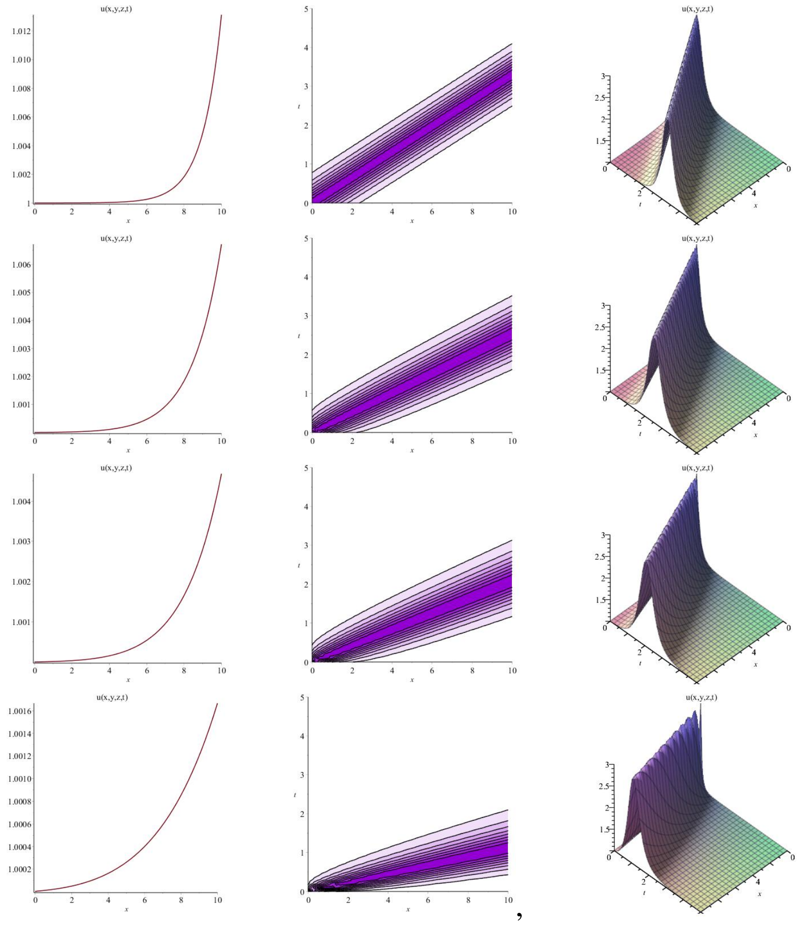

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Figure 7. Solitary wave solutions in 2D contour plot and 3D representation of Jacobi elliptic cosine function solutions for Eq (2.25) are shown for $\beta = 0.99, 0.8, 0.7, 0.5$, $\mu_3 = \mu_4 = \mu_5 = c = a = e_0 = e_1 = e_2 = m = 1$, and $b = 2$.

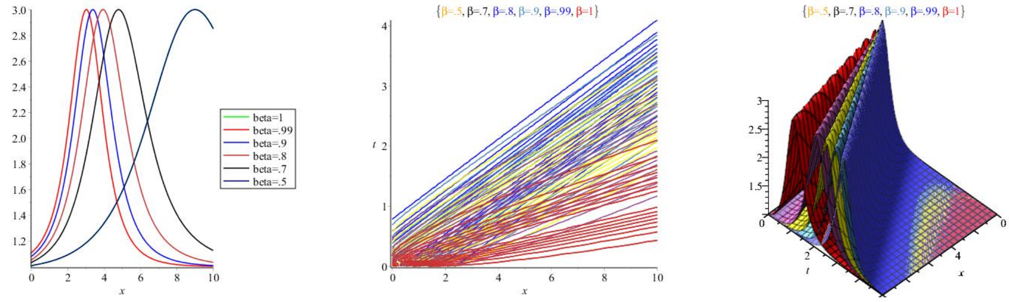

Figure 8. Shock wave solutions in 2D contour plot and 3D representation for Jacobi elliptic function solutions in Eq (2.25) for different $\beta = 1, 0.99, 0.9, 0.8, 0.7, 0.5$.

Case 3. If $d_0 = m^2$, then $d_1 = d_3 = 0$. Two variables, d_2 and d_4, are equal to negative infinity and one, respectively. The Jacobi elliptic solution for Eq (2.16) may be obtained by adding the solutions $H = ns(\xi, m)$ and $H = dc(\xi, m)$ to Eq (2.7).

$$U(x,y,z,t) = ns\left(a\frac{x^\beta}{\beta} + b\frac{y^\beta}{\beta} + c\frac{z^\beta}{\beta} - k\frac{t^\beta}{\beta}\right) + ns^2\left(a\frac{x^\beta}{\beta} + b\frac{y^\beta}{\beta} + c\frac{z^\beta}{\beta} - k\frac{t^\beta}{\beta}\right), \tag{2.26}$$

$$u(x,y,z,t) = dc\left(a\frac{x^\beta}{\beta} + b\frac{y^\beta}{\beta} + c\frac{z^\beta}{\beta} - k\frac{t^\beta}{\beta}\right) + dc^2\left(a\frac{x^\beta}{\beta} + b\frac{y^\beta}{\beta} + c\frac{z^\beta}{\beta} - k\frac{t^\beta}{\beta}\right), \tag{2.27}$$

illustrated in Figures 9 and 10 in their normal shape.

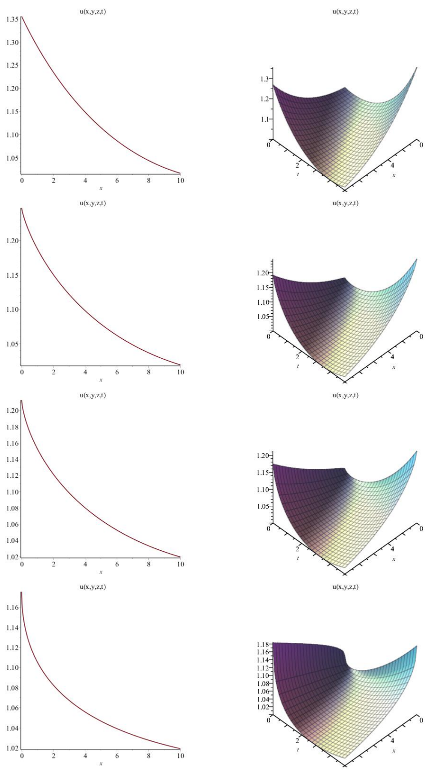

Figure 9. The Jacobi elliptic function solutions for U in Eq (2.27) are shown in 2D and 3D at different values of $\beta = 0.99, 0.8, 0.7, 0.5$, $e_0 = e_1 = e_2 = 1, b = .1, c = .005$, and $m = .02$.

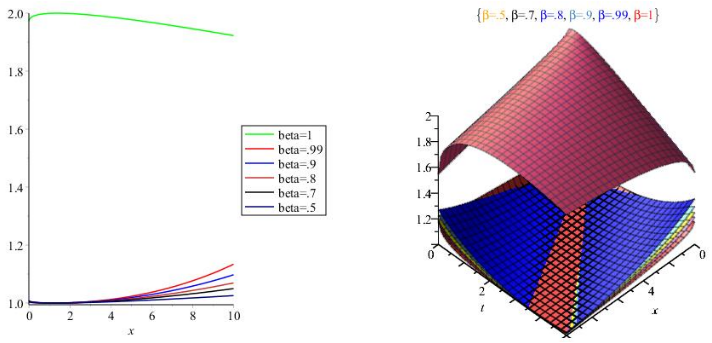

Figure 10. The 2D and 3D representations of the Jacobi elliptic function solutions for U in Eq (2.27) at various values of $\beta = 1, 0.99, 0.9, 0.8, 0.7, 0.5$, respectively.

As the parameter m approaches zero in Eq (2.27), the equation simplifies significantly, leading to periodic-like solutions

$$U(x, y, z, t) = 1 + \sec(a\frac{x^\beta}{\beta} + b\frac{y^\beta}{\beta} + c\frac{z^\beta}{\beta} - k\frac{t^\beta}{\beta}) + \sec^2(a\frac{x^\beta}{\beta} + b\frac{y^\beta}{\beta} + c\frac{z^\beta}{\beta} - k\frac{t^\beta}{\beta}), \qquad (2.28)$$

where $k = \frac{\mu_4 b^2 - \mu_5 c^2}{a}$. The solitary wave solutions are depicted in 2D and 3D, for which the typical shape is illustrated in Figures 11 and 12.

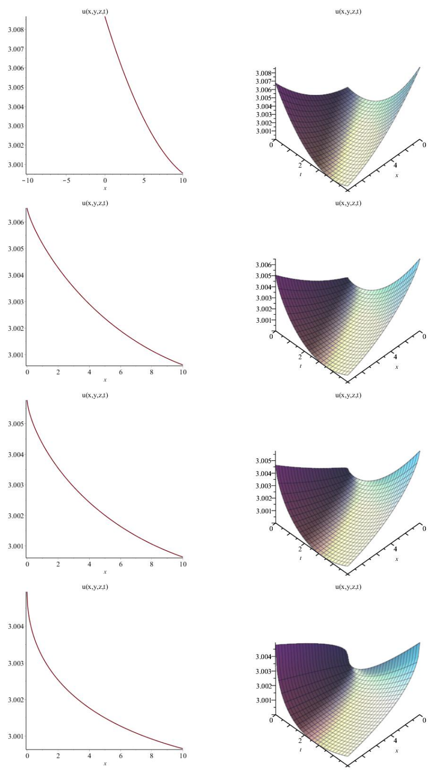

Figure 11. The solitary wave solutions as 2D and 3D representations of U in Eq (2.28) at various values of $\beta = 0.99, 0.8, 0.7, 0.5$, $e_0 = e_1 = e_2 = 1$, $b = .01$, $c = .0005$, and $m = 0$.

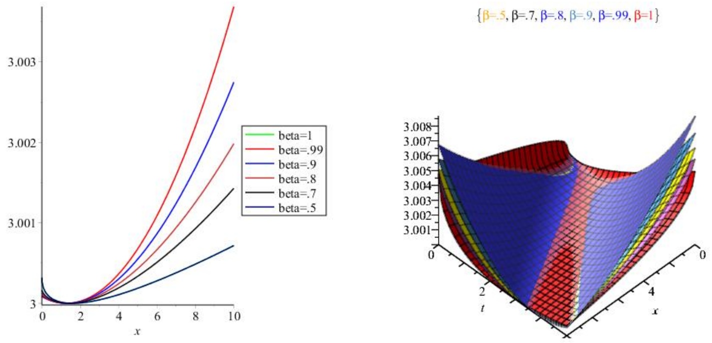

Figure 12. Solitary wave solutions in 2D and 3D for periodic wave solutions in Eq (2.28) for U with various $\beta = 1, 0.99, 0.9, 0.8, 0.7, 0.5$.

The solitary and periodic wave structures obtained in this work have significant implications for nonlinear physical systems. Specifically, the (3+1)-dimensional space-time fractional KdV-BBM has connections to certain physical behaviors of nonlinear dispersive waves, namely to waves of long wavelength in physical systems like shallow water waves, acoustic waves of long wavelength in inhomogeneous crystals, and particularly to magneto-acoustic waves propagating in plasmas. The transition of Jacobi elliptic periodic solutions to solitary wave solutions is physically meaningful as $m- > 1$; that is to say, localization occurs and has implications similar to those found in fiber optics, where waves or pulses (solitons) maintain their profile throughout long distances of propagation. The introduction of the fractional-order derivative β has strong implications to sub-diffusive behavior and memory effects, indicating that the velocity and damping of these types of solutions depend on the historical state of the medium, which is not apparently taken into account in classic integer-order models.

From the above plot, it is clear that a unique periodic solution exists. Here also, the regions of localized energy concentration or the blow-up points are represented by the terms *sec* and sec^2, which arise due to the (3+1)-dimensional medium. This is due to the constant offset represented by the term. Observing the regions during the transition from $\beta = 0.99$ to 0.5, it is quite obvious that the fractional order affects the positioning of the singular points observed in the solution, representing the high-intensity regions of the wave solution using the KdV-BBM equation.

The motivation for investigating these particular structures is graphically highlighted by the evolution of the wave profile. By tuning the fractional order along with the Jacobi modulus m, one can, in effect, model the physical deformation of a wave as one sends it through a complex medium. Our figures illustrate the fact that our model doesn't provide just one solution but, instead, a continuum of wave morphologies ranging from high-frequency periodic oscillations to stable high-energy solitary pulses.

2.1.1. Physical explanations of fractional solitary and periodic wave solutions of the KdV-BBM equation

The mathematical descriptions of various wave regimes may be found in Eqs (2.21)–(2.28). The outcomes of our application of the Jacobi elliptic function expansion technique to the space-time fractional KdV-BBM equation are shown in Figures 1–12. Solutions to traveling waves can be expressed as periodic functions, such as sin, cos, and tan. Solitons maintain their identity while interacting with other solitons, and they contain particle-like characteristics such as extended structures and magnetic monopoles, which distinguish them from other kinds of solitary waves.

The impact of fractional derivatives is to alter the dispersion relation, which gives the relation of wave frequency to wave number and modifies phase and group speeds of waves, and the speed of solitary and periodic waves that is a source of slower or faster propagation as shown in Figures 1–12. The form of the wave can become either more spread out or sharper, and the amplitude can decrease or increase, reflecting the nonlocal dispersive effects. Indeed, depending on its value, fractional orders can support either the stabilization or destabilization of wave solutions in ways that affect how long waves retain their shape.

As the parameter β changes, it changes the elliptic modulus m for the Jacobi elliptic sine function $sn(\xi|m)$. Further, as the parameter β increases while the elliptic modulus approaches 1, the waveform changes from a periodic sinusoid to a localized solitary wave with the Jacobi elliptic sine function defined as a hyperbolic tangent. Furthermore, as the parameter β decreases while the elliptic modulus decreases, the sinusoid becomes more periodic with a broader waveform structure.

Our investigation of the fractional order parameter β reveals its crucial role in governing wave characteristics: (i) Solution amplitudes exhibit monotonic growth with increasing β, affecting all Jacobi function types ($sn\xi, cn\xi$, and $dc\xi$ in Figures 1, 5, and 9, respectively), periodic wave trains (Figures 3, 4, 6, 8), and localized solitary waves (Figures 2, 7, 11, and 12); and (ii) wavepacket width displays significant β-dependent expansion, particularly for $sn\xi$-type solutions and shock waves. These coupled amplitude-width changes, with differing scaling methods, lead to β, a critical control parameter for nonlinear wave dynamics in fractional systems. It has direct control over energy distribution as well as spatiotemporal localization aspects. There are several forms of traveling wave solutions, including periodic, solitary, and bell-shaped soliton solutions.

Also, to understand the impact of the fractional order, the evolution of the solitary wave solution with various values of the fractional order $\beta \in (0, 1]$ is studied. It is noticed that the wave behavior with decreasing values of the fractional order β evolves with a decay/growth profile of the wave amplitude and 'expansion/contraction' of the characteristic width of the Jacobi elliptic function. From the above discussions, it is evident that the memory effect of the space-time fractional derivatives plays a major role in the balance between the nonlinearity and dispersion of the (3+1)-dimensional KdV-BBM equation. Unlike the integer-order results, the fractional-order results are found to show greater adaptability.

3. Conclusions

This study analyzes the space-time fractional KdV-BBM problem using Jacobi elliptic function expansion. Numerical analysis yields fractional solitary wave and periodic solutions, among others. In fractional calculus-governed systems, Figures 1–12 and Eqs (2.21)–(2.28) demonstrate several complex nonlinear wave behaviors. This technique enhances comprehension of phenomena, including solitary wave propagation, memory effects, and nonlinear dynamics in systems. This method uses fractional calculus to incorporate previous impacts and historical dependencies into system models, unlike standard methods. This allows detailed investigation of kink waves, pulse waves, and periodically moving waves, as well as nonlinear behavior. Our investigation reveals that the fractional order β significantly influences these wave patterns. This is why left and right waves propagate concurrently. Several solutions have graphs to show it. We observe several key effects as β changes: Solution amplitudes increase, especially in Jacobi elliptic functions like $sn\xi$, $cn\xi$, and $dc\xi$ (Figures 1, 5, and 9); periodic wave trains and solitary waveforms become more prominent; and wavepackets broaden, especially in $sn\xi$-type solutions. We see from this analysis that different values of β control wave amplitude, wave propagation direction, and wave energy distribution. Here, we see again the important role played by fractional calculus in the effective inclusion of memory and history effects. Notably, in the solution, changing β causes more intense wave packets, more spread-out periodic waves, and the shifting localization of waves.

While this study provides a comprehensive analysis of solitary and periodic wave structures for the (3+1)-dimensional space-time fractional KdV-BBM equation, several avenues for future research remain open. First, the application of the Hamiltonian-based method could be extended to explore the stochastic stability of these solutions under external perturbations. Furthermore, investigating the interaction dynamics between multiple solitons (N-soliton solutions) in fractional media may reveal complex phenomena such as pulse compression or energy exchange. Finally, extending the current framework to include non-local fractional operators, such as the Atangana-Baleanu or Caputo-Fabrizio derivatives, would provide a more nuanced understanding of the memory effects in higher-dimensional nonlinear systems.

Author contributions

Fahad Sameer Alshammari: Conceptualization, formal analysis, writing-review & editing, critical review; A. A. Elsadany: Conceptualization, graphic, drawing writing-review & editing, critical review; A. Aldurayhim: Methodology; formal analysis, writing-review & editing; Mohammed. K. Elboree: Formal analysis, writing-review & editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Funding information

The authors extend their appreciation to Prince Sattam bin Abdulaziz University for funding this research work through the project number (PSAU/2025/01/34258).

Availability of data and materials

The data used in this research are available/mentioned within the manuscript.

Acknowledgments

The authors extend their appreciation to Prince Sattam bin Abdulaziz University for funding this research work through the project number (PSAU/2025/01/34258).

Conflict of interest

The authors declare that they have no conflict of interest.

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