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Research article

Chaotic dynamics and stability analysis of the Von Foerster–Lasota PDE with conformable space–time derivatives in Orlicz space

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Abstract: This study rigorously investigates the asymptotic dynamics of systems governed by the space–time conformable Von Foerster–Lasota partial differential equation of order $\kappa \in (0, 1)$, where both the temporal and spatial variables are defined in the conformable sense. We establish sufficient conditions for the emergence of Devaney chaos and provide a detailed analysis of the strong stability of the associated solution κ-semigroup within Orlicz spaces induced by a convex Ψ-function. By employing Matuszewska–Orlicz indices, we offer a precise characterization of the long-term behavior of the generated κ-semigroup. Notably, this work bridges the gap between the studies of Dawidowicz and Poskrobko (2016), who considered Orlicz spaces for classical derivatives, and Elkhalloufy et al. (2025), who investigated conformable equations in Lebesgue spaces. By extending the conformable framework to the more general setting of Orlicz spaces, we offer new insights into the interplay between chaotic dynamics and stability in structured population models.

Keywords: chaos; stability; conformable derivative; κ-semigroup; Von Foerster Lasota equation; Orlicz space; Matuszewska-Orlicz indices
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1. Introduction

Fractional derivatives have been widely used to model memory-dependent and anomalous phenomena in physics, biology, and engineering. However, many classical definitions of fractional derivatives, such as the Riemann–Liouville or Caputo forms, are nonlocal and integral-based, which makes them computationally demanding and less intuitive for direct applications. To overcome these limitations, Khalil et al. [1] introduced the conformable derivative, a simple local operator that extends the classical derivative to fractional orders while preserving many of its fundamental properties. In particular, it satisfies linearity as well as the product, quotient, and chain rules, which makes it a convenient tool for the study of differential equations [1–3].

An important advantage of the conformable derivative is that it allows the definition of fractional derivatives at points where the classical derivative does not exist. For example, the function $f(\nu) = 2\sqrt{\nu}$ is not classically differentiable at $\nu = 0$, but it has a well-defined conformable derivative of order $1/2$, which equals 1. This observation has led to the interpretation of a "conformable tangent" at singular points and supports the use of conformable calculus in the analysis of dynamical systems involving local growth or velocity effects [4].

Another motivation for adopting the conformable derivative arises from the modeling diffusion processes in heterogeneous or complex media. In classical models, the mean square displacement of a particle satisfies $\langle \nu^2(\tau)\rangle \sim \tau$, whereas experimental studies in various systems, including biological tissues and porous media, report anomalous diffusion characterized by $\langle \nu^2(\tau)\rangle \sim \tau^{\kappa}$, with $\kappa \in (0, 1)$. Since standard diffusion equations are unable to describe such behavior accurately, conformable derivatives offer a flexible and effective framework for modeling subdiffusive processes [5]. Standard diffusion equations fail to capture this behavior, whereas conformable derivatives provide a convenient framework for describing such subdiffusive processes [5]. This makes conformable calculus relevant for both theoretical analysis and applied modeling in physics and biology.

Chaos, as a mathematical concept, describes the fact that deterministic systems may exhibit highly irregular and unpredictable behavior. Several notions of chaos have been proposed in the literature; in this work, we adopt the definition introduced by Devaney [6], which is based on three properties: Sensitivity to initial conditions, topological transitivity, and the density of periodic points. Banks et al. [7] later proved that topological transitivity together with a dense set of periodic points implies sensitivity to initial conditions, although the converse implication does not necessarily hold.

In the setting of operator theory, Desch, Schappacher, and Webb [8] initiated the study of linear chaos for strongly continuous semigroups of bounded operators by establishing spectral criteria for chaoticity. Their results stimulated further research on chaotic semigroups, hypercyclicity, and related dynamical properties in functional analysis [9–11]. More recently, these ideas have been extended to the framework of conformable semigroups. In particular, Menchih et al. [12] developed a theory of linear chaos within conformable calculus and derived necessary and sufficient conditions for chaos and hypercyclicity in conformable semigroups (or κ-semigroups). They also investigated chaotic dynamics in maturity-structured population models governed by conformable partial differential equations. Furthermore, they analyzed chaos in time and space conformable partial differential equations (PDEs) in specific Lebesgue spaces [13]. Related results on spatially conformable PDEs further confirmed the presence of hypercyclic and chaotic dynamics in the associated solution semigroup [14].

One of the most prominent models in structured population dynamics is the Von Foerster–Lasota equation, originally introduced by McKendrick [15] in 1926. This PDE describes the evolution of an age-structured population under growth and mortality effects and has played a central role in the mathematical theory of population dynamics. In its simplest form, the model is given by

$$\frac{\partial \breve{\omega}(\tau,\nu)}{\partial \tau} + \nu\frac{\partial \breve{\omega}(\tau,\nu)}{\partial \nu} = \aleph\breve{\omega}(\tau,\nu), \quad \nu \in [0,1],\ \tau \geq 0,\ \aleph \in \mathbb{R},$$

where $\breve{\omega}(\tau,\nu)$ denotes the population density of individuals of age ν at time τ, and $\aleph$ is a growth parameter. The Von Foerster–Lasota equation has been extensively studied in various functional settings, and it has been shown that chaotic behavior may arise under suitable conditions [16–18].

In our previous work [19], we investigated chaotic dynamics and strong stability for a conformable version of the Von Foerster–Lasota equation given by

$$\frac{\partial^\kappa \breve{\omega}}{\partial \tau^\kappa} + \Upsilon \nu^\kappa \frac{\partial^\kappa \breve{\omega}}{\partial \nu^\kappa} = \wp(\nu)\breve{\omega}, \quad \tau \geq 0,\ 0 \leq \nu \leq 1.$$

For a constant growth rate function $\wp(\nu) = \aleph$, we derived an explicit representation of the associated C_0-κ-semigroup in the Lebesgue space $L^p([0,1])$ and applied a conformable spectral criterion to obtain sufficient conditions for chaos. The analysis was subsequently extended to nonconstant growth rates, where it was shown that both chaotic behavior and strong stability persist under appropriate assumptions.

From a biological point of view, the interplay between chaos and stability in age-structured models has important implications. Chaotic dynamics may reflect unpredictable fluctuations in population sizes, leading to cycles of overpopulation and sudden decline. On the other hand, strong stability ensures long-term persistence and convergence to equilibrium states. Understanding the balance between these two behaviors is crucial for explaining real-world phenomena in ecology, epidemiology, and cellular dynamics. For instance, chaotic patterns may model sudden outbreaks in epidemics or irregular growth in cell populations, and stability conditions can describe extinction thresholds or controlled regulation mechanisms. This biological relevance provides strong motivation for extending the analysis to more general functional frameworks.

The primary objective of this work is to extend the dynamical analysis previously developed in the Lebesgue spaces $L^p([0,1])$ to the more general setting of Orlicz spaces $L^\Psi(0,1)$, where the classical L^p framework becomes insufficient. Unlike L^p spaces, which impose polynomial-type growth and integrability constraints, Orlicz spaces allow a much broader class of admissible population densities, including those exhibiting heavy-tailed age distributions or super-polynomial growth [20–23]. Such profiles naturally arise in biological systems characterized by nonstandard survival mechanisms, long-lived subpopulations, or rapid proliferation phases, for which the L^p norm may be infinite while the Orlicz norm remains finite and well defined [24, 25]. This functional flexibility is essential for capturing population dynamics that cannot be adequately described within L^p or Hilbert space settings. For instance, age distributions with exponential or near-exponential tails, which may occur in models of microbial growth, cancer cell populations, or epidemiological processes with persistent carriers, fall outside the scope of standard L^p spaces but can be naturally accommodated in suitable Orlicz spaces. To address these issues, we employ the Matuszewska–Orlicz indices [26, 27] as a key analytical tool to describe the asymptotic growth of the generating function Ψ and to identify sharp thresholds governing strong stability and chaotic dynamics. Although these indices have been successfully applied to the asymptotic analysis of the classical Von Foerster–Lasota equation [24], their use in the study of conformable fractional models and conformable solution semigroups is new. By combining conformable derivatives, which are well-suited for describing subdiffusive effects [5] and singular local dynamics [4], with the Orlicz space framework, the present approach offers a flexible and mathematically rigorous tool for modeling structured population dynamics beyond the limitations of classical functional settings.

The paper is structured as follows. Section 2 introduces the preliminary concepts, including conformable derivatives, κ-semigroups, chaos, and Orlicz spaces. In Section 3, we establish sufficient conditions for chaos and strong stability in the case of a constant growth rate function. Section 4 generalizes the analysis related to nonconstant growth functions, highlighting the influence of local behavior near zero. Finally, Section 5 concludes the paper by summarizing the main results and discussing their implications for structured population dynamics and fractional-type conformable systems.

2. Preliminaries

2.1. Conformable fractional derivatives and κ-semigroups

Definition 2.1. [1] The conformable derivative of a function $\breve{\omega} : (0, \infty) \to \mathbb{R}$ of order $\kappa \in (0, 1]$ at $\tau > 0$ is defined by

$$\frac{d^\kappa \breve{\omega}(\tau)}{d\tau^\kappa} := \lim_{\delta \to 0} \frac{\breve{\omega}(\tau + \delta\tau^{1-\kappa}) - \breve{\omega}(\tau)}{\delta},$$

provided the limit exists. Moreover, if the limit

$$\lim_{\tau \to 0^+} \frac{d^\kappa \breve{\omega}(\tau)}{d\tau^\kappa}$$

exists, it is called the conformable derivative of $\breve{\omega}$ at the point $\tau = 0$.

Theorem 2.2. *[1] Let $\kappa \in (0, 1]$. Assume that $d^\kappa \breve{\omega}(\tau)/d\tau^\kappa$ and $d^\kappa \upsilon(\tau)/d\tau^\kappa$ exist at $\tau \geq 0$. Then,*

(*i*) $d^\kappa(\breve{a}\breve{\omega} + \breve{b}\upsilon)(\tau)/d\tau^\kappa = \breve{a}(d^\kappa \breve{\omega}(\tau)/d\tau^\kappa) + \breve{b}(d^\kappa \upsilon(\tau)/d\tau^\kappa)$ *for all* $\breve{a}, \breve{b} \in \mathbb{R}$.
(*ii*) $d^\kappa(\breve{\omega}\upsilon)(\tau)/d\tau^\kappa = \upsilon(d^\kappa \breve{\omega}(\tau)/d\tau^\kappa) + \breve{\omega}(d^\kappa \upsilon(\tau)/d\tau^\kappa)$.
(*iii*) $d^\kappa(\breve{\omega}/\upsilon)(\tau)/d\tau^\kappa = \left(\upsilon(d^\kappa \breve{\omega}(\tau)/d\tau^\kappa) - \breve{\omega}(d^\kappa \upsilon(\tau)/d\tau^\kappa)\right)/\upsilon^2$.
(*iv*) $d^\kappa \mu/d\tau^\kappa = 0$ *for all* $\mu \in \mathbb{R}$.
(*v*) *If* $\breve{\omega}$ *is differentiable, then* $d^\kappa \breve{\omega}(\tau)/d\tau^\kappa = \tau^{1-\kappa}(d\breve{\omega}(\tau)/d\tau)$.

Definition 2.3. [3] Let $\breve{\omega} : \mathbb{R}^n \to \mathbb{R}$. The conformable partial derivative of order $0 < \kappa \leq 1$ with respect to $\tau_{\breve{\imath}}$ for $\breve{\imath} \in \{1, 2, \ldots, n\}$ is given by

$$\frac{\partial^\kappa \breve{\omega}}{\partial \tau_{\breve{\imath}}^\kappa}(\tau_1, \ldots, \tau_n) = \lim_{\delta \to 0} \frac{1}{\delta}\left[\breve{\omega}\left(\tau_1, \ldots, \tau_{\breve{\imath}-1}, \tau_{\breve{\imath}} + \delta\tau_{\breve{\imath}}^{1-\kappa}, \tau_{\breve{\imath}+1}, \ldots, \tau_n\right) - \breve{\omega}(\tau_1, \ldots, \tau_n)\right].$$

In particular, at $\tau_{\breve{\imath}} = 0$,

$$\frac{\partial^\kappa \breve{\omega}}{\partial \tau_{\breve{\imath}}^\kappa}(\tau_1, \ldots, 0, \ldots, \tau_n) = \lim_{\tau_{\breve{\imath}} \to 0} \frac{\partial^\kappa \breve{\omega}}{\partial \tau_{\breve{\imath}}^\kappa}(\tau_1, \ldots, \tau_{\breve{\imath}}, \ldots, \tau_n).$$

The concept of conformable semigroups was first introduced in 2015 [2]. In that work, the authors employed the newly proposed definition of the conformable derivative to establish a foundational notion of conformable semigroups of linear operators, which naturally extends the classical semigroup framework in the following way:

Definition 2.4. [2] Let $\kappa \in (0, \breve{a}]$, where $\breve{a} > 0$, and let $\breve{B}$ be a Banach space. A family of bounded linear operators $\{\top_\kappa(\tau)\}_{\tau \geq 0} \subset L(\breve{B}; \breve{B})$ is said to be a conformable κ-semigroup (or briefly a κ-semigroup) if the following conditions hold:

- $\top_\kappa(0) = I_{\breve{B}}$;
- For every $\breve{s}, \tau \in \mathbb{R}^+$, $\top_\kappa((\breve{s} + \tau)^{1/\kappa}) = \top_\kappa(\breve{s}^{1/\kappa})\top_\kappa(\tau^{1/\kappa})$.

Moreover, the κ-semigroup is called a strongly continuous κ-semigroup (or C_0-κ-semigroup) if for every $b \in \check{B}$,

$$\lim_{\tau\to 0^+} \top_\kappa(\tau)b = b.$$

It is evident that when $\kappa = 1$, the 1-semigroups are just the traditional semigroups.

In the theory of dynamical systems, quasi-conjugacy and conjugacy are fundamental equivalence relations. They allow one to analyze complex dynamical behaviors by relating a given system to another, often simpler and more tractable.

Definition 2.5. [11] Let $\{\top_\kappa(\tau)\}_{\tau\geq 0}$ and $\{\tilde{\top}_\kappa(\tau)\}_{\tau\geq 0}$ be two C_0-κ-semigroups acting on separable Banach spaces $\check{B}$ and $\check{A}$, respectively. We say that these κ-semigroups are quasi-conjugate if there exists a continuous map $\xi : \check{B} \longrightarrow \check{A}$ with a dense range such that

$$\top_\kappa(\tau) \circ \xi \;=\; \xi \circ \tilde{\top}_\kappa(\tau).$$

If, in addition, ξ can be chosen as a homeomorphism, then the two κ-semigroups are said to be conjugate.

2.2. *Conformable linear chaos*

In [12], Menchih et al. adopted the definition of Devaney's chaos for a C_0-κ-semigroup and proposed the following definition.

Definition 2.6. [12] In a Banach space $\check{B}$, the C_0-κ-semigroup $\{\top_\kappa(\tau)\}_{\tau\geq 0}$ is said to be chaotic if it satisfies the following properties:

(i) The existence of a dense set of periodic points $\check{B}_p = \{\breve{\omega} \in \check{B};$ there exists some $\tau_p > 0$ such that $\top_\kappa(\tau_p)\breve{\omega} = \breve{\omega}\}$.
(ii) Topological Transitivity, that is, if for every pair of open subsets $\mathcal{V}_1, \mathcal{V}_2 \subset \check{B}$, there exists $\tau > 0$ such that $\top_\kappa(\tau)\mathcal{V}_1 \cap \mathcal{V}_2 \neq \emptyset$.

Proposition 2.7. *[11] Under quasi-conjugacy and conjugacy, chaos is preserved.*

2.3. *Orlicz space*

Orlicz spaces L^Ψ, a natural extension of the classical Lebesgue spaces L^p, are generated by a function known as an Orlicz function or Ψ-function, which generalizes power functions. We review here the basic characteristics of Orlicz spaces L^Ψ and refer the reader to [28, 29] for more details.

Definition 2.8. [28, 29] Let $\check{X}$ be a real vector space. A convex modular is a functional $\varrho : \check{X} \to [0, \infty]$ that fulfills

- $\varrho(\check{x}) = 0$ if and only if $\check{x} = 0$,
- $\varrho(-\check{x}) = \varrho(\check{x})$,
- $\varrho(\check{a}\check{x} + \check{b}\check{y}) \leqslant \check{a}\varrho(\check{x}) + \check{b}\varrho(\check{y})$ for $\check{x}, \check{y} \in X, \check{a}, \check{b} \geqslant 0, \check{a} + \check{b} = 1$.

The subspace that ϱ generates is the modular space, defined by

$$\check{X}_\varrho = \left\{\check{x} \in \check{X} : \lim_{\theta\to 0} \varrho(\theta\check{x}) = 0\right\}.$$

Definition 2.9. [28, 29] A function $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ is called a Ψ-*function* if it is continuous, nondecreasing, and satisfies the following conditions:

- $\Psi(0) = 0$,
- $\Psi(\nu) > 0$ for $\nu > 0$,
- $\lim_{\nu\to\infty} \Psi(\nu) = +\infty$.

For Orlicz functions, the $\triangle_2$-condition is a growth condition. It is crucial to the study of Orlicz space geometry.

Definition 2.10. [28, 29] A Ψ-function is said to satisfy the $\triangle_2$-condition if, for some $c > 0$, we have

$$\Psi(2\nu) \leqslant c\Psi(\nu), \quad \text{for any } 0 \leqslant \nu.$$

Proposition 2.11. *[28, 29] Let $\check{X}$ be the set of all real-valued, γ-measurable, and finite μ-almost everywhere functions on $\check{\Omega}$, with equality μ-almost everywhere. Then, for every $\check{x} \in \check{X}$,*

$$\varrho(\check{x}) = \int_{\check{\Omega}} \Psi(|\check{x}(\tau)|)d\mu \tag{2.1}$$

is a modular in $\check{X}$. Moreover, if Ψ is a convex function, then ϱ is a convex modular in $\check{X}$.

Definition 2.12. [28, 29] Let ϱ be the modular given by (2.1). The corresponding modular space $\check{X}_\varrho$ is referred to as an Orlicz space and is denoted by $L^\Psi(\check{\Omega}, \gamma, \mu)$ (or simply L^Ψ), where

$$L^\Psi = \left\{ \check{x} \in \check{X} : \int_{\check{\Omega}} \Psi(\theta|\check{x}(\tau)|)\, d\mu \longrightarrow 0 \quad \text{as } \theta \to 0^+ \right\}. \tag{2.2}$$

In addition, the subset

$$L_0^\Psi = \left\{ \check{x} \in \check{X} : \int_{\check{\Omega}} \Psi(|\check{x}(\tau)|)\, d\mu < \infty \right\} \tag{2.3}$$

is called the Orlicz class.

Moreover, $L_0^\Psi = L^\Psi$ if and only if the Ψ-function Ψ satisfies the $\triangle_2$-condition. To illustrate the functional versatility of the Orlicz framework, we present several concrete examples of Ψ-functions commonly employed in the literature:

- Polynomial and exponential power functions: The most standard Ψ-functions include the power functions $\Psi(\nu) = \nu^p$ for $p \geq 1$. In more rapid growth scenarios, exponential power functions defined by $\Psi_\alpha(\nu) := \exp(\nu^\alpha) - 1$ for $\alpha \geq 1$ are utilized. Specifically, Ψ_2 and Ψ_1 characterize sub-Gaussian and subexponential tail behaviors, respectively.
- For certain deviation inequalities in empirical process theory, van de Geer and Lederer [22] utilized a more complex construction:

$$\Psi(\nu) = \exp\left(\frac{(\sqrt{1 + 2K\nu} - 1)^2}{K} \right),$$

where K is a positive constant.

- To derive maximal inequalities of the Bennett type, Wellner [23] employed the following function:

$$\Psi(v) = \exp\left(2K^{-2}\mathfrak{h}(Kv)\right) - 1,$$

where the auxiliary function $\mathfrak{h}(v)$ is defined as $\mathfrak{h}(v) := (1+v)\log(1+v) - v$.

Theorem 2.13. *[28, 29] Assume that Ψ is convex. Then, the functional*

$$\|\check{x}\|^L = \inf\left\{\check{s} > 0 : \int_{\check{\Omega}} \Psi\left(\frac{|\check{x}(\tau)|}{\check{s}}\right) d\mu \leq 1\right\}$$

defines a norm on L^Ψ, referred to as the Luxemburg norm. Moreover, the space $(L^\Psi, \|\cdot\|^L)$ is a Banach space.

Theorem 2.14. [28] *Let $(\check{x}_k)_{k\in\mathbb{N}}$ be a sequence in L^Ψ equipped with the Luxemburg norm. Then, $(\check{x}_k)$ converges to zero in norm if and only if*

$$\lim_{k\to\infty} \varrho(\theta\check{x}_k) = 0 \qquad \textit{for all } \theta > 0.$$

Remark 1. [24, 28] Assume that $\mu(\check{\Omega}) < \infty$. We then have the inclusion $L_0^\Psi(\check{\Omega}) \subset L_0^\varphi(\check{\Omega})$ if and only if

$$\limsup_{u\to\infty} \frac{\varphi(u)}{\Psi(u)} < \infty.$$

Consequently, two Ψ-functions satisfying the $\triangle_2$-condition generate the same Orlicz space whenever they differ only on a finite subset of $\check{\Omega}$. As an illustration, we have $L^\varphi = L^\Psi$, where

$$\varphi(\tau) = \begin{cases} \Psi(\tau), & \tau \geqslant 1, \\ \Psi(1)\,\tau^p, & \tau < 1, \end{cases}$$

with $p \geqslant 0$ and Ψ satisfying the $\triangle_2$-condition. Therefore, replacing a Ψ-function by another that differs only on a finite subset of $\check{\Omega}$ does not affect the asymptotic behavior of the associated Orlicz space L^Ψ.

The growth and scaling properties of a Ψ-function are effectively characterized by the Matuszewska–Orlicz indices [26, 27]. To ensure a clear and consistent notation throughout this study, we adopt the following unified formulation as introduced in [30].

Definition 2.15. [30] Let Ψ be a Ψ-function. The lower Matuszewska–Orlicz index of Ψ is defined as

$$p_\Psi = \sup\left\{\check{p} : \exists\, C > 0 \text{ such that } \Psi(\check{a}\tau) \geq C\,\check{a}^{\check{p}}\,\Psi(\tau), \quad \forall\,\tau \geq 0,\ \check{a} \geq 1\right\}.$$

Similarly, the upper Matuszewska–Orlicz index of Ψ is defined by

$$q_\Psi = \inf\left\{\check{q} : \exists\, C < \infty \text{ such that } \Psi(\check{a}\tau) \leq C\,\check{a}^{\check{q}}\,\Psi(\tau), \quad \forall\,\tau \geq 0,\ \check{a} \geq 1\right\}.$$

It is immediate that the Matuszewska–Orlicz indices satisfy $0 \leq p_\Psi \leq q_\Psi \leq \infty$.

Proposition 2.16. [28] *An Orlicz function Ψ satisfies the $\triangle_2$-condition if and only if $q_\Psi < \infty$.*

3. Chaotic dynamics and strong stability in the case of a constant growth rate function

In this section, we restrict our attention to Devaney's definition of chaos and generalize the results previously obtained in [19] for the Lebesgue space $L^p([0,1])$ to the framework of Orlicz spaces. More precisely, we investigate the Orlicz space $L^\Psi(0,1)$, where the convex function Ψ is assumed to satisfy the $\triangle_2$-condition. Our objective is to analyze the occurrence of chaotic dynamics within the κ-semigroup associated with the conformable model under consideration and its strong stability, focusing on the case of a constant growth coefficient $\wp(\nu) = \aleph \in \mathbb{R}$. The dynamics are governed by the following conformable partial differential equation:

$$\begin{cases} \dfrac{\partial^\kappa \breve{\omega}(\tau,\nu)}{\partial \tau^\kappa} + \Upsilon \nu^\kappa \dfrac{\partial^\kappa \breve{\omega}(\tau,\nu)}{\partial \nu^\kappa} = \aleph\, \breve{\omega}(\tau,\nu), \quad \nu \in [0,1],\ 0<\kappa\le 1,\ \tau,\Upsilon \in \mathbb{R}^+, \\ \breve{\omega}(0,\nu) = \breve{x}(\nu), \quad \breve{x} \in L^\Psi(0,1). \end{cases} \tag{3.1}$$

In [19], it was established that the solution of (3.1) is given by

$$\top_\kappa(\tau)\breve{x}(\nu) =: \breve{\omega}(\tau,\nu) = e^{\aleph \frac{\tau^\kappa}{\kappa}}\, \breve{x}\left(\nu e^{-\Upsilon \frac{\tau^\kappa}{\kappa}}\right), \quad \breve{x} \in L^p(0,1) \tag{3.2}$$

and that the family $\{\top_\kappa(\tau)\}_{\tau \ge 0}$ defines a C_0-κ-semigroup in $L^p(0,1)$. The next lemma extends this result to Orlicz spaces.

Lemma 3.1. *The operator family $\{\top_\kappa(\tau)\}_{\tau\ge 0}$ is defined on the space $L^\Psi(0,1)$ by*

$$\top_\kappa(\tau)\breve{x}(\nu) = e^{\aleph \frac{\tau^\kappa}{\kappa}}\, \breve{x}\left(\nu e^{-\Upsilon \frac{\tau^\kappa}{\kappa}}\right), \tag{3.3}$$

which is a C_0-κ-semigroup on $L^\Psi(0,1)$.

Proof. Letting $\breve{x} \in L^\Psi(0,1)$ and $\theta > 0$, we have

$$\begin{aligned} \varrho_{[0,1]}(\theta \top_\kappa(\tau)\breve{x}) &= \int_0^1 \Psi\left(\theta \left|\top_\kappa(\tau)\breve{x}(\nu)\right|\right) d\nu \\ &= \int_0^1 \Psi\left(\theta \left| e^{\aleph \frac{\tau^\kappa}{\kappa}}\, \breve{x}\left(\nu e^{-\Upsilon \frac{\tau^\kappa}{\kappa}}\right)\right|\right) d\nu \\ &= e^{\Upsilon \frac{\tau^\kappa}{\kappa}} \int_0^{e^{-\Upsilon \frac{\tau^\kappa}{\kappa}}} \Psi\left(\theta e^{\aleph \frac{\tau^\kappa}{\kappa}} |\breve{x}(\nu)|\right) d\nu. \end{aligned}$$

We distinguish two cases:

Case $\aleph < 0$. Since $e^{\aleph \tau^\kappa/\kappa} < 1$, and Ψ is nondecreasing, we obtain

$$\varrho_{[0,1]}(\theta \top_\kappa(\tau)\breve{x}) \le e^{\Upsilon \frac{\tau^\kappa}{\kappa}} \int_0^1 \Psi(\theta |\breve{x}(\nu)|) d\nu.$$

Case $\aleph \ge 0$. As $e^{\aleph \tau^\kappa/\kappa} \ge 1$, Definition 2.15 ensures the existence of $C > 0$ such that

$$\Psi\left(\theta e^{\aleph \frac{\tau^\kappa}{\kappa}} |\breve{x}(\nu)|\right) \le C e^{\aleph q_\Psi \frac{\tau^\kappa}{\kappa}} \Psi(\theta |\breve{x}(\nu)|).$$

Hence,

$$\varrho_{[0,1]}(\theta\top_\kappa(\tau)\check{x}) \le Ce^{(\Upsilon+\aleph q_\Psi)\frac{\tau^\kappa}{\kappa}}\int_0^1 \Psi(\theta|\check{x}(\nu)|)d\nu.$$

Since $\check{x} \in L^\Psi(0,1)$, it follows that

$$\lim_{\theta\to 0^+}\varrho_{[0,1]}(\theta\top_\kappa(\tau)\check{x}) = 0,$$

which shows that $\top_\kappa(\tau)\check{x} \in L^\Psi(0,1)$. Moreover,

$$\|\top_\kappa(\tau)\check{x}\|^L_{[0,1]} \le \begin{cases} Ce^{(\Upsilon+\aleph q_\Psi)\frac{\tau^\kappa}{\kappa}}\|\check{x}\|^L_{[0,1]} & \text{if } \aleph \ge 0,\\ e^{\Upsilon\frac{\tau^\kappa}{\kappa}}\|\check{x}\|^L_{[0,1]} & \text{if } \aleph < 0\end{cases}$$

so that $\top_\kappa(\tau) \in \mathcal{L}(L^\Psi(0,1); L^\Psi(0,1))$. It remains to prove strong continuity. Let $\check{x} \in L^\Psi(0,1)$ and $\varepsilon > 0$. Since Ψ satisfies the Δ_2-condition, by Remark 8.15 in [29], the set S of simple, integrable functions is dense in $L^\Psi([0,1])$. Given that the measure is finite on $[0,1]$, the space of continuous functions $C[0,1]$ is dense in set S, so $C[0,1]$ is also dense in $L^\Psi([0,1])$ with respect to the Luxemburg norm. Then, there exists $\phi \in C([0,1])$ such that

$$\|\check{x}-\phi\|^L_{[0,1]} < \frac{\varepsilon}{6}.$$

As the function ϕ is continuous on $[0,1]$, it is uniformly continuous on $[0,1]$. Then, by using the dominated convergence theorem and the properties of the Ψ-function, one shows that

$$\begin{aligned}\lim_{\tau\to 0^+}\int_0^1 \Psi\left(\theta\left|\top_\kappa(\tau)\phi(\nu)-\phi(\nu)\right|\right)d\nu &= \int_0^1 \lim_{\tau\to 0^+}\Psi\left(\theta\left|e^{\aleph\frac{\tau^\kappa}{\kappa}}\phi(xe^{-\Upsilon\frac{\tau^\kappa}{\kappa}})-\phi(\nu)\right|\right)d\nu\\ &= \int_0^1 \Psi(0)d\nu\\ &= 0.\end{aligned}$$

Then, $\lim_{\tau\to 0^+}\|\top_\kappa(\tau)\phi-\phi\|^L_{[0,1]} = 0$. Consequently, there exists $\delta > 0$ such that, for all $\tau \in (0,\delta)$,

$$\|\top_\kappa(\tau)\phi-\phi\|^L_{[0,1]} < \frac{\varepsilon}{2}.$$

There are two cases to distinguish. If $\aleph < 0$, we have

$$\begin{aligned}\|\top_\kappa(\tau)\check{x}-\check{x}\|^L_{[0,1]} &\le \|\top_\kappa(\tau)(\check{x}-\phi)\|^L_{[0,1]} + \|\top_\kappa(\tau)\phi-\phi\|^L_{[0,1]} + \|\phi-\check{x}\|^L_{[0,1]}\\ &\le e^{\Upsilon\frac{\tau^\kappa}{\kappa}}\|\check{x}-\phi\|^L_{[0,1]} + \|\top_\kappa(\tau)\phi-\phi\|^L_{[0,1]} + \|\check{x}-\phi\|^L_{[0,1]}\\ &\le \left(e^{\Upsilon\frac{\tau^\kappa}{\kappa}}+1\right)\|\check{x}-\phi\|^L_{[0,1]} + \|\top_\kappa(\tau)\phi-\phi\|^L_{[0,1]}\\ &< (2+1)\frac{\varepsilon}{6}+\frac{\varepsilon}{2} = \varepsilon\end{aligned}$$

for $0 < \tau < \min\{\delta, (\kappa\ln(2)/\Upsilon)^{1/\kappa}\}$.
If $\aleph \ge 0$, we have

$$\begin{aligned}\|\top_\kappa(\tau)\check{x}-\check{x}\|^L_{[0,1]} &\le Ce^{(\Upsilon+\aleph q_\Psi)\frac{\tau^\kappa}{\kappa}}\|\check{x}-\phi\|^L_{[0,1]} + \|\top_\kappa(\tau)\phi-\phi\|^L_{[0,1]} + \|\check{x}-\phi\|^L_{[0,1]}\\ &\le \left(Ce^{(\Upsilon+\aleph q_\Psi)\frac{\tau^\kappa}{\kappa}}+1\right)\|\check{x}-\phi\|^L_{[0,1]} + \|\top_\kappa(\tau)\phi-\phi\|^L_{[0,1]}\\ &< (2+1)\frac{\varepsilon}{6}+\frac{\varepsilon}{2} = \varepsilon,\end{aligned}$$

for $0<\tau<\min\{\delta,(\kappa\ln(2/C)/(\Upsilon+\aleph q_\Psi))^{1/\kappa}\}$.
Thus, $\{\top_\kappa(\tau)\}_{\tau\geq 0}$ is a strongly continuous κ-semigroup on $L^\Psi(0,1)$. This completes the proof. □

Theorem 3.2. *Assume that $\aleph>-\Upsilon/q_\Psi$, where q_Ψ denotes the upper Matuszewska–Orlicz index associated with the Ψ-function. Then, for every $\tau_p>0$, the κ-semigroup $\{\top_\kappa(\tau)\}_{\tau\geq 0}$ defined in* (3.3) *admits a periodic point $\check{x}_p\in L^\Psi(0,1)$.*

Proof. Let us consider a function $\check{x}\in L^\Psi(e^{-\Upsilon\tau_p^\kappa/\kappa},1)$. We construct $\check{x}_p$ on $(0,1]$ by decomposing the interval as $(0,1]=\bigcup_{m=0}^{\infty}(e^{-\Upsilon(m+1)\tau_p^\kappa/\kappa},e^{-\Upsilon m\tau_p^\kappa/\kappa}]$ and by rescaling $\check{x}$ appropriately inside each subinterval. More precisely, we define

$$\check{x}_p(\nu)=\begin{cases} e^{-m\aleph\frac{\tau_p^\kappa}{\kappa}}\,\check{x}\Big(\nu e^{-\Upsilon m\frac{\tau_p^\kappa}{\kappa}}\Big) & \text{if } \nu\in(e^{-\Upsilon(m+1)\frac{\tau_p^\kappa}{\kappa}},\, e^{-\Upsilon m\frac{\tau_p^\kappa}{\kappa}}],\\ \check{x}(\nu) & \text{if } \nu\in(e^{-\Upsilon m\frac{\tau_p^\kappa}{\kappa}},\,1]. \end{cases} \tag{3.4}$$

It remains to check that $\check{x}_p$ indeed belongs to $L^\Psi(0,1)$. For $\theta>0$, we compute

$$\begin{aligned}\varrho_{[0,1]}(\theta\check{x}_p)&=\int_0^1\Psi(\theta|\check{x}_p(\nu)|)d\nu\\ &=\sum_{m=0}^{\infty}\int_{e^{-\Upsilon(m+1)\frac{\tau_p^\kappa}{\kappa}}}^{e^{-\Upsilon m\frac{\tau_p^\kappa}{\kappa}}}\Psi(\theta|\check{x}_p(\nu)|)\,d\nu\\ &=\sum_{m=0}^{\infty}\int_{e^{-\Upsilon(m+1)\frac{\tau_p^\kappa}{\kappa}}}^{e^{-\Upsilon m\frac{\tau_p^\kappa}{\kappa}}}\Psi\Big(\theta e^{-m\aleph\frac{\tau_p^\kappa}{\kappa}}\left|\check{x}\,(\nu e^{\Upsilon m\frac{\tau_p^\kappa}{\kappa}})\right|\Big)\,d\nu\\ &=\sum_{m=0}^{\infty}e^{-\Upsilon m\frac{\tau_p^\kappa}{\kappa}}\int_{e^{-\Upsilon\frac{\tau_p^\kappa}{\kappa}}}^{1}\Psi\Big(\theta e^{-m\aleph\frac{\tau_p^\kappa}{\kappa}}|\check{x}(\nu)|\Big)d\nu.\end{aligned}$$

In light of the Definition 2.15, if $-\Upsilon/q_\Psi<\aleph<0$, then there exists $\check{C}>0$ such that

$$\Psi\Big(\theta e^{-m\aleph\frac{\tau_p^\kappa}{\kappa}}|\check{x}(\nu)|\Big)\leq\check{C}e^{-m\aleph q_\Psi\frac{\tau_p^\kappa}{\kappa}}\Psi(\theta|\check{x}(\nu)|).$$

Consequently,

$$\varrho_{[0,1]}(\theta\check{x}_p)\leq\check{C}\varrho_{\left[e^{-\Upsilon\frac{\tau_p^\kappa}{\kappa}},1\right]}(\theta\check{x})\sum_{m=0}^{\infty}e^{-m\frac{\tau_p^\kappa}{\kappa}(\Upsilon+\aleph q_\Psi)}.$$

However, for $\aleph\geq 0$ we simply obtain

$$\varrho_{[0,1]}(\theta\check{x}_p)\leq\varrho_{\left[e^{-\Upsilon\frac{\tau_p^\kappa}{\kappa}},1\right]}(\theta\check{x})\sum_{m=0}^{\infty}e^{-\Upsilon m\frac{\tau_p^\kappa}{\kappa}}.$$

In both cases, the infinite series is convergent. Since $\check{x}\in L^\Psi(e^{-\Upsilon\tau_p^\kappa/\kappa},1)$, it follows that $\lim_{\theta\to 0^+}\varrho_{[0,1]}(\theta\check{x}_p)=0$, which ensures that $\check{x}_p\in L^\Psi(0,1)$. This completes the proof. □

Theorem 3.3. *Suppose that $\aleph>-\Upsilon/q_\Psi$. Then, the set of periodic points of the κ-semigroup $\{\top_\kappa(\tau)\}_{\tau\geq 0}$ forms a dense subset of the Orlicz space $L^\Psi(0,1)$.*

Proof. Let $\check{x} \in L^{\Psi}(0,1)$ and fix $\varepsilon > 0$. We consider the function $\check{x}_p$ defined in (3.4), which is periodic by construction. Next, choose $\tau_p > 0$ sufficiently large so that

$$\|\check{x}\|^{L}_{[0,\, e^{-\Upsilon \frac{\tau_p^{\kappa}}{\kappa}}]} < \frac{\varepsilon}{2} \quad \text{and} \quad \|\check{x}_p\|^{L}_{[0,\, e^{-\Upsilon \frac{\tau_p^{\kappa}}{\kappa}}]} < \frac{\varepsilon}{2}.$$

Notice that on the interval $[e^{-\Upsilon t_0^{\kappa}/\kappa}, 1]$, we have $\check{x}_p(\nu) = \check{x}(\nu)$. Hence,

$$\begin{aligned}\|\check{x}_p - \check{x}\|^{L}_{[0,1]} &= \|\check{x}_p - \check{x}\|^{L}_{[0,\, e^{-\Upsilon \frac{t_0^{\kappa}}{\kappa}}]} \\ &\leq \|\check{x}_p\|^{L}_{[0,\, e^{-\Upsilon \frac{\tau_p^{\kappa}}{\kappa}}]} + \|\check{x}\|^{L}_{[0,\, e^{-\Upsilon \frac{\tau_p^{\kappa}}{\kappa}}]} \\ &< \varepsilon.\end{aligned}$$

Thus, given any $\check{x} \in L^{\Psi}(0,1)$ and any $\varepsilon > 0$, we can find a periodic function $\check{x}_p$ arbitrarily close to $\check{x}$ in the Luxemburg norm. This establishes that periodic points are dense in $L^{\Psi}(0,1)$. □

Theorem 3.4. *Suppose that $\aleph > -\Upsilon/q_{\Psi}$. Then, the κ-semigroup $\{\top_{\kappa}(\tau)\}_{t\geq 0}$ is topologically transitive on $L^{\Psi}(0,1)$.*

Proof. Let $\mathcal{V}(\check{x}_{\check{\imath}}, \varepsilon_{\check{\imath}}) = \{\gamma \in L^{\Psi}(0,1) : \|\gamma - \check{x}_{\check{\imath}}\|^{L} < \varepsilon_{\check{\imath}}\}$ for $\check{\imath} = 1,2$ be two nonempty open balls. Our objective is to demonstrate that, at a time $\tau > 0$, there exists a function $\check{y} \in \mathcal{V}(\check{x}_1, \varepsilon_1)$ such that $\top_{\kappa}(\tau)g \in \mathcal{V}(\check{x}_2, \varepsilon_2)$. Fix $\tau > 0$ and define $\check{y} : (0,1] \to \mathbb{R}$ by

$$\check{y}(\nu) = \begin{cases} e^{-\aleph \frac{\tau^{\kappa}}{\kappa}}\, \check{x}_2(\nu\, e^{\Upsilon \frac{\tau^{\kappa}}{\kappa}}), & \nu < e^{-\Upsilon \frac{\tau^{\kappa}}{\kappa}}, \\ \check{x}_1(\nu), & \nu \geq e^{-\Upsilon \frac{\tau^{\kappa}}{\kappa}}. \end{cases}$$

For any $\theta > 0$,

$$\varrho_{[0, e^{-\Upsilon \frac{\tau^{\kappa}}{\kappa}}]}(\theta \check{y}) = e^{-\Upsilon \frac{\tau^{\kappa}}{\kappa}} \int_0^1 \Psi\big(\theta e^{-\aleph \frac{\tau^{\kappa}}{\kappa}}\, |\check{x}_2(\nu)|\big)\, d\nu.$$

If $-\Upsilon/q_{\Psi} < \aleph < 0$, then Definition 2.15 of upper Matuszewska–Orlicz index yields a constant $\tilde{C} > 0$ with

$$\Psi(\theta e^{-\aleph \frac{\tau^{\kappa}}{\kappa}} |\check{x}_2(\nu)|) \leq \tilde{C}\, e^{\aleph q_{\Psi} \frac{\tau^{\kappa}}{\kappa}} \Psi(\theta |\check{x}_2(\nu)|).$$

Since $\varrho_{[0,1]}(\theta\check{y}) \leqslant \varrho_{[0, e^{-\Upsilon \frac{\tau^{\kappa}}{\kappa}}]}(\theta\check{y}) + \varrho_{[e^{-\Upsilon \frac{\tau^{\kappa}}{\kappa}}, 1]}(\theta\check{y})$, then

$$\varrho_{[0,1]}(\theta\check{y}) \leq \tilde{C}\, e^{-(\Upsilon + \aleph q_{\Psi}) \frac{\tau^{\kappa}}{\kappa}} \varrho_{[0,1]}(\theta\check{x}_2) + \varrho_{[0,1]}(\theta\check{x}_1).$$

If $\aleph \geq 0$, by monotonicity of Ψ,

$$\varrho_{[0,1]}(\theta\check{y}) \leq e^{-\Upsilon \frac{\tau^{\kappa}}{\kappa}} \varrho_{[0,1]}(\theta\check{x}_2) + \varrho_{[0,1]}(\theta\check{x}_1).$$

In either case, since $\check{x}_1, \check{x}_2 \in L^{\Psi}(0,1)$, we have $\varrho_{[0,1]}(\theta\check{y}) \to 0$ as $\theta \to 0^+$. Thus, $\check{y} \in L^{\Psi}(0,1)$. Moreover, from the preceding estimate, we deduce the following inequality:

$$\|\check{y}\|^{L}_{[0, e^{-\Upsilon \frac{\tau^{\kappa}}{\kappa}}]} \leqslant K(\tau),$$

where the function $K(\tau)$ can be made arbitrarily small by choosing τ sufficiently large. Then, we obtain

$$\begin{aligned}
\|\check{x}_1 - \check{y}\|^L_{[0,1]} &= \|\check{x}_1 - \check{y}\|^L_{[0,e^{-\Upsilon\frac{\tau^\kappa}{\kappa}}]} \\
&\leqslant \|\check{x}_1\|^L_{[0,e^{-\Upsilon\frac{\tau^\kappa}{\kappa}}]} + \|\check{y}\|^L_{[0,e^{-\Upsilon\frac{\tau^\kappa}{\kappa}}]} \\
&= \|\check{x}_1\|^L_{[0,e^{-\Upsilon\frac{\tau^\kappa}{\kappa}}]} + K(\tau).
\end{aligned}$$

Therefore, for sufficiently large values of τ, we obtain $|\check{x}_1 - \check{y}|^L_{[0,1]} < \varepsilon_1$, which implies that $\check{y} \in \mathcal{V}(\check{x}_1, \varepsilon_1)$. Finally, for $\nu \in (0, e^{-\Upsilon\frac{\tau^\kappa}{\kappa}}]$,

$$\top_\kappa(\tau)\check{y}(\nu) = e^{\aleph\frac{\tau^\kappa}{\kappa}}\, \check{y}(\nu e^{-\Upsilon\frac{\tau^\kappa}{\kappa}}) = \check{y}_2(\nu),$$

so $\|\top_\kappa(\tau)\check{y} - \check{x}_2\|^L = 0 < \varepsilon_2$, that is, $\top_\kappa(\tau)\check{y} \in \mathcal{V}(\check{x}_2, \varepsilon_2)$. Therefore,

$$\top_\kappa(\tau)(\mathcal{V}(\check{x}_1, \varepsilon_1)) \cap \mathcal{V}(\check{x}_2, \varepsilon_2) \neq \emptyset,$$

which proves topological transitivity. □

Corollary 3.5. *If $\aleph > -\Upsilon/q_\Psi$, then the κ-semigroup $\{\top_\kappa(\tau)\}_{\tau\geq 0}$ is chaotic in the space $L^\Psi(0,1)$ according to Devaney's definition.*

Next, we study the strong stability of the conformable Von Foerster–Lasota equation given by (3.1).

Definition 3.6. A κ-semigroup $\{\top_\kappa(\tau)\}_{\tau\geq 0}$ is said to be strongly stable in L^Ψ if

$$\lim_{\tau\to\infty} \|\top_\kappa(\tau)\check{x}\|^L = 0 \qquad \text{for every } \check{x} \in L^\Psi.$$

Theorem 3.7. *If $\aleph \leq -\Upsilon/p_\Psi$, where p_Ψ denotes the lower Matuszewska–Orlicz index of Ψ, then the κ-semigroup $\{\top_\kappa(\tau)\}_{\tau\geq 0}$ is strongly stable in $L^\Psi(0,1)$.*

Proof. Let $\check{x} \in L^\Psi(0,1)$ and $\theta > 0$. By definition,

$$\begin{aligned}
\varrho_{[0,1]}(\theta\top_\kappa(\tau)\check{x}) &= \int_0^1 \Psi\left(\theta\left|\top_\kappa(\tau)\check{x}(\nu)\right|\right) d\nu \\
&= \int_0^1 \Psi\left(\theta e^{\aleph\frac{\tau^\kappa}{\kappa}}\left|\check{x}(\nu e^{-\Upsilon\frac{\tau^\kappa}{\kappa}})\right|\right) d\nu \\
&= e^{\Upsilon\frac{\tau^\kappa}{\kappa}} \int_0^{e^{-\Upsilon\frac{\tau^\kappa}{\kappa}}} \Psi\left(\theta e^{\aleph\frac{\tau^\kappa}{\kappa}}|\check{x}(\nu)|\right) d\nu.
\end{aligned}$$

Since $e^{\aleph\tau^\kappa/\kappa} < 1$ for sufficiently large τ, there exists a constant $\check{R} > 0$ such that

$$\Psi\left(\theta e^{\aleph\frac{\tau^\kappa}{\kappa}}|\check{x}(\nu)|\right) \leq \check{R}\left(e^{\aleph\frac{\tau^\kappa}{\kappa}}\right)^{p_\Psi} \Psi(\theta|\check{x}(\nu)|).$$

Consequently,

$$\begin{aligned}
\varrho_{[0,1]}(\theta\top_\kappa(\tau)\check{x}) &\leq \check{R}\, e^{(\Upsilon+\aleph p_\Psi)\frac{\tau^\kappa}{\kappa}} \int_0^{e^{-\Upsilon\frac{\tau^\kappa}{\kappa}}} \Psi(\theta|\check{x}(\nu)|)\, d\nu \\
&= \check{R}\, e^{(\Upsilon+\aleph p_\Psi)\frac{\tau^\kappa}{\kappa}}\, \varrho_{[0,\, e^{-\Upsilon\frac{\tau^\kappa}{\kappa}}]}(\theta\check{x}).
\end{aligned}$$

Since $\check{x} \in L^{\psi}(0,1)$, we have $\lim\limits_{\tau\to\infty} \varrho_{[0,\, e^{-\Upsilon\frac{\tau^\kappa}{\kappa}}]}(\theta\check{x}) = 0$. Moreover, under the assumption $\aleph \le -\Upsilon/p_\Psi$, we obtain $e^{(\Upsilon+\aleph p_\Psi)\tau^\kappa/\kappa} \to 0$ as $\tau \to \infty$. Therefore, $\varrho_{[0,1]}(\theta\top_\kappa(\tau)\check{x}) \to 0$ as $\tau \to \infty$, which implies

$$\lim_{\tau\to\infty} \|\top_\kappa(\tau)\check{x}\|^L = 0.$$

This proves the strong stability of $\{\top_\kappa(\tau)\}_{\tau\ge 0}$ in $L^\Psi(0,1)$. □

We now provide an illustrative example showing that the κ-semigroup $\{\top_\kappa(\tau)\}_{\tau\ge 0}$ fails to be strongly stable when the parameter $\aleph$ belongs to the interval $(-\Upsilon/p_\Psi, 0)$.

Example 3.8. Consider the conformable Von Foerster–Lasota equation (3.1) with the initial condition

$$\check{x}(\nu) = \Psi^{-1}\left(\sigma\nu^{\sigma-1}\right),$$

where the parameters satisfy $-\Upsilon/p_\Psi < \aleph < 0$ and $\sigma = 1 + \aleph p_\Psi/\Upsilon > 0$.

Clearly, $\check{x}$ is positive on $(0,1)$, and $\varrho_{[0,1]}(\check{x}) = \int_0^1 \Psi(|\check{x}(\nu)|)\,d\nu = \int_0^1 \sigma\nu^{\sigma-1}\,d\nu = 1$. Thus, $\check{x} \in L^\Psi(0,1)$. Next, let us evaluate the modular of the κ-semigroup $\{\top_\kappa(\tau)\}_{\tau\ge 0}$:

$$\begin{aligned}
\varrho_{[0,1]}\left(\top_\kappa(\tau)\check{x}\right) &= \int_0^1 \Psi\left(\left|e^{\aleph\frac{\tau^\kappa}{\kappa}}\,\check{x}\left(\nu e^{-\Upsilon\frac{\tau^\kappa}{\kappa}}\right)\right|\right)d\nu \\
&= e^{\Upsilon\frac{\tau^\kappa}{\kappa}} \int_0^{e^{-\Upsilon\frac{\tau^\kappa}{\kappa}}} \Psi(\check{x}(\nu))\,\frac{\Psi\left(e^{\aleph\frac{\tau^\kappa}{\kappa}}\,\check{x}(\nu)\right)}{\Psi(\check{x}(\nu))}d\nu \\
&= e^{\Upsilon\frac{\tau^\kappa}{\kappa}} \int_0^{e^{-\Upsilon\frac{\tau^\kappa}{\kappa}}} \sigma\nu^{\sigma-1}\,\frac{\Psi\left(e^{\aleph\frac{\tau^\kappa}{\kappa}}\,\check{x}(\nu)\right)}{\Psi(\check{x}(\nu))}d\nu \\
&\ge e^{\Upsilon\frac{\tau^\kappa}{\kappa}} \int_0^{e^{-\Upsilon\frac{\tau^\kappa}{\kappa}}} \sigma\nu^{\sigma-1}\Phi_\infty\left(e^{\aleph\frac{\tau^\kappa}{\kappa}},\Psi\right)d\nu \\
&\ge e^{\Upsilon\frac{\tau^\kappa}{\kappa}} \int_0^{e^{-\Upsilon\frac{\tau^\kappa}{\kappa}}} \sigma\nu^{\sigma-1}e^{\aleph p_\Psi\frac{\tau^\kappa}{\kappa}}\,d\nu \\
&= e^{(\Upsilon+\aleph p_\Psi-\Upsilon\sigma)\frac{\tau^\kappa}{\kappa}} \\
&= 1.
\end{aligned}$$

Consequently,

$$\lim_{\tau\to+\infty} \|\top_\kappa(\tau)\|^L_{[0,1]} \ne 0.$$

Hence, the κ-semigroup $\{\top_\kappa(\tau)\}_{\tau\ge 0}$ is not strongly stable.

Remark 2. In the classical L^p framework, the asymptotic behavior of the solution κ-semigroup is completely understood. More precisely, depending on the value of the parameter $\aleph$, the κ-semigroup exhibits either chaotic behavior or strong stability, yielding a sharp dichotomy that fully characterizes the dynamics. These results rely on the fact that, in L^p spaces, the lower and upper Matuszewska–Orlicz indices coincide and are equal to p. Since every L^p space ($1 \le p < \infty$) can be seen as a particular Orlicz space with $p_\Psi = q_\Psi = p$, all the results previously obtained in the L^p setting can be recovered from the present work as special cases. In particular, within this framework, the dynamics of the κ-semigroup can be completely determined for any value of the parameter $\aleph$.

By contrast, when working in a general Orlicz space L^{Ψ}, the possible discrepancy between the indices p_{Ψ} and q_{Ψ} leads to a fundamentally different situation. Although chaotic behavior and stability can still be established in certain parameter ranges, there appears an intermediate region $\aleph \in (-\Upsilon/p_{\Psi}, -\Upsilon/q_{\Psi}]$ for which the nature of the dynamics cannot be classified using current techniques. This indeterminate regime has no analogue in the L^p theory and highlights the intrinsic richness and added value of the Orlicz space framework.

3.1. Numerical illustration

In this subsection, we provide a concrete example to illustrate our theoretical results regarding stability and chaos in Orlicz spaces. We show how the Matuszewska–Orlicz indices determine the dynamical behavior of the κ-semigroup $\{\top_{\kappa}(\tau)\}_{\tau\geq 0}$.

Consider the function $\Psi_{(p;q)} : [0, \infty) \to [0, \infty)$ defined by

$$\Psi_{(p;q)}(\nu) = \nu^p + \nu^q, \quad \text{with } 1 \leq p < q < \infty. \tag{3.5}$$

First, we verify that $\Psi_{(p;q)}$ is a valid Ψ-function. It is continuous, nondecreasing, $\Psi_{(p;q)}(0) = 0$, and its second derivative $\Psi''_{(p;q)}(\nu) = p(p-1)\nu^{p-2} + q(q-1)\nu^{q-2} > 0$, for $\nu > 0$ and $p, q \geq 1$, ensures strict convexity. Furthermore, $\Psi_{(p;q)}$ satisfies the Δ_2-condition. Indeed, for any $\nu \geq 0$, we have

$$\Psi_{(p;q)}(2\nu) = (2\nu)^p + (2\nu)^q \leq 2^q(\nu^p + \nu^q) = 2^q\Psi_{(p;q)}(\nu).$$

By setting the constant $c = 2^q$, the condition $\Psi_{(p;q)}(2\nu) \leq c\Psi_{(p;q)}(\nu)$ is satisfied. For this specific function, the lower and upper Matuszewska–Orlicz indices are $p_{\Psi} = p$ and $q_{\Psi} = q$, respectively.

The comparison between the Ψ-function $\Psi_{(p;q)}(\nu) = \nu^p + \nu^q$, witch generates the Orlicz space, and the Ψ-function $\Psi_p(\nu) = \nu^p$, witch generates the Lebesque space L^p, highlights the fundamental advantage of our approach. Whereas L^p spaces impose a homogeneous scaling ($p_{\Psi} = q_{\Psi} = p$), the Orlicz framework allows for multiscale dynamics. As illustrated in Figure 1, the Orlicz profile deviates significantly from the L^p curve as ν increases. The shaded region in the figure highlights the extended growth domain provided by the Orlicz framework, illustrating its capacity to accommodate densities with growth rates exceeding the standard L^p constraints. This leads to the identification of a transition zone between stability and chaos that is invisible in the standard L^p setting.

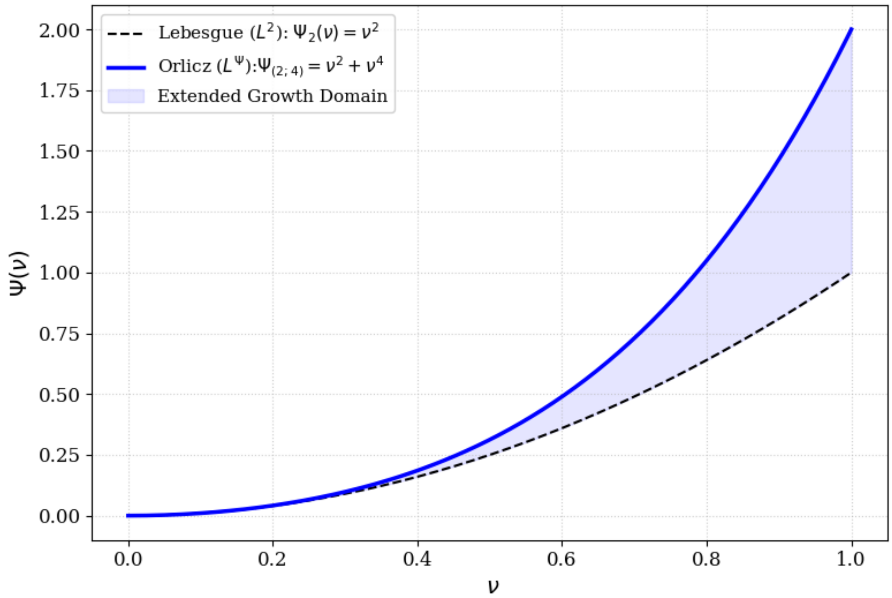

Figure 1. Growth comparison of Ψ-functions for the Lebesgue space L^p and the Orlicz space L^Ψ with $p = 2$ and $q = 4$.

Now, let us consider the following conformable Van Forester–Lasota equation of order $\kappa = 0.5$ in the Orlicz space $L^{\Psi_{(2;4)}}([0, 1])$ with a transport velocity $\Upsilon = 1$:

$$\frac{\partial^{0.5}\breve{\omega}}{\partial\tau^{0.5}} + \sqrt{\nu}\frac{\partial^{0.5}\breve{\omega}}{\partial\nu^{0.5}} = \aleph\breve{\omega}, \quad \tau \geqslant 0, \ 0 \leqslant \nu \leqslant 1, \tag{3.6}$$

with the initial condition defined by:

$$\breve{\omega}(0, \nu) = 16\nu^2(1 - \nu)^2. \tag{3.7}$$

The exact solution is given by:

$$\breve{\omega}(\tau, \nu) = 16e^{2\aleph\sqrt{\tau}}\left[\nu e^{-2\sqrt{\tau}}\right]^2\left[1 - \nu e^{-2\sqrt{\tau}}\right]^2.$$

We distinguish three dynamical regimes determined by the Matuszewska–Orlicz indices $p_\pi = 2$ and $q_\pi = 4$ and the transport velocity $\Upsilon = 1$. The critical thresholds are given by $-\Upsilon/p_\Psi = -0.5$ and $-\Upsilon/q_\Psi = -0.25$.

- Strong stability ($\aleph \leq -0.5$): Using $\aleph = -0.6$ as a representative value, we observe in Figure 2 that the density peak is initially 1.0 at $\nu = 0.5$ for $\tau = 0$. By the time $\tau = 1$, the density drops significantly across the whole interval. From $\tau = 2$ onwards, the population effectively vanishes across all maturity levels ($\nu \in [0, 1]$). Scientifically, the aggressive decay coefficient $\aleph$ is so dominant that it prevents the population from surviving or developing at any stage. Biologically, this represents rapid extinction, where a highly controlled environment neutralizes the population entirely before it can establish any presence.

- For the indeterminate region, choosing $\aleph = -0.26$ as a representative value reveals a more resilient dynamical behavior, as shown in Figure 3. At $\tau = 1$, the density successfully reaches full maturity ($\nu = 1$) with a significantly higher magnitude than in the strong stability case. By $\tau = 2$, the density decreases considerably, yet a small and visible trace still persists across the entire maturity range. It is only starting from $\tau = 3$ onwards that the population effectively vanishes across all maturity levels. Scientifically, the weaker decay coefficient $\aleph$ allows the transport effect to be more visible, enabling individuals to reach advanced maturity stages before extinction. Biologically, this represents temporary persistence or resistance; the community is robust enough to complete its development cycle and reach full maturity, but it eventually fails to sustain itself, leading to a slow and final decline.
- Chaotic regime ($\aleph > -0.25$): By selecting $\aleph = 1.5$ as a representative value, we observe a dramatic shift in the system's dynamics, as illustrated in Figure 4. Unlike the previous cases, the density peak, initially at 1.0 for $\nu = 0.5$, exhibits an explosive increase in magnitude as time progresses from $\tau = 1$ to $\tau = 5$. While the peak shifts toward full maturity ($\nu = 1$), its height grows exponentially, far exceeding the initial density levels. Scientifically, this behavior is a hallmark of hypercyclicity, where the high growth coefficient $\aleph$ completely overcomes the stabilizing effects of the system, causing the solution to become unbounded. Biologically, this corresponds to a pathological state similar to cancer cell proliferation. In this scenario, cells divide uncontrollably and bypass all regulatory signals, leading to a malignant expansion that eventually dominates the entire system.

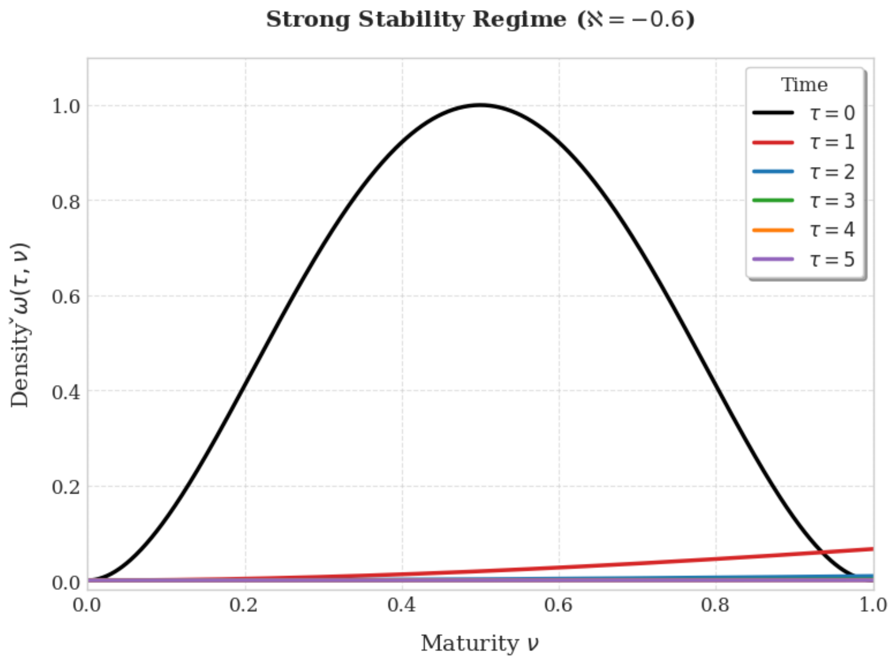

Figure 2. Numerical evolution of the density $\check{\omega}(\tau, \nu)$ for $\aleph = -0.6$.

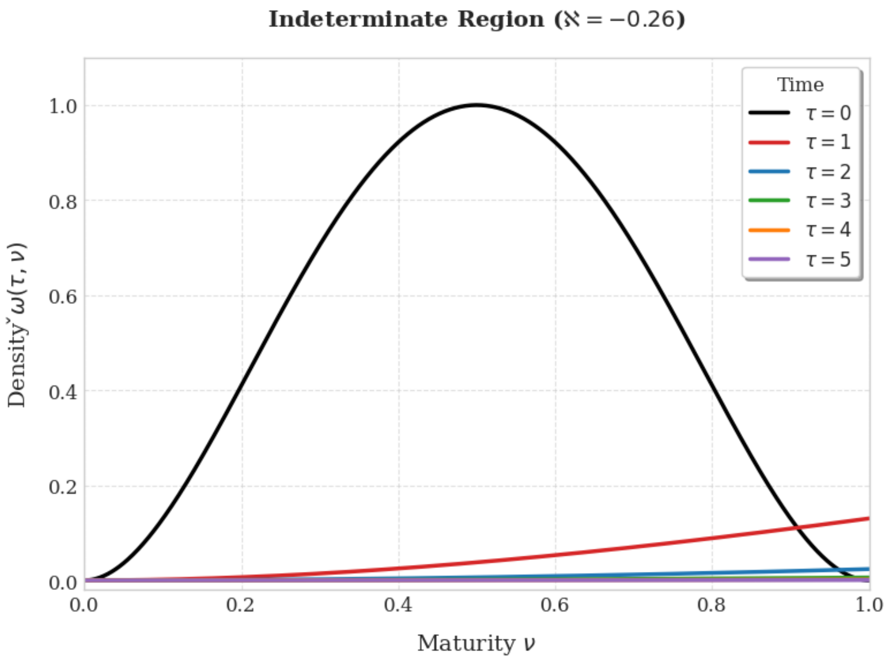

Figure 3. Numerical evolution of the density $\breve{\omega}(\tau, \nu)$ for $\aleph = -0.26$.

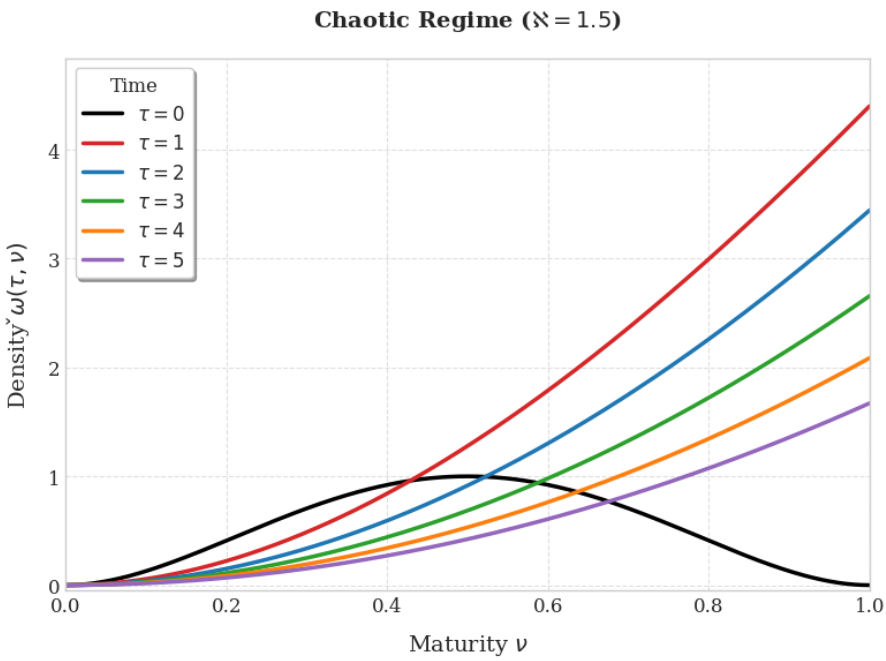

Figure 4. Numerical evolution of the density $\breve{\omega}(\tau, \nu)$ for $\aleph = 1.5$.

4. Chaotic dynamics and strong stability in the case of a nonconstant growth rate function

In this section, we investigate the chaotic dynamics and the strong stability of the conformable Von Foerster–Lasota equation in the presence of a nonconstant growth rate function $\wp(\nu)$, which introduces an additional degree of complexity into the behavior of the system. The model is given by

$$\frac{\partial^{\kappa}\breve{\omega}}{\partial\tau^{\kappa}} + \Upsilon\nu^{\kappa}\frac{\partial^{\kappa}\breve{\omega}}{\partial\nu^{\kappa}} = \wp(\nu)\breve{\omega}, \quad \tau \geqslant 0,\ 0 \leqslant \nu \leqslant 1, \tag{4.1}$$

with

$$\breve{\omega}(0,\nu) = \check{x}(\nu), \tag{4.2}$$

where $\wp : [0,1] \to \mathbb{R}$ is a continuous function, and $\check{x} \in L^{\Psi}(0,1)$. Based on the representation formula of the classical solution of this problem, we define family of operator

$$S(\tau)\check{x}(\nu) = \exp\left\{\int_0^{\tau} \wp\left(\nu e^{-\frac{\Upsilon}{\kappa}(\tau^{\kappa}-s^{\kappa})}\right) s^{\kappa-1} ds\right\} \check{x}\left(\nu e^{-\Upsilon\frac{\tau^{\kappa}}{\kappa}}\right). \tag{4.3}$$

Lemma 4.1. *The operator family $\{S_{\kappa}(\tau)\}_{\tau\geq 0}$ defined on the space $L^{\Psi}(0,1)$ by* (4.3) *is a C_0-κ-semigroup on $L^{\Psi}(0,1)$.*

Proof. Since $\wp$ is continuous on $[0,1]$, it is bounded. Hence, there exists a constant $M > 0$ such that $|\wp(\nu)| \leq M$ for all $\nu \in [0,1]$. Let $\check{x} \in L^{\Psi}(0,1)$ and $\theta > 0$. The Orlicz modular satisfies

$$\begin{aligned}
\varrho_{[0,1]}(\theta S_{\kappa}(\tau)\check{x}) &= \int_0^1 \Psi\left(\theta\left|S_{\kappa}(\tau)\check{x}(\nu)\right|\right) d\nu \\
&= \int_0^1 \Psi\left(\theta \exp\left\{\int_0^{\tau} \wp\left(\nu e^{-\frac{\Upsilon}{\kappa}(\tau^{\kappa}-s^{\kappa})}\right) s^{\kappa-1} ds\right\} \left|\check{x}\left(\nu e^{-\Upsilon\frac{\tau^{\kappa}}{\kappa}}\right)\right|\right) d\nu \\
&\leq \int_0^1 \Psi\left(\theta \exp\left\{\int_0^{\tau} M s^{\kappa-1} ds\right\} \left|\check{x}\left(\nu e^{-\Upsilon\frac{\tau^{\kappa}}{\kappa}}\right)\right|\right) d\nu \\
&= e^{\Upsilon\frac{\tau^{\kappa}}{\kappa}} \int_0^{e^{-\Upsilon\frac{\tau^{\kappa}}{\kappa}}} \Psi\left(\theta e^{M\frac{\tau^{\kappa}}{\kappa}}|\check{x}(\nu)|\right) d\nu \\
&\leq e^{(\Upsilon+Mq_{\Psi})\frac{\tau^{\kappa}}{\kappa}} \int_0^1 \Psi\left(\theta|\check{x}(\nu)|\right) d\nu.
\end{aligned}$$

Since $\check{x} \in L^{\Psi}(0,1)$, we have

$$\lim_{\theta\to 0^+} \int_0^1 \Psi\left(\theta|\check{x}(\nu)|\right) d\nu = 0,$$

and therefore,

$$\lim_{\theta\to 0^+} \int_0^1 \Psi\left(\theta\left|S_{\kappa}(\tau)\check{x}(\nu)\right|\right) d\nu = 0.$$

This proves that $S_{\kappa}(\tau)\check{x} \in L^{\Psi}(0,1)$ for every $\check{x} \in L^{\Psi}(0,1)$. Moreover,

$$\|S_{\kappa}(\tau)\check{x}\|_{[0,1]}^{L} \leq e^{(\Upsilon+Mq_{\Psi})\frac{\tau^{\kappa}}{\kappa}} \|\check{x}\|_{[0,1]}^{L},$$

showing that $\{S_{\kappa}(\tau)\}_{\tau\geq 0}$ is well-defined and bounded on $L^{\Psi}(0,1)$.

To establish strong continuity, we proceed as in the proof of Lemma 3.1. Since Ψ satisfies the Δ_2-condition, the space of continuous functions $C[0,1]$ is dense in $L^{\Psi}(0,1)$ with respect to the Luxemburg norm. Thus, there exists a continuous function $\phi \in C[0,1]$ such that

$$\|\check{x} - \phi\|^{L}_{[0,1]} < \frac{\epsilon}{6}.$$

Let $E(\tau,\nu) = \exp\left\{\int_0^{\tau} \wp\left(\nu e^{-\frac{\Upsilon}{\kappa}(\tau^{\kappa}-s^{\kappa})}\right) s^{\kappa-1} ds\right\}$. Since $\wp$ is continuous on the compact set $[0,1]$, it is bounded by some $R > 0$. Consequently,

$$|E(\tau,\nu) - 1| \leq e^{R\frac{\tau^{\kappa}}{\kappa}} - 1,$$

which converges to 0 as $\tau \to 0^{+}$ independently of ν. Furthermore, the uniform continuity of ϕ on $[0,1]$ implies that for any $\eta > 0$, there exists $\delta_1 > 0$ such that $|\phi(\nu e^{-\Upsilon\frac{\tau^{\kappa}}{\kappa}}) - \phi(\nu)| < \eta$ for all $\tau < \delta_1$ and all $\nu \in [0,1]$. Combining these, we obtain

$$\begin{aligned} |S_{\kappa}(\tau)\phi(\nu) - \phi(\nu)| &= |E(\tau,\nu)\phi(\nu e^{-\Upsilon\frac{\tau^{\kappa}}{\kappa}}) - \phi(\nu)| \\ &\leq E(\tau,\nu)|\phi(\nu e^{-\Upsilon\frac{\tau^{\kappa}}{\kappa}}) - \phi(\nu)| + |E(\tau,\nu) - 1||\phi(\nu)|. \end{aligned}$$

Choosing η and τ sufficiently small, we have $|S_{\kappa}(\tau)\phi(\nu) - \phi(\nu)| < \epsilon'$ uniformly on $[0,1]$. By using the dominated convergence theorem and the properties of the Ψ-function, one obtains

$$\begin{aligned} \lim_{\tau\to 0^{+}} \int_0^1 \Psi\left(\theta\,|S_{\kappa}(\tau)\phi(\nu) - \phi(\nu)|\right) d\nu &= \int_0^1 \lim_{\tau\to 0^{+}} \Psi\left(\theta\,|S_{\kappa}(\tau)\phi(\nu) - \phi(\nu)|\right) d\nu \\ &= \int_0^1 \Psi(0) d\nu \\ &= 0. \end{aligned}$$

Then, there exists $\delta_2 > 0$ such that for all $\tau < \delta_2$, one has

$$\|S_{\kappa}(\tau)\phi - \phi\|^{L}_{[0,1]} < \frac{\epsilon}{3}.$$

Finally, by taking $\delta = \min\{\delta_2, (\kappa \ln(2)/(\Upsilon + Mq_{\Psi}))^{1/\kappa}\}$, we conclude that for all $\tau \in (0,\delta)$,

$$\begin{aligned} \|S_{\kappa}(\tau)\check{x} - \check{x}\|^{L}_{[0,1]} &\leq e^{(\Upsilon+Mq_{\Psi})\frac{\tau^{\kappa}}{\kappa}} \|\check{x} - \phi\|^{L}_{[0,1]} + \|S_{\kappa}(\tau)\phi - \phi\|^{L}_{[0,1]} + \|\check{x} - \phi\|^{L}_{[0,1]} \\ &\leq \left(e^{(\Upsilon+Mq_{\Psi})\frac{\tau^{\kappa}}{\kappa}} + 1\right) \|\check{x} - \phi\|^{L}_{[0,1]} + \|S_{\kappa}(\tau)\phi - \phi\|^{L}_{[0,1]} \\ &< (2+1)\frac{\varepsilon}{6} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Hence, $\{S_{\kappa}(\tau)\}_{\tau\geq 0}$ is indeed a C_0-κ-semigroup on $L^{\Psi}(0,1)$. □

Before presenting our main theorem, we note that the long-term behavior of the C_0-κ-semigroup $\{S_{\kappa}(\tau)\}_{\tau\geq 0}$ depends solely on the behavior of the function $\wp$ near 0. More precisely, we can establish the following result in a similar way as in the paper [19].

Lemma 4.2. *Let* $\wp$ *and* $\tilde{\wp}$ *be two continuous functions defined on* $[0, 1]$ *with values in* $\mathbb{C}$*. Assume that there exists* $\wedge > 0$ *such that*

$$\wp(\nu) = \tilde{\wp}(\nu) \qquad \textit{for all } \nu \in [0, \wedge].$$

Consider the conformable partial differential equation

$$\frac{\partial^{\kappa}\breve{\omega}(\tau,\nu)}{\partial\tau^{\kappa}} + \Upsilon\nu^{\kappa}\frac{\partial^{\kappa}\breve{\omega}(\tau,\nu)}{\partial\nu^{\kappa}} = \tilde{\wp}(\nu)\,\breve{\omega}(\tau,\nu). \tag{4.4}$$

Suppose further that Eqs (4.1) *and* (4.4) *generate* C_0*-*κ*-semigroups on* $L^{\Psi}[0,1]$*, denoted by* $\{S_{\kappa}(\tau)\}_{\tau\geq 0}$ *and* $\{\tilde{S}_{\kappa}(\tau)\}_{\tau\geq 0}$*, respectively.*

Then, there exist a continuous function $q : [0,1] \to \mathbb{C}^{\star}$ *and some* $\tau_0 > 0$ *such that*

$$q(\nu) = 1, \qquad \textit{for all } \nu \in \left[0, e^{-\Upsilon\frac{\tau_0^{\kappa}}{\kappa}}\right], \tag{4.5}$$

and

$$S_{\kappa}(\tau)\breve{x} \;=\; q\,\tilde{S}_{\kappa}(\tau)\breve{x}, \qquad \textit{for all } \tau \geq \tau_0. \tag{4.6}$$

Proof. For any $\breve{x} \in L^{\Psi}(0,1)$ and $\nu \in [0,1]$ we have

$$S_{\kappa}(\tau)\breve{x}(\nu) = \exp\left\{\int_{-\frac{\tau^{\kappa}}{\kappa}}^{0} \wp(\nu e^{\kappa s})ds\right\}\breve{x}\left(\nu e^{-\Upsilon\frac{\tau^{\kappa}}{\kappa}}\right) \tag{4.7}$$

and

$$\tilde{S}_{\kappa}(\tau)\breve{x}(\nu) = \exp\left\{\int_{-\frac{\tau^{\kappa}}{\kappa}}^{0} \tilde{\wp}(\nu e^{\kappa s})ds\right\}\breve{x}\left(\nu e^{-\frac{\Upsilon}{\kappa}\tau^{\kappa}}\right). \tag{4.8}$$

Let $\tau_0 > 0$ such that $e^{-\Upsilon\tau_0^{\kappa}/\kappa} < \wedge$. For $\tau > \tau_0$ and $\nu \in [0,1]$, we obtain

$$\begin{aligned}
S_{\kappa}(\tau)\breve{x}(\nu) &= \exp\left\{\int_{-\frac{\tau^{\kappa}}{\kappa}}^{-\frac{\tau_0^{\kappa}}{\kappa}} \wp(\nu e^{\kappa s})ds + \int_{-\frac{\tau_0^{\kappa}}{\kappa}}^{0} \wp(\nu e^{\kappa s})ds\right\}\breve{x}\left(\nu e^{-\Upsilon\frac{\tau^{\kappa}}{\kappa}}\right)\\
&= \exp\left\{\int_{-\frac{\tau^{\kappa}}{\kappa}}^{-\frac{\tau_0^{\kappa}}{\kappa}} \tilde{\wp}(\nu e^{\kappa s})ds + \int_{-\frac{\tau_0^{\kappa}}{\kappa}}^{0} \tilde{\wp}(\nu e^{\kappa s})ds - \int_{-\frac{\tau_0^{\kappa}}{\kappa}}^{0} \tilde{\wp}(\nu e^{\kappa s})ds\right.\\
&\qquad \left. + \int_{-\frac{\tau_0^{\kappa}}{\kappa}}^{0} \wp(\nu e^{\kappa s})ds\right\}\breve{x}\left(\nu e^{-\Upsilon\frac{\tau^{\kappa}}{\kappa}}\right)\\
&= \exp\left\{\int_{-\frac{\tau_0^{\kappa}}{\kappa}}^{0} \left(\wp(\nu e^{\kappa s}) - \tilde{\wp}(\nu e^{\kappa s})\right)ds\right\}\exp\left\{\int_{-\frac{\tau^{\kappa}}{\kappa}}^{0} \tilde{\wp}(\nu e^{\kappa s})ds\right\}\breve{x}\left(\nu e^{-\Upsilon\frac{\tau^{\kappa}}{\kappa}}\right).
\end{aligned}$$

This implies that, with a function q defined by the formula

$$q(\nu) = \exp\left\{\int_{-\frac{\tau_0^{\kappa}}{\kappa}}^{0} \left(\wp(\nu e^{\kappa s}) - \tilde{\wp}(\nu e^{\kappa s})\right)ds\right\}, \quad \nu \in [0,1], \tag{4.9}$$

we have, for any $\tau > \tau_0$, $S_{\kappa}(\tau)\breve{x}(\nu) = q(\nu)\tilde{S}_{\kappa}(\tau)\breve{x}(\nu)$. This completes the proof. □

Theorem 4.3. *Consider the conformable Von Foerster–Lasota equation* (4.1) *with the initial condition* (4.2) *in* $L^{\Psi}(0,1)$. *Assume that the function* $\wp : [0,1] \to \mathbb{R}$ *is continuous and satisfies the following condition:*

(H) *There exist constants* $\check{c} > 0$ *and* $\aleph > -\Upsilon/q_{\Psi}$ *such that*

$$\wp(\nu) > \aleph \quad \textit{for all } \nu \in [0,\check{c}].$$

Then, the C_0-κ-*semigroup* $\{S_{\kappa}(\tau)\}_{\tau\geq 0}$ *generated by* (4.1) *is chaotic in* $L^{\Psi}(0,1)$.

Proof. Define the function $\xi : [0,1] \to \mathbb{R}$ by

$$\xi(\nu) = \exp\left(\frac{-1}{\Upsilon}\int_{\nu}^{1}\frac{\wp(s)-\aleph}{s}\,ds\right).$$

Under assumption (H), this function is continuous, strictly positive on $(0,1]$, and satisfies $\xi(0) = 0$. Moreover, the multiplication operator $\mathcal{M}_{\xi} : \check{\omega} \to \xi\check{\omega}$ defines a bounded and injective linear operator on $L^{\Psi}(0,1)$.
Let $\check{\omega}$ be a solution of the conformable equation with constant coefficient $\wp(\nu) = \aleph$ (3.1) and define

$$\widetilde{\omega}(\tau,\nu) := \xi(\nu)\check{\omega}(\tau,\nu).$$

Then, $\widetilde{\omega}$ is a solution to the nonconstant coefficient problem (4.1), and we have the following commutative diagram:

$$\begin{array}{ccc} L^{\Psi} & \xrightarrow{T_{\kappa}(\tau)} & L^{\Psi} \\ {\scriptstyle \mathcal{M}_{\xi}}\downarrow & & \downarrow{\scriptstyle \mathcal{M}_{\xi}} \\ L^{\Psi} & \xrightarrow[S_{\kappa}(\tau)]{} & L^{\Psi} \end{array}$$

Let $\check{\omega} \in L^{\Psi}(0,1)$, and for each $n \in \mathbb{N}$, define

$$\check{\omega}_n(\nu) = \begin{cases} \check{\omega}(\nu) & \text{for } \nu \in \left(\frac{1}{n},1\right], \\ 0 & \text{for } \nu \in \left[0,\frac{1}{n}\right]. \end{cases}$$

Since $\xi(\nu) > 0$ on $(0,1]$, it follows that $\check{\omega}_n/\xi$ is well-defined and belongs to $L^{\Psi}(0,1)$. Then,

$$\check{\omega}_n = \mathcal{M}_{\xi}\left(\frac{\check{\omega}_n}{\xi}\right) \in \mathcal{M}_{\xi}(L^{\Psi}(0,1)).$$

Moreover, let $\theta > 0$,

$$\begin{aligned} \lim_{n\to\infty}\varrho\left(\theta(\check{\omega}_n - \check{\omega})\right) &= \lim_{n\to\infty}\int_0^1 \Psi\left(\theta|\check{\omega}_n(\nu) - \check{\omega}(\nu)|\right)d\nu \\ &= \lim_{n\to\infty}\int_0^{\frac{1}{n}} \Psi\left(\theta|\check{\omega}(\nu)|\right)d\nu \\ &= \lim_{n\to\infty}\int_0^1 \mathbb{1}_{[0,\frac{1}{n}]}\Psi\left(\theta|\check{\omega}(\nu)|\right)d\nu. \end{aligned}$$

Since $\mathbb{k}_{[0,1/n]}\Psi(\theta|\check{\omega}(\nu)|) \to 0$ as $n \to \infty$ and is dominated by the integrable function $\Psi(\theta|\check{\omega}|) \in L^1(0,1)$, we apply the dominated convergence theorem to obtain

$$\lim_{n\to\infty} \int_0^1 \mathbb{k}_{[0,\frac{1}{n}]}\Psi\left(\theta|\check{\omega}(\nu)|\right) d\nu = 0.$$

Hence, $\lim_{n\to+\infty} \|\check{\omega}_n - \check{\omega}\|_{L^\Psi} = 0$, which proves that the range $\mathcal{M}_\xi(L^\Psi(0,1))$ is dense in $L^\Psi(0,1)$.

Since the multiplication operator $\mathcal{M}_\xi$ is bounded, injective, and has a dense range, it defines a quasi-conjugacy between the semigroups $\{\top_\kappa(\tau)\}_{\tau\geq 0}$ and $\{S_\kappa(\tau)\}_{\tau\geq 0}$.

Finally, since $\aleph > -\Upsilon/q_\Psi$, the constant-coefficient κ-semigroup $\{\top_\kappa(\tau)\}_{\tau\geq 0}$ is chaotic in $L^\Psi(0,1)$. It is a well-known result in linear dynamics that chaos (specifically hypercyclicity and the density of periodic points) is preserved under quasi-conjugacy by a continuous mapping with dense range [11]. Therefore, the κ-semigroup $\{S_\kappa(\tau)\}_{\tau\geq 0}$ is also chaotic in $L^\Psi(0,1)$. □

Now, we present a sufficient condition that ensures the strong stability of the C_0-κ-semigroup $\{S_\kappa(\tau)\}_{\tau\geq 0}$.

Theorem 4.4. *Consider the conformable Von Foerster–Lasota equation given by* (4.1)*, with initial condition* (4.2)*. Suppose that the function* $\wp : [0,1] \to \mathbb{R}$ *is continuous and satisfies the following condition:*

($\check{H}$) *There exist two constants* $c > 0$ *and* $\aleph \leq -\Upsilon/p_\Psi$ *such that*

$$\wp(\nu) < \aleph \quad \textit{for } \nu \in [0,c].$$

Then, the C_0-κ*-semigroup* $\{S_\kappa(\tau)\}_{\tau\geq 0}$ *generated by Eq* (4.1) *is strongly stable in the space* $L^\Psi(0,1)$.

Proof. Letting $\theta > 0$ and $\check{x} \in L^\Psi(0,1)$, we have

$$\begin{aligned}
\varrho\left(\theta_\kappa(\tau)\check{x}\right) &= \int_0^1 \Psi\left(\theta|S_\kappa(\tau)\check{x}(\nu)|\right) d\nu \\
&= \int_0^1 \Psi\left(\theta|\xi(\nu)\top_\kappa(\tau)\check{x}(\nu)|\right) d\nu \\
&= \int_0^1 \Psi\left(\theta \exp\left\{\frac{-1}{\Upsilon}\int_\nu^1 \frac{\wp(s)-\aleph}{s}\, ds\right\}|\top_\kappa(\tau)\check{x}(\nu)|\right) d\nu.
\end{aligned}$$

Since $\wp(s) > \aleph$ and Ψ is a nondecreasing function, then

$$\Psi\left(\theta \exp\left\{\frac{-1}{\Upsilon}\int_\nu^1 \frac{\wp(s)-\aleph}{s}\, ds\right\}|\top_\kappa(\tau)\check{x}(\nu)|\right) \leq \Psi\left(\theta\,|\top_\kappa(\tau)\check{x}(\nu)|\right).$$

This implies that

$$\varrho(\theta S_\kappa(\tau)\check{x}) \leq \varrho(\theta\top_\kappa(\tau)\check{x}).$$

By using Theorem 3.7, we get

$$\lim_{\tau\to+\infty} \varrho(\theta\top_\kappa(\tau)\check{x}) = 0, \qquad \text{for all } \theta > 0.$$

Therefore,

$$\lim_{\tau\to+\infty} \varrho(\theta S_\kappa(\tau)\check{x}) = 0, \quad \text{for all } \theta > 0,$$

which implies

$$\lim_{\tau\to+\infty} \|\top_\kappa(\tau)\check{x}\|^L = 0.$$

This completes the proof. □

5. Conclusions

In this paper, we have presented a rigorous asymptotic analysis of the conformable Von Foerster–Lasota equation within the framework of Orlicz spaces $L^\Psi(0,1)$. By exploiting the analytical properties of the Matuszewska–Orlicz indices p_Ψ and q_Ψ, we successfully characterized the long-term behavior of the associated κ-semigroup. Our results reveal that the transition between stability and chaos is intrinsically governed by the parameter $\aleph$ relative to specific critical thresholds; specifically, the system exhibits Devaney's chaos for $\aleph > -\Upsilon/q_\Psi$ and ensures strong stability for $\aleph \le -\Upsilon/p_\Psi$. A key finding of this study is the elucidation of a transitional regime within the interval $\aleph \in (-\Upsilon/p_\Psi, -\Upsilon/q_\Psi]$, where the criteria for strong stability are no longer satisfied, yet the presence of chaotic dynamics remains an unresolved mathematical challenge. Future research will focus on the introduction of nonlinearity into the model, particularly through the incorporation of density-dependent growth or mortality rates via a nonlinear function $\wp(\nu, \breve{\omega})$ to represent resource-limited environments and saturation effects. Additionally, we plan to study this model using nonlocal derivative operators in various functional spaces to explore the influence of nonlocal effects on the dynamics and stability of the system.

Author contributions

Khadija Elkhalloufy: Conceptualization, formal analysis, writing–original draft; Manal Menchih: Investigation, validation, writing–review & editing; Khalid Hilal: Investigation, supervision; Ahmed Kajouni: Investigation, supervision. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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