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Research article

A non-zero-sum reinsurance-investment game informed by alpha-maximin ambiguity-averse preferences

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Abstract: This paper investigates the optimal reinsurance and investment problem with delay in the framework of the non-zero-sum stochastic differential game, where the game is considered between two insurers characterized by similar ambiguity-averse preferences. Both insurers maximize their respective α-maxmin mean-variance criterion in the market. The criterion is time-inconsistent and we derive the equilibrium reinsurance-investment strategies and value functions by the extended HJB equations. Finally, some numerical examples and sensitivity analysis are presented to demonstrate the effects of model parameters on the equilibrium strategy. We find the delay factor and the various attitudes of decision-makers toward ambiguity have a great influence on the final strategy. Moreover, the insurer's investment behavior will be more aggressive the more intense its competition with other insurers and the greater its ambiguity-seeking.

Keywords: α-maxmin mean-variance criterion; non-zero-sum differential game; delay; time-consistent strategy; similar ambiguity-averse preferences
Mathematics Subject Classification: 62P05, 91B30

1. Introduction

The optimal reinsurance-investment problem has consistently captivated the interest of researchers in actuarial science. As evidenced by the works of [13, 17, 32], among others, this area has been richly explored and continues to evolve. However, most of these literature assume that the decision-makers are aware of the reference model parameters, which is inconsistent with the actual situation. Therefore, in subsequent research, some scholars have studied the α-maxmin expected utility, especially [18] proposed the α-maxmin mean-variance criterion, which is different from the general mean-variance criterion, and can consider a more diverse range of ambiguity attitudes, wider than the traditional robust model and closer to the real situation. There are relatively few subsequent studies on α-robustness, for example, [14] studied the Stackelberg game of insurers and reinsurers under the stochastic volatility model in the α-maxmin mean-variance criterion. For other references, see [5, 19, 37].

At the same time, there is competition between multiple insurers in the real-world, so some literature has been studied for non-zero-sum games. For example, [20] considered the non-zero-sum stochastic differential game problem between two ambiguity-averse insurers with a common shock. For other related literature, see [2, 10, 16, 27, 29]. In addition, [3, 22, 36, 40] suggested that many systematic cognitive biases in people's judgment and decision-making may be the result of the cognitive system's ability to optimize information with limited resources, i.e., the cognitive biases that people exhibit in judgment and decision-making tasks are not simply a reflection of personal preferences, but rather, they are related to an individual's cognitive resources. Considering that the cognitive resources of the two insurers that have competed for a long time are similar, the cognitive bias and the attitude toward ambiguity are similar. Based on this, this paper focuses on the non-zero-sum game problem between two insurers with similar ambiguity-averse preferences.

Compared with the existing literature, the two main innovations of this paper are as follows: First, this paper combines the attitude of insurers toward ambiguity and their competition. Under normal circumstances, insurers' attitudes towards ambiguity are either similar or opposite. By incorporating the respective ambiguous attitudes of the two competing insurers into the game model and analyzing their equilibrium reinsurance-investment strategies from both subjective and objective perspectives, it is possible to better balance the subjectivity of the decision-makers of the two companies and the objectivity of the model's content. Second, this paper investigates the non-zero-sum game model with delay, which is based on the α-maxmin mean-variance utility function. Considering the above two points and the fact that the attitudes towards ambiguity of two insurers in a long-term competitive relationship are mostly similar, this paper further studies the α-robust optimal reinsurance-investment problem under the non-zero-sum game framework influenced by delay.

The remainder of the paper is structured as follows. In Section 2, the financial model with delay effect and uncertainty is introduced. In Section 3, a non-zero-sum game model between two insurers is constructed based on the α-maxmin mean-variance criterion. In Section 4, by solving the HJB equation, we obtain time-consistent reinsurance-investment strategies for the two insurers. Section 5 presents a numerical analysis of the results that studies the impact of each parameter on the reinsurance-investment strategy. Section 6 concludes this paper.

2. Financial model

Let $(\mathbf{\Omega}, \mathcal{F}, \mathcal{F}_t = \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbf{P})$ be a filtered complete probability space satisfying the usual conditions. $\mathbf{P}$ represents the reference measure. All the processes below are assumed to be well-defined and adapted to $\mathcal{F}_t$, both insurers adjusted the strategies within the time horizon $[0, T]$. Assume that there are two types of financial assets in a continuous-time financial market: risk-free asset and risky asset. The price of the risk-free asset at moment t is $S_0(t)$ which satisfies the following ordinary differential equation

$$dS_0(t) = rS_0(t)dt, \tag{2.1}$$

where $r > 0$ is a constant, denoting the return on the risk-free asset.

The price of the risky asset at moment t is $S_1(t)$, and the price process satisfies

$$dS_1(t) = \mu_1 S_1(t)dt + \sigma_1 S_1(t)dW_1(t), \tag{2.2}$$

where $\mu_1 > r$ is a constant denoting the average return of the risky asset, $\sigma_1 > 0$ is a constant denoting the volatility of the risky asset, and $W_1(t)$ is the standard Brownian motion on the probability space $(\mathbf{\Omega}, \mathcal{F}, \mathcal{F}_t, \mathbf{P})$.

For analytical simplicity, consider an insurance market consisting of two insurers and one reinsurer, and use the classical Cramer-Lundberg risk model to describe the surplus process of the two insurers:

$$R_i(t) = x_i^0 + c_i(t) - \sum_{n=1}^{G_i(t)} Z_i^n, \ i \in \{1, 2\}, \tag{2.3}$$

where $x_i^0 \geq 0$ is the initial surplus of insurer i, $c_i(t)$ is the premium charged by insurer i, $\sum_{n=1}^{G_i(t)} Z_i^n$ is a compound Poisson process, $\{Z_i^n, n = 1, 2, ...\}$ is a list of positive, independent and identically distributed random variable denoting the amount claimed by insurer i in the nth claim, with finite first moment $\mu_{z_i} \in (0, \infty)$ and finite second moment $\sigma_{z_i}^2 \in (0, \infty)$. $\{G_i(t), t \geq 0\}$ is a homogeneous Poisson process of intensity λ_i, denoting the cumulative number of claims for insurer i at time $[0, t]$. $\{G_i(t), t \geq 0\}$ and $\{Z_i^n, n = 1, 2, ...\}$ are independent of each other.

In this paper, we assume that insurer i adopts the expected value premium principle when conducting premium collection, such that $c_i(t) = (1+\eta_i)\lambda_i\mu_{z_i}t$, where the constant $\eta_i > 0$ is the safety loading factor of the insurer i. Insurers often purchase proportional reinsurance for their benefit to spread risk and stabilize their business. Assume that insurers 1 and 2 conduct proportional reinsurance business with the same reinsurer, their retention levels are $\{q_1(t), t > 0\}$ and $\{q_2(t), t > 0\}$ respectively. The premium to be paid by insurer i to the reinsurer at time t is $\lambda_i\mu_{z_i}(1 + \theta_i)(1 - q_i(t))$, where $\theta_i > \eta_i$ is the safety loading factor of the reinsurer relative to the insurer i. Thus, for the reinsurance strategy $q_i(t)$, the surplus process for insurer i can be expressed as follows

$$dR_i^{q_i}(t) = \lambda_i\mu_{z_i} \left[\eta_i + q_i(t) - \theta_i + \theta_i q_i(t)\right] dt - q_i(t)d\sum_{n=1}^{G_i(t)} Z_i^n. \tag{2.4}$$

In summary, the reinsurance-investment strategy for insurer i is $u_i(t) = (q_i(t), \pi_i(t))$: where it is assumed that at time t the insurer i invests the amount of $\pi_i(t)$ in risky asset, and the remaining $X_i^{u_i}(t) - \pi_i(t)$ is invested in risk-free asset, and short-selling is allowed in the investment, i.e. $\pi_i(t) \in \mathbb{R}$. Therefore, the wealth process of insurer i is modeled as

$$\begin{aligned} dX_i^{u_i}(t) &= \pi_i(t)\frac{dS_1(t)}{S_1(t)} + (X_i^{u_i}(t) - \pi_i(t))\frac{dS_0(t)}{S_0(t)} + dR_i^{q_i}(t) \\ &= \Big[\pi_i(t)(\mu_1 - r) + rX_i^{u_i}(t) + \lambda_i\mu_{z_i}(\eta_i + q_i(t) - \theta_i + \theta_i q_i(t))\Big]dt \\ &\quad + \pi_i(t)\sigma_1 dW_1(t) - q_i(t)d\sum_{n=1}^{G_i(t)} Z_i^n, \end{aligned} \tag{2.5}$$

with $X_i^{u_i}(0) = x_i^0$.

In addition, it can be found that most of the literatures in recent years only rely on the information possessed by insurers when making decisions to select the optimal strategy, while the insurer's reinsurance-investment strategy is affected by the inflow and outflow of capital in the past. See, [21, 26, 30, 33, 35, 38] for related studies. Therefore, when describing the wealth process of an insurer, the delay effect should be taken into account, i.e., the average performance and the point performance in the past period should be included in the wealth process. It is assumed that there is a flow of funds associated with historical performance in the wealth equation of insurer i, denoted by the function $f_i(t, X_i^{u_i}(t) - \overline{M}_i(t), X_i^{u_i}(t) - N_i(t))$. Then the wealth process $\{X_i^{u_i}(t) : t \in [0, T]\}$ of the insurer i satisfies a delayed stochastic differential equation

$$\begin{aligned} dX_i^{u_i}(t) = &\Big[\pi_i(t)(\mu_1 - r) + rX_i^{u_i}(t) + \lambda_i\mu_{z_i}(\eta_i + q_i(t) - \theta_i + \theta_i q_i(t))\Big]dt \\ &+ \pi_i(t)\sigma_1 dW_1(t) - q_i(t)d\sum_{n=1}^{G_i(t)} Z_i^n - f_i(t, X_i^{u_i}(t) - \overline{M}_i(t), X_i^{u_i}(t) - N_i(t))dt, \end{aligned} \tag{2.6}$$

where $M_i(t) = \int_{-h_i}^0 e^{\delta_i s} X_i^{u_i}(t+s)ds, \overline{M}_i(t) = \frac{M_i(t)}{\int_{-h_i}^0 e^{\delta_i s}ds}, N_i(t) = X_i^{u_i}(t - h_i)$ and $t \in [0, T]$. Here the constant $\delta_i \geq 0$ is the average parameter of insurer i, the constant $h_i > 0$ is the delay parameter of insurer i, and $e^{\delta_i s}$ is the exponential decay factor. $M_i(t), \overline{M}_i(t), N_i(t)$ reflect the aggregate, average, and point-by-point performance of insurer $i's$ capital over the past time interval $[t - h_i, t]$. $X_i^{u_i}(t) - N_i(t)$ is the historical absolute performance over the time interval $[t-h_i, t]$, $X_i^{u_i}(t) - \overline{M}_i(t)$ is the historical average performance over the time interval $[t - h_i, t]$.

To have an analytical solution to the delayed control problem, similar to [26], we assume that the portion of the capital flow associated with historical performance is proportional to the historical performance of the insurer, i.e.,

$$f_i(t, X_i^{u_i}(t) - \overline{M}_i(t), X_i^{u_i}(t) - N_i(t)) = \overline{\tau}_i(X_i^{u_i}(t) - \overline{M}_i(t)) + \varphi_i(X_i^{u_i}(t) - N_i(t)), \tag{2.7}$$

where $\overline{\tau}_i, \varphi_i$ are non-negative constants. Additionally, in the study of the classical Cramer-Lundberg model, it is often difficult to obtain analytical solutions, and [15, 23] proposed approximating the compound Poisson process of claims with general Brownian motion. Therefore, this paper also approximates $d\sum_{n=1}^{G_i(t)} Z_i^n$ using the following diffusion process

$$d\sum_{n=1}^{G_i(t)} Z_i^n \approx \lambda_i\mu_{z_i}dt - \sqrt{\lambda_i}\sigma_{z_i}dW_{R_i}(t),\ i \in \{1, 2\}, \tag{2.8}$$

where $W_{R_1}(t)$ and $W_{R_2}(t)$ are two standard Brownian motions on the probability space $(\mathbf{\Omega}, \mathcal{F}, \mathcal{F}_t, \mathbf{P})$, with correlation coefficient ρ_R, and $(W_{R_1}(t), W_{R_2}(t))$ and $W_1(t)$ are independent. Then the wealth process of insurer i becomes

$$\begin{aligned} dX_i^{u_i}(t) = &\Big[\pi_i(t)(\mu_1 - r) + X_i^{u_i}(t)(r - \overline{\tau}_i - \varphi_i) + \lambda_i\mu_{z_i}(\eta_i - \theta_i + \theta_i q_i(t)) \\ &+ \tau_i M_i(t) + \varphi_i N_i(t)\Big]dt + \pi_i(t)\sigma_1 dW_1(t) + q_i(t)\sqrt{\lambda_i}\sigma_{z_i}dW_{R_i}(t), \end{aligned} \tag{2.9}$$

$$dM_i(t) = [X_i^{u_i}(t) - \delta_i M_i(t) - e^{-\delta_i h_i}N_i(t)]dt, \tag{2.10}$$

where $\tau_i = \frac{\overline{\tau}_i}{\int_{-h_i}^0 e^{\delta_i s}ds}$.

It is difficult to accurately estimate the parameters of the reference model under the probability measure $\mathbf{P}$, the insurer is not in a position to ascertain whether or not the reference model is the true model. Here a set of alternative measures $\mathbf{Q}$ is set up, satisfying $\mathbf{Q} \sim \mathbf{P}$, i.e., the substitutable measure $\mathbf{Q}$ is equivalent to the reference probability measure $\mathbf{P}$, so the set of probability measures $\mathcal{Q} = \{\mathbf{Q}|\mathbf{Q} \sim \mathbf{P}\}$ is set up to characterize a range of alternative models. The following definition is given to introduce the concept of ambiguity.

Definition 2.1. The probability distortion function $\{\phi_i(t) = (\phi_{1i}(t), \phi_{2i}(t)) \mid t \in [0, T]\}$, which represents the cognitive bias of the insurer i towards probability information in judgment and decision-making tasks, satisfies

(1) $\phi_{1i}(t), \phi_{2i}(t)$ are measurable w.r.t. $\mathcal{F}_t$;

(2) $\phi_{1i}(t), \phi_{2i}(t)$ satisfies $E\left[e^{\frac{1}{2}\int_t^T \phi_{1i}^2(s)+\phi_{2i}^2(s)ds}\right] < \infty$, and this condition is called Novikov condition, which guarantees that the following $\Lambda^{\phi}(t)$ is a martingale under the reference measure $\mathbf{P}$.

Denote by Φ_i the space consisting of all probability distortion processes $\{\phi_i(t)\}$, and for every probability measure $\mathbf{Q}^{\phi_i} \in \mathcal{Q}$, there exists a process $\phi_i(t) \in \Phi_i$ such that the Radon-Nikodym derivative process $\frac{d\mathbf{Q}^{\phi_i}}{d\mathbf{P}} |_{\mathcal{F}_t} := \Lambda^{\phi_i}(t)$ satisfies

$$\Lambda^{\phi_i}(t) = \exp\left\{-\int_0^t \phi_{1i}(s)dW_{R_i}(s) - \frac{1}{2}\int_0^t \phi_{1i}^2(s)ds - \int_0^t \phi_{2i}(s)dW_1(s) - \frac{1}{2}\int_0^t \phi_{2i}^2(s)ds\right\}, \tag{2.11}$$

and $E\left[\Lambda^{\phi_i}(t)\right] = 1$.

According to Girsanov theorem, the standard Brownian motion of the alternative measure $\mathbf{Q}^{\phi_i} \in \mathcal{Q}$ can be expressed as:

$$\begin{aligned} dW_{R_i}^{\phi_i}(t) &= dW_{R_i}(t) + \phi_{1i}(t)dt, \\ dW_1^{\phi_i}(t) &= dW_1(t) + \phi_{2i}(t)dt. \end{aligned} \tag{2.12}$$

Then, under the alternative measure $\mathbf{Q}^{\phi_i}$, the wealth process of the insurer $i \in \{1, 2\}$ is given by

$$\begin{aligned} dX_i^{\phi_i}(t) = &\Big[\pi_i(t)(\mu_1 - r) + X_i^{\phi_i}(t)(r - \overline{\tau}_i - \varphi_i) + \lambda_i\mu_{z_i}(\eta_i - \theta_i + \theta_i q_i(t)) \\ &+ \tau_i M_i(t) + \varphi_i N_i(t) - \pi_i(t)\sigma_1\phi_{2i}(t) - q_i(t)\sqrt{\lambda_i}\sigma_{z_i}\phi_{1i}(t)\Big]dt \\ &+ \pi_i(t)\sigma_1 dW_1^{\phi_i}(t) + q_i(t)\sqrt{\lambda_i}\sigma_{z_i}dW_{R_i}^{\phi_i}(t). \end{aligned} \tag{2.13}$$

3. Optimization problem based on α-maximin ambiguity-averse preferences

Motivated by [2], this paper considers the competition problem between two insurers under the relative performance concern and introduces the α-robust theory to analyze the investment and reinsurance problems of insurers under the diversified ambiguity attitudes.

Considering that historical investment performance will have an impact on the wealth of insurers, this paper assumes that insurer i not only focuses on its terminal wealth $X_i^{\phi_i}(T)$, but also its average wealth value $\overline{M}_i(T)$ over a time interval $[T - h_i, T]$, i.e., it focuses on the value of $X_i^{\phi_i}(T) + \overline{\omega}_i\overline{M}_i(T)$, where $\overline{\omega}_i \in (0, 1)$ is the weight of $\overline{M}_i(T)$, which represents the degree of influence of the average wealth $\overline{M}_i(T)$ on the terminal wealth $X_i^{\phi_i}(T)$ over the period $[T - h_i, T]$. To make the model consistent with the literature on classical delay models, i.e., similar to [26], this paper uses cumulative wealth $M_i(t)$ instead of average wealth $\overline{M}_i(t)$ in the subsequent analysis, and $\omega_i = \frac{\overline{\omega}_i}{\int_{-h_i}^{0} e^{\delta_i s} ds}$, we have $\overline{\omega}_i \overline{M}_i(T) = \omega_i M_i(T)$.

In this paper, we assume that the two insurers have homogeneous ambiguity-averse preferences for instance, if insurer i is extremely ambiguity-averse, insurer j will exhibit the same degree of extreme ambiguity aversion. Specifically, both insurers adopt the α-maxmin mean-variance criterion and have symmetrically aligned distortion functions: when the lower bound of insurer i is $\underline{\phi}_i$, that of insurer j is $\underline{\phi}_j$, and when the upper bound of insurer i is $\overline{\phi}_i$, that of insurer j is $\overline{\phi}_j$. Under this structure, the α-robust mean-variance return function for insurer i takes the following form:

$$\begin{aligned} &J_i(t, x_i, x_j, m_i, m_j, u_i, u_j) \\ &:= \alpha_i \inf_{\phi_i \in \Phi_i} \underline{J}_i^{\phi_i, \underline{\phi}_j}(t, x_i, x_j, m_i, m_j, u_i, u_j) + \hat{\alpha}_i \sup_{\phi_i \in \Phi_i} \overline{J}_i^{\phi_i, \overline{\phi}_j}(t, x_i, x_j, m_i, m_j, u_i, u_j) \\ &= \alpha_i \underline{J}_i^{\underline{\phi}_i, \underline{\phi}_j}(t, x_i, x_j, m_i, m_j, u_i, u_j) + \hat{\alpha}_i \overline{J}_i^{\overline{\phi}_i, \overline{\phi}_j}(t, x_i, x_j, m_i, m_j, u_i, u_j), \end{aligned} \tag{3.1}$$

where $i \neq j \in \{1, 2\}, \alpha_i \in [0, 1], \hat{\alpha}_i = 1 - \alpha_i$,

$$\begin{aligned} &\underline{J}_i^{\phi_i, \underline{\phi}_j}(t, x_i, x_j, m_i, m_j, u_i, u_j) \\ &= E_{t, x_i, x_j, m_i, m_j}^{\phi_i, \underline{\phi}_j}[(X_i^{\phi_i}(T) + \omega_i M_i(T)) - \xi_i(X_j^{\underline{\phi}_j}(T) + \omega_j M_j(T))] \\ &- \frac{\gamma_i}{2} Var_{t, x_i, x_j, m_i, m_j}^{\phi_i, \underline{\phi}_j}[(X_i^{\phi_i}(T) + \omega_i M_i(T)) - \xi_i(X_j^{\underline{\phi}_j}(T) + \omega_j M_j(T))] + \int_t^T h^{\psi_i}(\phi_i(s)) ds, \end{aligned} \tag{3.2}$$

$$\begin{aligned} &\overline{J}_i^{\phi_i, \overline{\phi}_j}(t, x_i, x_j, m_i, m_j, u_i, u_j) \\ &= E_{t, x_i, x_j, m_i, m_j}^{\phi_i, \overline{\phi}_j}[(X_i^{\phi_i}(T) + \omega_i M_i(T)) - \xi_i(X_j^{\overline{\phi}_j}(T) + \omega_j M_j(T))] \\ &- \frac{\gamma_i}{2} Var_{t, x_i, x_j, m_i, m_j}^{\phi_i, \overline{\phi}_j}[(X_i^{\phi_i}(T) + \omega_i M_i(T)) - \xi_i(X_j^{\overline{\phi}_j}(T) + \omega_j M_j(T))] - \int_t^T h^{\psi_i}(\phi_i(s)) ds, \end{aligned} \tag{3.3}$$

where $E_{t, x_i, x_j, m_i, m_j}[\cdot]$ and $Var_{t, x_i, x_j, m_i, m_j}[\cdot]$ are the conditional expectation and conditional variance under $\{X_i^{\phi_i}(t) = x_i, X_j^{\phi_j}(t) = x_j, M_i(t) = m_i, M_j(t) = m_j\}$, respectively. $\gamma_i > 0$ is the financial risk aversion coefficient for insurer i, $\xi_i \in [0, 1]$ measures the sensitivity of insurer i to the performance of insurer $j (j \neq i \in \{1, 2\})$, h^{ψ_i} is the penalty function

$$h^{\psi_i}(\phi_i(t)) = \frac{\phi_{1i}^2(t)}{2\psi_{1i}(t, x_i, m_i)} + \frac{\phi_{2i}^2(t)}{2\psi_{2i}(t, x_i, m_i)}, \tag{3.4}$$

where $\psi_{1i}(t, x_i, m_i), \psi_{2i}(t, x_i, m_i)$ denote the ambiguity level of the surplus process and the stock price process of insurer i, respectively. Here let $\psi_i(t, x_i, m_i) = (\psi_{1i}(t, x_i, m_i), \psi_{2i}(t, x_i, m_i))$, and assume that $\psi_{1i}(t, x_i, m_i) = \psi_{1i}$ and $\psi_{2i}(t, x_i, m_i) = \psi_{2i}$, where ψ_{1i}, ψ_{2i} are non-negative constants. When the value of ψ_{1i}, ψ_{2i} is larger, the smaller the value of the penalty function, the higher the level of ambiguity aversion of the insurer i. In particular, $\alpha_i = 0$ indicates that insurer i is ambiguity-seeking, $\alpha_i = \frac{1}{2}$ indicates that insurer i is ambiguity-neutral, and $\alpha_i = 1$ indicates that insurer i is ambiguity-averse.

Definition 3.1. For any $t \in [0, T]$, a reinsurance-investment strategy $u_i(t) = (q_i(t), \pi_i(t))$ is said to be admissible if it satisfies the following condition:

(1) $u_i(t)$ is $\mathcal{F}_t$-progressively measurable, and $q_i(t) \in [0, +\infty)$;

(2) $E^{\underline{\phi}_i}_{t,x_i,m_i}\left[\int_0^T (q_i^2(s) + \pi_i^2(s))ds\right] < +\infty$, and $E^{\overline{\phi}_i}_{t,x_i,m_i}\left[\int_0^T (q_i^2(s) + \pi_i^2(s))ds\right] < +\infty$ for any $(t, x_i, m_i) \in [0, T] \times \mathbb{R}^2$;

(3) The delayed stochastic differential equations (2.13) has a unique strong solution $X_i^{\phi_i}(t)$ for any $(t, x_i, m_i) \in [0, T] \times \mathbb{R}^2$, and satisfies $E[\sup_{t\in T} |X_i^{\phi_i}(t)|^2] < +\infty, i \in \{1, 2\}$.

Denote by $\mathcal{U}_i = \{u_i(t)|t \in [0, T]\}$ the set of all admissible strategies.

The optimization problem for insurer $i \in \{1, 2\}$ under the α-maxmin mean-variance criterion is

$$V_i(t, x_i, x_j, m_i, m_j) := \sup_{u_i\in\mathcal{U}_i} J_i(t, x_i, x_j, m_i, m_j, u_i, u_j^*) = J_i(t, x_i, x_j, m_i, m_j, u_i^*, u_j^*). \tag{3.5}$$

In addition, the problem of time inconsistency in optimization was first studied by [24]. The problem (3.5) belongs to this category, where Bellman's optimal criterion is no longer applicable. For this problem, there are generally two methods to solve it. The first is the so-called pre-commitment, which means that the optimal strategy we obtain is only optimal at the initial point and not necessarily optimal at later points. Relevant literature see [4, 7–9, 28, 39]. The second method was proposed by [24], who provided the original idea of time-consistent strategies for the time-inconsistent problem. Specifically, [24] regarded the initial control problem as a game with participants at each time point, and then sought the subgame perfect Nash equilibrium. For relevant literature, see [25, 31, 34, 41].

Meanwhile, [42] indicates that the time-consistency strategy is more stable and safer than the pre-commitment strategy under the generalized mean-variance criterion. Therefore, based on this literature and solvability, this paper selects the equilibrium strategy to solve the problem of time inconsistency, thereby reducing the possibility of insurers changing their decisions in the future due to time inconsistency.

According to classical game theory, the problem (3.5) is equivalent to the following non-zero-sum game problem.

Non-zero-sum game problem: Find Nash equilibrium strategy $(u_1^*, u_2^*) \in \mathcal{U}_1 \times \mathcal{U}_2$ such that

$$\begin{cases} J_1(t, x_1, x_2, m_1, m_2, u_1, u_2^*) \le J_1(t, x_1, x_2, m_1, m_2, u_1^*, u_2^*), & \forall u_1 \in \mathcal{U}_1, \\ J_2(t, x_1, x_2, m_1, m_2, u_1^*, u_2) \le J_2(t, x_1, x_2, m_1, m_2, u_1^*, u_2^*), & \forall u_2 \in \mathcal{U}_2. \end{cases} \tag{3.6}$$

Definition 3.2. For any admissible strategy $u_i^*(t) \in \mathcal{U}_i(t)$, we consider the following strategy:

$$u_i^{\varepsilon_i}(s, x_i) := \begin{cases} (\tilde{q}_i(s), \tilde{\pi}_i(s)), & \forall (s, x_i) \in [t, t + \varepsilon_i) \times \mathbb{R}, \\ u_i^*(s, x_i), & \forall (s, x_i) \in [t + \varepsilon_i, T) \times \mathbb{R}, \end{cases} \tag{3.7}$$

where $\tilde{q}_i(t) \in [0, +\infty), \tilde{\pi}_i(t) \in \mathbb{R}$, and $\varepsilon_i > 0$. If for any $(q_i^*(t), \pi_i^*(t)) \in [0, +\infty)\times\mathbb{R}$, and $(t, x_i, x_j, m_i, m_j) \in [0, T] \times \mathbb{R}^4$ satisfying

$$\liminf_{\varepsilon_i \downarrow 0} \frac{J_i(t, x_i, x_j, m_i, m_j, u_i^*, u_j^*) - J_i(t, x_i, x_j, m_i, m_j, u_i^{\varepsilon_i}, u_j^*)}{\varepsilon} \ge 0, \tag{3.8}$$

then u_i^* is called an equilibrium reinsurance-investment strategy of insurer i, $V_i(t, x_i, x_j, m_i, m_j) = J_i(t, x_i, x_j, m_i, m_j, u_i^*, u_j^*)$ is the corresponding equilibrium value function.

4. Main results

For non-zero-sum game problem (3.6), let $\tilde{X}_i(t) = X_i^{\phi_i}(t) - \xi_i X_j^{\phi_j}(t)$, where $i \neq j \in \{1, 2\}$, then we have

$$
\begin{aligned}
d\tilde{X}_i(t) = \Big[&(\mu_1 - r)(\pi_i(t) - \xi_i\pi_j(t)) + r(X_i^{\phi_i} - \xi_i X_j^{\phi_j}) - (\overline{\tau}_i + \varphi_i)X_i^{\phi_i} + \xi_i(\overline{\tau}_j + \varphi_j)X_j^{\phi_j} \\
&+ \lambda_i\mu_{z_i}(\eta_i - \theta_i + \theta_i q_i(t)) - \xi_i\lambda_j\mu_{z_j}\left(\eta_j - \theta_j + \theta_j q_j(t)\right) + \tau_i M_i(t) - \xi_i\tau_j M_j(t) \\
&+ \varphi_i N_i(t) - \xi_i\varphi_j N_j(t) - \sigma_1(\pi_i(t)\phi_{2i}(t) - \xi_i\pi_j(t)\phi_{2j}(t)) - q_i(t)\sqrt{\lambda_i}\sigma_{z_i}\phi_{1i}(t) \\
&+ \xi_i q_j(t)\sqrt{\lambda_j}\sigma_{z_j}\phi_{1j}(t)\Big]dt + \sigma_1(\pi_i(t)dW_1^{\phi_i}(t) - \xi_i\pi_j(t)dW_1^{\phi_j}(t)) \\
&+ q_i(t)\sqrt{\lambda_i}\sigma_{z_i}dW_{R_i}^{\phi_i}(t) - \xi_i q_j(t)\sqrt{\lambda_j}\sigma_{z_j}dW_{R_j}^{\phi_j}(t).
\end{aligned} \tag{4.1}
$$

Denote $\tilde{x}_i = x_i - \xi_i x_j$, for any $(t, \tilde{x}_i, m_i, m_j) \in [0, T] \times \mathbb{R}^3$, the problem (3.5) becomes

$$
V_i(t, \tilde{x}_i, m_i, m_j) := \sup_{u_i \in \mathcal{U}_i} J_i(t, \tilde{x}_i, m_i, m_j, u_i, u_j^*), i \neq j \in \{1, 2\}, \tag{4.2}
$$

where

$$
\begin{aligned}
&J_i(t, \tilde{x}_i, m_i, m_j, u_i, u_j^*) \\
&:= \alpha_i \inf_{\phi_i \in \Phi_i} \underline{J}_i^{\phi_i, \underline{\phi}_j^*}(t, \tilde{x}_i, m_i, m_j, u_i, u_j^*) + \hat{\alpha}_i \sup_{\phi_i \in \Phi_i} \overline{J}_i^{\phi_i, \overline{\phi}_j^*}(t, \tilde{x}_i, m_i, m_j, u_i, u_j^*) \\
&= \alpha_i \underline{J}_i^{\underline{\phi}_i, \underline{\phi}_j^*}(t, \tilde{x}_i, m_i, m_j, u_i, u_j^*) + \hat{\alpha}_i \overline{J}_i^{\overline{\phi}_i, \overline{\phi}_j^*}(t, \tilde{x}_i, m_i, m_j, u_i, u_j^*),
\end{aligned} \tag{4.3}
$$

$$
\begin{aligned}
&\underline{J}_i^{\phi_i, \underline{\phi}_j^*}(t, \tilde{x}_i, m_i, m_j, u_i, u_j^*) \\
&= E_{t, \tilde{x}_i, m_i, m_j}^{\phi_i, \underline{\phi}_j^*}[\tilde{X}_i(T) + \omega_i M_i(T) - \xi_i\omega_j M_j(T)] \\
&- \frac{\gamma_i}{2} Var_{t, \tilde{x}_i, m_i, m_j}^{\phi_i, \underline{\phi}_j^*}[\tilde{X}_i(T) + \omega_i M_i(T) - \xi_i\omega_j M_j(T)] + \int_t^T h^{\psi_i}(\phi_i(s))ds,
\end{aligned} \tag{4.4}
$$

$$
\begin{aligned}
&\overline{J}_i^{\phi_i, \overline{\phi}_j^*}(t, \tilde{x}_i, m_i, m_j, u_i, u_j^*) \\
&= E_{t, \tilde{x}_i, m_i, m_j}^{\phi_i, \overline{\phi}_j^*}[\tilde{X}_i(T) + \omega_i M_i(T) - \xi_i\omega_j M_j(T)] \\
&- \frac{\gamma_i}{2} Var_{t, \tilde{x}_i, m_i, m_j}^{\phi_i, \overline{\phi}_j^*}[\tilde{X}_i(T) + \omega_i M_i(T) - \xi_i\omega_j M_j(T)] - \int_t^T h^{\psi_i}(\phi_i(s))ds.
\end{aligned} \tag{4.5}
$$

Let $C^{1,2,1,1}([0, T] \times \mathbb{R}^3)$ be the space of all continuous functions $\Psi^i(t, \tilde{x}_i, m_i, m_j)$, where $\Psi^i(t, \cdot, \cdot, \cdot)$ is first-order continuously differentiable on $[0, T]$, $\Psi^i(\cdot, \tilde{x}_i, \cdot, \cdot)$ is second-order continuously differentiable on $\mathbb{R}$, and $\Psi^i(\cdot, \cdot, m_i, \cdot)$ and $\Psi^i(\cdot, \cdot, \cdot, m_j)$ are first-order continuously differentiable on $\mathbb{R}$. For $\Psi^i(t, \tilde{x}_i, m_i, m_j) \in C^{1,2,1,1}([0, T] \times \mathbb{R}^3)$ and $u_i \in \mathcal{U}_i$, the differential operator corresponding to

Eq (4.1) can be expressed as

$$\begin{aligned}
&\mathcal{A}_i^{u_i,\phi_i,\phi_j}\Psi^i(t,\tilde{x}_i,m_i,m_j) = \Psi_t^i(t,\tilde{x}_i,m_i,m_j) + \Big[(\mu_1 - r)(\pi_i - \xi_i\pi_j) + r(x_i - \xi_i x_j) \\
&- (\overline{\tau}_i + \varphi_i)x_i + \xi_i(\overline{\tau}_j + \varphi_j)x_j + \lambda_i\mu_{z_i}(\eta_i - \theta_i + \theta_i q_i) - \xi_i\lambda_j\mu_{z_j}\left(\eta_j - \theta_j + \theta_j q_j\right) \\
&+ \tau_i m_i - \xi_i\tau_j m_j + \varphi_i n_i - \xi_i\varphi_j n_j - \sigma_1(\pi_i\phi_{2i} - \xi_i\pi_j\phi_{2j}) - q_i\sqrt{\lambda_i}\sigma_{z_i}\phi_{1i} \\
&+ \xi_i q_j\sqrt{\lambda_j}\sigma_{z_j}\phi_{1j}\Big]\Psi_{\tilde{x}_i}^i(t,\tilde{x}_i,m_i,m_j) + \frac{1}{2}\Big[\sigma_1^2\pi_i^2 - 2\sigma_1^2\xi_i\pi_i\pi_j + \sigma_1^2\xi_i^2\pi_j^2 + q_i^2\lambda_i\sigma_{z_i}^2 \\
&- 2\rho_R\xi_i q_i q_j\sqrt{\lambda_i\lambda_j}\sigma_{z_i}\sigma_{z_j} + \xi_i^2 q_j^2\lambda_j\sigma_{z_j}^2\Big]\Psi_{\tilde{x}_i\tilde{x}_i}^i(t,\tilde{x}_i,m_i,m_j) \\
&+ (x_i - \delta_i m_i - e^{-\delta_i h_i}n_i)\Psi_{m_i}^i(t,\tilde{x}_i,m_i,m_j) + (x_j - \delta_j m_j - e^{-\delta_j h_j}n_j)\Psi_{m_j}^i(t,\tilde{x}_i,m_i,m_j).
\end{aligned} \tag{4.6}$$

The proof of the following verification theorem is similar to [18], so we omit it here.

Theorem 4.1. *Suppose there exist functions $V^i(t,\tilde{x}_i,m_i,m_j), \underline{g}^i(t,\tilde{x}_i,m_i,m_j), \overline{g}^i(t,\tilde{x}_i,m_i,m_j) \in C^{1,2,1,1}([0,T]\times\mathbb{R}^3)$ satisfying the following conditions:*

(1) For any $(t,\tilde{x}_i,m_i,m_j) \in [0,T]\times\mathbb{R}^3$,

$$\begin{aligned}
0 = \sup_{u_i\in\mathcal{U}_i}\Bigg\{&\alpha_i\inf_{\phi_i\in\Phi_i}\Big[\mathcal{A}_i^{u_i,\phi_i,\underline{\phi}_j^*}V^i(t,\tilde{x}_i,m_i,m_j) - \frac{\gamma_i}{2}\mathcal{A}_i^{u_i,\phi_i,\underline{\phi}_j^*}\underline{g}^{i2}(t,\tilde{x}_i,m_i,m_j) \\
&+\gamma_i\underline{g}^i(t,\tilde{x}_i,m_i,m_j)\mathcal{A}_i^{u_i,\phi_i,\underline{\phi}_j^*}\underline{g}^i(t,\tilde{x}_i,m_i,m_j) + h^{\psi_i}(\phi_i(t))\Big] \\
&+\hat{\alpha}_i\sup_{\phi_i\in\Phi_i}\Big[\mathcal{A}_i^{u_i,\phi_i,\overline{\phi}_j^*}V^i(t,\tilde{x}_i,m_i,m_j) - \frac{\gamma_i}{2}\mathcal{A}_i^{u_i,\phi_i,\overline{\phi}_j^*}\overline{g}^{i2}(t,\tilde{x}_i,m_i,m_j) \\
&+\gamma_i\overline{g}^i(t,\tilde{x}_i,m_i,m_j)\mathcal{A}_i^{u_i,\phi_i,\overline{\phi}_j^*}\overline{g}^i(t,\tilde{x}_i,m_i,m_j) - h^{\psi_i}(\phi_i(t))\Big]\Bigg\};
\end{aligned} \tag{4.7}$$

(2) For any $(t,\tilde{x}_i,m_i,m_j) \in [0,T]\times\mathbb{R}^3$,

$$\begin{aligned}
&V^i(T,\tilde{x}_i,m_i,m_j) = \tilde{x}_i + \omega_i m_i - \xi_i\omega_j m_j, \\
&\mathcal{A}_i^{u_i^*,\underline{\phi}_i^*,\underline{\phi}_j^*}\underline{g}^i(t,\tilde{x}_i,m_i,m_j) = \mathcal{A}_i^{u_i^*,\overline{\phi}_i^*,\overline{\phi}_j^*}\overline{g}^i(t,\tilde{x}_i,m_i,m_j) = 0, \\
&\underline{g}^i(T,\tilde{x}_i,m_i,m_j) = \overline{g}^i(T,\tilde{x}_i,m_i,m_j) = \tilde{x}_i + \omega_i m_i - \xi_i\omega_j m_j,
\end{aligned} \tag{4.8}$$

where

$$\begin{aligned}
&\underline{g}^i(t,\tilde{x}_i,m_i,m_j) = E_{t,\tilde{x}_i,m_i,m_j}^{\underline{\phi}_i^*,\underline{\phi}_j^*}[\tilde{X}_i(T) + \omega_i M_i(T) - \xi_i\omega_j M_j(T)], \\
&\overline{g}^i(t,\tilde{x}_i,m_i,m_j) = E_{t,\tilde{x}_i,m_i,m_j}^{\overline{\phi}_i^*,\overline{\phi}_j^*}[\tilde{X}_i(T) + \omega_i M_i(T) - \xi_i\omega_j M_j(T)];
\end{aligned} \tag{4.9}$$

(3) $\underline{\phi}_i^* = \underline{\phi}_i^{u_i^*}, \overline{\phi}_i^* = \overline{\phi}_i^{u_i^*}$,
then u_i^* is the equilibrium strategy for insurer i, and $V_i(t,\tilde{x}_i,m_i,m_j) = J_i(t,\tilde{x}_i,m_i,m_j,u_i^*,u_j^*), i \neq j \in \{1,2\}$ is the equilibrium value function to the α-robust reinsurance-investment problem.

Theorem 4.2. *Consider the α-robust non-zero-sum game problem under the influence of similar ambiguity-averse preferences and delay effects.*

(1) The Nash equilibrium reinsurance-investment strategy (u_1^*, u_2^*) *is*

$$u_1^*: \begin{cases} q_1^*(t) &= \dfrac{-\lambda_1\mu_{z_1}\theta_1\sigma_{z_2}(-\gamma_2+(1-2\alpha_2)\psi_{12})+\rho_R\xi_1\sqrt{\lambda_1\lambda_2}\sigma_{z_1}\gamma_1\mu_{z_2}\theta_2}{e^{H_1(T-t)}\lambda_1\sigma_{z_1}^2\sigma_{z_2}\left[(-\gamma_1+(1-2\alpha_1)\psi_{11})(-\gamma_2+(1-2\alpha_2)\psi_{12})-\rho_R^2\xi_1\xi_2\gamma_1\gamma_2\right]} \vee 0, \\ \pi_1^*(t) &= \dfrac{(r-\mu_1)\left[-\gamma_2+(1-2\alpha_2)\psi_{22}-\xi_1\gamma_1\right]}{e^{H_1(T-t)}\sigma_1^2\left[(-\gamma_1+(1-2\alpha_1)\psi_{21})(-\gamma_2+(1-2\alpha_2)\psi_{22})-\xi_1\xi_2\gamma_1\gamma_2\right]}, \end{cases} \tag{4.10}$$

$$u_2^*: \begin{cases} q_2^*(t) &= \dfrac{-\lambda_2\mu_{z_2}\theta_2\sigma_{z_1}(-\gamma_1+(1-2\alpha_1)\psi_{11})+\rho_R\xi_2\sqrt{\lambda_1\lambda_2}\sigma_{z_2}\gamma_2\mu_{z_1}\theta_1}{e^{H_2(T-t)}\lambda_2\sigma_{z_2}^2\sigma_{z_1}\left[(-\gamma_1+(1-2\alpha_1)\psi_{11})(-\gamma_2+(1-2\alpha_2)\psi_{12})-\rho_R^2\xi_1\xi_2\gamma_1\gamma_2\right]} \vee 0, \\ \pi_2^*(t) &= \dfrac{(r-\mu_1)\left[-\gamma_1+(1-2\alpha_1)\psi_{21}-\xi_2\gamma_2\right]}{e^{H_2(T-t)}\sigma_1^2\left[(-\gamma_1+(1-2\alpha_1)\psi_{21})(-\gamma_2+(1-2\alpha_2)\psi_{22})-\xi_1\xi_2\gamma_1\gamma_2\right]}. \end{cases} \tag{4.11}$$

(2) The probability distortion function for extreme ambiguity aversion for insurer $i \in \{1, 2\}$ *is* $\underline{\phi}_i^* = (\underline{\phi}_{1i}^*, \underline{\phi}_{2i}^*)$ *with*

$$\underline{\phi}_1^* = (\underline{\phi}_{11}^*, \underline{\phi}_{21}^*): \begin{cases} \underline{\phi}_{11}^* = q_1^*(t)\sqrt{\lambda_1}\sigma_{z_1}\psi_{11}e^{H_1(T-t)}, \\ \underline{\phi}_{21}^* = \sigma_1\pi_1^*(t)\psi_{21}e^{H_1(T-t)}, \end{cases} \tag{4.12}$$

$$\underline{\phi}_2^* = (\underline{\phi}_{12}^*, \underline{\phi}_{22}^*): \begin{cases} \underline{\phi}_{12}^* = q_2^*(t)\sqrt{\lambda_2}\sigma_{z_2}\psi_{12}e^{H_2(T-t)}, \\ \underline{\phi}_{22}^* = \sigma_1\pi_2^*(t)\psi_{22}e^{H_2(T-t)}, \end{cases} \tag{4.13}$$

and the probability distortion function of extreme ambiguity-seeking is $\overline{\phi}_i^* = (\overline{\phi}_{1i}^*, \overline{\phi}_{2i}^*)$ *with*

$$\overline{\phi}_1^* = (\overline{\phi}_{11}^*, \overline{\phi}_{21}^*): \begin{cases} \overline{\phi}_{11}^* = -q_1^*(t)\sqrt{\lambda_1}\sigma_{z_1}\psi_{11}e^{H_1(T-t)}, \\ \overline{\phi}_{21}^* = -\sigma_1\pi_1^*(t)\psi_{21}e^{H_1(T-t)}, \end{cases} \tag{4.14}$$

$$\overline{\phi}_2^* = (\overline{\phi}_{12}^*, \overline{\phi}_{22}^*): \begin{cases} \overline{\phi}_{12}^* = -q_2^*(t)\sqrt{\lambda_2}\sigma_{z_2}\psi_{12}e^{H_2(T-t)}, \\ \overline{\phi}_{22}^* = -\sigma_1\pi_2^*(t)\psi_{22}e^{H_2(T-t)}. \end{cases} \tag{4.15}$$

(3) When $q_i^*(t) > 0$*, the corresponding equilibrium value function is*

$$V^i(t, \tilde{x}_i, m_i, m_j) = e^{H_i(T-t)}(\tilde{x}_i + \omega_i m_i - \xi_i\omega_j m_j) + B^i(t), \quad i = 1, 2, \tag{4.16}$$

where $B^i(t)$ *is shown in Eq* (4.36) *and* $H_i = r - \overline{\tau}_i - \varphi_i + \omega_i$.

Proof. The formula (4.7) can be written as

$$\begin{aligned} 0 = \sup_{u_i \in \mathcal{U}_i} \Bigg\{ & V_t^i + \Big[(\mu_1 - r)(\pi_i - \xi_i\pi_j^*) + r(x_i - \xi_i x_j) - (\overline{\tau}_i + \varphi_i)x_i + \xi_i(\overline{\tau}_j + \varphi_j)x_j \\ & + \lambda_i\mu_{z_i}(\eta_i - \theta_i + \theta_i q_i) - \xi_i\lambda_j\mu_{z_j}\left(\eta_j - \theta_j + \theta_j q_j^*\right) + \tau_i m_i - \xi_i\tau_j m_j + \varphi_i n_i - \xi_i\varphi_j n_j\Big]V_{\tilde{x}_i}^i \\ & + \frac{1}{2}\Big[\sigma_1^2\pi_i^2 - 2\sigma_1^2\xi_i\pi_i\pi_j^* + \sigma_1^2\xi_i^2\pi_j^{*2} + q_i^2\lambda_i\sigma_{z_i}^2 - 2\rho_R\xi_i q_i q_j^*\sqrt{\lambda_i\lambda_j}\sigma_{z_i}\sigma_{z_j} + \xi_i^2 q_j^{*2}\lambda_j\sigma_{z_j}^2\Big] \times \\ & \left(V_{\tilde{x}_i\tilde{x}_i}^i - \alpha_i\gamma_i\underline{g}_{\tilde{x}_i}^{i2} - \hat{\alpha}_i\gamma_i\overline{g}_{\tilde{x}_i}^{i2}\right) + (x_i - \delta_i m_i - e^{-\delta_i h_i}n_i)V_{m_i}^i + (x_j - \delta_j m_j - e^{-\delta_j h_j}n_j)V_{m_j}^i \\ & + \alpha_i \inf_{\phi_i \in \Phi_i}\Big\{\big[-\sigma_1\pi_i\phi_{2i} - q_i\sqrt{\lambda_i}\sigma_{z_i}\phi_{1i} + \sigma_1\xi_i\pi_j^*\underline{\phi}_{2j}^* + \xi_i q_j^*\sqrt{\lambda_j}\sigma_{z_j}\underline{\phi}_{1j}^*\big]V_{\tilde{x}_i}^i + \big[\frac{\phi_{1i}^2}{2\psi_{1i}} + \frac{\phi_{2i}^2}{2\psi_{2i}}\big]\Big\} \\ & + \hat{\alpha}_i \sup_{\phi_i \in \Phi_i}\Big\{\big[-\sigma_1\pi_i\phi_{2i} - q_i\sqrt{\lambda_i}\sigma_{z_i}\phi_{1i} + \sigma_1\xi_i\pi_j^*\overline{\phi}_{2j}^* + \xi_i q_j^*\sqrt{\lambda_j}\sigma_{z_j}\overline{\phi}_{1j}^*\big]V_{\tilde{x}_i}^i - \big[\frac{\phi_{1i}^2}{2\psi_{1i}} + \frac{\phi_{2i}^2}{2\psi_{2i}}\big]\Big\}\Bigg\}. \end{aligned} \tag{4.17}$$

Applying the first-order condition on (4.17) with respect to ϕ, the infimum and the supremum of ϕ are achieved respectively at:

$$\begin{cases}\underline{\phi}^*_{1i} = q_i\sqrt{\lambda_i}\sigma_{z_i}\psi_{1i}V^i_{\tilde{x}_i},\\ \underline{\phi}^*_{2i} = \sigma_1\pi_i\psi_{2i}V^i_{\tilde{x}_i},\end{cases} \tag{4.18}$$

and

$$\begin{cases}\overline{\phi}^*_{1i} = -q_i\sqrt{\lambda_i}\sigma_{z_i}\psi_{1i}V^i_{\tilde{x}_i},\\ \overline{\phi}^*_{2i} = -\sigma_1\pi_i\psi_{2i}V^i_{\tilde{x}_i}.\end{cases} \tag{4.19}$$

Inspired by the boundary condition, we guess that the solution to Eq (4.17) takes the following form

$$\begin{cases}V^i(t,\tilde{x}_i,m_i,m_j) = A^i(t)(\tilde{x}_i+\omega_i m_i-\xi_i\omega_j m_j)+B^i(t),\\ \underline{g}^i(t,\tilde{x}_i,m_i,m_j) = \underline{a}^i(t)(\tilde{x}_i+\omega_i m_i-\xi_i\omega_j m_j)+\underline{b}^i(t),\\ \overline{g}^i(t,\tilde{x}_i,m_i,m_j) = \overline{a}^i(t)(\tilde{x}_i+\omega_i m_i-\xi_i\omega_j m_j)+\overline{b}^i(t),\end{cases} \tag{4.20}$$

and satisfies the boundary conditions

$$\begin{aligned}&A^i(T) = \underline{a}^i(T) = \overline{a}^i(T) = 1,\\ &B^i(T) = \underline{b}^i(T) = \overline{b}^i(T) = 0.\end{aligned} \tag{4.21}$$

Then we have

$$\begin{aligned}&V^i_t = A^i_t(\tilde{x}_i+\omega_i m_i-\xi_i\omega_j m_j)+B^i_t,\ V^i_{\tilde{x}_i} = A^i,\ V^i_{\tilde{x}_i\tilde{x}_i} = 0,\ V^i_{m_i} = A^i\omega_i,\ V^i_{m_j} = -A^i\xi_i\omega_j,\\ &\underline{g}^i_t = \underline{a}^i_t(\tilde{x}_i+\omega_i m_i-\xi_i\omega_j m_j)+\underline{b}^i_t,\ \underline{g}^i_{\tilde{x}_i} = \underline{a}^i,\ \underline{g}^i_{\tilde{x}_i\tilde{x}_i} = 0,\ \underline{g}^i_{m_i} = \underline{a}^i\omega_i,\ \underline{g}^i_{m_j} = -\underline{a}^i\xi_i\omega_j,\\ &\overline{g}^i_t = \overline{a}^i_t(\tilde{x}_i+\omega_i m_i-\xi_i\omega_j m_j)+\overline{b}^i_t,\ \overline{g}^i_{\tilde{x}_i} = \overline{a}^i,\ \overline{g}^i_{\tilde{x}_i\tilde{x}_i} = 0,\ \overline{g}^i_{m_i} = \overline{a}^i\omega_i,\ \overline{g}^i_{m_j} = -\overline{a}^i\xi_i\omega_j.\end{aligned} \tag{4.22}$$

Substituting Eqs (4.18), (4.19) and (4.22) into (4.17), we have

$$\begin{aligned}0 = \sup_{u_i\in\mathcal{U}_i}\Bigg\{&A^i_t(\tilde{x}_i+\omega_i m_i-\xi_i\omega_j m_j)+B^i_t+\Big[(\mu_1-r)(\pi_i-\xi_i\pi^*_j)+r(x_i-\xi_i x_j)\\ &-(\overline{\tau}_i+\varphi_i)x_i+\xi_i(\overline{\tau}_j+\varphi_j)x_j+\lambda_i\mu_{z_i}(\eta_i-\theta_i+\theta_i q_i)-\xi_i\lambda_j\mu_{z_j}\left(\eta_j-\theta_j+\theta_j q^*_j\right)\\ &+\tau_i m_i-\xi_i\tau_j m_j+\varphi_i n_i-\xi_i\varphi_j n_j\Big]A^i+\frac{1}{2}\Big[\sigma_1^2\pi_i^2-2\sigma_1^2\xi_i\pi_i\pi^*_j+\sigma_1^2\xi_i^2\pi^{*2}_j+q_i^2\lambda_i\sigma^2_{z_i}\\ &-2\rho_R\xi_i q_i q^*_j\sqrt{\lambda_i\lambda_j}\sigma_{z_i}\sigma_{z_j}+\xi_i^2 q^{*2}_j\lambda_j\sigma^2_{z_j}\Big](-\alpha_i\gamma_i\underline{a}^{i2}-\hat{\alpha}_i\gamma_i\overline{a}^{i2})+(x_i-\delta_i m_i-e^{-\delta_i h_i}n_i)A^i\omega_i\\ &-(x_j-\delta_j m_j-e^{-\delta_j h_j}n_j)A^i\xi_i\omega_j+\frac{1-2\alpha_i}{2}\Big[\sigma_1^2\pi_i^2\psi_{2i}+q_i^2\lambda_i\sigma^2_{z_i}\psi_{1i}\Big]A^{i2}\\ &+\sigma_1\xi_i\pi^*_j(\alpha_i\underline{\phi}^*_{2j}+\hat{\alpha}_i\overline{\phi}^*_{2j})A^i+\xi_i q^*_j\sqrt{\lambda_j}\sigma_{z_j}(\alpha_i\underline{\phi}^*_{1j}+\hat{\alpha}_i\overline{\phi}^*_{1j})A^i\Bigg\}.\end{aligned} \tag{4.23}$$

Using the first-order condition of Eq (4.23) with respect to q_i and π_i, we get

$$q^*_i = \frac{-\lambda_i\mu_{z_i}\theta_i A^i+\rho_R\xi_i q^*_j\sqrt{\lambda_i\lambda_j}\sigma_{z_i}\sigma_{z_j}(-\alpha_i\gamma_i\underline{a}^{i2}-\hat{\alpha}_i\gamma_i\overline{a}^{i2})}{\lambda_i\sigma^2_{z_i}\left[-\alpha_i\gamma_i\underline{a}^{i2}-\hat{\alpha}_i\gamma_i\overline{a}^{i2}+(1-2\alpha_i)\psi_{1i}A^{i2}\right]}, \tag{4.24}$$

$$\pi_i^* = \frac{-A^i(\mu_1 - r) + \sigma_1^2\xi_i\pi_j^*(-\alpha_i\gamma_i\underline{a}^{i2} - \hat{\alpha}_i\gamma_i\overline{a}^{i2})}{\sigma_1^2\left[-\alpha_i\gamma_i\underline{a}^{i2} - \hat{\alpha}_i\gamma_i\overline{a}^{i2} + (1-2\alpha_i)\psi_{2i}A^{i2}\right]}. \tag{4.25}$$

Substituting (4.24) and (4.25) into (4.23), we have

$$\begin{aligned}
0 = {} & A_t^i(\tilde{x}_i + \omega_i m_i - \xi_i\omega_j m_j) + B_t^i + \Big[-(\mu_1 - r)\xi_i\pi_j^* + r(x_i - \xi_i x_j) - (\overline{\tau}_i + \varphi_i)x_i \\
& + \xi_i(\overline{\tau}_j + \varphi_j)x_j + \lambda_i\mu_{z_i}(\eta_i - \theta_i) - \xi_i\lambda_j\mu_{z_j}\left(\eta_j - \theta_j + \theta_j q_j^*\right) + \tau_i m_i - \xi_i\tau_j m_j + \varphi_i n_i \\
& - \xi_i\varphi_j n_j\Big]A^i(t) + \frac{1}{2}\Big[\sigma_1^2\xi_i^2\pi_j^{*2} + \xi_i^2 q_j^{*2}\lambda_j\sigma_{z_j}^2\Big]\big(-\alpha_i\gamma_i\underline{a}^{i2} - \hat{\alpha}_i\gamma_i\overline{a}^{i2}\big) + (x_i - \delta_i m_i - e^{-\delta_i h_i}n_i)A^i\omega_i \\
& - (x_j - \delta_j m_j - e^{-\delta_j h_j}n_j)A^i\xi_i\omega_j + \sigma_1\xi_i\pi_j^*(\alpha_i\underline{\phi}_{2j}^* + \hat{\alpha}_i\overline{\phi}_{2j}^*)A^i + \xi_i q_j^*\sqrt{\lambda_j}\sigma_{z_j}(\alpha_i\underline{\phi}_{1j}^* + \hat{\alpha}_i\overline{\phi}_{1j}^*)A^i \\
& - \frac{[\lambda_i\mu_{z_i}\theta_i A^i - \rho_R\xi_i q_j^*\sqrt{\lambda_i\lambda_j}\sigma_{z_i}\sigma_{z_j}(-\alpha_i\gamma_i\underline{a}^{i2} - \hat{\alpha}_i\gamma_i\overline{a}^{i2})]^2}{2\lambda_i\sigma_{z_i}^2\left[-\alpha_i\gamma_i\underline{a}^{i2} - \hat{\alpha}_i\gamma_i\overline{a}^{i2} + (1-2\alpha_i)\psi_{1i}A^{i2}\right]} \\
& - \frac{[A^i(\mu_1 - r) - \sigma_1^2\xi_i\pi_j^*(-\alpha_i\gamma_i\underline{a}^{i2} - \hat{\alpha}_i\gamma_i\overline{a}^{i2})]^2}{2\sigma_1^2\left[-\alpha_i\gamma_i\underline{a}^{i2} - \hat{\alpha}_i\gamma_i\overline{a}^{i2} + (1-2\alpha_i)\psi_{2i}A^{i2}\right]}.
\end{aligned} \tag{4.26}$$

[6, 11, 12] mentioned that optimal control problems considering delay are often infinite-dimensional problems. In order to obtain an optimal solution in analytic form so that the control problem becomes finite-dimensional and solvable, similar to [1, 26], the following assumptions are made about the parameters, i.e.,

$$\begin{aligned}
& \varphi_i = \omega_i e^{-\delta_i h_i},\ \tau_i = (\delta_i + r - \overline{\tau}_i - \varphi_i + \omega_i)\omega_i, \\
& r - \overline{\tau}_i - \varphi_i + \omega_i = r - \overline{\tau}_j - \varphi_j + \omega_j, i \neq j \in \{1, 2\}.
\end{aligned} \tag{4.27}$$

Then Eq (4.26) becomes

$$\begin{aligned}
0 = {} & [A_t^i + (r - \overline{\tau}_i - \varphi_i + \omega_i)A^i](\tilde{x}_i + \omega_i m_i - \xi_i\omega_j m_j) + B_t^i \\
& + \Big[-(\mu_1 - r)\xi_i\pi_j^* + \lambda_i\mu_{z_i}(\eta_i - \theta_i) - \xi_i\lambda_j\mu_{z_j}\left(\eta_j - \theta_j + \theta_j q_j^*\right)\Big]A^i \\
& + \frac{1}{2}\Big[\sigma_1^2\xi_i^2\pi_j^{*2} + \xi_i^2 q_j^{*2}\lambda_j\sigma_{z_j}^2\Big]\big(-\alpha_i\gamma_i\underline{a}^{i2} - \hat{\alpha}_i\gamma_i\overline{a}^{i2}\big) \\
& + \sigma_1\xi_i\pi_j^*(\alpha_i\underline{\phi}_{2j}^* + \hat{\alpha}_i\overline{\phi}_{2j}^*)A^i + \xi_i q_j^*\sqrt{\lambda_j}\sigma_{z_j}(\alpha_i\underline{\phi}_{1j}^* + \hat{\alpha}_i\overline{\phi}_{1j}^*)A^i \\
& - \frac{[\lambda_i\mu_{z_i}\theta_i A^i - \rho_R\xi_i q_j^*\sqrt{\lambda_i\lambda_j}\sigma_{z_i}\sigma_{z_j}(-\alpha_i\gamma_i\underline{a}^{i2} - \hat{\alpha}_i\gamma_i\overline{a}^{i2})]^2}{2\lambda_i\sigma_{z_i}^2\left[-\alpha_i\gamma_i\underline{a}^{i2} - \hat{\alpha}_i\gamma_i\overline{a}^{i2} + (1-2\alpha_i)\psi_{1i}A^{i2}\right]} \\
& - \frac{[A^i(\mu_1 - r) - \sigma_1^2\xi_i\pi_j^*(-\alpha_i\gamma_i\underline{a}^{i2} - \hat{\alpha}_i\gamma_i\overline{a}^{i2})]^2}{2\sigma_1^2\left[-\alpha_i\gamma_i\underline{a}^{i2} - \hat{\alpha}_i\gamma_i\overline{a}^{i2} + (1-2\alpha_i)\psi_{2i}A^{i2}\right]}.
\end{aligned} \tag{4.28}$$

Combining (4.6) and substituting (4.18), (4.19), (4.22), (4.24) and (4.25) into the second equation of (4.8), i.e.,

$$\mathcal{A}_i^{u_i^*, \underline{\phi}_i^*, \underline{\phi}_j^*}\underline{g}^i(t, \tilde{x}_i, m_i, m_j) = 0,$$

$$\mathcal{A}_i^{u_i^*, \overline{\phi}_i^*, \overline{\phi}_j^*}\overline{g}^i(t, \tilde{x}_i, m_i, m_j) = 0,$$

and separating the variables, as well as separating the variables of Eq (4.28), we get

$$A_t^i + (r - \overline{\tau}_i - \varphi_i + \omega_i)A^i = 0, \tag{4.29}$$

$$\begin{aligned}
&B_t^i + \Big[(r-\mu_1)\xi_i\pi_j^* + \lambda_i\mu_{z_i}(\eta_i - \theta_i) - \xi_i\lambda_j\mu_{z_j}\left(\eta_j - \theta_j + \theta_j q_j^*\right)\Big]A^i \\
&+ \frac{1}{2}\Big[\sigma_1^2\xi_i^2\pi_j^{*2} + \xi_i^2 q_j^{*2}\lambda_j\sigma_{z_j}^2\Big](-\alpha_i\gamma_i\underline{a}^{i2} - \hat{\alpha}_i\gamma_i\overline{a}^{i2}) \\
&+ \sigma_1\xi_i\pi_j^*(\alpha_i\underline{\phi}_{2j}^* + \hat{\alpha}_i\overline{\phi}_{2j}^*)A^i + \xi_i q_j^*\sqrt{\lambda_j}\sigma_{z_j}(\alpha_i\underline{\phi}_{1j}^* + \hat{\alpha}_i\overline{\phi}_{1j}^*)A^i \\
&- \frac{[\lambda_i\mu_{z_i}\theta_i A^i - \rho_R\xi_i q_j^*\sqrt{\lambda_i\lambda_j}\sigma_{z_i}\sigma_{z_j}(-\alpha_i\gamma_i\underline{a}^{i2} - \hat{\alpha}_i\gamma_i\overline{a}^{i2})]^2}{2\lambda_i\sigma_{z_i}^2\left[-\alpha_i\gamma_i\underline{a}^{i2} - \hat{\alpha}_i\gamma_i\overline{a}^{i2} + (1-2\alpha_i)\psi_{1i}A^{i2}\right]} \\
&- \frac{[A^i(\mu_1 - r) - \sigma_1^2\xi_i\pi_j^*(-\alpha_i\gamma_i\underline{a}^{i2} - \hat{\alpha}_i\gamma_i\overline{a}^{i2})]^2}{2\sigma_1^2\left[-\alpha_i\gamma_i\underline{a}^{i2} - \hat{\alpha}_i\gamma_i\overline{a}^{i2} + (1-2\alpha_i)\psi_{2i}A^{i2}\right]} = 0,
\end{aligned} \tag{4.30}$$

$$\underline{a}_t^i + (r - \overline{\tau}_i - \varphi_i + \omega_i)\underline{a}^i = 0, \tag{4.31}$$

$$\begin{aligned}
&\underline{b}_t^i + \underline{a}^i(t)\Big[(\mu_1 - r)\frac{-A^i(\mu_1 - r) + \sigma_1^2\xi_i\pi_j^*(-\alpha_i\gamma_i\underline{a}^{i2} - \hat{\alpha}_i\gamma_i\overline{a}^{i2})}{\sigma_1^2\left[-\alpha_i\gamma_i\underline{a}^{i2} - \hat{\alpha}_i\gamma_i\overline{a}^{i2} + (1-2\alpha_i)\psi_{2i}A^{i2}\right]} \\
&- (\mu_1 - r)\xi_i\pi_j^* + \lambda_i\mu_{z_i}(\eta_i - \theta_i) + \sigma_1\xi_i\pi_j^*\underline{\phi}_{2j}^* + \xi_i q_j^*\sqrt{\lambda_j}\sigma_{z_j}\underline{\phi}_{1j}^* \\
&+ \mu_{z_i}\theta_i\frac{-\lambda_i\mu_{z_i}\theta_i A^i + \rho_R\xi_i q_j^*\sqrt{\lambda_i\lambda_j}\sigma_{z_i}\sigma_{z_j}(-\alpha_i\gamma_i\underline{a}^{i2} - \hat{\alpha}_i\gamma_i\overline{a}^{i2})}{\sigma_{z_i}^2\left[-\alpha_i\gamma_i\underline{a}^{i2} - \hat{\alpha}_i\gamma_i\overline{a}^{i2} + (1-2\alpha_i)\psi_{1i}A^{i2}\right]} - \xi_i\lambda_j\mu_{z_j}\times \\
&\left(\eta_j - \theta_j + \theta_j q_j^*\right) - \psi_{2i}A^i\frac{[-A^i(\mu_1 - r) + \sigma_1^2\xi_i\pi_j^*(-\alpha_i\gamma_i\underline{a}^{i2} - \hat{\alpha}_i\gamma_i\overline{a}^{i2})]^2}{\sigma_1^2\left[-\alpha_i\gamma_i\underline{a}^{i2} - \hat{\alpha}_i\gamma_i\overline{a}^{i2} + (1-2\alpha_i)\psi_{2i}A^{i2}\right]^2} \\
&- \psi_{1i}A^i\frac{[-\lambda_i\mu_{z_i}\theta_i A^i + \rho_R\xi_i q_j^*\sqrt{\lambda_i\lambda_j}\sigma_{z_i}\sigma_{z_j}(-\alpha_i\gamma_i\underline{a}^{i2} - \hat{\alpha}_i\gamma_i\overline{a}^{i2})]^2}{\lambda_i\sigma_{z_i}^2\left[-\alpha_i\gamma_i\underline{a}^{i2} - \hat{\alpha}_i\gamma_i\overline{a}^{i2} + (1-2\alpha_i)\psi_{1i}A^{i2}\right]^2}\Big] = 0,
\end{aligned} \tag{4.32}$$

$$\overline{a}_t^i + (r - \overline{\tau}_i - \varphi_i + \omega_i)\overline{a}^i = 0, \tag{4.33}$$

$$\begin{aligned}
&\overline{b}_t^i + \overline{a}^i(t)\Big[(\mu_1 - r)\frac{-A^i(\mu_1 - r) + \sigma_1^2\xi_i\pi_j^*(-\alpha_i\gamma_i\underline{a}^{i2} - \hat{\alpha}_i\gamma_i\overline{a}^{i2})}{\sigma_1^2\left[-\alpha_i\gamma_i\underline{a}^{i2} - \hat{\alpha}_i\gamma_i\overline{a}^{i2} + (1-2\alpha_i)\psi_{2i}A^{i2}\right]} \\
&- (\mu_1 - r)\xi_i\pi_j^* + \lambda_i\mu_{z_i}(\eta_i - \theta_i) + \sigma_1\xi_i\pi_j^*\overline{\phi}_{2j}^* + \xi_i q_j^*\sqrt{\lambda_j}\sigma_{z_j}\overline{\phi}_{1j}^* \\
&+ \mu_{z_i}\theta_i\frac{-\lambda_i\mu_{z_i}\theta_i A^i + \rho_R\xi_i q_j^*\sqrt{\lambda_i\lambda_j}\sigma_{z_i}\sigma_{z_j}(-\alpha_i\gamma_i\underline{a}^{i2} - \hat{\alpha}_i\gamma_i\overline{a}^{i2})}{\sigma_{z_i}^2\left[-\alpha_i\gamma_i\underline{a}^{i2} - \hat{\alpha}_i\gamma_i\overline{a}^{i2} + (1-2\alpha_i)\psi_{1i}A^{i2}\right]} - \xi_i\lambda_j\mu_{z_j}\times \\
&\left(\eta_j - \theta_j + \theta_j q_j^*\right) + \psi_{2i}A^i\frac{[-A^i(\mu_1 - r) + \sigma_1^2\xi_i\pi_j^*(-\alpha_i\gamma_i\underline{a}^{i2} - \hat{\alpha}_i\gamma_i\overline{a}^{i2})]^2}{\sigma_1^2\left[-\alpha_i\gamma_i\underline{a}^{i2} - \hat{\alpha}_i\gamma_i\overline{a}^{i2} + (1-2\alpha_i)\psi_{2i}A^{i2}\right]^2} \\
&+ \psi_{1i}A^i\frac{[-\lambda_i\mu_{z_i}\theta_i A^i + \rho_R\xi_i q_j^*\sqrt{\lambda_i\lambda_j}\sigma_{z_i}\sigma_{z_j}(-\alpha_i\gamma_i\underline{a}^{i2} - \hat{\alpha}_i\gamma_i\overline{a}^{i2})]^2}{\lambda_i\sigma_{z_i}^2\left[-\alpha_i\gamma_i\underline{a}^{i2} - \hat{\alpha}_i\gamma_i\overline{a}^{i2} + (1-2\alpha_i)\psi_{1i}A^{i2}\right]^2}\Big] = 0.
\end{aligned} \tag{4.34}$$

Using the boundary conditions of (4.21), then (4.29), (4.31) and (4.33) yield

$$A^i(t) = \underline{a}^i(t) = \overline{a}^i(t) = e^{H_i(T-t)}, \tag{4.35}$$

where $H_i = r - \overline{\tau}_i - \varphi_i + \omega_i$. Repeating the above steps, we can also calculate the $\underline{\phi}_j^*, \overline{\phi}_j^*, q_j^*, \pi_j^*$ corresponding to the insurer j. Under the assumption (4.27), it can be obtained that $A^i(t) = A^j(t) = \underline{a}^i(t) = \underline{a}^j(t) = \overline{a}^i(t) = \overline{a}^j(t)$. Substituting this equation and $\underline{\phi}_j^*, \overline{\phi}_j^*, q_j^*, \pi_j^*$ back into (4.30), (4.32), (4.34), with the boundary conditions of (4.21), we get

$$\begin{aligned}B^i(t) = (T-t)\Big\{&\frac{(r-\mu_1)\xi_i k_1}{\sigma_1^2} - \frac{\xi_i \mu_{z_j}\theta_j k_3}{\sigma_{z_j}^2\sigma_{z_i}} + \xi_i\Big[(2\alpha_i - 1)\psi_{2j} - \frac{1}{2}\gamma_i\xi_i\Big]\frac{k_1^2}{\sigma_1^2} + \frac{\xi_i k_3^2}{\lambda_j \sigma_{z_j}^2\sigma_{z_i}^2} \times \\ &\Big[(2\alpha_i - 1)\psi_{1j} - \frac{1}{2}\gamma_i\xi_i\Big] - \frac{k_2^2(-\gamma_i + (1-2\alpha_i)\psi_{2i})}{2\sigma_1^2} - \frac{k_4^2(-\gamma_i + (1-2\alpha_i)\psi_{1i})}{2\lambda_i\sigma_{z_i}^2\sigma_{z_j}^2}\Big\} \\ &+ \frac{1}{H_i}\Big(e^{H_i(T-t)} - 1\Big)\Big[\lambda_i\mu_{z_i}(\eta_i - \theta_i) - \xi_i\lambda_j\mu_{z_j}(\eta_j - \theta_j)\Big],\end{aligned} \tag{4.36}$$

$$\begin{aligned}\underline{b}^i(t) = \Big\{&\frac{(\mu_1 - r)k_2}{\sigma_1^2} + \frac{\xi_i(r-\mu_1)k_1}{\sigma_1^2} - \frac{\mu_{z_i}\theta_i k_4}{\sigma_{z_i}^2\sigma_{z_j}} - \frac{\xi_i\mu_{z_j}\theta_j k_3}{\sigma_{z_j}^2\sigma_{z_i}} + \frac{\xi_i\psi_{1j}k_3^2}{\lambda_j\sigma_{z_j}^2\sigma_{z_i}^2} + \frac{\xi_i\psi_{2j}k_1^2}{\sigma_1^2} \\ &- \frac{\psi_{2i}k_2^2}{\sigma_1^2} - \frac{\psi_{1i}k_4^2}{\lambda_i\sigma_{z_i}^2\sigma_{z_j}^2}\Big\}(T-t) + \frac{1}{H_i}\Big(e^{H_i(T-t)} - 1\Big)\Big[\lambda_i\mu_{z_i}(\eta_i - \theta_i) - \xi_i\lambda_j\mu_{z_j}(\eta_j - \theta_j)\Big],\end{aligned} \tag{4.37}$$

$$\begin{aligned}\overline{b}^i(t) = \Big\{&\frac{(\mu_1 - r)k_2}{\sigma_1^2} + \frac{\xi_i(r-\mu_1)k_1}{\sigma_1^2} - \frac{\mu_{z_i}\theta_i k_4}{\sigma_{z_i}^2\sigma_{z_j}} - \frac{\xi_i\mu_{z_j}\theta_j k_3}{\sigma_{z_j}^2\sigma_{z_i}} - \frac{\xi_i\psi_{1j}k_3^2}{\lambda_j\sigma_{z_j}^2\sigma_{z_i}^2} - \frac{\xi_i\psi_{2j}k_1^2}{\sigma_1^2} \\ &+ \frac{\psi_{2i}k_2^2}{\sigma_1^2} + \frac{\psi_{1i}k_4^2}{\lambda_i\sigma_{z_i}^2\sigma_{z_j}^2}\Big\}(T-t) + \frac{1}{H_i}\Big(e^{H_i(T-t)} - 1\Big)\Big[\lambda_i\mu_{z_i}(\eta_i - \theta_i) - \xi_i\lambda_j\mu_{z_j}(\eta_j - \theta_j)\Big],\end{aligned} \tag{4.38}$$

where

$$\begin{aligned}k_1 &= \frac{(r-\mu_1)[-\gamma_i + (1-2\alpha_i)\psi_{2i} - \xi_j\gamma_j]}{(-\gamma_i + (1-2\alpha_i)\psi_{2i})(-\gamma_j + (1-2\alpha_j)\psi_{2j}) - \xi_i\xi_j\gamma_i\gamma_j}, \\ k_2 &= \frac{(r-\mu_1)[-\gamma_j + (1-2\alpha_j)\psi_{2j} - \xi_i\gamma_i]}{(-\gamma_i + (1-2\alpha_i)\psi_{2i})(-\gamma_j + (1-2\alpha_j)\psi_{2j}) - \xi_i\xi_j\gamma_i\gamma_j}, \\ k_3 &= \frac{-\lambda_j\mu_{z_j}\theta_j\sigma_{z_i}(-\gamma_i + (1-2\alpha_i)\psi_{1i}) + \rho_R\xi_j\sqrt{\lambda_i\lambda_j}\sigma_{z_j}\gamma_j\mu_{z_i}\theta_i}{(-\gamma_i + (1-2\alpha_i)\psi_{1i})(-\gamma_j + (1-2\alpha_j)\psi_{1j}) - \rho_R^2\xi_i\xi_j\gamma_i\gamma_j}, \\ k_4 &= \frac{\lambda_i\mu_{z_i}\theta_i\sigma_{z_j}(-\gamma_j + (1-2\alpha_j)\psi_{1j}) - \rho_R\xi_i\sqrt{\lambda_i\lambda_j}\sigma_{z_i}\gamma_i\mu_{z_j}\theta_j}{(-\gamma_i + (1-2\alpha_i)\psi_{1i})(-\gamma_j + (1-2\alpha_j)\psi_{1j}) - \rho_R^2\xi_i\xi_j\gamma_i\gamma_j}.\end{aligned} \tag{4.39}$$

Note that if (4.24) yields $q_i(t) \le 0$, then we have $q_i^*(t) = 0$. However, since this paper considers non-cheap reinsurance ($\theta_0 > \eta_0$), insurers usually do not introduce this kind of insurance and reinsurance business, so we do not solve the value function in this case.

Finally, by substituting (4.22) and (4.35)–(4.39) into (4.18)–(4.20), (4.24) and (4.25), and combining them with $\underline{\phi}_j^*, \overline{\phi}_j^*, q_j^*, \pi_j^*$, we get the specific expressions of $\underline{\phi}_i^*, \overline{\phi}_i^*, q_i^*, \pi_i^*$ and $V^i(t, \tilde{x}, m_i, m_j)$. It can be verified that the above result satisfies the conditions of the verification theorem. Thus, the theorem is proved. □

Remrk 4.1. *If the delay information is not taken into account, i.e., $\overline{\tau}_i = \varphi_i = \omega_i = 0, i = 1, 2$, then the Nash equilibrium strategy and the equilibrium value function for $q_i^*(t) > 0$ are*

$$u_1^*:\begin{cases} q_1^*(t) &= \dfrac{-\lambda_1\mu_{z_1}\theta_1\sigma_{z_2}(-\gamma_2+(1-2\alpha_2)\psi_{12})+\rho_R\xi_1\sqrt{\lambda_1\lambda_2}\sigma_{z_1}\gamma_1\mu_{z_2}\theta_2}{e^{r(T-t)}\lambda_1\sigma_{z_1}^2\sigma_{z_2}\Big[(-\gamma_1+(1-2\alpha_1)\psi_{11})(-\gamma_2+(1-2\alpha_2)\psi_{12})-\rho_R^2\xi_1\xi_2\gamma_1\gamma_2\Big]} \vee 0,\\ \pi_1^*(t) &= \dfrac{(r-\mu_1)\Big[-\gamma_2+(1-2\alpha_2)\psi_{22}-\xi_1\gamma_1\Big]}{e^{r(T-t)}\sigma_1^2\Big[(-\gamma_1+(1-2\alpha_1)\psi_{21})(-\gamma_2+(1-2\alpha_2)\psi_{22})-\xi_1\xi_2\gamma_1\gamma_2\Big]}, \end{cases} \tag{4.40}$$

$$u_2^*:\begin{cases} q_2^*(t) &= \dfrac{-\lambda_2\mu_{z_2}\theta_2\sigma_{z_1}(-\gamma_1+(1-2\alpha_1)\psi_{11})+\rho_R\xi_2\sqrt{\lambda_1\lambda_2}\sigma_{z_2}\gamma_2\mu_{z_1}\theta_1}{e^{r(T-t)}\lambda_2\sigma_{z_2}^2\sigma_{z_1}\Big[(-\gamma_1+(1-2\alpha_1)\psi_{11})(-\gamma_2+(1-2\alpha_2)\psi_{12})-\rho_R^2\xi_1\xi_2\gamma_1\gamma_2\Big]} \vee 0,\\ \pi_2^*(t) &= \dfrac{(r-\mu_1)\Big[-\gamma_1+(1-2\alpha_1)\psi_{21}-\xi_2\gamma_2\Big]}{e^{r(T-t)}\sigma_1^2\Big[(-\gamma_1+(1-2\alpha_1)\psi_{21})(-\gamma_2+(1-2\alpha_2)\psi_{22})-\xi_1\xi_2\gamma_1\gamma_2\Big]}, \end{cases} \tag{4.41}$$

$$V^i(t,\tilde{x}_i,m_i,m_j) = e^{r(T-t)}(\tilde{x}_i + \omega_i m_i - \xi_i\omega_j m_j) + B^{i'}(t), \quad i = 1, 2, \tag{4.42}$$

where

$$\begin{aligned} B^{i'}(t) = (T-t)\Big\{&\frac{(r-\mu_1)\xi_i k_1}{\sigma_1^2} - \frac{\xi_i\mu_{z_j}\theta_j k_3}{\sigma_{z_j}^2\sigma_{z_i}} + \xi_i\Big[(2\alpha_i-1)\psi_{2j} - \frac{1}{2}\gamma_i\xi_i\Big]\frac{k_1^2}{\sigma_1^2} + \frac{\xi_i k_3^2}{\lambda_j\sigma_{z_j}^2\sigma_{z_i}^2}\times\\ &\Big[(2\alpha_i-1)\psi_{1j} - \frac{1}{2}\gamma_i\xi_i\Big] - \frac{k_2^2(-\gamma_i+(1-2\alpha_i)\psi_{2i})}{2\sigma_1^2} - \frac{k_4^2(-\gamma_i+(1-2\alpha_i)\psi_{1i})}{2\lambda_i\sigma_{z_i}^2\sigma_{z_j}^2}\Big\}\\ &+\frac{1}{r}\Big(e^{r(T-t)}-1\Big)\Big[\lambda_i\mu_{z_i}(\eta_i-\theta_i) - \xi_i\lambda_j\mu_{z_j}(\eta_j-\theta_j)\Big]. \end{aligned} \tag{4.43}$$

Remrk 4.2. *For general robust model, i.e., $\alpha_i = 1, i \in \{1, 2\}$, then the Nash equilibrium strategy and the equilibrium value function for $q_i^*(t) > 0$ are*

$$u_1^*:\begin{cases} q_1^*(t) &= \dfrac{\lambda_1\mu_{z_1}\theta_1\sigma_{z_2}(\gamma_2+\psi_{12})+\rho_R\xi_1\sqrt{\lambda_1\lambda_2}\sigma_{z_1}\gamma_1\mu_{z_2}\theta_2}{e^{H_1(T-t)}\lambda_1\sigma_{z_1}^2\sigma_{z_2}\Big[(\gamma_1+\psi_{11})(\gamma_2+\psi_{12})-\rho_R^2\xi_1\xi_2\gamma_1\gamma_2\Big]} \vee 0,\\ \pi_1^*(t) &= \dfrac{-(r-\mu_1)\Big[\gamma_2+\psi_{22}+\xi_1\gamma_1\Big]}{e^{H_1(T-t)}\sigma_1^2\Big[(\gamma_1+\psi_{21})(\gamma_2+\psi_{22})-\xi_1\xi_2\gamma_1\gamma_2\Big]}, \end{cases} \tag{4.44}$$

$$u_2^*:\begin{cases} q_2^*(t) &= \dfrac{\lambda_2\mu_{z_2}\theta_2\sigma_{z_1}(\gamma_1+\psi_{11})+\rho_R\xi_2\sqrt{\lambda_1\lambda_2}\sigma_{z_2}\gamma_2\mu_{z_1}\theta_1}{e^{H_2(T-t)}\lambda_2\sigma_{z_2}^2\sigma_{z_1}\Big[(\gamma_1+\psi_{11})(\gamma_2+\psi_{12})-\rho_R^2\xi_1\xi_2\gamma_1\gamma_2\Big]} \vee 0,\\ \pi_2^*(t) &= \dfrac{-(r-\mu_1)\Big[\gamma_1+\psi_{21}+\xi_2\gamma_2\Big]}{e^{H_2(T-t)}\sigma_1^2\Big[(\gamma_1+\psi_{21})(\gamma_2+\psi_{22})-\xi_1\xi_2\gamma_1\gamma_2\Big]}. \end{cases} \tag{4.45}$$

$$V^i(t,\tilde{x}_i,m_i,m_j) = e^{H_i(T-t)}(\tilde{x}_i + \omega_i m_i - \xi_i\omega_j m_j) + B^{i''}(t), \quad i = 1, 2, \tag{4.46}$$

where

$$\begin{aligned} B^{i''} = (T-t)\Big\{&\frac{(r-\mu_1)\xi_i k_{1'}}{\sigma_1^2} - \frac{\xi_i\mu_{z_j}\theta_j k_{3'}}{\sigma_{z_j}^2\sigma_{z_i}} + \xi_i(\psi_{2j} - \frac{1}{2}\gamma_i\xi_i)\frac{k_{1'}^2}{\sigma_1^2} + \frac{\xi_i k_{3'}^2}{\lambda_j\sigma_{z_j}^2\sigma_{z_i}^2}(\psi_{1j} - \frac{1}{2}\gamma_i\xi_i)\\ &+\frac{k_{2'}^2(\gamma_i+\psi_{2i})}{2\sigma_1^2} + \frac{k_{4'}^2(\gamma_i+\psi_{1i})}{2\lambda_i\sigma_{z_i}^2\sigma_{z_j}^2}\Big\} + \frac{1}{H_i}(e^{H_i(T-t)}-1)\Big[\lambda_i\mu_{z_i}(\eta_i-\theta_i) - \xi_i\lambda_j\mu_{z_j}(\eta_j-\theta_j)\Big], \end{aligned} \tag{4.47}$$

$$\begin{aligned} k_{1'} &= \frac{-(r-\mu_1)[\gamma_i+\psi_{2i}+\xi_j\gamma_j]}{(\gamma_i+\psi_{2i})(\gamma_j+\psi_{2j})-\xi_i\xi_j\gamma_i\gamma_j}, \\ k_{2'} &= \frac{-(r-\mu_1)[\gamma_j+\psi_{2j}+\xi_i\gamma_i]}{(\gamma_i+\psi_{2i})(\gamma_j+\psi_{2j})-\xi_i\xi_j\gamma_i\gamma_j}, \\ k_{3'} &= \frac{\lambda_j\mu_{z_j}\theta_j\sigma_{z_i}(\gamma_i+\psi_{1i})+\rho_R\xi_j\sqrt{\lambda_i\lambda_j}\sigma_{z_j}\gamma_j\mu_{z_i}\theta_i}{(\gamma_i+\psi_{1i})(\gamma_j+\psi_{1j})-\rho_R^2\xi_i\xi_j\gamma_i\gamma_j}, \\ k_{4'} &= \frac{-\lambda_i\mu_{z_i}\theta_i\sigma_{z_j}(\gamma_j+\psi_{1j})-\rho_R\xi_i\sqrt{\lambda_i\lambda_j}\sigma_{z_i}\gamma_i\mu_{z_j}\theta_j}{(\gamma_i+\psi_{1i})(\gamma_j+\psi_{1j})-\rho_R^2\xi_i\xi_j\gamma_i\gamma_j}. \end{aligned} \tag{4.48}$$

5. Sensitivity analysis

In this section, the influence of parameters on the optimal reinsurance-investment strategy is succinctly described through the trend of curve changes. Unless otherwise specified, the basic parameters are given by $T = 10$, $t = 5$, $r = 0.3$, $\mu_1 = 0.4$, $\sigma_1 = 0.5$, $\rho_R = 0.0816$, and the rest of the parameters are given in Table 1.

Table 1. Parameter setting.

Parameters of insurer 1				Parameters of insurer 2			
Symbol	value	Symbol	value	Symbol	value	Symbol	value
λ_1	2	μ_{z_1}	2	λ_2	3	μ_{z_2}	3
θ_1	0.7	σ_{z_1}	5	θ_2	0.8	σ_{z_2}	6
γ_1	1	α_1	0.8	γ_2	1.5	α_2	0.6
ξ_1	0.5	ψ_{11}	1	ξ_2	0.7	ψ_{12}	2
ψ_{21}	1.2	h_1	1	ψ_{22}	2.2	h_2	0.5
ω_1	0.5	$\overline{\tau}_1$	0.7	ω_2	0.5	$\overline{\tau}_2$	0.7
δ_1	0.05			δ_2	0.1		

5.1. Equilibrium reinsurance strategy

Figure 1 shows that for a fixed competition parameter ξ_i, the larger the risk aversion parameter γ_i, the lower the retention ratio q_i^*, i.e., the more risk averse an insurer is, the more it tends to increase reinsurance to diversify its risk. For a fixed risk aversion parameter γ_i, the larger the competition parameter ξ_i, the larger q_i^*, suggesting that if the competition between the two firms intensifies, the insurer will increase its retention ratio and decrease its reinsurance ratio.

Figure 2 shows that as the delay parameters h_i and ω_i increase, the optimal reinsurance strategy q_i^* decreases. For a fixed ω_i, the larger h_i is, the longer the time interval the insurer needs to consider the past wealth, i.e., more historical information is taken into account. Therefore to reduce the potential risk, the insurer will increase the purchase of reinsurance so that the retention ratio q_i^* decreases. For a fixed h_i, the larger the delay parameter ω_i is, the larger the weight of historical average performance on terminal wealth is, and the risk to the insurer increases, so the insurer will increase the reinsurance ratio to reduce its risk of claims.

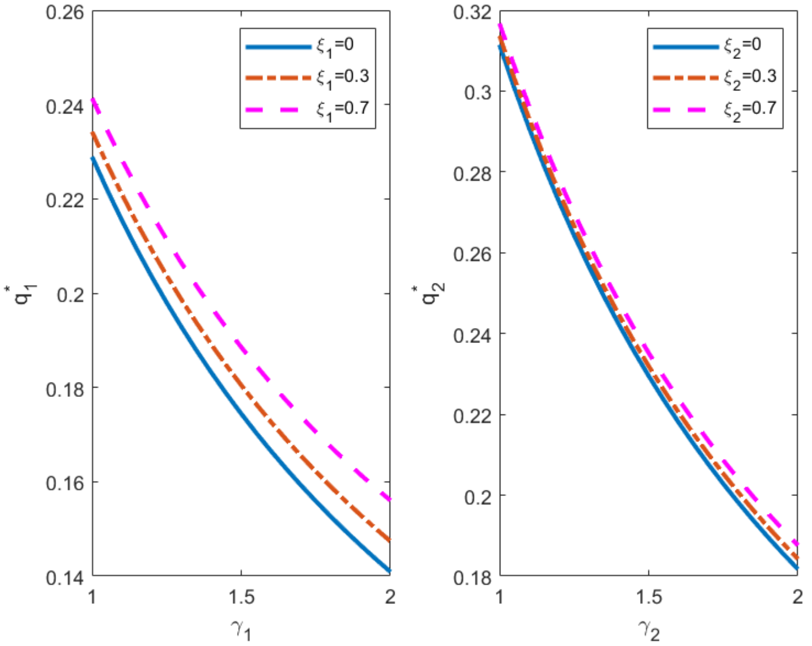

Figure 1. Effects of parameters γ_i, ξ_i on q_i^*.

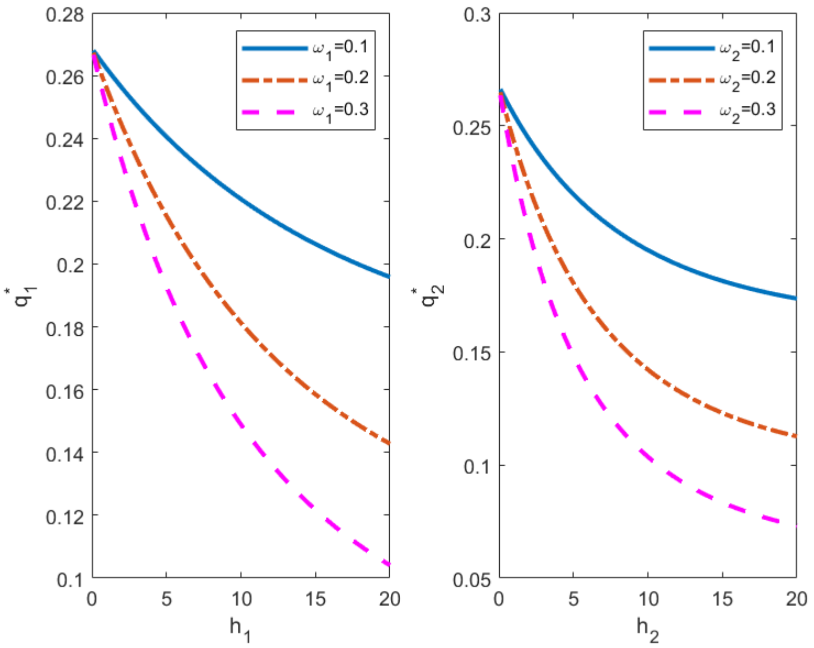

Figure 2. Effects of parameters h_i, ω_i on q_i^*.

Figure 3 shows that as the parameter α_i and the average parameter δ_i increase, the optimal reinsurance strategy q_i^* decreases. When the average parameter δ_i is determined, the larger α_i is, the more ambiguity-averse the insurer is, the more conservative it is in risk-taking, and thus the more it increases reinsurance purchases. When α_i is certain, the larger δ_i is, the smaller the proportion of wealth value in an earlier time, which indicates that the insurer pays more attention to the wealth closer to the current time, and to control the greater risk it may face, the insurer tends to reduce the proportion of self-retention q_i^*, which makes practical economic sense.

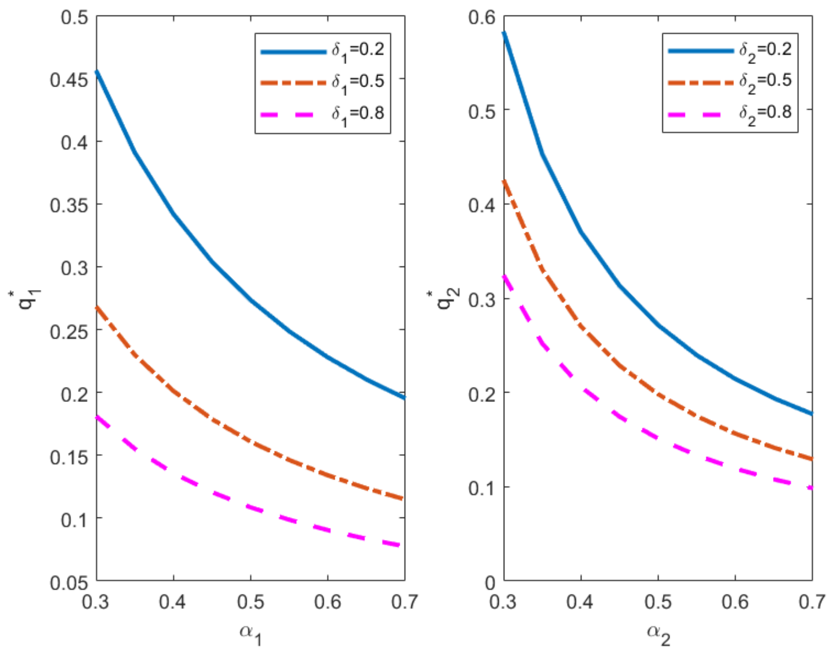

Figure 3. Effects of parameters α_i, δ_i on q_i^*.

5.2. Equilibrium investment strategy

Figure 4 shows that for a fixed competitive parameter ξ_i, the larger the risk aversion parameter γ_i, the smaller the amount of risky assets invested by the insurer i. For a fixed γ_i, the larger the ξ_i, the more the insurer tends to increase the investment in risky assets, mainly because the larger the ξ_i means the more intense the competition, the insurer wants to achieve better results, it will make its investment behavior more adventurous, and is inclined to increase the share of wealth invested in risky assets.

Figure 5 shows that the larger the delay parameters h_i and ω_i are, the smaller the share of investment in risky assets π_i^* is. For a fixed ω_i, the larger h_i is, the more stable the average historical performance of the insurer is, and the insurer will conservatively reduce its investment in risky assets in a more stable financial market. For a fixed h_i, the larger the delay parameter ω_i is, the larger the weight of the historical average performance on the terminal wealth is, resulting in the insurer needing to bear more risk, and therefore the insurer will reduce the investment in risky assets to reduce the risk of claims. Meanwhile, Figure 5 also shows that incorporating the delay information into the decision maker's decision will greatly affect the equilibrium investment strategy of the insurer, so it is of practical significance to take the delay effect into account in the study of the non-zero-sum game problem in this paper.

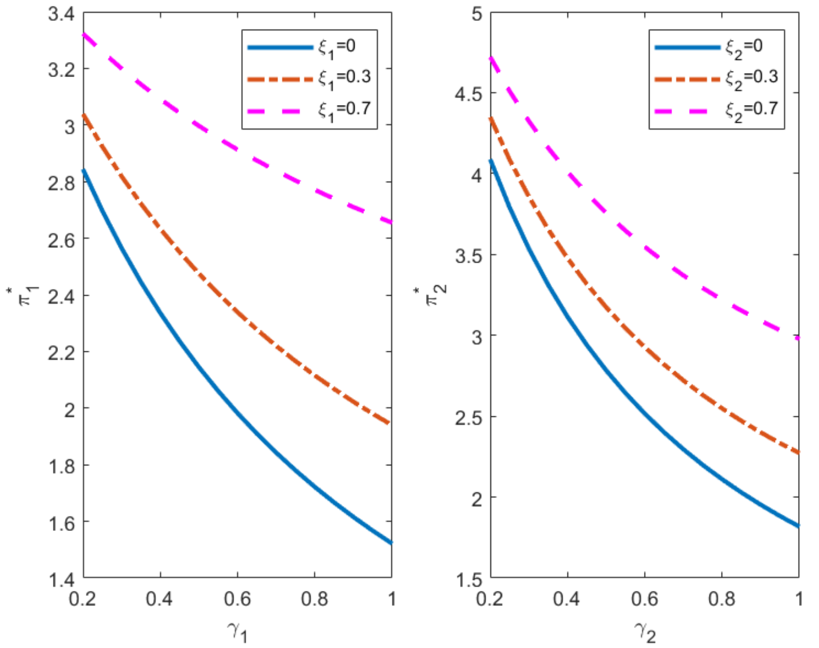

Figure 4. Effects of parameters γ_i, ξ_i on π_i^*.

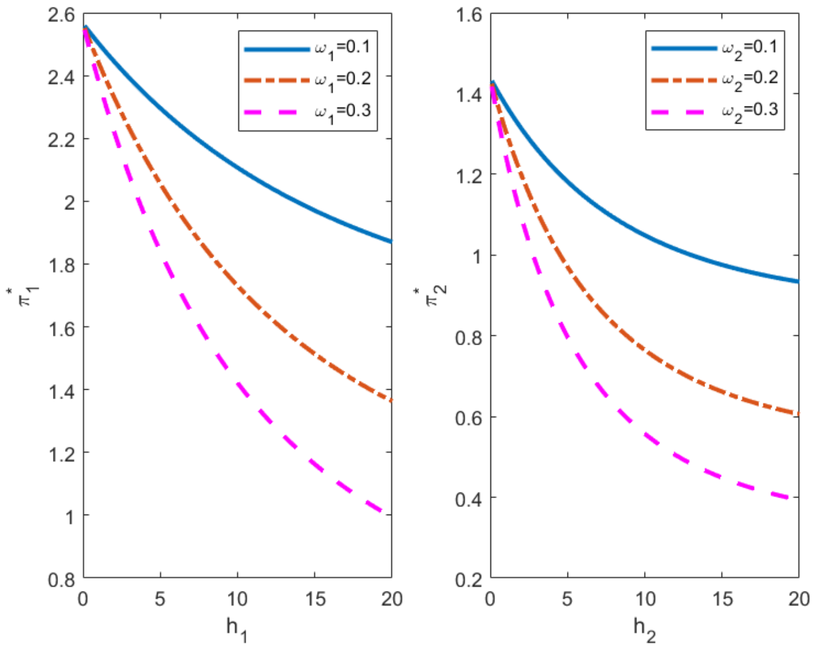

Figure 5. Effects of parameters h_i,ω_i on π_i^*.

Figure 6 shows that as the parameter α_i and the average parameter δ_i increase, the optimal investment strategy π_i^* decreases. When the average parameter δ_i is certain, the larger α_i is, the more ambiguity-averse the insurer is and the more conservative its investment behavior is. On the contrary, the smaller α_i is, the more aggressive the insurer's investment behavior will be, so it will increase the purchase of risky assets. When α_i is certain, the larger δ_i is, the smaller the effect of the wealth value at an earlier time on $M(t)$, which leads to an increase in the risk to be borne by the insurer, and thus the insurer tends to reduce the purchase of risky assets to reduce the risk.

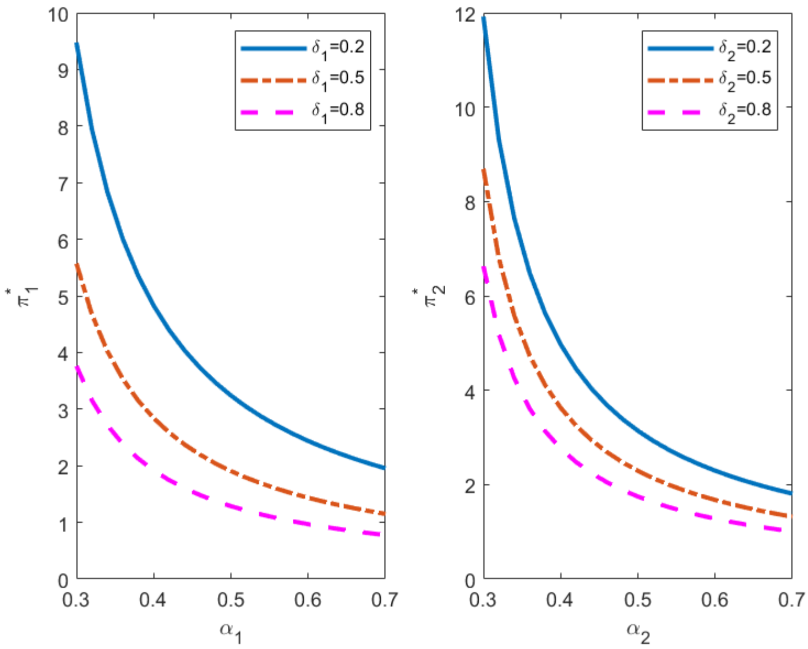

Figure 6. Effects of parameters α_i, δ_i on π_i^*.

Figures 7 and 8 compare the equilibrium reinsurance-investment strategies of the general robust problem with extreme ambiguity aversion and the α-robust problem with non-extreme ambiguity aversion. These two figures show that the equilibrium reinsurance-investment strategies of general robustness are smaller than that of α-robustness, because insurers with extreme ambiguity aversion are more risk-averse and ambiguity-averse than other non-extreme ambiguity-averse insurers, and thus more inclined to reduce risk.

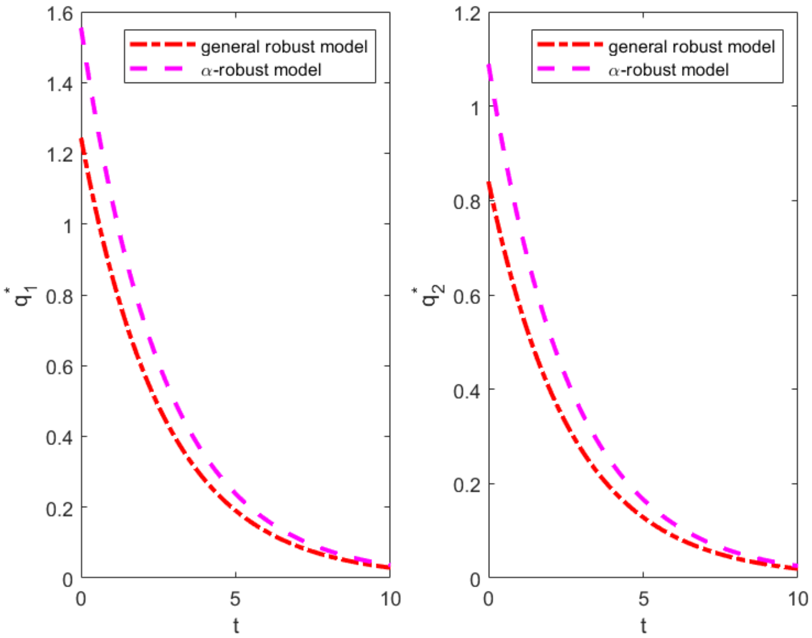

Figure 7. Comparison of reinsurance strategies for α-robust and general robust.

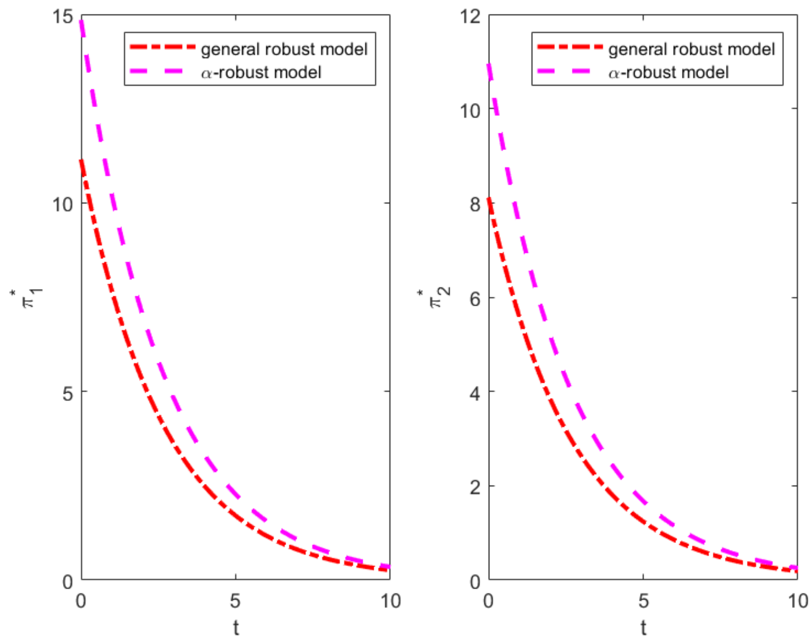

Figure 8. Comparison of investment strategies for α-robust and general robust.

6. Conclusions

Based on the delay effect, this paper investigates a non-zero-sum game between two insurers with similar ambiguity-averse preferences under the α-maxmin mean-variance criterion. The surplus process of the two insurers is described by the classical Cramer-Lundberg model, and part of the risk is transferred to the same reinsurer through proportional reinsurance. In addition, a delayed stochastic differential equation is used to describe the wealth process of the insurer. Since the optimization problem considered is time-inconsistent, we study equilibrium strategies and present the extended HJB equations of the α-maxmin mean-variance problem. The equilibrium reinsurance-investment strategies of the two insurers and the corresponding value functions are obtained by solving the HJB equation. Finally, the numerical results show that taking into account the delay effect and the different attitudes of decision-makers towards ambiguity have a significant impact on the equilibrium reinsurance-investment strategy. The more intense the competition between two insurers and the more the insurers seek ambiguity, the more aggressive their investment behavior.

In future research, it is possible to further consider excess-of-loss reinsurance strategies or combinations of reinsurance strategies, as well as to research the corresponding problem under more general premium guidelines such as mean-variance premium or loss-dependent premium principle.

Author contributions

Y. Chen conceptualized the research idea, developed the mathematical model, carried out the theorem proofs, and prepared the original draft of the manuscript. M. Chen and X. Hu supervised the research and critically revised the manuscript for intellectual content. All authors have read and approved the final version of the paper.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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