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Research article

Two disjoint cycles with prescribed lengths and arcs in digraphs

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Abstract: Let $D = (V, A)$ be an n-vertex digraph with $n \geq 6$. For each vertex $v \in V$, let $a(v)$ be the degree of v in D. Assume that for every pair of distinct vertices $x, y \in V$, the sum of their degrees satisfies $a(x) + a(y) \geq 3n + 1$. For any two independent arcs $f_1, f_2 \in A(D)$ and any integer partition $n = n_1 + n_2$ where $n_1 \geq 3$ and $n_2 \geq 3$, D contains two mutually vertex-disjoint dicycles C_1 and C_2 such that $|V(C_1)| = n_1$, $|V(C_2)| = n_2$, $f_1 \in E(C_1)$, and $f_2 \in E(C_2)$. Moreover, the condition is sharp.

Keywords: directed graph; disjoint dicycles; cycle coverings
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1. Introduction

All graphs and digraphs in this paper are finite and simple. Unless otherwise specified, we follow the terminology and notation of [2]. A directed graph is termed a digraph, and a directed cycle is termed a dicycle. In particular, a directed cycle of order 3 is explicitly denoted as a ditriangle. A family of graphs or digraphs is said to be disjoint if no two of its members have a vertex in common. Let X and Y be subsets of V, sequences consisting of distinct vertices from V, or subdigraphs of D. For any vertex $x \in V$, we use $a^+(x)$, $a^-(x)$, and $a(x)$ to denote the out-degree, in-degree, and degree of x in D, respectively.

The study of cycle decompositions in graphs has a rich history, beginning with Dirac's classical theorem on Hamiltonian cycles. In 1993, Aigner and Brandt [1] made a significant extension to Dirac's result by establishing the following theorem on disjoint cycles of specified lengths:

Theorem 1.1. *[1] Let G be an n-vertex graph. Suppose that G is a graph with minimum degree $\delta(G) \geq (2n-1)/3$. For any k integers $n_1, n_2, \ldots, n_k$ satisfying $n_i \geq 3$ for each $1 \leq i \leq k$ and $\sum_{i=1}^{k} n_i \leq n$, G contains a set of k pairwise vertex-disjoint cycles $C_1, C_2, \ldots, C_k$ such that $|V(C_i)| = n_i$ for all $1 \leq i \leq k$.*

This line of research was naturally extended to directed graphs. Molla [5] initiated the study of the directed analogue, proving the following result for the decomposition into directed cycles of equal length:

Theorem 1.2. *[5] For each integer $k \geq 2$, there exists a positive integer n_0 with the following property: If D is an n-vertex digraph where $n \geq n_0$, k divides n, and the minimum degree $\delta(D) \geq (n+2)/2$, then D contains n/k pairwise vertex-disjoint dicycles each of order k.*

Wang [6] subsequently improved the degree condition by removing the minimum degree requirement, obtaining a more general result:

Theorem 1.3. *[6] Let $D = (V, A)$ be an n-vertex digraph with $n \geq 4$. For each vertex $v \in V$, let $a(v)$ be the degree of v in D. Assume that for every pair of distinct vertices $x, y \in V$, the sum of their degrees satisfies $a(x) + a(y) \geq 3n - 4$. Then for any k positive integers $n_1, n_2, \ldots, n_k$ such that $n_i \geq 2$ for all $1 \leq i \leq k$ and $\sum_{i=1}^{k} n_i \leq n$, D contains k mutually vertex-disjoint dicycles $C_1, C_2, \ldots, C_k$ where $|V(C_i)| = n_i$ for each $1 \leq i \leq k$, unless D falls into one known exceptional family of digraphs.*

A further direction was explored by Wang [7], who considered the problem of embedding disjoint cycles containing prescribed edges. The following result was established for the undirected case, along with a corresponding conjecture for digraphs:

Theorem 1.4. *[7] Let G be an n-vertex graph with minimum degree $\delta(G) \geq (n+4)/2$. For any two independent edges $e_1, e_2 \subseteq E(G)$ and any integer partition $n = n_1 + n_2$ satisfying $n_1 \geq 3$ and $n_2 \geq 3$, G contains two pairwise vertex-disjoint cycles C_1 and C_2, where $|V(C_1)| = n_1$, $|V(C_2)| = n_2$, $e_1 \in E(C_1)$, and $e_2 \in E(C_2)$.*

Conjecture 1.5. *[7] Let k and n be two positive integers with $k \geq 2$ and $n \geq 9$. Let D be an n-vertex digraph such that every vertex of D has a degree at least $(3n + 2k - 4)/2$. Then for any k independent arcs $f_1, \ldots, f_k$ of D and any integer partition $n = n_1 + \cdots + n_k$ with $n_i \geq 4$ for all $1 \leq i \leq k$, D has k disjoint directed cycles $C_1, \ldots, C_k$ of order $n_1, \ldots, n_k$, respectively, such that f_i is an arc of C_i for all $1 \leq i \leq k$.*

Recently, Liu and Wang [4] made progress on this conjecture by resolving the case $k = 2$:

Theorem 1.6. *[4] Let $D = (V, A)$ be an n-vertex digraph with $n \geq 9$. Assume that D has a minimum degree $\delta(D) \geq 3n/2$. For any two independent arcs $f, g \in A(D)$ and any integer partition $n = n_1 + n_2$ satisfying $n_1 \geq 4$ and $n_2 \geq 4$, D contains two mutually vertex-disjoint dicycles C_1 and C_2 where $|V(C_1)| = n_1$, $|V(C_2)| = n_2$, $f \subseteq E(C_1)$, and $g \subseteq E(C_2)$.*

In this paper, we consider the problem of embedding two disjoint directed cycles of specified lengths containing prescribed arcs, under a degree-sum condition. Our main result is as follows:

Theorem 1.7. *Let $D = (V, A)$ be an n-vertex digraph with $n \geq 6$. For each vertex $v \in V$, let $a(v)$ denote the degree of v in D. Assume that for every pair of distinct vertices $x, y \in V$, the sum of their degrees satisfies $a(x) + a(y) \geq 3n + 1$. For any two independent arcs $f_1, f_2 \in A(D)$ and any integer partition $n = n_1 + n_2$ where $n_1 \geq 3$ and $n_2 \geq 3$, D contains two mutually vertex-disjoint dicycles C_1 and C_2 such that $|V(C_1)| = n_1$, $|V(C_2)| = n_2$, $f_1 \in E(C_1)$, and $f_2 \in E(C_2)$.*

The degree condition in Theorem 1.7 is sharp. To verify this, let K_n^* represent the complete directed graph with n vertices (where n is an even integer), which means that each vertex in K_n^* has a degree of $2(n-1)$. Consider a partition of the vertex set $V(K_n^*)$ into three disjoint subsets X, Y, and Z, i.e., $V(K_n^*) = X \cup Y \cup Z$, such that the order of X and Y satisfies $|X| = |Y| = (n-4)/2$. Let $f_1 = (u_1, v_1)$ and

$f_2 = (u_2, v_2)$ be two arcs, and let $Z = \{u_1, v_1, u_2, v_2\}$. From K_n^*, delete all arcs directed from X to $\{u_1, u_2\}$ and all arcs directed from $\{v_1, v_2\}$ to Y, as illustrated in Figure 1. In the resulting digraph D, we have $a(x) + a(y) \geq 3n$ for every pair of distinct vertices $x, y \in V$, with equality attained for some $\{x, y\} \subseteq V$, $x \neq y$. However, there exists no directed triangle in $D - V(f_{3-i})$ that contains f_i for $i = 1, 2$. This shows that the degree condition in Theorem 1.7 is the best possible.

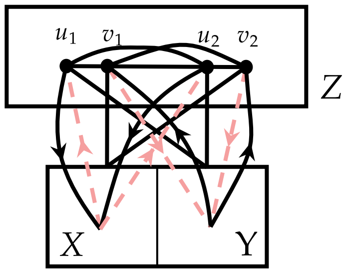

Figure 1. Counterexample.

We introduce the following terminology and notation. Let $D = (V, A)$ be a digraph and $G = (V, E)$ be a graph sharing the same vertex set V, i.e., $V(D) = V(G) = V$. The underlying graph of D is the graph $G = (V, E)$ where $V(G) = V$ and $E(G) = \{xy : (x, y) \in A \text{ and } (y, x) \in A\}$.

Define $N_D^+(x, Y)$ as the set of vertices $y \in Y$ for which $(x, y) \in A(D)$. Similarly, define $N_D^-(x, Y)$, and let $N_D(x, Y) = N_D^+(x, Y) \cup N_D^-(x, Y)$. When the context clearly indicates the digraph in question, the subscript D can be omitted. We further define the following cardinalities: $a^+(x, Y) = |N^+(x, Y)|$, $a^-(x, Y) = |N^-(x, Y)|$, and $a(x, Y) = a^+(x, Y) + a^-(x, Y)$. For a given set or sequence X, let $a(X, Y) = \sum_{x \in X} a(x, Y)$, where the sum extends over all vertices contained in X. In cases where $X = \{x_1, x_2, \ldots, x_p\}$ and $Y = \{y_1, y_2, \ldots, y_q\}$, we occasionally write $a(X, Y) = a(x_1 x_2 \ldots x_p, y_1 y_2 \ldots y_q)$ for brevity.

In the graph G, $N_G(x, Y)$ is defined as the set of vertices $y \in Y$ such that $xy \in E(G)$ and $d(x, Y) = |N_G(x, Y)|$. For a set or sequence X, define $d(X, Y) = \sum_{u \in X} d(u, Y)$.

For a vertex subset $U \subseteq V$, let $D[U]$ denote the subdigraph induced by U. The digraph $D - U$ is obtained by removing all vertices in U and all arcs incident to them. If $S = \{v\}$, we simply denote $D - S$ by $D - v$ rather than $D - \{v\}$. If $X_1, \ldots, X_t$ are subsets, sequences, subgraphs, or subdigraphs, we write $D[X_1, \ldots, X_t]$ for the subdigraph of D induced by the union of the vertices in $X_1, \ldots, X_t$. When no confusion arises, we abbreviate this as $[X_1, \ldots, X_t]$.

The length of a cycle or path C is denoted by $\ell(C)$; similarly, the length of a directed cycle or directed path P is denoted by $\ell(P)$. If $C = (x_1, x_2, \ldots, x_m, x_1)$ is a directed cycle or $C = x_1 x_2 \cdots x_m x_1$ is a cycle, all subscript arithmetic is performed modulo m.

An f-dicycle (or f-dipath) in D is a directed cycle (or directed path) containing the arc f. For a cycle C (directed or undirected) and a vertex x on C, let x^+ and x^- denote the successor and predecessor of x along C, respectively. Given a dicycle $C = (x_1, \ldots, x_k)$ and a vertex $z \notin V(C)$, if there is a vertex $x_i \in V(C)$ such that $\{(x_{i-1}, z), (z, x_{i+1})\} \subseteq A(D)$, then the cycle $C' = (x_{i+1}, \ldots, x_{i-1}, z, x_{i+1})$, obtained by replacing x_i with z, is denoted by $C - x_i + z$. Further, $x \rightarrow y$ denotes that $(x, y) \in A$, and $x \nrightarrow y$ denotes $(x, y) \notin A$.

2. Lemmas

Let $D = (V, A)$ be a directed graph. The following four lemmas will be employed in our argument. Specially, in our proof, we will frequently work with the underlying graph of D. Lemmas 2.3 and 2.4 are crucial for our analysis. Before presenting Lemma 2.4, we introduce some specialized terminology that is specific to that result.

Lemma 2.1. *[4] Let $(x, y) \in A(D)$, and let C be an (x, y)-dicycle on t vertices. For any vertex $z \notin V(C)$, if $a(z, C) \geq t + 2$, then the subdigraph induced by $C \cup \{z\}$ contains a Hamiltonian cycle C' that passes through the arc (x, y).*

Lemma 2.2. *[4] Let $f \in A(D)$ and C be an f-dicycle of order t in D. Take $x, y \in V(D) \setminus V(C)$. If $a(xy, C) \geq 3t + 3$, then there exists $z \in V(C)$ such that $C - z + (z^-, x) + (x, z^+)$ is an f-dicycle in D with $a(y, z) = 2$.*

Lemma 2.3. *[4] Let $P = (x_1, x_2, \ldots, x_t)$ be a path in the graph G, and let u be a vertex in $V(G) \setminus V(P)$. Assume that $d(ux_1, P) \geq t$. Then either the induced subgraph $G[V(P) \cup \{u\}]$ contains a path P' of order $t + 1$ with u as the starting vertex and x_t as the ending vertex, or $d(ux_1, P) = t$ and the edge $ux_t \in E(G)$.*

For a path $L = v_1 v_2 \cdots v_k$ in the graph G, define $\alpha(L, v_k)$ as the order of the longest subpath $v_i v_{i+1} \cdots v_k$ of L satisfying $v_i v_k \in E(G)$. Let u be a vertex in $V(G)$, and let P be a longest path in G with u as its starting vertex. Denote the terminal vertex of P by v. We say that a path P in graph G is optimal at vertex u if it satisfies the following condition: For every longest path P' starting at u with terminal vertex v', the inequality $\alpha(P', v') \leq \alpha(P, v)$ holds, where v denotes the terminal vertex of P.

Lemma 2.4. *[3] Let $P = x_t x_{t-1} \cdots x_1$ be an optimal path at x_t in G, with $r = \alpha(P, x_1)$ and $c \geq r/2$. Suppose that for every vertex $v \in V(G)$ reachable from x_t by a longest path, we have $d(v) \geq c$. Then the following conclusions hold: (i) $N(x_i) \subseteq \{x_1, x_2, \ldots, x_r\}$ for all $i \in \{1, 2, \ldots, r-1\}$; (ii) the subgraph $[P]$ contains an x_t-x_i Hamiltonian path for each $i \in \{1, 2, \ldots, r-1\}$; and (iii) $d(x_i) \geq c$ for all $i \in \{1, 2, \ldots, r-1\}$. Furthermore, if $t > r$, then x_r is a cut-vertex of G.*

3. Proof of Theorem 6

Let $D = (V, A)$ be a digraph of order $n \geq 6$ satisfying $a(x) + a(y) \geq 3n + 1$ for every pair of distinct vertices $x, y \in V$. Suppose Theorem 6 does not hold. That is, there exist two independent arcs f_1 and f_2 in D and an integer partition $n = n_1 + n_2$ with $n_1, n_2 \geq 3$ such that D does not contain two disjoint directed cycles C_1 and C_2 of lengths n_1 and n_2, respectively, where C_1 contains f_1 and C_2 contains f_2. Let $G = (V, E)$ denote the underlying graph of D.

If $n = 6$, then D is necessarily isomorphic to the complete digraph K_6^*, which yields a contradiction. We therefore assume $n \geq 7$ in what follows.

Claim 1. *For each $i \in \{1, 2\}$, D contains at least one f_i-ditriangle.*

Proof of Claim 1. Let $f_1 = (u, v)$ and $f_2 = (x, y)$, and define $V_1 = V \setminus \{u, v, x, y\}$ so that $|V_1| = n - 4$. Without loss of generality, consider the arc f_1. We have the following inequality:

$$
\begin{aligned}
&a^-(u, V_1) + a^+(v, V_1) \\
&\geq a(u) + a(v) - a^+(u) - a^-(v) - a^-(u, vxy) - a^+(v, uxy) \\
&\geq (3n + 1) - 2(n - 1) - 3 - 3 = n - 3 > n - 4.
\end{aligned}
$$

Since $a^-(u, V_1) + a^+(v, V_1) > |V_1|$, there exists a vertex $w \in V_1$ such that $(w, u) \in A(D)$ and $(v, w) \in A(D)$, implying that $T_1 = (u, v, w, u)$ is an f_1-ditriangle. By symmetry, there also exists a vertex $z \in V_1$ such that $T_2 = (x, y, z, x)$ is an f_2-ditriangle. □

Claim 2. *For each $i \in \{1, 2\}$, there exists an f_i-ditriangle T_i such that $D - V(T_i)$ contains an f_{3-i}-dicycle of order* 4.

Proof of Claim 2. Assume, for contradiction, that the claim fails for some $i \in \{1, 2\}$, say $i = 1$. By Claim 1, D contains an f_1-ditriangle $T_1 = (u, v, w, u)$ with $f_1 = (u, v)$. Then $D - V(T_1)$ contains no f_2-dicycle of order 4. Let $f_2 = (x, y)$, and suppose $D - V(T_1)$ contains $m \geq 0$ distinct f_2-ditriangles. Denote by $Z = \{z_1, \ldots, z_m\}$ the set of vertices such that (x, y, z_i, x) is an f_2-ditriangle for each $i = 1, \ldots, m$. Let $V_2 = V \setminus (Z \cup \{u, v, w, x, y\})$. Then $N^-(x, V_2) \cap N^+(y, V_2) = \emptyset$.

Suppose first that $m \geq 2$. Without loss of generality, consider z_1 and z_2. Under the contradiction hypothesis, we have $a(z_i, z_j) = 0$ for $i \neq j$, $N^-(z_1, V_2) \cap N^+(y, V_2) = \emptyset$, and $N^-(x, V_2) \cap N^+(z_1, V_2) = \emptyset$. Then

$$
\begin{aligned}
3n + 1 &\leq a(z_1) + a(z_2) \\
&= a(z_1, D - V_2) + a(z_2, D - V_2) + a(z_1, V_2) + a(z_2, V_2) \\
&\leq 10 + 10 + 2(n - m - 5) + 2(n - m - 5) = 4n - 4m,
\end{aligned}
$$

which implies $m \leq (n - 1)/4$. On the other hand,

$$
\begin{aligned}
9n + 3 &\leq 2a(x) + 2a(y) + a(z_1) + a(z_2) \\
&= 2(a^+(x) + a^-(y)) + (2a^+(y) + a^-(z_1) + a^-(z_2)) + \\
&\quad (2a^-(x) + a^+(z_1) + a^+(z_2)) \\
&\leq 4(n - 1) + 2(n - m - 5 + m + 9) + 2(n - m - 5 + m + 9) = 8n + 12.
\end{aligned}
$$

Hence, $n \leq 9$. Then $2 \leq m \leq \lfloor (9 - 1)/4 \rfloor = 2$, so all inequalities hold. It follows that (u, v, z_1, u) is an f_1-ditriangle and (x, y, z_2, u, x) is an f_2-dicycle of order 4, a contradiction.

Therefore, $m \leq 1$. Suppose $m = 1$, and let $z \in Z$ be such that $T_2 = (x, y, z, x)$ is an f_2-ditriangle. Assume that $D - V(T_2)$ contains at least two f_1-ditriangles. Then

$$
\begin{aligned}
(9n + 3)/2 &\leq a(x) + a(y) + a(z) \\
&= a(xyz, uvw) + a(xyz, T_2) + a(xyz, V_2) \\
&\leq (6 + 6 + 4) + (4 + 4 + 4) + 4(n - 6) = 4n + 4,
\end{aligned}
$$

which gives $n \leq 5$, a contradiction. Thus, $D - V(T_2)$ contains exactly one f_1-ditriangle, that is, T_1. Next, observe that

$$
\begin{aligned}
(9n + 3)/2 &\leq a(x) + a(y) + a(z) \\
&= (a^+(x) + a^-(y)) + (a^+(y) + a^-(z)) + (a^+(z) + a^-(x)) \\
&\leq 2(n - 1) + 2(n - 6 + 10) = 4n + 6,
\end{aligned}
$$

which implies $n \leq 9$. However, for any $q \in V_2$, we have

$$\begin{aligned}(9n+3)/2 &\leq a(x)+a(y)+a(q)\\ &= (a^+(x)+a^-(y)) + (a^+(y)+a^-(x)) + a(q)\\ &\leq 2(n-1)+(n-6+10)+(2(n-7)+8) = 5n-4,\end{aligned}$$

which implies $n \geq 11$, a contradiction.

Hence, $m = 0$. It follows immediately that $Z = \emptyset$ and $|V_2| = n-5$. By Claim 1, (x, y, w, x) forms an f_1-dicycle. We thus derive the following chain of inequalities:

$$\begin{aligned}3n+1 &\leq a(x)+a(y)\\ &= (a^+(x)+a^-(y)) + (a^+(y)+a^-(x))\\ &\leq 2(n-1)+(n-5+8) = 3n+1,\end{aligned}$$

which forces equality throughout. Consequently, $a^-(x, V_2) + a^+(y, V_2) = n-5$. From this equality and the preceding assertion, D contains exactly one f_1-ditriangle. We now establish that $a^-(x, V_2) \geq 1$ and $a^+(y, V_2) \geq 1$. Suppose, for the sake of contradiction, that $a^-(x, V_2) = 0$. Then

$$\begin{aligned}3(3n+1)/2 &\leq a(u)+a(v)+a(x)\\ &= (a^-(v)+a^+(u)+a^+(x)) + (a^-(u)+a^+(v)) + a^-(x)\\ &\leq 3(n-1)+(n-5+8)+4 = 4n+4,\end{aligned}$$

which simplifies to $n \leq 1$, a contradiction.

As a direct consequence, there exist two distinct vertices $x_1, y_1 \in V_2$ such that $(x_1, x) \in A$ and $(y, y_1) \in A$. Combining this with our contradiction hypothesis, we obtain

$$\begin{aligned}3n+1 &\leq a(x_1)+a(y_1)\\ &= (a^+(x_1)+a^-(y_1)) + (a^-(x_1)+a^+(y_1))\\ &\leq 2(n-1)+(n-7+8) = 3n-1,\end{aligned}$$

a contradiction. □

By Claim 2, there is a contradiction when $n = 7$. So, we only need to consider the case of $n \geq 8$ in the following.

Claim 3. *For every $i \in \{1, 2\}$ and every integer t satisfying $4 \leq t \leq n-3$, we can find an f_i-ditriangle T_i for which the remaining graph $D - V(T_i)$ contains a directed cycle of length t that includes the arc f_{3-i}.*

Proof of Claim 3. Suppose, by way of contradiction, that the statement does not hold for some index $i \in \{1, 2\}$; without loss of generality, take $i = 1$. Define r as the minimum value in $\{4, \ldots, n-3\}$, and let t be the maximum integer in $\{r, \ldots, n-3\}$ with the property that for all $j \in \{r, \ldots, t\}$, there exists an f_1-ditriangle T_j in D such that $D - V(T_j)$ contains an f_2-dicycle of order j. It follows from Claim 2 that $r \geq 4$.

Let $T = (u, v, w, u)$ be an f_1-ditriangle, and let $C = (x_1, \ldots, x_t, x_1)$ be an f_2-dicycle of order t in $D - V(T)$, where $f_2 = (x_{t-1}, x_t)$. Choose T and C such that $\ell(C)$ is maximal. Let $H = D - V(C)$, and let $h = n - t$. If $t = n-3$, the result holds. Thus, assume $t < n-3 = n - \ell(T)$. Define $S = V \setminus (V(T) \cup V(C))$.

Suppose $|S| = h - 3 \geq 2$, and let s_1, s_2 be two distinct vertices in S. Then

$$\begin{aligned} 3n + 1 &\leq a(s_1) + a(s_2) \\ &= a(s_1, T) + a(s_2, T) + a(s_1, C) + a(s_2, C) + a(s_1, S) + a(s_2, S) \\ &\leq 6 + 6 + (t + 1) + (t + 1) + 2(h - 4) + 2(h - 4) = 2t + 4h - 2, \end{aligned}$$

which implies $t + 3 \leq h$. Now suppose $a^+(x_t, S) = 0$. Then for any $y \in S$,

$$\begin{aligned} 3n + 1 &\leq a(x_t) + a(y) \\ &= a(x_t, T) + a(y, T) + a(x_t, C) + a(y, C) + a(x_t, S) + a(y, S) \\ &\leq 6 + 6 + 2(t - 1) + (t + 1) + (h - 3) + 2(h - 4) = 3n, \end{aligned}$$

a contradiction. Thus, $a^+(x_t, S) \geq 1$, and there exists $y \in S$ such that $(x_t, y) \in A$.

By the maximality of t, we have $N^+(y, S) \cap N^-(x_2, S) = \emptyset$. Then

$$\begin{aligned} 3n + 1 &\leq a(x_2) + a(y) \\ &= a(x_2, T) + a(y, T) + a(x_2, C) + a(y, C) + a(x_2, y) \\ &\quad + (a^+(x_2, S \setminus \{y\}) + a^-(y, S)) + (a^-(x_2, S \setminus \{y\}) + a^+(y, S)) \\ &\leq 6 + 6 + 2(t - 1) + (t + 1) + 2 + 2(h - 4) + (h - 4) = 3n + 1. \end{aligned}$$

Thus, the equality holds, implying $a(x_2, T) = 6$, $a(y, T) = 6$. Now change the role of y and x_1, and we can get $a(x_1, T) = 6$. Then $T' = (u, v, x_1)$ is an f_1-ditriangle, and $C' = (x_2, x_3, \ldots, x_t, y, w, x_2)$ is an f_2-dicycle of order $t + 1$ in $D - V(T')$.

Therefore, $|S| = 1$, and hence $n = t + 4$. Let $S = \{s\}$. From the maximality of $\ell(C)$ and by Lemma 2.1, we have $a(s, C) \leq t + 1$. If (u, v, s, u) is an f_1-ditriangle, then $a(w, C) \leq t + 1$, and $3n + 1 \leq a(s) + a(w) \leq (6 + t + 1) + (6 + t + 1) = 2n + 6$, which implies $n \leq 5$, a contradiction. Hence, $a(s, T_1) \leq 5$, and $3n + 1 \leq a(s) + a(w) \leq (5 + t + 1) + (6 + 2t) = 3n$, again a contradiction. □

Now, let $T = (u_1, u_2, u_3, u_1)$ be an f_1-ditriangle with $f_1 = (u_1, u_2)$ and $C = (x_1, x_2, \ldots, x_{n_2}, x_1)$ an f_2-dicycle of order n_2, where $f_2 = (x_{n_2-1}, x_{n_2})$. Denote by H the subdigraph $D[V \setminus V(C)]$ and by R its underlying graph. Consider a longest path $P = y_t y_{t-1} \ldots y_1$ in $R \setminus \{u_1, u_2\}$ with an initial vertex $y_t = u_3$, assuming $n_1 \geq n_2$. The objects C, T, and P are chosen to maximize $\ell(P)$.

We now prove that $\ell(P) = n_1 - 2$. Assume, for contradiction, that $t < n_1 - 2$. Define $R' = R \setminus (V(T) \cup V(P))$, and let $r' = n_1 - t - 2$. For any vertex $w \in R'$, Lemma 2.2 gives $a(wy_1, C) \leq 3n_2 + 2$, so we have $a(wy_1, H) \geq (3n + 1) - (3n_2 + 2) = 3n_1 - 1$. This implies $d(wy_1, R) \geq a(wy_1, H) - |N(w, H)| - |N(y_1, H)| \geq n_1 + 1$. Note that $d(y_1, R') = 0$ and $d(w, R') \leq r' - 1$. By Lemma 2.3, $d(wy_1, P) \leq t$, with equality holding only when $wy_t \in E$. Therefore, we obtain that $4 \geq d(wy_1, f_1) \geq (n_1 + 1) - t - (r' - 1) \geq 4$, which forces $d(wy_1, f_1) = 4$ and $R' \cong K_{r'}$. Consequently, H contains an f_1-dicycle of order n_1, contradicting our assumption.

Now, let $P = y_1 \ldots y_t$ be an arbitrary Hamiltonian path in $R \setminus \{u_1, u_2\}$ such that $T = (u_1, u_2, y_t, u_1)$. We claim that $H - y_1$ is not f_1-Hamiltonian. Suppose, to the contrary, that $H - y_1$ is f_1-Hamiltonian. Since H itself is not f_1-Hamiltonian, we have $a(y_1, H) \leq n_1$. Assume there exist distinct indices $i, j \in \{1, 2, \ldots, n_2 - 2\}$ such that $a^-(y_1, x_i^-) + a^+(y_1, x_i^+) = 2$ and $a^-(y_1, x_j^-) + a^+(y_1, x_j^+) = 2$. By Lemma 2.1, $a(x_i, H - y_1) \leq n_1$. Since $a(x_i, C) \leq 2(n_2 - 1)$, we obtain $a(x_i, D) \leq n_1 + 2 + 2(n_2 -$

$1) = n + n_2$. Similarly, $a(x_j, D) \leq n + n_2$. Therefore, $3n + 1 \leq a(x_i) + a(x_j) \leq 2(n + n_2) \leq 3n$, a contradiction. Hence, there is at most one vertex $x_i \in V(C)$ satisfying $a^-(y_1, x_i^-) + a^+(y_1, x_i^+) = 2$. It follows that $a(y_1, C) = a(y_1, C - V(f_2)) + a(y_1, f_2) \leq (n_2 - 2 + 1) + 4 = n_2 + 3$. Consequently, $3n + 1 \leq a(x_i, D) + a(y_1, D) \leq (n + n_2) + (n_2 + 3 + n_1) < 3n$, again a contradiction.

It follows that $n_1 \geq 5$. Since $H - y_1$ contains no f_1-Hamiltonian dicycle, we have $(y_2, u_1) \notin A$ and $(u_2, y_2) \notin A$. Moreover, for each $i \in \{3, \ldots, t\}$, if $(y_2, y_i) \in A$, then $(u_2, y_{i-1}) \notin E$, and if $(y_i, y_2) \in A$, then $(y_{i-1}, u_1) \notin E$. Consequently, we get $a^+(y_2, H - y_1) + d(u_2, R - y_1) \leq n_1 - 1$ and $a^-(y_2, H - y_1) + d(u_1, R - y_1) \leq n_1 - 1$. Thus, $a(y_2, H - y_1) + d(u_1u_2, R - y_1) \leq 2(n_1 - 1)$. Since $(y_1, u_1) \notin A$ and $(u_2, y_1) \notin A$, we have $d(y_1, u_1u_2) = 0$, and hence $a(y_2, H) + d(u_1u_2, R) \leq 2n_1$.

On the other hand, for each $i \in \{1, \ldots, n_2\}$, define

$$s_i = a^-(y_1, x_i^-) + a^+(y_1, x_i^+) + a^+(y_2, x_i) + d(u_1, x_i),$$
$$s_i' = a^-(y_1, x_i^-) + a^+(y_1, x_i^+) + a^-(y_2, x_i) + d(u_2, x_i).$$

For $i \in \{1, \ldots, n_2 - 2\}$, we have $s_i \leq 3$ and $s_i' \leq 3$; otherwise, if $s_i = 4$ or $s_i' = 4$, then $C - x_i + y_1$ is an f_2-Hamiltonian dicycle of order n_2, and $H - y_1 + x_i$ is an f_1-Hamiltonian dicycle of order n_1, a contradiction. For $i \in \{n_2 - 1, n_2\}$, we have $s_i \leq 4$ and $s_i' \leq 4$. Therefore, $2a(y_1, C) + a(y_2, C) + d(u_1u_2, C) = \sum_{i=1}^{n_2} s_i + \sum_{i=1}^{n_2} s_i' \leq 2(3n_2 + 2) = 6n_2 + 4$. It follows that $2a(y_1, H) + a(y_2, H) + d(u_1u_2, R) \geq \lceil 5(3n + 1)/2 - 2(n - 1) - (6n_2 + 4) \rceil \geq 5n_1 + 1$. Consequently, $a(y_1, H) \geq (3n_1 + 1)/2$, and hence $d(y_1, R) \geq (n_1 + 3)/2$. By Lemma 2.4, there exists $b \in \{2, \ldots, t\}$ such that $N_G(y_i, R) = N_G(y_1, R) \subseteq \{y_1, \ldots, y_b\}$, the graph $G[V(P)]$ has a y_t-y_i Hamiltonian path, and $d(y_i, P) \geq (n_1 + 3)/2$ for each $i \in \{1, \ldots, b - 1\}$. This implies $b \geq (n_1 + 3)/2$, and $(y_i, u_1) \notin A$, $(u_2, y_i) \notin A$ for each $i \in \{1, \ldots, b - 1\}$.

Suppose $b = n_1 - 2$. Then H contains at most one f_1-ditriangle. For each $i \in \{1, \ldots, n_2\}$ and $p \in \{1, \ldots, b - 1\}$, let

$$\alpha_{i,p} = a^-(y_p, x_i^-) + a^+(y_p, x_i^+) + d(x_i, u_1u_2).$$

For $i \in \{1, \ldots, n_2 - 2\}$, we have $\alpha_{i,p} \leq 3$; for $i \in \{n_2 - 1, n_2\}$, we have $\alpha_{i,p} \leq 4$. Therefore, $a(y_p, C) + d(u_1u_2, C) \leq 3(n_2 - 2) + 8 = 3n_2 + 2$. Since $d(u_1u_2, G) \geq a(u_1u_2, D) - |N(u_1, D)| - |N(u_2, D)| \geq (3n+1) - (n-1) - (n-1) = n + 3$, we have $a(y_p, H) + d(u_1u_2, R) \geq 3(3n+1)/2 - 2(n-1) - (3n_2+2) \geq 2n_1 + 1$ for each $p \in \{1, \ldots, b - 1\}$, a contradiction.

Thus, $b < n_1 - 2$, and y_b is a cut-vertex of $G[P]$. By the above argument, there exists $y_k \in P$ such that $T' = (u_1, u_2, y_k, u_1)$ is also an f_1-ditriangle with $k \in \{b, b + 1, \ldots, t - 1\}$; otherwise, by the above argument, H contains at most one f_1-ditriangle. Choose T, P, and T' such that $t - k$ is minimized. Define

$$\beta_i = a^-(y_{k+1}, x_i^-) + a^+(y_{k+1}, x_i^+) + d(x_i, y_1y_{k+2}),$$

where $y_{k+2} = u_2$ when $k = t - 1$. For $i \in \{1, \ldots, n_2 - 2\}$, we have $\beta_i \leq 3$; for $i \in \{n_2 - 1, n_2\}$, we have $\beta_i \leq 4$. Hence, $a(y_{k+1}, C) + d(y_1y_{k+2}, C) \leq 3(n_2 - 2) + 8 = 3n_2 + 2$. Since $d(y_1y_{k+2}, G) \geq a(y_1y_{k+2}, D) - |N(y_1, D)| - |N(y_{k+2}, D)| \geq (3n + 1) - (n - 1) - (n - 1) = n + 3$, we obtain $a(y_{k+1}, H) + d(y_1y_{k+2}, R) \geq 3(3n + 1)/2 - 2(n - 1) - (3n_2 + 2) \geq 2n_1 + 1$. Recall that y_k is a cut-vertex of $G[P]$, so $a(y_{k+1}, H) + d(y_1y_{k+2}, R) \leq 2(n_1 - b) + (b - 1) + (n_1 - b) = 3n_1 - 2b - 1$. Thus, $n_1 \geq 2b + 2 \geq n_1 + 5$, a contradiction.

This completes the proof of the theorem.

Author contributions

Siyue Liu: Methodology, conceptualization, writing-original draft preparation, writing, formal analysis, validation, investigation, visualization; Gang Chen: Methodology, conceptualization, supervision, writing, rormal analysis, resources, validation, reviewing, editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict to interest.

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