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Research article

Dynamical bifurcations of nontrivial traveling waves in the Zakharov-Ito equation: a Hamiltonian system approach with parameter dependence

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Abstract: This paper proposes a Hamiltonian framework to systematically classify nontrivial traveling wave classical solutions for the Zakharov-Ito (ZI) equation defined on $\mathbb{R}$, revealing their parameter dependence and dynamic bifurcation features. The rigorous analysis proves that only two types of smooth traveling wave solutions exist for the ZI equation in the real domain: periodic solutions and unimodal solutions (which include solitary waves), with both left-going and right-going waves being identified. These findings differ from a previous result that asserted the existence of compactons and kink-type solutions. This work not only establishes complete existence theorems for ZI traveling waves but also highlights a subtlety in certain coordinate transformation techniques which may inadvertently introduce solutions that do not correspond to genuine traveling waves of the original ZI system. This work establishes the foundation for future investigations into the stability of traveling wave solutions.

Keywords: Zakharov-Ito equation; traveling wave solutions; periodic solution; soliton solution; Hamiltonian system
Mathematics Subject Classification: 34C37, 34C60, 35B32, 35C07

1. Introduction

The study of water waves propagating over ambient currents is a fundamental problem in coastal oceanography and environmental fluid dynamics. The Zakharov-Ito (ZI) equation, discovered by Zakharov [1] and Ito [2], is a significant integrable system whose physical relevance is most pronounced in the context of rotational shallow water wave modeling. Traditionally, the theoretical foundation for water waves has relied on the assumption of irrotational flow [3]. Although this approximation works well in many cases, incorporating vorticity is essential to model the ubiquitous effects of background currents and wave-current interactions. Rotational effects are significant in numerous geophysical contexts, such as wind-driven waves, waves propagating on sheared currents, and waves near coastal structures [4].

The modern theory for periodic surface water waves with a general vorticity distribution was established by Constantin and Strauss [5], an investigation which initiated intense study of waves with vorticity [6]. A fruitful intermediate approach is to prescribe a vorticity distribution, the simplest being constant vorticity. This choice is not merely a mathematical simplification. For long waves propagating in shallow water over a nearly flat bed, the existence of a nonzero mean vorticity is more dynamically important than its specific distribution. Flows with constant vorticity are inherently two-dimensional and provide a pertinent model for tidal flows, where the current shear is often approximately linear [7, 8].

Within this physical framework, the two-component ZI system with constant vorticity emerges as a key asymptotic model for shallow water waves on a linear shear current [4, 9]. It can be written as

$$\begin{cases} u_t - 4ku_x + u_{xxx} + 3uu_x + \rho\rho_x = 0, \\ \rho_t + (u\rho)_x = 0, \end{cases} \tag{1.1}$$

where u and ρ are related to the spatial variable, $x \in \mathbb{R}$ and temporal variable t represent the depth-averaged horizontal velocity and the free surface elevation respectively, and $k \in \mathbb{R}$ is a parameter directly related to the constant vorticity of the background shear flow.

More precisely, for a two-dimensional shallow water wave under gravity with a background shear flow of the form

$$u_{\text{background}}(y) = Ay,$$

where A is constant, the analysis in [9] establishes the relation $A = 4k$ within Eq (1.1). A representative profile of the corresponding shallow water waves is shown in Figure 1. The free surface elevation $\eta(t, x)$ depicted in the figure can be reconstructed from solutions to (1.1); further details are provided in [9, Section 5].

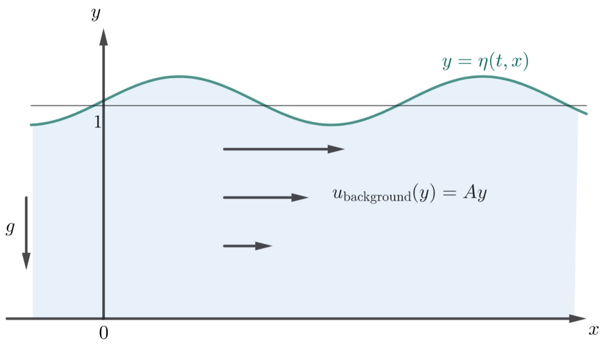


Figure 1. Shallow water waves.

The system (1.1) is integrable with the following Lax pair [9–11]:

$$\Psi_{xx} = \mathcal{A}\Psi, \quad \Psi_t = -\frac{1}{2}\mathcal{B}_x\Psi + \mathcal{B}\Psi_x,$$

with

$$\mathcal{A} = \zeta - \frac{u}{2} + k - \frac{\rho^2}{16\zeta}, \quad \mathcal{B} = -4\zeta - u,$$

where ζ denotes the spectral parameter.

The forward and inverse Miura transforms for the ZI system were explored in [12]. New auto-Bäcklund transformations and nonlinear superposition formulas for a multicomponent form of ZI system were constructed in [13], and the soliton solutions were also obtained. In [14], the authors constructed a zero-curvature representation and bi-Hamiltonian structure for the ZI system, then deduced a hierarchy and proved the hierarchy's Liouville integrability. More recently, the authors in [15] developed action-angle variables and expressed infinite conservation laws through scattering data for the integrable hierarchy associated with the ZI system using inverse scattering techniques, further elaborating the Hamiltonian formalism and canonical Poisson brackets.

This study focuses on traveling wave solutions to (1.1), that is, the wave profiles moving at a constant speed while preserving their shape. Localized traveling waves, first systematically documented by Russell in his seminal hydrodynamic observations [16], have been classified into two fundamental categories based on their interaction dynamics: solitons and solitary waves. The critical distinction lies in their post-collision behavior: solitons exhibit elastic interactions characterized by complete preservation of waveform integrity and kinematic properties; otherwise, it is a solitary wave. A third class of traveling waves, periodic traveling wave solutions exemplified by cnoidal waves, emerged through rigorous mathematical derivation in the groundbreaking analytical work of Korteweg and de Vries [17].

Since the foundational work of Zabuski and Kruskal [18], the investigation of solitary wave solutions for various partial differential equations (PDEs) has attracted considerable attention [19]. This burgeoning interest has naturally extended to the study of traveling wave solutions for the ZI system, inspiring a growing body of research dedicated to the stability of solitary waves [20–22], systematic construction of multisolitary wave solutions [23–25], and the in-depth investigation of their stability properties [26–28], among other related aspects. Parallel to these analytical efforts, significant advances have been made in numerical and data-driven methods for nonlinear waves, including the study of rogue waves in integrable systems [29] and the application of neural networks to nonintegrable models [30–32]. There are also some recent works on the stability of periodic traveling wave solutions and kink-type traveling wave solutions [33–35]. Furthermore, for the ZI equation with discontinuous initial data, we applied Whitham modulation theory to classify the nonlinear wave structures, arising from the Riemann problem [36].

In constructing exact dynamical solutions for nonlinear complex models, various numerical and analytical methods have emerged. Among these, bifurcation theory for dynamical systems has become an effective tool for studying traveling wave solutions in nonlinear evolution equations, owing to its capacity for qualitative analysis. This approach establishes correspondences between orbits and exact solutions by analyzing the evolution of topological structures in phase portraits under parameter variations, combining mathematical rigor with geometric intuition. For instance, bifurcation and chaos analysis have been applied to the Kadomtsev-Petviashvili-modified equal width equation to reveal diverse soliton patterns and dynamical transitions [37]; similarly, W-shaped solitons and other structured waves have been constructed for the modified Zakharov-Kuznetsov model using various expansion techniques [38]. The study of fractional-order models such as the Zakharov-Kuznetsov-Benjamin-Bona-Mahony equation further demonstrates how bifurcation analysis and novel solution methods can uncover complex wave patterns, including compacton-kink and peakon waves [39]. Beyond fluid dynamics, bifurcation methods are also instrumental in analyzing ecological systems, such as predator-prey models with Allee effects, highlighting the broad applicability of dynamical systems theory [40]. In recent years, significant progress has been made using this theory in solving nonlinear models. Akram, Nawaz, and Rehman [41] successfully analyzed dynamical behaviors of cubic nonlinear differential systems via bifurcation analysis. Zhou, Zhuang, and Li [42] systematically investigated local bifurcation features and global structures of traveling wave solutions in the generalized Alexeyev's A± equation. Meng, Zheng, Zhang, and Wang [43] validated the completeness and consistency of solution sets derived from various numerical methods through bifurcation analysis of phase trajectories, and so on [44–46]. Compared to conventional techniques, this theory provides intuitive insights into how solution branches are generated and how their stability changes with parameters, by tracking topological evolution in phase diagrams. This offers a new paradigm for studying nonlinear system dynamics.

In this study, we present a systematic classification of nontrivial traveling wave solutions for system (1.1), establishing rigorous mathematical characterizations through a unified Hamiltonian framework. This work builds upon and extends the foundational Hamiltonian approach developed by Dutykh and Ionescu [4] for analyzing traveling wave solutions. We emphasize that the Hamiltonian framework is not merely an alternative technique but the most natural and powerful lens through which to analyze the ZI system for several compelling reasons. First, from a physical standpoint, the ZI equation describes a conservative fluid system. Its Hamiltonian structure is not an artifact of analysis but a direct mathematical expression of fundamental physical conservation laws (e.g., energy). Therefore, preserving and analyzing this intrinsic structure is paramount to ensuring that all derived solutions remain physically admissible and consistent with the system's mechanics. Second, mathematically, the Hamiltonian formulation transforms the problem of finding traveling waves into the study of orbits in a two-dimensional phase plane, governed by a conserved quantity. This geometrization is exceptionally powerful: It allows for a complete qualitative classification of all possible solution types based on the level sets of the Hamiltonian, directly linking parameter variations to bifurcations in the solution structure. Third, this framework offers unparalleled clarity and rigor in distinguishing genuine solutions from mathematical artifacts. By working directly with the Hamiltonian system derived from the original PDE, we avoid the pitfalls of coordinate transformations that may alter the underlying dynamics. It is noteworthy that this Hamiltonian approach does not intrinsically depend on the complete integrability of the ZI system; rather, it leverages the mechanical analogy provided by traveling wave reduction, a technique applicable to a broad class of translation-invariant wave equations. The integrability of the ZI equation is reflected in the particularly simple and tractable algebraic structure of the resulting Hamiltonian, which is what permits the exhaustive analytical classification achieved in this work.

Whereas their work established the Hamiltonian structure and provided initial phase portrait analysis, our research addresses several important extensions to their framework. Specifically, by further developing the Hamiltonian method introduced in [4], we achieve a complete classification of all possible traveling wave solutions across the entire parameter space, including both periodic and solitary wave families. Our methodology adopts a more fundamental perspective, focusing directly on the original Hamiltonian formulation outlined in [4]. This comprehensive analysis not only resolves the classification problem in its full generality but also yields new insights into solution behaviors and their parametric dependence, extending the preliminary observations made in [4] regarding the phase space structure.

We also note the recent work of Wen [47], which proposed a classification of all bounded traveling wave solutions for (1.1), including solitary wave, periodic, kink-like (antikink), and compacton solutions under specific parameter conditions. Our findings, however, establish that within the class of smooth solutions defined on $\mathbb{R}$, the admissible traveling waves for the original ZI system consist solely of periodic and unimodal (including solitary wave) types. This conclusion aligns with and substantiates the indications from the phase portrait analysis presented in [4]. This critical discrepancy stems from a profound methodological distinction and underscores the importance of our Hamiltonian approach: The analysis in [47] introduces a nonlinear coordinate transformation $d\xi = (\varphi - c)^2 d\tau$ into the dynamical system for traveling waves. Although such a transformation can sometimes simplify the algebraic form of the equations, it renders the analyzed system no longer equivalent to the Hamiltonian system of the original ZI equation. More precisely, solutions which are globally defined in the new coordinate τ may not be globally defined in the original coordinate ξ. See Remark 2.2 for discussions on the importance of "global definedness". Consequently, solution types such as kinks and compactons that arise from the transformed system do not correspond to genuine solutions of the original ZI system (1.1), a fact we rigorously establish in this work. In contrast, our work maintains a direct analysis of the original Hamiltonian system (2.9) first identified in [4], ensuring that all derived solutions are strictly equivalent to those of (1.1). Our analysis thus provides a rigorous confirmation and significant extension of the solution structure suggested by the Hamiltonian approach in [4]. Furthermore, unlike the approach in [47], which imposes restrictions on the wave propagation speed (see [47, formula (2)]), our framework imposes no such constraints a priori. As a result, we are able to demonstrate that the ZI equation admits traveling wave solutions propagating in both leftward and rightward directions, a distinctive feature that contrasts with the unidirectional propagation characteristic of classical Korteweg-de Vries (KdV) solitary waves and further elaborates on the wave propagation properties indicated in [4]. See Section 5 for more discussions on the physical meaning of excluding kink-type traveling wave solutions and compactons as well as the physical meaning of the existence of bi-directional solitary waves.

Given a classical solution $(\rho, u) = (\rho(t,x), u(t,x))$ to (1.1), if there exists some constant $\widetilde{c} \in \mathbb{R}$ such that

$$\rho(t,x) = \tilde{\rho}(\xi), \quad u(t,x) = U(\xi), \quad \xi := x - \widetilde{c}t, \tag{1.2}$$

for some smooth profiles $\tilde{\rho}$ and U, then (ρ, u) is called a traveling wave solution of (1.1), with $\widetilde{c}$ termed the traveling speed. If, moreover,

- $\tilde{\rho}$ and U are both periodic with a same period, then we say that (ρ, u) is a periodic traveling wave solution;
- U is a unimodal function, that is, there exists $\xi_0 \in \mathbb{R}$ such that U is strictly monotone on $(-\infty, \xi_0)$ and $(\xi_0, +\infty)$, respectively, and $\lim_{\xi\to-\infty} U(\xi) = \lim_{\xi\to+\infty} U(\xi) = U_\infty \in \mathbb{R}$, then we say that (ρ, u) is a unimodal traveling wave solution;
- U is a unimodal function with $U_\infty = 0$, then we say that (ρ, u) is a solitary wave solution.

Here, "classical" means that all derivatives appearing in the ZI system (1.1) are meaningful. In particular, $\rho \in C^1_{t,x}$, and $u \in C^1_t C^3_x$.

Remark 1.1. *We reiterate that our study of the ZI system* (1.1) *is conducted on the real line* $\mathbb{R}$. *This choice of domain is not arbitrary or merely for convenience. To clarify, recall that a solution* $(u(t,x), \rho(t,x))$ *is called a traveling wave solution if it can be expressed as* $u(t,x) = U(x - \tilde{c}t)$ *and*

$\rho(t,x) = \tilde{\rho}(x-\tilde{c}t)$ for some wave speed $\tilde{c} \in \mathbb{R}$. If one instead considers the system on a bounded interval (a,b) with $-\infty < a < b < +\infty$, an inconsistency arises. At time $t = 0$, the profile $U(x) = u(0,x)$ is defined on (a,b). At time $t = 1$, however, the relation $U(x-\tilde{c}) = u(1,x)$ requires U to be defined on the shifted interval $(a-\tilde{c}, b-\tilde{c})$. Unless $\tilde{c} = 0$, these domains are distinct, indicating that the concept of a traveling wave is inherently incompatible with a bounded spatial domain. The same analysis in this remark applies also to the semi-infinity domain $(a,+\infty)$. This demonstrates that the whole line is the natural setting for analyzing such solutions.

Our main result is articulated as follows.

Theorem 1.1. *For any $\widetilde{c} \in \mathbb{R}$, there are infinitely many nontrivial periodic traveling wave solutions to* (1.1) *with the traveling speed $\widetilde{c}$, which form a three-parameter family, up to translation; and there are infinitely many nontrivial unimodal traveling wave solutions to* (1.1) *with the traveling speed $\widetilde{c}$, which form a two-parameter family, up to translation. In particular, for any $\widetilde{c} \in \mathbb{R} \setminus \{0\}$, there is a one-parameter family (up to translation) of nontrivial solitary wave solutions to* (1.1) *with the traveling speed $\widetilde{c}$. Moreover, these solutions constitute all the traveling wave solutions to* (1.1).

Remark 1.2. *Here, "triviality" means that the solution is either a constant, or solves the classical KdV equation for u (where ρ is constant). See Section 6 for more detailed discussions on the differences between "trivial" and "nontrivial" solutions. Therefore, Theorem 1.1 does not imply the non-existence of solitary wave solutions to* (1.1) *with traveling speed being zero. Indeed, there are such solitary wave solutions where $\rho \equiv 0$, and u solves the classical KdV equation with traveling speed being zero.*

While the primary aim of this work is to provide a rigorous mathematical classification of traveling waves, the resulting solution families, periodic wave trains and solitary waves, correspond directly to observable, coherent structures in shear flows. Moreover, the Hamiltonian framework we employ is intrinsically linked to the system's energy conservation laws. Consequently, our classification not only clarifies the admissible wave types but also establishes a foundation for analyzing their energy distribution and stability, with direct implications for modeling wave-current interactions in realistic geophysical settings.

Remark 1.3. *We are concerned with some special periodic solutions and some appropriately decaying solutions to* (1.1). *One should observe that the total energy of these solutions, whose definition varies depending on the situation, is preserved in the course of evolution.*

- *If (ρ, u) is a C^3 periodic classical solution to* (1.1) *with period $L > 0$, then the total energy is defined by*
$$E_{per}(t) := \frac{1}{2}\int_{\mathbb{T}_L} \left(|u(t,x)|^2 + \rho(t,x)^2\right) \mathrm{d}x,$$
where $\mathbb{T}_L := \mathbb{R}/(L\mathbb{Z})$.
- *If (ρ, u) is a C^3 classical solution to* (1.1) *such that for all t we have $\rho(t,x) \to \rho_\infty$ and $u(t,x) \to u_\infty$ as $|x| \to +\infty$ for some constants ρ_∞ and u_∞, then the total energy is defined by*
$$E(t) := \frac{1}{2}\int_{\mathbb{R}} \left(|u(t,x) - u_\infty|^2 + |\rho(t,x) - \rho_\infty|^2\right) \mathrm{d}x.$$

The energy definitions highlight a key physical distinction between solution types. Periodic traveling waves carry finite energy per period, distributed uniformly in space. In contrast, solitary waves are characterized by a finite total energy localized within a propagating pulse, with the energy density decaying exponentially away from the core. This localization property is crucial for understanding how solitary waves can transport energy and momentum over long distances without significant dispersion in shear currents. The absence of kink-type and compacton solutions in our classification further implies that the ZI system does not support traveling structures with finite energy concentrated on a strictly compact support or with energy asymptotically locked between two distinct background states.

Methodology statements. This article employs a Hamiltonian dynamical systems approach to rigorously classify all nontrivial traveling wave solutions of the ZI system. The method proceeds through the following steps:

- The traveling wave ansatz $u(t,x) = U(\xi)$, $\rho(t,x) = \tilde{\rho}(\xi)$, $\xi = x - \tilde{c}t$ is substituted into the ZI system (1.1), yielding the ordinary differential equation (ODE) system (2.1).
- After one integration, the system is reduced to a single second-order ODE (2.6) for U; then, a re-normalization $v = 3(U-\tilde{c})/2$ transforms (2.1) the central equation (2.8): $v'' = -v^2 - cv + g_1 - g_2/v^2$. This is formulated as the Hamiltonian system (2.9):

$$\frac{\mathrm{d}v}{\mathrm{d}\xi} = \varphi, \quad \frac{\mathrm{d}\varphi}{\mathrm{d}\xi} = -v^2 - cv + g_1 - \frac{g_2}{v^2},$$

  with Hamiltonian $H(v,\varphi) = \varphi^2/2 + F(v)$, where $F(v) = -v^3/3 - cv^2/2 + g_1 v + g_2/v$.
- Before proceeding to a detailed analysis of the Hamiltonian system (2.9), we note that it contains five free parameters: c, g_1, g_2, v_0, and φ_0. A direct study of the system in such a high-dimensional parameter space is cumbersome. The first key simplification arises from a symmetry consideration: We may set $\varphi_0 = 0$ without loss of generality. Lemma 2.1 establishes that any global smooth solution of (2.9) can be translated so that $\varphi(0) = 0$. This corresponds to the fact that if (u,ρ) is a traveling wave solution of (1.1), then for any $x_0 \in \mathbb{R}$, the translated pair $(u(t,x-x_0), \rho(t,x-x_0))$ is also a solution; by choosing x_0 appropriately, we can arrange for the corresponding phase variable φ to satisfy $\varphi(0) = 0$. Consequently, we set $\varphi_0 = 0$, and we can focus on (2.21).
- The core analysis of (2.21) is a detailed phase-plane study. The parameter space (c, g_1, g_2) is partitioned into five regimes $\mathcal{P}_0 \sim \mathcal{P}_4$ based on the root structure of $g(v) = v^2 f(v)$ (see details in Lemma 3.1), which dictates the topology of the level sets of H.
- Lemma 3.2 establishes nonglobal existence for parameters in $\mathcal{P}_0 \cup \mathcal{P}_1$ by contradiction: Assuming a solution of (2.21) exists for all ξ but $\varphi(\xi) \neq 0$, the integral representation

$$\phi_1(v) = \int_v^{v_0} \frac{\mathrm{d}\tilde{v}}{\sqrt{G(\tilde{v})}}$$

  derived from the Hamiltonian conservation is shown to have finite limits. On the other hand, $\phi_1(v)$ is shown to diverge due to the asymptotic behavior $F(v) \to +\infty$ as $v \to 0+$ or $v \to -\infty$; thus, we reach at a contradiction.
- For regimes supporting bounded solutions, existence is proven constructively. Lemma 3.3 proves the existence of periodic orbits for parameters in $\mathcal{P}_2 \cup \mathcal{P}_3 \cup \mathcal{P}_4$ when v_0 lies between certain roots:

The period is defined via the integral

$$T_0 = \int_{v_0}^{v_0'} \frac{\mathrm{d}\widetilde{v}}{\sqrt{2(F(\widetilde{v}) - F(v_0))}},$$

and the periodic function $v(\xi)$ is constructed by inverting this integral, with smoothness verified by Taylor expansion near the endpoints.

- Lemma 3.4 proves the existence of homoclinic orbits (unimodal/solitary waves) by analyzing the improper integral

$$\Phi(v) = \int_{v}^{v_0} \frac{\mathrm{d}\widetilde{v}}{\sqrt{2(F(\widetilde{v}) - F(v_0))}},$$

showing it maps $(v_0', v_0]$ onto $(0, +\infty)$ and demonstrating exponential decay to the equilibrium v_0' via the asymptotic relation

$$\Phi(v) \sim \frac{1}{\sqrt{2f'(v_0')}} \ln \frac{1}{|v - v_0'|}.$$

The primary advantages of this approach are:

- Fidelity to physics: It analyzes the untransformed Hamiltonian system derived directly from the ZI equation, avoiding nonequivalent coordinate transformations that can introduce spurious solutions (e.g., compactons or kinks).
- Complete geometric classification: The method provides an exhaustive, parameter-dependent classification of all smooth traveling waves on $\mathbb{R}$ by directly linking algebraic conditions to the topology of phase portraits.
- Mathematical rigorous: The use of constructive proofs and contradiction arguments grounded in phase-plane analysis and the properties of $F(v)$ yields a rigorous, unambiguous existence theory.

2. The corresponding Hamiltonian system

Plugging (1.2) into (1.1) gives the following system for the profiles $(\tilde{\rho}, U)$:

$$\begin{cases} -\widetilde{c}U' - 4kU' + U''' + 3UU' + \tilde{\rho}\tilde{\rho}' = 0, \\ -\widetilde{c}\tilde{\rho}' + (U\tilde{\rho})' = 0, \end{cases} \tag{2.1}$$

where the derivatives are taking with respective to the traveling wave coordinate $\xi = x - \widetilde{c}t$. Recall that our goal is to seek traveling wave solutions of (1.1). This is equivalent to find all global smooth solutions $(\tilde{\rho}, U) = (\tilde{\rho}, U)(\xi)$ to the (steady) system (2.1). Here, the term "*global*" means that the solution $(\tilde{\rho}, U)(\xi)$ is defined on the whole line $\xi \in \mathbb{R}$.

Remark 2.1. *According to the definition above Theorem 1.1, for a traveling wave solution (ρ, u) of the ZI system, it only has the regularity $\rho \in C^1_{t,x}$ and $u \in C^1_t C^3_x$. Consequently, the corresponding profile $(\tilde{\rho}, U)$ only has the regularity $\tilde{\rho} \in C^1$ and $U \in C^3$. Nevertheless, it is easy to see the following fact: If $(\tilde{\rho} \in C^1, U \in C^3)$ solves* (2.1)*, then $\tilde{\rho}$ and U are both smooth. Hence, it suffices to find all global smooth solutions to* (2.1).

Remark 2.2. *Another point that we want to explain here is: Why do we emphasize "global"? The reason is simple: We consider the ZI system* (1.1) *on the domain* $\mathbb{R}$*. Obviously, the ZI system* (1.1) *is an evolution equation. If* (ρ, u) *is a classical solution to* (1.1)*, then of course the initial data* $(\rho(0,x), u(0,x))$ *are defined on* $x \in \mathbb{R}$*. Plugging* $t = 0$ *into* (1.2) *gives that* $\rho(0,x) = \tilde{\rho}(x)$ *and* $u(0,x) = U(x)$*; hence,* $\tilde{\rho}$ *and* U *are both defined on* $\mathbb{R}$*. Therefore, we focus on solutions to* (2.1)*, which are defined on the whole real line* $\mathbb{R}$*. This property (called "global") helps us to exclude many "illegal" (i.e., "non-global") solutions of* (2.1)*, it therefore needs to be emphasized.*

Integrating once gives that

$$\begin{cases} -(\widetilde{c}+4k)U + U'' + \frac{3}{2}U^2 + \frac{1}{2}\tilde{\rho}^2 = \widetilde{g}_1, \\ (-\widetilde{c}+U)\tilde{\rho} = \widetilde{g}_2, \end{cases} \tag{2.2}$$

where $\widetilde{g}_1 \in \mathbb{R}$ and $\widetilde{g}_2 \in \mathbb{R}$ are integration constants. If $\widetilde{g}_2 = 0$, then one of the following two cases holds:

- If $U = \widetilde{c}$ is constant, then the first equation of (2.2) implies that $\tilde{\rho}$ is also constant; hence, this case gives trivial solutions to (1.1).
- If $\tilde{\rho} = 0$, then $\rho = 0$, and thus (1.1) degenerates to the classical KdV equation for u,

$$u_t - 4ku_x + u_{xxx} + 3uu_x = 0. \tag{2.3}$$

The construction of traveling wave solutions of (2.3) is classical and has been obtained in [17, 18, 48]. Specifically, the traveling wave solutions to KdV consist of solitons and cnoidal elliptic waves. See also Section 6 for more details on trivial solutions given by the classical KdV equation.

As a consequence, we assume from now that

$$\widetilde{g}_2 \neq 0. \tag{2.4}$$

Assuming (2.4), we have $U(\xi) \neq \widetilde{c}$ for all $\xi \in \mathbb{R}$, so

$$\tilde{\rho} = \frac{\widetilde{g}_2}{U - \widetilde{c}}. \tag{2.5}$$

Plugging (2.5) into the first equation of (2.2) gives that

$$-(\widetilde{c}+4k)\,U + U'' + \frac{3}{2}U^2 + \frac{\widetilde{g}_2^2}{2\,(U-\widetilde{c})^2} = \widetilde{g}_1. \tag{2.6}$$

To simplify the analysis, we introduce a renormalization,

$$v := \frac{3}{2}(U - \widetilde{c}) \neq 0, \quad g_1 := \frac{3}{2}\widetilde{g}_1 + 6k\widetilde{c} - \frac{3}{4}\widetilde{c}^2, \quad g_2 := \frac{27}{16}\widetilde{g}_2^2 > 0, \quad c := 2\,(\widetilde{c} - 2k). \tag{2.7}$$

Then, (2.6) is converted to the following second-order ODE for v:

$$v'' = -v^2 - cv + g_1 - \frac{g_2}{v^2}. \tag{2.8}$$

Now, we can convert (2.8) to the following Hamiltonian system by introducing $\varphi = v'$:

$$\begin{cases} \frac{\mathrm{d}v}{\mathrm{d}\xi} = \varphi, \\ \frac{\mathrm{d}\varphi}{\mathrm{d}\xi} = f(v) := -v^2 - cv + g_1 - \frac{g_2}{v^2}, \end{cases} \quad \text{with} \quad \begin{cases} v(0) = v_0 \in \mathbb{R}\setminus\{0\}, \\ \varphi(0) = \varphi_0 \in \mathbb{R}. \end{cases} \tag{2.9}$$

Remark 2.3. *The Hamiltonian system (2.9) coincides with the Hamiltonian system derived in [4, (19)]. We shall analyze it directly, without performing the nonlinear time rescaling $d\xi = (\varphi - c)^2 d\tau$ used in [47]. The justification for this approach is central to our work. That rescaling, although sometimes useful for phase portrait classification, alters the original flow and can lead to solution types (such as compactons or kinks) that are not necessarily present in the original system (1.1). Critically, such a transformation breaks the direct correspondence with the Hamiltonian of the physical system, potentially analyzing a mathematically related but physically distinct dynamical system. By sticking to the untransformed system (2.9), we ensure that all solutions we obtain are genuine traveling waves of (1.1).*

Remark 2.4. *Here, we consider the equation for U. An equivalent analysis can be performed for $\tilde{\rho}$ using the relation $U = \widetilde{c} + \widetilde{g}_2/\tilde{\rho}$ from* (2.2). *Indeed, multiplying the first equation of* (2.2) *by U' and integrating once again, we obtain*

$$-\frac{\widetilde{c}+4k}{2}U^2 + \frac{1}{2}(U')^2 + \frac{1}{2}U^3 - \frac{1}{2}\widetilde{g}_2\tilde{\rho} = \widetilde{g}_1 U + \widetilde{g}_3, \tag{2.10}$$

where $\widetilde{g}_3$ is the integration constant. Plugging $U = \widetilde{c} + \widetilde{g}_2/\tilde{\rho}$ into (2.10) *gives that*

$$-2k\widetilde{c}^2 - \widetilde{g}_1\widetilde{c} - \widetilde{g}_3 + \frac{\widetilde{g}_2}{\tilde{\rho}}\left(\frac{1}{2}\widetilde{c}^2 - 4k\widetilde{c} - \widetilde{g}_1\right) + \frac{\widetilde{g}_2^2}{\tilde{\rho}^2}(\widetilde{c} - 2k) + \frac{1}{2}\frac{\widetilde{g}_2^3}{\tilde{\rho}^3} - \frac{1}{2}\widetilde{g}_2\tilde{\rho} + \frac{1}{2}\frac{\widetilde{g}_2^2(\tilde{\rho}')^2}{\tilde{\rho}^4} = 0.$$

For simplicity, let $w := \tilde{\rho}/\widetilde{g}_2$, $c_4^{\tilde{\rho}} := 2(2k\widetilde{c}^2 + \widetilde{g}_1\widetilde{c} + \widetilde{g}_3)$, $c_3^{\tilde{\rho}} = 8k\widetilde{c} + 2\widetilde{g}_1 - \widetilde{c}^2$, $c_2^{\tilde{\rho}} := 4k - 2\widetilde{c}$; then, w solves

$$(w')^2 = \widetilde{g}_2^2 w^5 + c_4^{\tilde{\rho}} w^4 + c_3^{\tilde{\rho}} w^3 + c_2^{\tilde{\rho}} w^2 - w. \tag{2.11}$$

One can study (2.11) *instead of* (2.9)*, and this leads to the same conclusion as ours. See the analysis of the equivalent equation in [4, 49].*

To find all traveling wave solutions to (1.1), it suffices to figure out when the solution to (2.9) is defined for all $\xi \in \mathbb{R}$, recalling the discussion that followed (2.1).

Let $(v, \varphi) = (v, \varphi)(\xi)$ be the unique solution to (2.9). The the function $\xi \mapsto H(v(\xi), \varphi(\xi))$ is constant, where the Hamiltonian H is defined by

$$H(v, \varphi) := \frac{1}{2}\varphi^2 - F(v), \quad F(v) := -\frac{1}{3}v^3 - \frac{1}{2}cv^2 + g_1 v + \frac{g_2}{v}. \tag{2.12}$$

Thanks to $g_2 > 0$, we note that F, which is the primitive function of f, has the property that

$$\lim_{v\to-\infty} F(v) = +\infty, \quad \lim_{v\to 0-} F(v) = -\infty, \quad \lim_{v\to 0+} F(v) = +\infty, \quad \lim_{v\to+\infty} F(v) = -\infty; \tag{2.13}$$

hence, the intermediate value theorem implies that F has at least one negative root and at least one positive root. Based on this observation, we have the following:

Lemma 2.1. *Assume that $v_0 \in \mathbb{R} \setminus \{0\}$ and $\varphi_0 \in \mathbb{R}$. Let $(v, \varphi) = (v, \varphi)(\xi)$ be the unique solution to* (2.9)*. If it holds that the maximal interval of existence of* (v, φ) *is* $\mathbb{R}$*, then there exists $\xi_0 \in \mathbb{R}$ such that $\varphi(\xi_0) = 0$.*

Proof. We assume that $v_0 > 0$. If $\varphi_0 = 0$, then there is nothing to prove. Assuming that $\varphi_0 > 0$, and we suppose on contrary that

$$\varphi(\xi) > 0, \quad \forall \xi \in \mathbb{R}, \tag{2.14}$$

it follows from $H(v(\xi), \varphi(\xi)) = H(v_0, \varphi_0)$ that $\varphi(\xi)^2 = 2F(v(\xi)) + 2H(v_0, \varphi_0)$ for all $\xi \in \mathbb{R}$, and hence

$$\varphi(\xi) = \sqrt{2F(v(\xi)) + 2H(v_0, \varphi_0)}, \quad \forall \xi \in \mathbb{R}. \tag{2.15}$$

Plugging the above into the first equation of (2.9), we obtain the following ODE for v:

$$\frac{\mathrm{d}v}{\mathrm{d}\xi} = \sqrt{2F(v(\xi)) + 2H(v_0, \varphi_0)}, \ \forall \xi \in \mathbb{R} \quad \text{with} \quad v(\xi = 0) = v_0 > 0. \tag{2.16}$$

In particular, the function $\xi \mapsto v(\xi)$ is strictly increasing, and $v(\xi) > 0$ for all $\xi \in \mathbb{R}$.

By (2.13), the map $v \mapsto F(v) + H(v_0, \varphi_0)$ has at least one positive root. Let $v_1 > 0$ be the smallest positive root of $v \mapsto F(v) + H(v_0, \varphi_0)$; then, $v_1 \neq v_0$. Define $G(v) := 2F(v) + 2H(v_0, \varphi_0)$. Suppose that $v_0 \in (0, v_1)$. Because $G(v) > 0$, $\forall v \in (0, v_1)$, $\lim_{v \to 0+} G(v) = +\infty$, and $v_0 \in (0, v_1)$, there exists a positive constant m_0 such that

$$G(v) \geq m_0^2, \quad \forall v \in (0, v_0]. \tag{2.17}$$

Let

$$\phi_1(v) := \int_v^{v_0} \frac{\mathrm{d}\widetilde{v}}{\sqrt{G(\widetilde{v})}}, \quad \forall v \in [0, v_0].$$

Then, (2.16) implies that

$$\phi_1(v(\xi)) = -\xi, \quad \forall \xi < 0. \tag{2.18}$$

On the other hand, by (2.17), we have $0 < \phi_1(v) \leq v_0/m_0$ for all $v \in [0, v_0]$, which gives a contradiction as long as we take $\xi = -2v_0/m_0$ in (2.18). As a consequence, we must have $v_0 > v_1$. Because $v_1 < v_0$, $G(v_0) = m_0^2 > 0$, and $\lim_{v \to +\infty} G(v) = -\infty$, there exists $v_2 > v_0 > v_1$ such that $G(v_2) = 0$ and $G(v) > 0$ for $v \in [v_0, v_2)$. Moreover, we note that $vG(v)$ is a polynomial of degree 4 with at least one negative root and two positive roots $v_2 > v_1 > 0$, which, combined with $vG(v) > 0$ for $v \in [v_0, v_2)$ and $\lim_{v \to +\infty} vG(v) = -\infty$, implies that v_2 is the largest root of vG, and this root is simple. We claim that

$$v(\xi) \in [v_0, v_2), \quad \forall \xi > 0. \tag{2.19}$$

Indeed, suppose not, and then by the strict increase of v, there exists $\xi_0 > 0$ such that $v(\xi_0) = v_2$, so $G(v(\xi_0)) = 0$, and thus (2.15) implies that $\varphi(\xi_0) = \sqrt{G(v(\xi_0))} = 0$, which contradicts our assumption (2.14). Therefore, (2.19) holds. Define

$$\phi_2(v) := \int_{v_0}^{v} \frac{\mathrm{d}\widetilde{v}}{\sqrt{G(\widetilde{v})}} > 0, \quad \forall v \in (v_0, v_2).$$

Then, (2.16) implies that

$$\phi_2(v(\xi)) = \xi, \quad \forall \xi > 0. \tag{2.20}$$

On the other hand, we recall that v_2 is a simple root of the polynomial $vG(v)$, so $\sqrt{G(\widetilde{v})} \sim \sqrt{v_2 - \widetilde{v}}$ as $\widetilde{v} \to v_2-$, and thus

$$0 < \phi_2(v) \leq \phi_2(v_2) := \int_{v_0}^{v_2} \frac{\mathrm{d}\widetilde{v}}{\sqrt{G(\widetilde{v})}} < +\infty.$$

This gives a contradiction if we take $\xi = 2\phi_2(v_2)$ in (2.20), recalling (2.19). This completes the proof for the case $v_0 > 0$.

One can prove the desired conclusion similarly for the case $v_0 < 0$. Assume $\varphi_0 > 0$ (the case $\varphi_0 < 0$ is symmetric). Then, v is strictly increasing and remains negative. Define $G(v) = 2F(v) + 2H(v_0, \varphi_0)$. By (2.13), $G(v) \to -\infty$ as $v \to 0-$ and as $v \to -\infty$, and $G(v_0) > 0$. Thus, there exist two roots $v_2 < v_0 < v_1 < 0$ of G. If $v_0 \in (v_1, 0)$, then the solution must reach v_1 in finite positive time, at which $\varphi = 0$. If $v_0 < v_1$, then the solution must reach v_2 in finite negative time, also forcing $\varphi = 0$. Both contradict the assumption that $\varphi > 0$ for all ξ. Hence, there exists $\xi_0 \in \mathbb{R}$ such that $\varphi(\xi_0) = 0$.

In view of Lemma 2.1, if the solution (v, φ) to (2.9) is globally defined, then $\varphi(\xi_0) = 0$ for some $\xi_0 \in \mathbb{R}$. Hence, (v, φ) solves the Hamiltonian system with $v(\xi_0) \in \mathbb{R} \setminus \{0\}$ and $\varphi(\xi_0) = 0$. Due to the translation invariance, it suffices to consider the following initial value problem:

$$\begin{cases} \frac{dv}{d\xi} = \varphi, \\ \frac{d\varphi}{d\xi} = f(v) := -v^2 - cv + g_1 - \frac{g_2}{v^2}, \end{cases} \quad \text{with} \quad \begin{cases} v(0) = v_0 \in \mathbb{R} \setminus \{0\}, \\ \varphi(0) = 0 \in \mathbb{R}. \end{cases} \tag{2.21}$$

Before stating our theorem on the initial value problem (2.21), we introduce some notations. First, if $9c^2 + 32g_1 > 0$, then we denote

$$v_\pm := \frac{-3c \pm \sqrt{9c^2 + 32g_1}}{8}. \tag{2.22}$$

Let $g(v) := v^2 f(v) = -v^4 - cv^3 + g_1 v^2 - g_2$. We also define the following sets for parameters (c, g_1, g_2):

$$\mathcal{P}_0 := \{(c, g_1, g_2) \in \mathbb{R} \times \mathbb{R} \times (0, +\infty) : 9c^2 + 32g_1 \le 0\},$$
$$\mathcal{P}_1 := \{(c, g_1, g_2) \in \mathbb{R} \times \mathbb{R} \times (0, +\infty) : 9c^2 + 32g_1 > 0, g(v_-) \le 0 \text{ and } g(v_+) \le 0\},$$
$$\mathcal{P}_2 := \{(c, g_1, g_2) \in \mathbb{R} \times \mathbb{R} \times (0, +\infty) : 9c^2 + 32g_1 > 0 \text{ and } g(v_-)g(v_+) < 0\},$$
$$\mathcal{P}_3 := \Big\{(c, g_1, g_2) \in \mathbb{R} \times \mathbb{R} \times (0, +\infty) : 9c^2 + 32g_1 > 0 \text{ and } \min\{g(v_-), g(v_+)\} = 0 < \max\{g(v_-), g(v_+)\}\Big\},$$
$$\mathcal{P}_4 := \{(c, g_1, g_2) \in \mathbb{R} \times \mathbb{R} \times (0, +\infty) : 9c^2 + 32g_1 > 0, g(v_-) > 0 \text{ and } g(v_+) > 0\}.$$

Then, clearly, $\mathbb{R} \times \mathbb{R} \times (0, +\infty) = \mathcal{P}_0 \cup \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3 \cup \mathcal{P}_4$.

We have the following theorem on the initial value problem (2.21), from which our Theorem 1.1 follows immediately. The proof of Theorem 1.1 using Theorem 2.1 can be found at the end of Section 3.

Theorem 2.1. *Let (v, φ) be the unique classical solution to (2.21). Then, $(v, \varphi) \in C^\infty$, v is an even function, and φ is an odd function, and we have the following conclusions:*

(i) If $(c, g_1, g_2) \in \mathcal{P}_0 \cup \mathcal{P}_1$, then for all $v_0 \in \mathbb{R} \setminus \{0\}$, the solution (v, φ) is not globally defined.

(ii) If $(c, g_1, g_2) \in \mathcal{P}_2$, then $\{v \in \mathbb{R} : f(v) = 0\} = \{v_1, v_2\}$, where $v_1 < v_2$ are simple roots of f satisfying $v_1 v_2 > 0$, and there exists a unique $v_1' > v_2$ such that $F(v_1') = F(v_1)$ and $v_1' v_1 > 0$. Moreover, if $v_0 \in \{v_1, v_2\}$, then (v, φ) is globally defined, and $(v, \varphi) \equiv (v_0, 0)$; if $v_0 \in (v_1, v_2) \cup (v_2, v_1')$, then (v, φ) is globally defined and periodic; if $v_0 = v_1'$, then (v, φ) is globally defined such that $\lim_{\xi \to -\infty} v(\xi) = \lim_{\xi \to +\infty} v(\xi) = v_1$, and v is strictly increasing on $\xi \in (-\infty, 0)$ and strictly decreasing on $\xi \in (0, +\infty)$; if $v_0 \in \mathbb{R} \setminus ([v_1, v_1'] \cup \{0\})$, then (v, φ) is not globally defined.

(iii) *If* $(c, g_1, g_2) \in \mathcal{P}_3$*, then* $\{v \in \mathbb{R} : f(v) = 0\} = \{v_1, v_2, v_3\}$*, where* $v_1 < v_2$ *are simple roots of* f *satisfying* $v_1 v_2 > 0$*,* v_3 *is a double root of* f *satisfying* $v_3 v_1 < 0$*, and there exists a unique* $v_1' > v_2$ *such that* $F(v_1') = F(v_1)$ *and* $v_1' v_1 > 0$*. Moreover, if* $v_0 \in \{v_1, v_2, v_3\}$*, then* (v, φ) *is globally defined, and* $(v, \varphi) \equiv (v_0, 0)$*; if* $v_0 \in (v_1, v_2) \cup (v_2, v_1')$*, then* (v, φ) *is globally defined and periodic; if* $v_0 = v_1'$*, then* (v, φ) *is globally defined such that* $\lim_{\xi \to -\infty} v(\xi) = \lim_{\xi \to +\infty} v(\xi) = v_1$*, and* v *is strictly increasing on* $\xi \in (-\infty, 0)$ *and strictly decreasing on* $\xi \in (0, +\infty)$*; if* $v_0 \in \mathbb{R} \setminus ([v_1, v_1'] \cup \{0, v_3\})$*, then* (v, φ) *is not globally defined.*

(iv) *If* $(c, g_1, g_2) \in \mathcal{P}_4$*, then* $\{v \in \mathbb{R} : f(v) = 0\} = \{v_1, v_2, v_3, v_4\}$*, where* $v_1 < v_2 < 0 < v_3 < v_4$ *are simple roots of* f*, and there exist unique* $v_1' \in (v_2, 0)$ *and* $v_3' \in (v_4, +\infty)$ *such that* $F(v_1') = F(v_1)$*, and* $F(v_3') = F(v_3)$*. Moreover, if* $v_0 \in \{v_1, v_2, v_3, v_4\}$*, then* (v, φ) *is globally defined, and* $(v, \varphi) \equiv (v_0, 0)$*; if* $v_0 \in (v_1, v_2) \cup (v_2, v_1') \cup (v_3, v_4) \cup (v_4, v_3')$*, then* (v, φ) *is globally defined and periodic; if* $v_0 = v_1'$ *(or* $v_0 = v_3'$*), then* (v, φ) *is globally defined such that* $\lim_{\xi \to -\infty} v(\xi) = \lim_{\xi \to +\infty} v(\xi) = v_1$ *(or* v_3*), and* v *is strictly increasing on* $\xi \in (-\infty, 0)$ *and strictly decreasing on* $\xi \in (0, +\infty)$*; if* $v_0 \in \mathbb{R} \setminus ([v_1, v_1'] \cup [v_3, v_3'] \cup \{0\})$*, then* (v, φ) *is not globally defined.*

See Table 1 for a summary table mapping parameter regime to solution types.

Table 1. Summary of Theorem 2.1: classifications of solutions to (2.21).

Parameter region	Roots of $f(v) = 0$	Range of initial v_0	Properties of solution (v, φ)
$\mathcal{P}_0 \cup \mathcal{P}_1$	$f(v) \leq 0$ for all $v \in \mathbb{R} \setminus \{0\}$	$\mathbb{R} \setminus \{0\}$	Not globally defined
$\mathcal{P}_2$	$v_1 < v_2$ (simple, $v_1 v_2 > 0$) Exists $v_1' > v_2$ with $F(v_1') = F(v_1)$, $v_1' v_1 > 0$	$v_0 \in \{v_1, v_2\}$	Globally defined, $(v, \varphi) \equiv (v_0, 0)$
		$v_0 \in (v_1, v_2) \cup (v_2, v_1')$	Globally defined and periodic
		$v_0 = v_1'$	Globally defined, $\lim_{\xi \to \pm\infty} v(\xi) = v_1$, v strictly inc. on $(-\infty, 0)$, strictly dec. on $(0, +\infty)$
		$v_0 \in \mathbb{R} \setminus ([v_1, v_1'] \cup \{0\})$	Not globally defined
$\mathcal{P}_3$	$v_1 < v_2$ (simple, $v_1 v_2 > 0$) Exists $v_1' > v_2$ with $F(v_1') = F(v_1)$, $v_1' v_1 > 0$	$v_0 \in \{v_1, v_2, v_3\}$	Globally defined, $(v, \varphi) \equiv (v_0, 0)$
		$v_0 \in (v_1, v_2) \cup (v_2, v_1')$	Globally defined and periodic
		$v_0 = v_1'$	Globally defined, $\lim_{\xi \to \pm\infty} v(\xi) = v_1$, v strictly inc. on $(-\infty, 0)$, strictly dec. on $(0, +\infty)$
		$v_0 \in \mathbb{R} \setminus ([v_1, v_1'] \cup \{0, v_3\})$	Not globally defined
$\mathcal{P}_4$	$v_1 < v_2 < 0 < v_3 < v_4$ (simple roots) Exist $v_1' \in (v_2, 0)$, $v_3' \in (v_4, +\infty)$ with $F(v_1') = F(v_1)$, $F(v_3') = F(v_3)$	$v_0 \in \{v_1, v_2, v_3, v_4\}$	Globally defined, $(v, \varphi) \equiv (v_0, 0)$
		$v_0 \in (v_1, v_2) \cup (v_2, v_1')$ $\cup (v_3, v_4) \cup (v_4, v_3')$	Globally defined and periodic
		$v_0 = v_1'$	Globally defined, $\lim_{\xi \to \pm\infty} v(\xi) = v_1$, v strictly inc. on $(-\infty, 0)$, strictly dec. on $(0, +\infty)$
		$v_0 = v_3'$	Globally defined, $\lim_{\xi \to \pm\infty} v(\xi) = v_3$, v strictly inc. on $(-\infty, 0)$, strictly dec. on $(0, +\infty)$
		$v_0 \in \mathbb{R} \setminus$ $([v_1, v_1'] \cup [v_3, v_3'] \cup \{0\})$	Not globally defined

Remark 2.5. *The sets $\mathcal{P}_0$ to $\mathcal{P}_4$ demarcate regimes in the physical parameter space (c, g_1, g_2), which originates from the background vorticity k, wave speed $\widetilde{c}$, and integration constants $\widetilde{g}_1, \widetilde{g}_2$, that dictate the possible types of bounded traveling waves. In regimes $\mathcal{P}_0$ and $\mathcal{P}_1$, the effective potential $F(v)$ lacks a local minimum, implying that no stable oscillatory base exists for persistent wave structures; consequently, no bounded traveling waves (periodic or localized) are physically admissible. The regime $\mathcal{P}_2$ supports a single potential well, giving rise to a continuous family of periodic waves (closed orbits in the phase portrait) and, at a critical energy level, a solitary wave (homoclinic orbit). In $\mathcal{P}_3$, a saddle-node bifurcation occurs at one boundary of the parameter space, creating a degenerate critical point; here, the solitary wave coexists with periodic waves but marks the threshold of this wave family's existence. Finally, $\mathcal{P}_4$ corresponds to a double-well potential, permitting two distinct families of periodic waves (around two separate centers) and two possible solitary waves, each connecting to a different asymptotic background. This classification thus maps directly onto the qualitative morphology of water waves under shear flow: from no sustained waves, to simple wave trains and single pulses, and finally to bistable systems capable of supporting compound or multimodal wave structures.*

To provide a more intuitive understanding of Theorem 2.1, we constructed five distinct parameter sets corresponding to $\mathcal{P}_0$ through $\mathcal{P}_4$. For each parameter set, we plotted the phase portrait of system (2.21). The phase diagrams demonstrate that when v_0 takes different values, the solutions of system (2.21) exhibit distinct characteristics; these solutions may be periodic, unimodal, or fail to be globally defined. Each streamline in the phase portraits represents a possible trajectory of the dynamical system (2.21), which corresponds to a traveling wave profile $v(\xi)$. Closed orbits correspond to periodic traveling wave solutions, where $v(\xi)$ oscillates between two extreme values. Homoclinic orbits (the curves connecting a saddle point to itself) correspond to unimodal (solitary wave) solutions, where $v(\xi)$ decays to a constant background value as $|\xi| \to \infty$. Trajectories escaping to infinity or toward the singularity at $v = 0$ indicate that the corresponding solutions to the original system (1.1) are not globally defined or physically admissible. Figures 2–6 systematically demonstrate how the solution landscape bifurcates across different parameter regimes $\mathcal{P}_0$ to $\mathcal{P}_4$. Specifically, in Figures 2 and 3, the solution pair (v, φ) is not globally defined. In Figures 4–6, both periodic and unimodal solutions are observed. Notably, periodic solutions exist for v_0 within certain parameter ranges, as detailed in the captions of the respective figures. The rest of this paper is devoted to the proof of Theorem 2.1. See Section 3 for full details. In Section 4, we will give an example to illustrate our result.

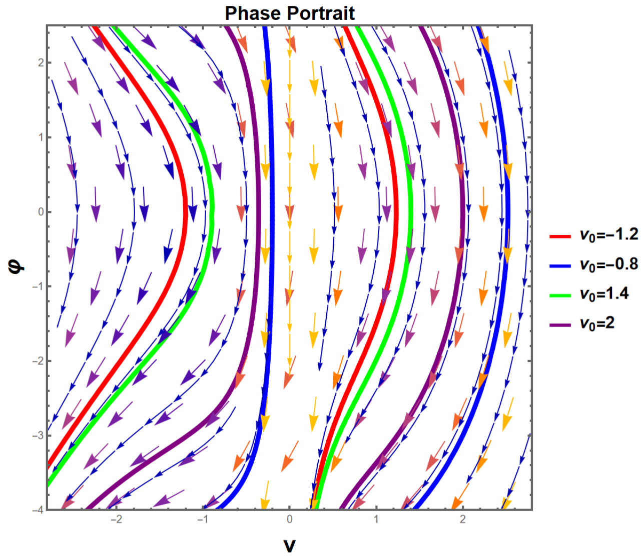


Figure 2. Phase portrait for the parameter set $\mathcal{P}_0$, where no bounded smooth traveling waves exist. The parameters are $c = 1$, $g_1 = -27/32$, and $g_2 = 2$. All streamlines either diverge to infinity or terminate at the singularity $v = 0$, indicating that no bounded smooth traveling wave solutions exist in this regime. This corresponds to the conclusion of Theorem 2.1(i).

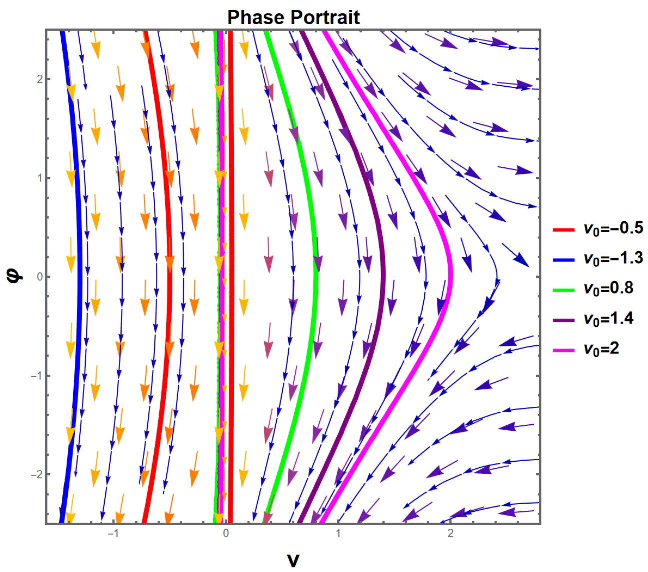


Figure 3. Phase portrait for the parameter set $\mathcal{P}_1$, also devoid of bounded traveling waves. The parameters are $c = -6$, $g_1 = -9$, and $g_2 = 4/15$. Similar to Figure 2, the absence of closed or homoclinic orbits indicates a lack of periodic or solitary wave solutions. All trajectories are unbounded, consistent with Theorem 2.1(i).

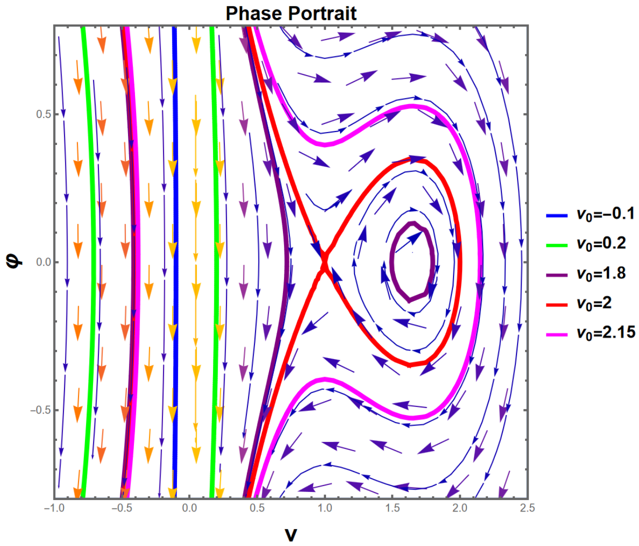


Figure 4. Phase portrait and streamlines of solutions to (2.21) for various $v_0 \neq 0$, where $c = -12/5$, $g_1 = -17/15$, and $g_2 = 4/15$. In this case, $(c, g_1, g_2) \in \mathscr{P}_2$. There holds $\{v \in \mathbb{R} : f(v) = 0\} = \{1, v_2\}$, where $v_2 \approx 1.658$, and $F(2) = F(1) = 0$. Here, $v_2 \approx 1.658$ is given by Mathematica, using the code `NSolve[-v*v + 12/5*v - 17/15 - 4/(15*v*v) == 0, v]`. If $v_0 \in [1, 2)$, then the solution (v, φ) to (2.21) is periodic; if $v_0 = 2$, then (v, φ) is unimodal; and for other v_0, (v, φ) is not globally defined.

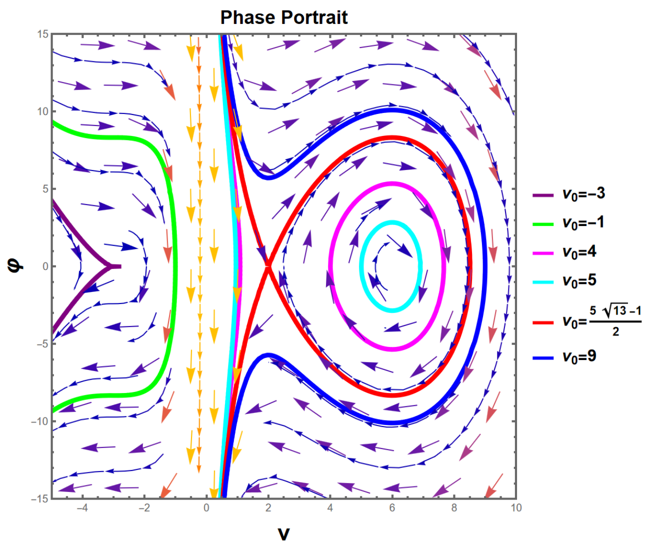


Figure 5. Phase portrait and streamlines of solutions to (2.21) for various $v_0 \neq 0$, where $c = -2$, $g_1 = 27$, and $g_2 = 108$. In this case, $(c, g_1, g_2) \in \mathscr{P}_3$. There holds $\{v \in \mathbb{R} : f(v) = 0\} = \{2, 6, -3\}$ and $F\left((5\sqrt{13} - 1)/2 \approx 8.51\right) = F(2)$. If $v_0 \in \left[2, (5\sqrt{13} - 1)/2\right)$, then the solution (v, φ) to (2.21) is periodic; if $v_0 = (5\sqrt{13} - 1)/2$, then (v, φ) is unimodal; and for other v_0, (v, φ) is not globally defined.

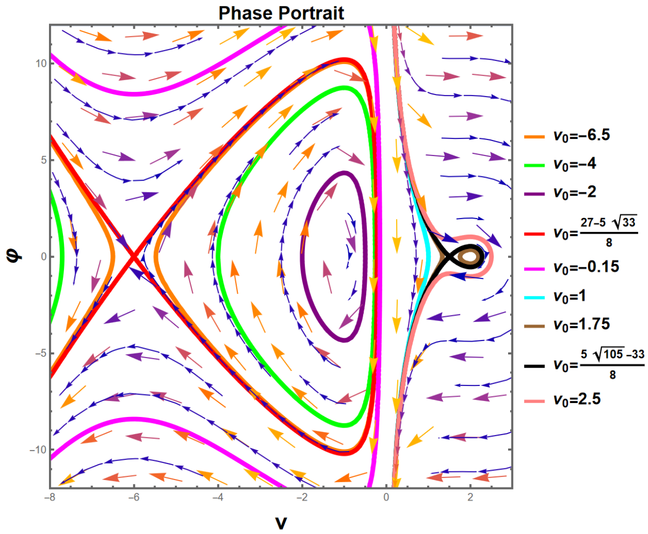


Figure 6. Phase portrait and streamlines of solutions to (2.21) for various $v_0 \neq 0$, where $c = 7/2$, $g_1 = 31/2$ and $g_2 = 18$. In this case, $(c, g_1, g_2) \in \mathscr{P}_4$. There holds $\{v \in \mathbb{R} : f(v) = 0\} = \{-6, -1, 3/2, 2\}$ and $F\left((27-5\sqrt{33})/8 \approx -0.21\right) = F(-6)$, $F\left((5\sqrt{105}-33)/8 \approx 2.28\right) = F(3/2)$. If $v_0 \in \left[-6, (27-5\sqrt{33})/8\right) \cup \left[3/2, (5\sqrt{105}-33)/8\right)$, then the solution (v, φ) to (2.21) is periodic; if $v_0 \in \left\{(27-5\sqrt{33})/8, (5\sqrt{105}-33)/8\right\}$, then (v, φ) is unimodal; and for other v_0, (v, φ) is not globally defined.

3. Proof of Theorems 2.1 and 1.1

Recall $f(v) = -v^2 - cv + g_1 - g_2/v^2$ and $g_2 > 0$. Then,

$$\lim_{v\to-\infty} f(v) = -\infty, \quad \lim_{v\to0-} f(v) = -\infty, \quad \lim_{v\to0+} f(v) = -\infty, \quad \lim_{v\to+\infty} f(v) = -\infty. \tag{3.1}$$

By (3.1) and the fact that $g(v) := v^2 f(v) = -v^4 - cv^3 + g_1 v^2 - g_2$ is a quartic polynomial, one of the following five cases holds:

(1) f has no real root;

(2) f has exactly one real root, which is a multiple root;

(3) f has exactly two different real roots;

(4) f has exactly three different real roots;

(5) f has four different real roots.

We start by exploring the ranges of parameters such that each of the above cases holds.

Lemma 3.1. *The following properties hold:*

- $f(v) \leq 0$ *for all* $v \in \mathbb{R} \setminus \{0\}$ *if and only if* $(c, g_1, g_2) \in \mathcal{P}_0 \cup \mathcal{P}_1$.
- $\mathcal{Z} := \{v \in \mathbb{R} : f(v) = 0\} = \{v_1, v_2\}$ *with simple roots* $v_1 < v_2$ *satisfying* $v_1 v_2 > 0$ *if and only if* $(c, g_1, g_2) \in \mathcal{P}_2$.

- $\mathcal{Z} = \{v_1, v_2, v_3\}$ *with a double root* v_3 *and simple roots* $v_1 < v_2$ *satisfying* $v_1v_2 > 0$ *if and only if* $(c, g_1, g_2) \in \mathcal{P}_3$, *where*

$$\mathcal{P}_3 = \{c > 0, g_1 > 0, g_2 > 0, g(v_-) > g(v_+) = 0\} \cup \{c < 0, g_1 > 0, g_2 > 0, g(v_-) = 0 < g(v_+)\}.$$

- $\mathcal{Z} := \{v_1, v_2, v_3, v_4\}$ *with simple roots* $v_1 < v_2 < 0 < v_3 < v_4$ *if and only if* $(c, g_1, g_2) \in \mathcal{P}_4$.

Moreover, the above enumeration comprises all possible cases that may occur.

Proof. We compute that $g'(v) = v(-4v^2 - 3cv + 2g_1)$. If $9c^2 + 32g_1 \le 0$, then $-4v^2 - 3cv + 2g_1 \le 0$ for all $v \in \mathbb{R}$; hence, g is increasing on $v \in (-\infty, 0)$, and g is decreasing on $v \in (0, +\infty)$; combining this with $g(0) = -g_2 < 0$ implies that g has no real root, that is, f has no real root. Now, we assume that $9c^2 + 32g_1 > 0$, in which case, the quadratic polynomial $-4v^2 - 3cv + 2g_1$ has two different roots v_-, v_+ defined by (2.22). For clarity, we consider the following six cases separately.

Case 1. $c = 0$. In this case, we have that $g(v) = -v^4 + g_1v^2 - g_2$ is a quadratic polynomial with respect to v^2, and $g_1 > 0$ due to the assumption $9c^2 + 32g_1 > 0$. Therefore,

- if $g_1^2 - 4g_2 < 0$, then g has no real root;
- if $g_1^2 - 4g_2 = 0$, then g has two distinct real roots $\pm\sqrt{g_1/2}$, each with a multiplicity of 2;
- if $g_1^2 - 4g_2 > 0$, then g has four different real roots, two negative and two positive.

Case 2. $c \neq 0$, and $g_1 = 0$. In this case, $g(v) = -v^4 - cv^3 - g_2$, and $g'(v) = -v^2(4v + 3c)$; hence, g is increasing on $v \in (-\infty, -3c/4)$, and g is decreasing on $v \in (-3c/4, +\infty)$. We compute that

$$\max g = g\left(-\frac{3}{4}c\right) = \frac{27}{256}c^4 - g_2.$$

Therefore,

- if $27c^4 - 256g_2 < 0$, then g has no real root;
- if $27c^4 - 256g_2 = 0$, then g has precisely one real root $-3c/4$, whose multiplicity is 2, and it is negative if $c > 0$ and positive if $c < 0$;
- if $27c^4 - 256g_2 > 0$, then g has precisely two distinct real roots, both of which are negative if $c > 0$ and are positive if $c < 0$.

Case 3. $c > 0$, and $g_1 < 0$. In this case, if $v_- < v_+ < 0$, then g is increasing on $v \in (-\infty, v_-)$ and $v \in (v_+, 0)$, and g is decreasing on $v \in (v_-, v_+)$ and $v \in (0, +\infty)$. Plugging $v_\pm^2 = (2g_1 - 3cv_\pm)/4$ into the expression of g, we compute that

$$g(v_\pm) = -\frac{c(9c^2 + 32g_1)}{16}v_\pm + \frac{g_1(3c^2 + 8g_1)}{32} - g_2. \tag{3.2}$$

Because g is increasing on $v \in (v_+, 0)$, and $g(0) = -g_2 < 0$, we have $g(v_+) < 0$. Therefore,

- if $g(v_-) < 0$, then g has no real root;
- if $g(v_-) = 0$, then g has precisely one real root $v_- < 0$, which is of multiplicity 2;
- if $g(v_-) > 0$, then g has precisely two distinct real roots, which are both negative.

Case 4. $c > 0$, and $g_1 > 0$. In this case, if $v_- < 0 < v_+$, then g is increasing on $v \in (-\infty, v_-)$ and $v \in (0, v_+)$, and g is decreasing on $v \in (v_-, 0)$ and $v \in (v_+, +\infty)$. Because $c > 0, 9c^2 + 32g_1 > 0$ and $v_- < 0 < v_+$, by (3.2), we have $g(v_-) > g(v_+)$. Therefore,

- if $g(v_-) < 0$, then $g(v_+) < 0$, and g has no real root;
- if $g(v_-) = 0$, then $g(v_+) < 0$, and g has precisely one real root v_-, which is negative and of multiplicity 2;
- if $g(v_-) > 0$, and $g(v_+) < 0$, then g has precisely two distinct real roots, both of which are negative;
- if $g(v_-) > 0$, and $g(v_+) = 0$, then g has precisely three distinct real roots, two of which are negative, and another one is $v_+ > 0$ with multiplicity 2;
- if $g(v_-) > 0$, and $g(v_+) > 0$, then g has four distinct real roots, two negative and two positive.

Case 5. $c < 0$, and $g_1 < 0$. In this case, if $0 < v_- < v_+$, then g is increasing on $v \in (-\infty, 0)$ and $v \in (v_-, v_+)$, and g is decreasing on $v \in (0, v_-)$ and $v \in (v_+, +\infty)$. Because $g(0) = -g_2 < 0$, we have $g(v_-) \leq g(0) < 0$. Therefore,

- if $g(v_+) < 0$, then g has no real root;
- if $g(v_+) = 0$, then g has precisely one real root $v_+ > 0$, which is of multiplicity 2;
- if $g(v_+) > 0$, then g has precisely two distinct real roots, which are both positive.

Case 6. $c < 0$, and $g_1 > 0$. In this case, if $v_- < 0 < v_+$, then g is increasing on $v \in (-\infty, v_-)$ and $v \in (0, v_+)$, and g is decreasing on $v \in (v_-, 0)$ and $v \in (v_+, +\infty)$. Because $c < 0$, $9c^2 + 32g_1 > 0$ and $v_- < 0 < v_+$, by (3.2) we have $g(v_-) < g(v_+)$. Therefore,

- if $g(v_-) < 0$, and $g(v_+) < 0$, then g has no real root;
- if $g(v_-) < 0$, and $g(v_+) = 0$, then g has precisely one real root $v_+ > 0$, which is of multiplicity 2;
- if $g(v_-) < 0$, and $g(v_+) > 0$, then g has precisely two distinct real roots, which are both positive;
- if $g(v_-) = 0$, then $g(v_+) > 0$, and g has precisely three distinct real roots, one of which is $v_- < 0$ with multiplicity 2, and another two are both positive;
- if $g(v_-) > 0$, then $g(v_+) > 0$, and g has four distinct real roots, two negative and two positive.

This completes the proof.

Lemma 3.2. *Let $v_0 \in \mathbb{R} \setminus \{0\}$ be such that $f(v_0) \neq 0$. Assume that one of the following holds:*

- *$v_0 < 0$ and $F(v) > F(v_0)$ for all $v < v_0$, or*
- *$v_0 > 0$ and $F(v) > F(v_0)$ for all $v \in (0, v_0)$.*

Then, the unique solution to (2.21) *is not defined globally, that is, the maximal interval of existence is not* $\mathbb{R}$.

Proof. **Case 1.** Assume that $v_0 < 0$ and $F(v) > F(v_0)$ for all $v < v_0$. Then, $f(v_0) = F'(v_0) < 0$ thanks to $f(v_0) \neq 0$. Suppose on contrary that the maximal interval of existence is $\mathbb{R}$. Define

$$\phi_1(v) := \int_v^{v_0} \frac{d\widetilde{v}}{\sqrt{2F(\widetilde{v}) - 2F(v_0)}} > 0, \quad \forall v \in (-\infty, v_0],$$

and then ϕ_1 is smooth and strictly decreasing on $v \in (-\infty, v_0)$ with $\phi_1(v_0) = 0$. Then, (2.21) implies that

$$\phi_1(v(\xi)) = -\xi, \quad \forall \xi < 0. \tag{3.3}$$

Because $F(\widetilde{v}) \sim -\widetilde{v}^3/3$ as $\widetilde{v} \to -\infty$, we know that

$$\xi_* := \lim_{v \to -\infty} \phi_1(v) = \int_{-\infty}^{v_0} \frac{d\widetilde{v}}{\sqrt{2F(\widetilde{v}) - 2F(v_0)}} \in (0, +\infty).$$

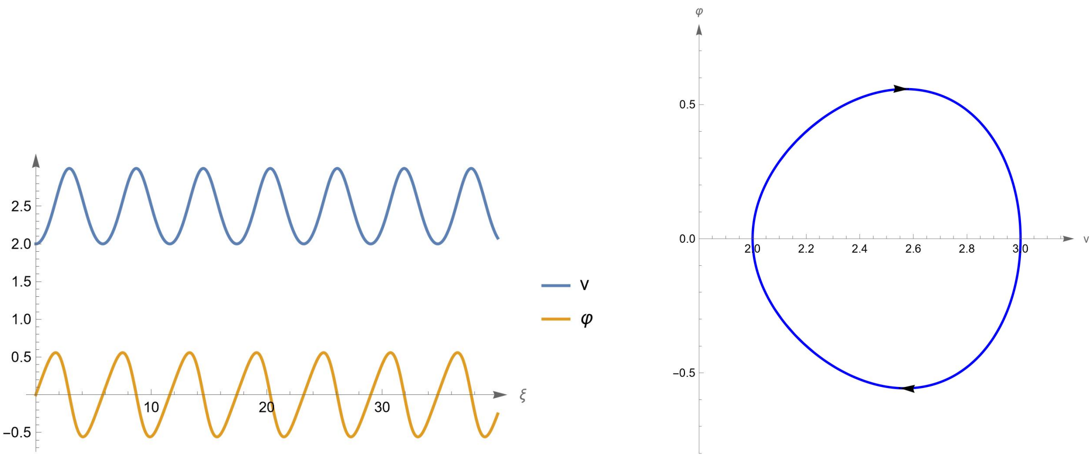
Thus, $\phi_1(v) \leq \xi_*$ for all $v \in (-\infty, v_0]$, which gives a contradiction as long as we take $\xi = -2\xi_*$ in (3.3).

Case 2. If $v_0 > 0$ and $F(v) > F(v_0)$ for all $v \in (0, v_0)$, then we consider

$$\phi_2(v) := \int_v^{v_0} \frac{\mathrm{d}\widetilde{v}}{\sqrt{2F(\widetilde{v}) - 2F(v_0)}} > 0, \quad \forall v \in (0, v_0],$$

replacing ϕ_1. The rest proof is similar with **Case 1**, using $\lim_{v\to 0-} \phi_2(v) \in (0, +\infty)$, due to $F(\widetilde{v}) \to +\infty$ as $\widetilde{v} \to 0+$.

Lemma 3.3. *(Periodic solutions) Let $v_0 \in \mathbb{R} \setminus \{0\}$ be such that $f(v_0) \neq 0$. If there exists $v_0' \in \mathbb{R} \setminus \{0, v_0\}$ such that $v_0 v_0' > 0$, $F(v_0') = F(v_0)$, $f(v_0') \neq 0$, and $F(v) > F(v_0)$ for all $v \in (\min\{v_0, v_0'\}, \max\{v_0, v_0'\})$, then the unique solution to* (2.21) *is global and periodic. See Figure 7.*



(a) Graphs of v and φ, where $v_0 = 2$ and $v_0' = 3$ (b) Phase portrait

Figure 7. The graph and phase portrait of the solution to (2.21) with $c = -40/11$, $g_1 = -85/33$, $g_2 = 12/11$, and $v_0 = 2$.

Proof. Assume without loss of generality that $v_0 < v_0'$ so that $f(v_0) > 0 > f(v_0')$. Define

$$\Phi_1(v) := \int_{v_0}^{v} \frac{\mathrm{d}\widetilde{v}}{\sqrt{2F(\widetilde{v}) - 2F(v_0)}}, \quad \forall v \in [v_0, v_0']; \tag{3.4}$$

then, Φ_1 is continuous on $[v_0, v_0']$, smooth on (v_0, v_0'), and strictly increasing on $[v_0, v_0']$ with $\Phi_1(v_0) = 0$ and $T_0 := \Phi_1(v_0') \in (0, +\infty)$ thanks to $F(\widetilde{v}) - F(v_0) = F(\widetilde{v}) - F(v_0') \sim f(v_0')(v_0' - \widetilde{v})$ as $\widetilde{v} \to v_0'-$. We also define

$$\Phi_2(v) := \int_{v}^{v_0'} \frac{\mathrm{d}\widetilde{v}}{\sqrt{2F(\widetilde{v}) - 2F(v_0)}}, \quad \forall v \in [v_0, v_0'];$$

then, Φ_2 is continuous on $[v_0, v_0']$, smooth on (v_0, v_0'), and strictly decreasing on $[v_0, v_0']$ with $\Phi_2(v_0') = 0$ and $\Phi_2(v_0) = T_0 = \Phi_1(v_0') \in (0, +\infty)$.

Let $v : \mathbb{R} \to [v_0, v_0']$ be defined by

$$v(\xi) := \begin{cases} \Phi_1^{-1}(\xi) & \text{for } \xi \in [0, T_0], \\ \Phi_2^{-1}(\xi - T_0) & \text{for } \xi \in [T_0, 2T_0], \end{cases} \quad \text{and} \quad v(\xi + 2T_0) = v(\xi) \text{ for all } \xi \in \mathbb{R}. \tag{3.5}$$

Then, $v : \mathbb{R} \to [v_0, v_0']$ is continuous, $v(0) = v_0 = v(2T_0)$, $v(T_0) = v_0'$, and v is $2T_0$-periodic. Now, we claim that $v : \mathbb{R} \to [v_0, v_0']$ is C^2. Indeed, it is clear that $v \in C^\infty((kT_0, (k+1)T_0))$ for any $k \in \mathbb{Z}$. Considering the periodicity, it suffices to prove the C^2 regularity at $\xi = 0$ and $\xi = T_0$. Below, we only give the proof of C^2 at $\xi = 0$ because the C^2 regularity at $\xi = T_0$ can be proved similarly. First of all, we note that $\Phi_1(v) + \Phi_2(v) = T_0$ for $v \in [v_0, v_0']$; hence, $\Phi_2^{-1}(\xi + T_0) = \Phi_1^{-1}(-\xi)$ for $\xi \in [-T_0, 0]$, and then by (3.5), we have $v(\xi) = \Phi_1^{-1}(-\xi) = v(-\xi)$ for $\xi \in [-T_0, 0]$. Using periodicity, we know that v is an even function on $\mathbb{R}$.

Claim 1. v is differentiable at $\xi = 0$ with $v'(0) = 0$, and v' is continuous at $\xi = 0$.

We first prove that

$$\lim_{\xi \to 0+} \frac{v(\xi) - v_0}{\xi} = \lim_{\xi \to 0+} \frac{\Phi_1^{-1}(\xi) - v_0}{\xi} = 0. \tag{3.6}$$

Indeed, because $F'(v_0) = f(v_0) > 0$, there exists $\varepsilon_0 \in (0, v_0' - v_0)$ such that $F(v) - F(v_0) < 2f(v_0)(v - v_0)$ for all $v \in (v_0, v_0 + \varepsilon_0)$. Then, by (3.4), we have

$$\Phi_1(v) \geq \frac{1}{\sqrt{4f(v_0)}} \int_{v_0}^{v} \frac{d\tilde{v}}{\sqrt{\tilde{v} - v_0}} = \frac{\sqrt{v - v_0}}{\sqrt{f(v_0)}}, \quad \forall v \in [v_0, v_0 + \varepsilon_0];$$

thus,

$$\xi \geq \frac{\sqrt{\Phi_1^{-1}(\xi) - v_0}}{\sqrt{f(v_0)}} \Longrightarrow 0 \leq \frac{\Phi_1^{-1}(\xi) - v_0}{\xi} = \frac{v(\xi) - v_0}{\xi} \leq f(v_0)\xi, \quad \forall \xi \in (0, \Phi_1(v_0 + \varepsilon_0)],$$

from which (3.6) follows. Because v is an even function, by (3.6), we have

$$\lim_{\xi \to 0-} \frac{v(\xi) - v_0}{\xi} = \lim_{\xi \to 0+} \frac{v(\xi) - v_0}{-\xi} = 0. \tag{3.7}$$

Hence, v is differentiable at $\xi = 0$, and $v'(0) = 0$. Moreover, for $\xi \in (0, T_0)$, we have

$$v'(\xi) = \left(\Phi_1^{-1}\right)'(\xi) = \frac{1}{\Phi_1'\left(\Phi_1^{-1}(\xi)\right)} = \sqrt{2F\left(\Phi_1^{-1}(\xi)\right) - 2F(v_0)} = \sqrt{2F(v(\xi)) - 2F(v_0)}, \tag{3.8}$$

and for $\xi \in (-T_0, 0)$, we have

$$v'(\xi) = -v'(-\xi) = -\sqrt{2F(v(-\xi)) - 2F(v_0)}; \tag{3.9}$$

hence, $\lim_{\xi \to 0+} v'(\xi) = \lim_{\xi \to 0-} v'(\xi) = 0 = v'(0)$. This proves the continuity of v' at $\xi = 0$.

Claim 2. v' is differentiable at $\xi = 0$ with $v''(0) = f(v_0)$, and v'' is continuous at $\xi = 0$.

Similarly with the proof of (3.6), for any $\delta \in (0, f(v_0)/2)$, there exists $\varepsilon_0 = \varepsilon_0(\delta) \in (0, v_0' - v_0)$ such that $(f(v_0) - \delta)(v - v_0) < F(v) - F(v_0) < (f(v_0) + \delta)(v - v_0)$ for $v \in (v_0, v_0 + \varepsilon_0)$, and then for $\xi \in (0, \Phi_1(v_0 + \varepsilon_0))$, we have

$$\frac{f(v_0) - \delta}{2}\xi^2 \leq v(\xi) - v_0 \leq \frac{f(v_0) + \delta}{2}\xi^2;$$

hence,

$$\frac{(f(v_0) - \delta)^2}{2}\xi^2 < F(v(\xi)) - F(v_0) < \frac{(f(v_0) + \delta)^2}{2}\xi^2, \quad \forall \xi \in (0, \Phi_1(v_0 + \varepsilon_0)).$$

Thus, (3.8) implies that

$$f(v_0) - \delta < v'(\xi)/\xi < f(v_0) + \delta, \quad \forall \xi \in (0, \Phi_1(v_0 + \varepsilon_0)).$$

So, $\lim_{\xi \to 0+} v'(\xi)/\xi = f(v_0)$, and then $\lim_{\xi \to 0-} \frac{v'(\xi)}{\xi} = \lim_{\xi \to 0-} \frac{-v'(-\xi)}{\xi} = \lim_{\xi \to 0+} \frac{v'(\xi)}{\xi} = f(v_0)$. Therefore, v' is differentiable at $\xi = 0$ with $v''(0) = f(v_0)$. Moreover, for $\xi \in (0, T_0)$, by (3.8), we have

$$v''(\xi) = \frac{\sqrt{2}}{2} \frac{F'(v(\xi))v'(\xi)}{\sqrt{F(v(\xi)) - F(v_0)}} = F'(v(\xi)) = f(v(\xi)),$$

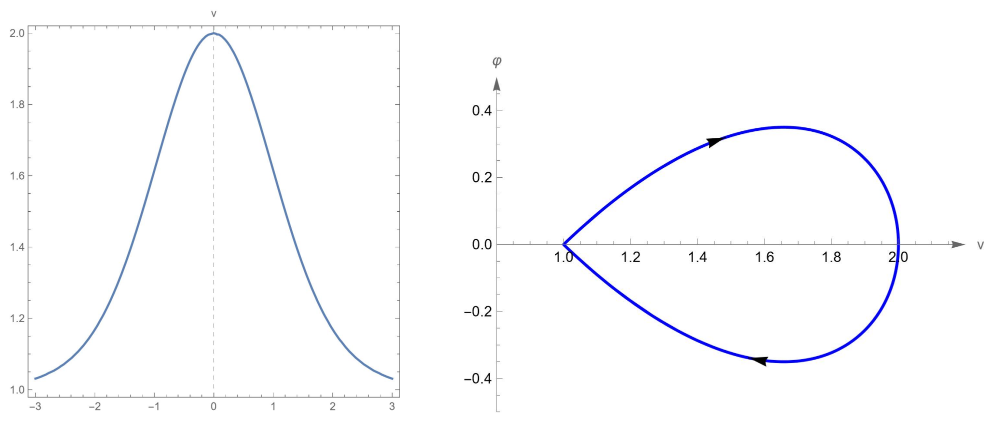
and for $\xi \in (-T_0, 0)$, by (3.9), we have $v''(\xi) = v''(-\xi) = f(v(-\xi)) = f(v(\xi))$. Hence, $\lim_{\xi \to 0+} v''(\xi) = \lim_{\xi \to 0-} v''(\xi) = f(v_0) = v''(0)$, which proves the continuity of v'' at $\xi = 0$.

So far, we have proved that $v \in C^2(\mathbb{R})$. Let $\varphi := v'$, and then $\varphi \in C^1(\mathbb{R})$ and $\varphi'(\xi) = f(v(\xi))$ for all $\xi \in \mathbb{R}$. Thus, (v, φ) is the unique C^1 solution to the Hamiltonian system (2.21), where the uniqueness follows from the classical theory of ODEs. Hence, the unique solution (v, φ) is smooth, globally defined, $2T_0$-periodic, $v(\xi) \in [v_0, v_0']$ for all $\xi \in \mathbb{R}$, $v|_{[0,T_0]} = \Phi_1^{-1}$, and v is an even function.

Lemma 3.4. *(Unimodal solutions) Let $v_0 \in \mathbb{R} \setminus \{0\}$ be such that $f(v_0) \neq 0$. If there exists $v_0' \in \mathbb{R} \setminus \{0, v_0\}$ such that $v_0 v_0' > 0$, $F(v_0') = F(v_0)$, $f(v_0') = 0$, and $F(v) > F(v_0)$ for all $v \in (\min\{v_0, v_0'\}, \max\{v_0, v_0'\})$, then the unique solution (v, φ) to (2.21) is global and satisfies the following properties:*

- *v is an even function, and φ is an odd function;*
- *v is strictly monotone on $(-\infty, 0)$ and $(0, +\infty)$, respectively;*
- *$\lim_{\xi \to -\infty} v(\xi) = \lim_{\xi \to +\infty} v(\xi) = v_0'$, and $\lim_{\xi \to -\infty} \varphi(\xi) = \lim_{\xi \to +\infty} \varphi(\xi) = 0$;*
- *$|v(\xi) - v_0'|$ converges to 0 exponentially as $|\xi| \to +\infty$.*

See Figure 8.



(a) Graph of v, where $v_0' = 1$ and $v_0 = 2$ (b) Phase portrait

Figure 8. The graph and phase portrait of the solution to (2.21) with $c = -12/5$, $g_1 = -17/15$, $g_2 = 4/15$, and $v_0 = 2$.

Proof. Assume without loss of generality that $v_0' < v_0$; then, $f(v_0) < 0 = f(v_0')$. Define

$$\Phi(v) := \int_v^{v_0} \frac{d\widetilde{v}}{\sqrt{2F(\widetilde{v}) - 2F(v_0)}}, \quad \forall v \in (v_0', v_0].$$

Then, Φ is smooth on (v_0', v_0) and strictly decreasing on $(v_0', v_0]$ with $\Phi(v_0) = 0$ and $\lim_{v\to v_0'+} \Phi(v) = +\infty$ because $F(\widetilde{v}) - F(v_0) = F(\widetilde{v}) - F(v_0') \leq C(v_0' - \widetilde{v})^2$ as $\widetilde{v} \to v_0'+$ due to the assumption $F'(v_0') = f(v_0') = 0$, where $C > 0$ is a constant independent of $\widetilde{v}$. We further claim that $f'(v_0') > 0$, and thus, $F(\widetilde{v}) - F(v_0) \sim f'(v_0')(\widetilde{v} - v_0')^2$ as $\widetilde{v} \to v_0'$. Indeed, a direct computation implies that

$$F(v) - F(v_0') = \frac{v - v_0'}{v_0' v} F_1(v),$$

where F_1 is a polynomial of degree 3 defined by

$$F_1(v) := -\frac{1}{3} v_0' v^3 - \left(\frac{1}{3} v_0' + \frac{1}{2} c\right) v_0' v^2 + \left(-\frac{1}{3} (v_0')^3 - \frac{1}{2} c (v_0')^2 + g_1 v_0'\right) v - g_2.$$

Taking $v \to v_0'$ in both sides of

$$v_0' v \left(F(v) - F(v_0')\right) = (v - v_0') F_1(v)$$

gives that $(v_0')^2 F'(v_0') = F_1(v_0')$; hence, $F_1(v_0') = (v_0')^2 f(v_0') = 0$, so $F_1(v) = (v - v_0') F_2(v)$ for some polynomial F_2 of degree 2. Taking $v \to v_0'$ in both sides of

$$v_0' v \left(F(v) - F(v_0')\right) = (v - v_0')^2 F_2(v) \tag{3.10}$$

gives that $(v_0')^2 f'(v_0') = (v_0')^2 F''(v_0') = F_2(v_0')$. On the other hand, the equation $F(v) = F(v_0')$ has another two roots v_0 and v_0'' which are different from v_0', where $v_0' v_0'' < 0$ due to (2.13); hence, by (3.10), we have $\{v \in \mathbb{R} : F_2(v) = 0\} = \{v_0, v_0''\}$, so $F_2(v_0') \neq 0$, and thus, $f'(v_0') \neq 0$. Moreover, because $F(v) > F(v_0) = F(v_0')$ for all $v \in (\min\{v_0, v_0'\}, \max\{v_0, v_0'\})$ and $F'(v_0') = f(v_0') = 0$, we deduce that $f'(v_0') > 0$. Consequently, we have

$$\Phi(v) \sim \frac{1}{\sqrt{2 f'(v_0')}} \ln \frac{1}{\left|v - v_0'\right|}, \quad \text{as } v \to v_0'. \tag{3.11}$$

Let $v(\xi) := \Phi^{-1}(|\xi|)$ for all $\xi \in \mathbb{R}$, and then similarly with the proof of Lemma 3.3, one can check that $v \in C^2(\mathbb{R})$, $v'(\xi) = -\sqrt{2F(v(\xi)) - 2F(v_0)}$ for all $\xi \geq 0$, and $v''(\xi) = f(v(\xi))$ for all $\xi \in \mathbb{R}$. Let $\varphi(\xi) := v'(\xi)$ for all $\xi \in \mathbb{R}$, and then $(v, \varphi) \in C^1(\mathbb{R})$ is the unique classical solution to (2.21); hence, $(v, \varphi) \in C^\infty(\mathbb{R})$, and all claimed properties are satisfied. Specifically, the exponential decay of $|v(\xi) - v_0'|$ to 0 as $|\xi| \to +\infty$ follows from (3.11).

Remark 3.1. *Under the assumptions of Lemma 3.4, only* $v_0' < v_0$ *is possible, using further the explicit expression of* f *in* (2.21). *Suppose that* $v_0 < v_0'$. *Without compromising generality, we can assume* $0 < v_0 < v_0'$, *then* $f(v_0) > 0$ *and* $f(v_0') = 0$; *because* $F(v) > F(v_0) = F(v_0')$ *for all* $v \in (v_0, v_0')$, *there exists* $\xi_1 \in (v_0, v_0')$ *such that* $f(\xi_1) = F'(\xi_1) = 0$; *moreover, because* $\lim_{v\to 0+} f(v) = -\infty$, *and* $f(v_0) > 0$, *there exists* $\xi_2 \in (0, v_0)$ *such that* $f(\xi_2) = 0$. *Hence,* $g := v^2 f$ *has three different positive roots* $0 < \xi_2 < \xi_1 < v_0'$. *Because* $g(v) = -v^4 - cv^3 + g_1 v^2 - g_2$ *is a polynomial of degree* 4 *with a zero first-order term, we know that* g *has another negative root* $\xi_3 < 0$ *satisfying* $\xi_3(\xi_1 v_0' + \xi_2 v_0' + \xi_1 \xi_2) = -\xi_1 \xi_2 v_0'$. *As a consequence, there holds* $g_2 = \xi_1 \xi_2 \xi_3 v_0' < 0$, *which contradicts with the fact* $g_2 > 0$; *see* (2.7).

Remark 3.2. *Despite Remark 3.1, we emphasize that the validity of Lemmas 3.3 and 3.4 does not depend on the explicit expressions of* F *and* $f = F'$.

As a consequence of Lemmas 3.1, 3.3, and 3.4, we obtain Theorem 2.1.
Proof of Theorem 2.1. We prove Theorem 2.1 by considering the following three cases:

- Suppose that $(c, g_1, g_2) \in \mathcal{P}_0 \cup \mathcal{P}_1$; then, for all $v_0 \in \mathbb{R} \setminus \{0\}$, the unique solution to (2.21) is not globally defined.
- Suppose that $(c, g_1, g_2) \in \mathcal{P}_2 \cup \mathcal{P}_3$; then, f has two simple roots $v_1 < v_2$ with $v_1 v_2 > 0$, and $f(v) \leq 0$ for all $v \in \mathbb{R}$ satisfying $vv_1 < 0$. Hence, F is strictly decreasing on $\{v \in \mathbb{R} : vv_1 > 0, v < v_1\}$ and $\{v \in \mathbb{R} : vv_1 > 0, v > v_2\}$, respectively, and F is strictly increasing on $v \in [v_1, v_2]$. It follows from (2.13) that $F(v_1') = F(v_1)$ for some unique $v_1' \in \{v \in \mathbb{R} : vv_1 > 0, v > v_2\}$. If $v_0 \in \mathcal{Z} := \{v \in \mathbb{R} : f(v) = 0\}$, then the unique solution to (2.21) is constant and thus trivial; if $v_0 \in (v_1, v_2) \cup (v_2, v_1')$, then Lemma 3.3 implies that the unique solution to (2.21) is globally defined and periodic; if $v_0 = v_1'$, then Lemma 3.4 implies that the unique solution to (2.21) is globally defined and unimodal; if $v_0 \in \mathbb{R} \setminus ([v_1, v_2] \cup \mathcal{Z} \cup \{0\})$, then Lemma 3.2 implies that the unique solution to (2.21) is not globally defined.
- Supposing that $(c, g_1, g_2) \in \mathcal{P}_4$, then f has four simple roots $v_1 < v_2 < 0 < v_3 < v_4$. Hence, F is strictly decreasing on $(-\infty, v_1)$, $(v_2, 0)$, $(0, v_3)$ and $(v_4, +\infty)$, respectively, and F is strictly increasing on $[v_1, v_2]$ and $[v_3, v_4]$, respectively. By (2.13), there exist unique $v_1' \in (v_2, 0)$ and $v_3' \in (v_4, +\infty)$ such that $F(v_1') = F(v_1)$ and $F(v_3') = F(v_3)$. If $v_0 \in \{v_1, v_2, v_3, v_4\}$, then the solution to (2.21) is trivial; if $v_0 \in (v_1, v_2) \cup (v_2, v_1') \cup (v_3, v_4) \cup (v_4, v_3')$, then Lemma 3.3 implies that the unique solution to (2.21) is globally defined and periodic; if $v_0 \in \{v_1', v_3'\}$, then Lemma 3.4 implies that the unique solution to (2.21) is globally defined and unimodal; if $v_0 \in \mathbb{R} \setminus ([v_1, v_2] \cup [v_3, v_4] \cup \{0\})$, then Lemma 3.2 implies that the unique solution to (2.21) is not globally defined.

Proof of Theorem 1.1. For fixed $k \in \mathbb{R}$ and fixed traveling speed $\widetilde{c} \in \mathbb{R}$, Theorem 2.1 implies that, for those g_1, g_2 satisfying $(c, g_1, g_2) \in \mathcal{P}_2 \cup \mathcal{P}_3 \cup \mathcal{P}_4$, if v_0 takes values in some intervals, then we obtain periodic traveling wave solutions to (1.1) through (2.7); hence, the three parameters are g_1, g_2, and v_0; if v_0 takes the right end-point of these intervals, then we obtain unimodal traveling wave solutions to (1.1) through (2.7), where the two parameters are thus g_1 and g_2. For these unimodal traveling wave solutions to be solitary waves, they should satisfy $\lim_{\xi \to +\infty} v(\xi) = \lim_{\xi \to -\infty} v(\xi) = -3\widetilde{c}/2$; hence, $v_1 = -3\widetilde{c}/2$ if $(c, g_1, g_2) \in \mathcal{P}_2 \cup \mathcal{P}_3$, and $-3\widetilde{c}/2 \in \{v_1, v_3\}$ if $(c, g_1, g_2) \in \mathcal{P}_4$, so we have $\widetilde{c} \neq 0$. As a consequence, g_1 and g_2 must satisfy $f(-3\widetilde{c}/2) = 0$ and $f'(-3\widetilde{c}/2) = F''(-3\widetilde{c}/2) > 0$, which is equivalent to

$$g_1 = \frac{4g_2}{9\widetilde{c}^2} + \frac{9}{4}\widetilde{c}^2 - \frac{3}{2}c\widetilde{c}, \qquad \frac{16g_2}{27\widetilde{c}^3} < 3\widetilde{c} - c;$$

hence, solitary wave solutions to (1.1) form a one-parameter family, up to translations, where the parameter is g_2 (or equivalently, g_1).

4. An example

Before proceeding to the specific calculations, we elucidate the physical rationale behind the choice of parameters $k = 1$ and $\widetilde{c} = 2$ for this illustrative example. In the ZI system (1.1), the parameter k is a dimensionless measure directly proportional to the constant vorticity of the underlying linear shear current [4, 9]. Selecting $k = 1$ represents a scenario where the vorticity-induced effects are of the same order of magnitude as the nonlinear and dispersive terms in the governing equations. This is not an ad hoc choice but a canonical scaling that captures the essential competition between these physical mechanisms. Such a balance is frequently encountered in geophysical shear flows, including tidal currents in estuaries, wind-driven surface layers, and river outflows, where the velocity profile often exhibits an approximately linear shear [7, 8]. Our general theorems are valid for any $k \in \mathbb{R}$, but the case $k = 1$ serves as a physically representative benchmark. It allows us to present a concrete, non-degenerate instance of our classification while maintaining the full complexity introduced by vorticity.

Consider $k = 1$ in (1.1):

$$\begin{cases} u_t - 4u_x + u_{xxx} + 3uu_x + \rho\rho_x = 0, \\ \rho_t + (u\rho)_x = 0. \end{cases} \tag{4.1}$$

Now, we seek all traveling wave solutions (ρ, u) to (4.1) with traveling speed $\widetilde{c} = 2$. We write

$$\rho(t,x) = \tilde{\rho}(\xi), \quad u(t,x) = U(\xi), \quad \xi = x - 2t; \tag{4.2}$$

then, the profile $(\tilde{\rho}, U)$ satisfies

$$-6U + U'' + \frac{3}{2}U^2 + \frac{1}{2}\tilde{\rho}^2 = \widetilde{g}_1, \quad \tilde{\rho} = \frac{\widetilde{g}_2}{U - 2}, \tag{4.3}$$

where $\widetilde{g}_1 \in \mathbb{R}, \widetilde{g}_2 \in \mathbb{R} \setminus \{0\}$ are constants. Let

$$v = \frac{3}{2}U - 3, \quad g_1 = \widetilde{g}_1 + 6, \quad g_2 = \frac{9}{8}\widetilde{g}_2^2 > 0, \tag{4.4}$$

and then v solves the following second order ODE:

$$v'' = -v^2 + g_1 - \frac{g_2}{v^2} = f(v). \tag{4.5}$$

This is equivalent to the Hamiltonian system (up to symmetry, by Lemma 2.1)

$$\begin{cases} \frac{dv}{d\xi} = \varphi, \\ \frac{d\varphi}{d\xi} = -v^2 + g_1 - \frac{g_2}{v^2} = f(v), \end{cases} \qquad \begin{cases} v(0) = v_0 \neq 0, \\ \varphi(0) = 0. \end{cases} \tag{4.6}$$

Let $g(v) = v^2 f(v) = -v^4 + g_1 v^2 - g_2$ and $F(v) = -v^3/3 + g_1 v + g_2/v$. If $g_1 > 0$, we denote $v_\pm = \pm\sqrt{g_1/2}$; then, $g(v_\pm) = g_1^2/4 - g_2$. Recall the definitions of $\mathcal{P}_j$ for $j \in \{0, 1, 2, 3, 4\}$ (where $c = 4(\widetilde{c} - 2k)/3 = 0$). We let

$$\widetilde{\mathcal{P}}_j := \left\{(g_1, g_2) \in \mathbb{R} \times (0, +\infty) : (0, g_1, g_2) \in \mathcal{P}_j\right\}, \quad j \in \{0, 1, 2, 3, 4\}.$$

Then $\widetilde{\mathcal{P}}_2 = \widetilde{\mathcal{P}}_3 = \emptyset$ and

$$\begin{aligned} \widetilde{\mathcal{P}}_0 &= \{(g_1, g_2) \in \mathbb{R} \times (0, +\infty) : g_1 \leq 0\}, \\ \widetilde{\mathcal{P}}_1 &= \left\{(g_1, g_2) \in \mathbb{R} \times (0, +\infty) : g_1 > 0, g_1^2 - 4g_2 \leq 0\right\}, \\ \widetilde{\mathcal{P}}_4 &= \left\{(g_1, g_2) \in \mathbb{R} \times (0, +\infty) : g_1 > 0, g_1^2 - 4g_2 > 0\right\}. \end{aligned}$$

By Theorem 2.1, if $(g_1, g_2) \in \widetilde{\mathcal{P}}_0 \cup \widetilde{\mathcal{P}}_1$, then for all $v_0 \in \mathbb{R} \setminus \{0\}$, the unique solution (v, φ) to (4.6) is not globally defined.

Now, we assume that $(g_1, g_2) \in \widetilde{\mathcal{P}}_4$, that is, $g_1 > 0$ and $g_1^2 - 4g_2 > 0$. By Theorem 2.1, we have $\{v \in \mathbb{R} : f(v) = 0\} = \{v_1, v_2, v_3, v_4\}$ with $v_1 < v_2 < 0 < v_3 < v_4$. One can compute that

$$v_1 = -v_4 = -\sqrt{\frac{g_1 + \sqrt{g_1^2 - 4g_2}}{2}}, \quad v_2 = -v_3 = \sqrt{\frac{g_1 - \sqrt{g_1^2 - 4g_2}}{2}}. \tag{4.7}$$

Theorem 2.1 also implies that there exist unique $v_1' \in (v_2, 0)$ and $v_3' \in (v_4, +\infty)$ such that $F(v_1') = F(v_1)$ and $F(v_3') = F(v_3)$. Moreover, $F(v) - F(v_1) > 0$ for $v \in (v_1, v_1')$ and $F(v) - F(v_3) > 0$ for $v \in (v_3, v_3')$. For further usage, we compute v_1'. Using the expression of F, we obtain

$$F(v) - F(v_1) = -\frac{1}{3}(v^3 - v_1^3) + g_1(v - v_1) + g_2\left(\frac{1}{v} - \frac{1}{v_1}\right) = \frac{v - v_1}{v_1 v}F_1(v),$$

where

$$F_1(v) := -\frac{1}{3}(v_1 v^3 + v_1^2 v^2 + v_1^3 v) + g_1 v_1 v - g_2$$

is a polynomial of degree 3. Note that $F_1(v_1) = v_1^2 f(v_1) = 0$; hence, $F_1(v) = (v - v_1)F_2(v)$ for some polynomial F_2 of degree 2. A direct computation gives that

$$F_2(v) = -\frac{1}{3}v_1 v^2 - \frac{2}{3}v_1^2 v - v_1^3 + g_1 v_1.$$

Because $-v_1/3 > 0$ and

$$\begin{aligned} F_2(0) = -v_1^3 + g_1 v_1 = v_1\left(-v_1^2 + g_1\right) = v_1\left(g_1 - \frac{g_1 + \sqrt{g_1^2 - 4g_2}}{2}\right) \\ = \frac{1}{2}v_1\left(g_1 - \sqrt{g_1^2 - 4g_2}\right) < 0, \end{aligned}$$

where we have also used (4.7), we know that F_2 has a unique negative zero, which is exactly v_1', thanks to $F(v_1') = F(v_1)$ and $v_1' \in (v_2, 0)$. Now, a direct computation gives that

$$v_1' = \frac{v_1^2 - \frac{\sqrt{2}}{2}\sqrt{g_1^2 + 4g_2 + g_1\sqrt{g_1^2 - 4g_2}}}{-v_1}. \tag{4.8}$$

Similarly, one can also compute v_3'.

Finally, we classify all traveling wave solutions to (4.1) as follows (assume $g_1 > 0$ and $g_1^2 - 4g_2 > 0$):

- If $v_0 \in \{v_1, v_2, v_3, v_4\}$, then $(v, \varphi) \equiv (v_0, 0)$; hence,

$$U(\xi) = 2 + \frac{2}{3}v_0, \quad \tilde{\rho}(\xi) = \frac{3}{2}\frac{\widetilde{g}_2}{v(\xi)} = \pm\frac{3}{2}\cdot\frac{2\sqrt{2}}{3}\frac{\sqrt{g_2}}{v_0} = \frac{\sqrt{2g_2}}{|v_0|}$$

  is a constant solution to (4.2). Here, we take $\rho \geq 0$ (i.e. take $|v_0|$) for simplicity. If $\rho < 0$, the analysis is similar. According to [9, formula (17)], taking $\rho \geq 0$ is also physically meaningful.

- If $v_0 \in (v_1, v_2) \cup (v_2, v_1') \cup (v_3, v_4) \cup (v_4, v_3')$, then the unique solution (v, φ) to (4.6) is periodic; hence,

$$U(\xi) = 2 + \frac{2}{3}v(\xi), \quad \tilde{\rho}(\xi) = \frac{\sqrt{2g_2}}{\mathrm{sgn}(v_0)v(\xi)}$$

is a periodic solution to (4.2), which gives a three-parameter family of periodic solutions to (4.1), with the parameters (g_1, g_2, v_0) belonging to

$$\left\{ (g_1, g_2, v_0) \in \mathbb{R}_+ \times \mathbb{R}_+ \times (\mathbb{R} \setminus \{0\}) : g_1^2 - 4g_2 > 0, \ \ v_0 \in (v_1, v_2) \cup (v_2, v_1') \cup (v_3, v_4) \cup (v_4, v_3') \right\}.$$

Moreover, these solutions are all nontrivial periodic traveling wave solutions to (4.1), up to translations.

- If $v_0 \in \{v_1', v_3'\}$, then the unique solution (v, φ) to (4.6) is unimodal; hence,

$$U(\xi) = 2 + \frac{2}{3}v(\xi), \quad \tilde{\rho}(\xi) = \frac{\sqrt{2g_2}}{\mathrm{sgn}(v_0)v(\xi)}$$

is a unimodal solution to (4.2), which gives two two-parameter families of unimodal traveling wave solutions to (4.1) (one given by $v_0 = v_1'$ and another by $v_0 = v_3'$), with the parameters (g_1, g_2) belonging to

$$\{(g_1, g_2) \in \mathbb{R}_+ \times \mathbb{R}_+ : g_1^2 - 4g_2 > 0\}.$$

Moreover, these solutions are all nontrivial unimodal traveling wave solutions to (4.1), up to translations.

- In particular, if $v_1 = v_1'$, and $v_1 = -3$, then $U(\xi) = 2 + 2v(\xi)/3 \to 0$ as $|\xi| \to +\infty$; hence, the unimodal solution is meanwhile a solitary wave solution. One checks that

$$v_1 = -3 \Leftrightarrow \sqrt{\frac{g_1 + \sqrt{g_1^2 - 4g_2}}{2}} = 3 \Leftrightarrow g_2 = 9g_1 - 81 \ (9 < g_1 < 18).$$

This gives a one-parameter family of nontrivial solitary wave solutions to (4.1), with the traveling profile given by

$$U(\xi) = 2 + \frac{2}{3}v(\xi), \quad \tilde{\rho}(\xi) = -\frac{3\sqrt{2(g_1 - 9)}}{v(\xi)},$$

where $g_1 \in (9, 18)$ is the parameter, and v solves the ODE

$$v'' = -v^2 + g_1 - \frac{9g_1 - 81}{v^2}, \quad v(0) = v_1', \quad v'(0) = 0.$$

Here, according to (4.8),

$$v_1' = \frac{9 - \frac{\sqrt{2}}{2}\sqrt{g_1^2 + 4(9g_1 - 81) + g_1\sqrt{(18 - g_1)^2}}}{3} = 3 - \sqrt{3(g_1 - 6)}.$$

Moreover, these solutions are all solitary wave solutions to (4.1), up to translations.

5. Discussions

5.1. A comparison of "trivial" solutions and "nontrivial" solutions

Recall that the ZI system reads

$$u_t - 4ku_x + u_{xxx} + 3uu_x + \rho\rho_x = 0, \qquad \rho_t + (u\rho)_x = 0. \tag{5.1}$$

If $\rho \equiv 0$, and $k = 0$, then the ZI system reduces to the classical KdV equation. Consequently, if $k = 0$, then (5.1) has a trivial class of traveling solutions:

- constant solutions: $(u, \rho) = (\text{constant}, 0)$;
- unimodal solutions:

$$u(t, x) = \frac{c - A}{3} + A\mathrm{sech}^2\left(\frac{\sqrt{A}}{2}(x - ct - x_0)\right), \quad \rho \equiv 0, \quad c \in \mathbb{R}, \ A > 0, \ x_0 \in \mathbb{R},$$

  where taking $c = A > 0$ gives the solitary wave solutions

$$u(t, x) = c\mathrm{sech}^2\left(\frac{\sqrt{c}}{2}(x - ct - x_0)\right), \quad \rho \equiv 0; \tag{5.2}$$

- periodic solutions:

$$u(t, x) = r_2 + (r_3 - r_2)\,\mathrm{cn}^2\left(\frac{\sqrt{r_3 - r_1}}{2}(x - ct - x_0), \frac{r_3 - r_2}{r_3 - r_1}\right), \quad \rho \equiv 0,$$

  where $r_1 < r_2 < r_3$, $c = r_1 + r_2 + r_3$, and $x_0 \in \mathbb{R}$. Here, cn is the Jacobi elliptic function.

We notice that, when $k = 0$, the above class of (trivial) traveling wave solutions has the following properties:

- For fixed wave speed $c \in \mathbb{R}$, up to translation (i.e., $x_0 \in \mathbb{R}$), periodic traveling wave solutions form a two-parameter family (taking r_1, r_2 as the parameters).
- For fixed wave speed $c \in \mathbb{R}$, up to translation (i.e., $x_0 \in \mathbb{R}$), unimodal traveling wave solutions form a one-parameter family (taking A as the parameter).
- For fixed positive wave speed $c > 0$, up to translation (i.e., $x_0 \in \mathbb{R}$), there is a unique solitary wave solution.

This clearly differs from the result in Theorem 1.1, where "nontrivial" traveling wave solutions are considered. The differences are mainly the following two aspects:

- For each type of traveling wave solutions, the "nontrivial" ones form a class with one more parameter than "trivial" ones, and the reason is that $\rho \neq 0$ introduces exactly one extra parameter.
- There exist "nontrivial" solitary waves with negative wave speed, but the wave speeds of all "trivial" solitary wave solutions are positive.

Unlike the classical KdV equation, which supports only right-going solitary waves due to its unidirectional balance between nonlinearity and dispersion, the ZI system admits solitary waves propagating in both directions. This bidirectional capability stems from the coupled dynamics of the horizontal velocity u and the free-surface elevation ρ. The coupling term $\rho\rho_x$ introduces an additional nonlinear mechanism that allows the wave profile to sustain propagation in either direction, independent of the sign of the wave speed. Moreover, the background vorticity parameter k further breaks the directional symmetry by adding a linear advective effect that can either oppose or reinforce the wave motion.

Physically, this reflects the richer wave-current interactions present in rotational shallow-water flows, such as tidal currents or wind-driven coastal streams, where waves can travel upstream or downstream depending on the interplay between the shear flow and the surface displacement. The coexistence of left- and right-going solitary waves in the ZI model thus provides a more realistic framework for describing coherent structures in environments where both vorticity and surface coupling play essential roles.

5.2. Physical interpretation and model implications

This short subsection elucidates the physical significance and broader modeling implications of our results. We reconnect the abstract solution types–periodic and solitary waves–to observable phenomena in geophysical shear flows, examine their characteristics through the lens of energy conservation inherent in the Hamiltonian framework, and discuss the consequences of our findings for the modeling of wave-current interactions. This synthesis aims to bridge the gap between the mathematical structure of the Zakharov-Ito equation and its practical relevance in fluid dynamics.

- Observable wave profiles: Our classification identifies two fundamental, observable wave types in rotational shallow water: periodic wave trains and solitary waves (including both elevation and depression pulses). These correspond to the frequently observed "cnoidal-like" wave trains and isolated "solitary wave" pulses in coastal zones with currents. The bidirectional propagation capability of ZI solitary waves is particularly relevant for environments like tidal inlets or river mouths [50, 51], where waves can be observed moving both with and against the prevailing current.
- Energy considerations: The Hamiltonian structure directly encodes the system's conserved energy [15, 52]. Our phase-plane analysis shows that bounded orbits (periodic and homoclinic) exist only for energy levels corresponding to local minima of the potential $F(v)$. This provides a selection mechanism: physically persistent waves occupy specific energy bands. The non-existence of kinks and compactons can be interpreted energetically: The ZI Hamiltonian lacks the double-well potential structure needed for kinks and does not permit the finite-energy, compact-support balance required for compactons, ensuring all localized disturbances have smooth, radiative tails.
- Implications for wave-current interaction models: Our work clarifies the legitimate wave structures within the ZI model, a key asymptotic model for waves on linear shear [36,53,54]. This serves as a benchmark for more complex or nonintegrable models. The finding that only smooth, delocalized structures persist underscores that simplified models aiming to capture vorticity effects must preserve the Hamiltonian structure to avoid spurious solutions. Furthermore, the parameter-dependent bifurcation between periodic and solitary waves (dictated by $k, \tilde{c}, \tilde{g}_1, \tilde{g}_2$) provides a predictive framework for how changes in background current (vorticity) or wave conditions can trigger a transition in the dominant wave regime observed in shear flows.

Our result that the ZI equation does not admit smooth, kink-type traveling waves reflects a fundamental physical constraint inherent in rotational shallow-water shear flows. Kinks require a heteroclinic orbit in the phase portrait, connecting two distinct equilibrium states–a topological structure that is absent from the Hamiltonian formulation derived directly from the original ZI system. This absence indicates that, within the conservative, vorticity-modified dynamics described by the ZI model, there exists no smooth, steady front that asymptotically connects two different uniform flow states. Physically, this means that in a shallow-water environment with constant background shear, wave-current interactions favor either periodic wave trains or localized solitary pulses but do not support permanent monotonic transitions between two distinct mean flows. The exclusion of kinks underscores the specificity of the balance between nonlinearity, dispersion, vorticity, and surface coupling in this system and clarifies that previously reported kink-like solutions arise only under non-conservative coordinate transformations that alter the original physical Hamiltonian structure.

The absence of compacton solutions in the original ZI system is equally revealing of its underlying physics. Compactons–solitary waves with compact support–arise mathematically through coordinate transformations that regularize singularities, but they do not survive as smooth traveling waves in the untransformed Hamiltonian framework. Physically, this implies that in a shallow-water shear flow governed by the ZI model, localized disturbances cannot maintain both finite energy and exactly sharp, nonanalytic edges while propagating unchanged. Any physically admissible localized wave must exhibit smooth, exponentially decaying tails, as captured by the homoclinic orbits in our phase portraits. The exclusion of compactons reinforces that the ZI dynamics, shaped by vorticity, nonlinearity, dispersion, and conservation laws, does not permit wave profiles with finite-length support; instead, localized structures inevitably interact with the background through infinitely extended, albeit rapidly decaying, oscillatory tails. This distinction highlights how faithful adherence to the original Hamiltonian structure filters out mathematically convenient but physically inadmissible waveform types.

6. Conclusions

In this paper, our work builds upon previous research and provides a comprehensive classification of the nontrivial traveling wave solutions for system (1.1), elucidating their parametric dependence and dynamical behaviors. We established that for any traveling speed $\widetilde{c} \in \mathbb{R}$, the ZI equation admits infinitely many nontrivial periodic traveling wave solutions, forming a three-parameter family, and infinitely many nontrivial unimodal traveling wave solutions, forming a two-parameter family. Notably, for $\widetilde{c} \neq 0$, there exists a one-parameter family of solitary wave solutions. Through the study of the corresponding Hamiltonian system and phase portraits, we identified the conditions under which periodic and unimodal solutions arise. The analysis revealed that the solutions' existence and properties are governed by the roots of the associated polynomial $f(v)$. Our approach diverged from prior methodologies by focusing on a more basic perspective, which allowed for a more detailed and rigorous analysis of the solution space. This included the introduction of renormalized variables and a thorough examination of the Hamiltonian structure. This study not only establishes complete existence theorems for traveling wave solutions of the ZI equation, but also develops Hamiltonian analysis methods applicable to traveling wave research in other integrable systems. These results lay a solid foundation for our future work focusing on the stability analysis of these traveling wave solutions.

Upon the construction of traveling wave solutions, a natural subsequent inquiry is to examine their stability. The orbital stability of solitary wave solutions for the KdV equation was established in [20]. Inspired by these results, we recently proved the orbital stability of the "trivial" solitary wave solution (5.2) for the ZI system. However, the orbital stability of the "nontrivial" solitary wave solutions constructed in the present work remains an open problem. In a broader context, the question of asymptotic stability is still widely open. For recent advances concerning the asymptotic stability of solitary waves or kinks in various dispersive PDEs, refer to [21] and the references therein. The techniques developed in these works may potentially be adaptable to prove the asymptotic stability of the "nontrivial" solitary wave solutions for the ZI system.

Author contributions

F. Wu: Writing–original draft, Software, Methodology, Investigation, Conceptualization, Validation; J. Zhuang: Software, Validation; Y. Liu: Visualization, Supervision, Writing–review & editing, Resources, Methodology. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no known competing financial interests.

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