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Research article

Representations of quantum superalgebra $U_{r,s}(\mathrm{osp}(1,2))$ at the root of unity and its restrictions

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Abstract: In this paper, the simple modules for the quantum superalgebra $U_{r,s}(\mathrm{osp}(1,2))$ with two parameters at the root of unity (i.e., $q = (rs^{-1})^{1/2}$ is a root of unity) are completely determined up to isomorphism. Furthermore, the classification of finite-dimensional simple modules over the restricted quantum superalgebra $\overline{U}_{r,s}(\mathrm{osp}(1,2))$ is given.

Keywords: two-parameter quantum group; quantum superalgebra; simple module
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1. Introduction

Quantum superalgebras, as quantum deformations of the universal enveloping algebras of Lie superalgebras, represent one of the most significant generalizations of Drinfeld–Jimbo quantum groups. Tolstoy [1] provided explicit constructions of quantum superalgebras associated with Kac–Moody superalgebras possessing symmetrizable Cartan matrices. Since then, their algebraic structures and representations have been investigated in various contexts; see, for instance, [2–6].

Among these, the quantum superalgebra $U_q(\mathrm{osp}(1,2n))$ was introduced in [7]. It is closely related to the quantum algebra $U_q(\mathrm{so}(2n+1))$. In the case when $n = 1$, the algebra $U_q(\mathrm{osp}(1,2))$ has been extensively studied in [8–11], where finite-dimensional representations, the center, and the Harish–Chandra homomorphism were determined. More recently, Sun, Wang, and Yang [12] constructed a quantum algebra $U_q(\mathrm{osp}(1,2,\mathrm{c}))$, established the Harish–Chandra homomorphism when q is not a root of unity, and classified all simple modules up to isomorphism for an arbitrary parameter q.

The theory of quantum groups has been generalized to the two-parameter and multi-parameter settings. Benkart and Witherspoon [13] initiated a systematic study of two-parameter quantum enveloping algebras for the general and special linear Lie algebras, further analyzed finite-dimensional representations in [14], and proved complete reducibility of weight modules when

rs^{-1} is not a root of unity. Subsequently, the definitions of two-parameter quantum groups of types B, C, and D were introduced in [15], where the Drinfel'd double was also established. The cases for exceptional types E and G_2 were treated in [16, 17], respectively.

Restricted two-parameter quantum groups $\mathfrak{u}_{r,s}(\mathfrak{g})$, where $\mathfrak{g}$ is a finite-dimensional semisimple Lie algebra at roots of unity, have attracted considerable attention. Benkart and Witherspoon [18] introduced the restricted quantum group $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$ of type A, and analogous constructions for other finite types were developed in [19–23]. In [24], Hu and Xu classified and enumerated the isomorphism classes of simple modules over $\mathfrak{u}_{r,s}(\mathfrak{g})$, assuming r and s are primitive roots of unity of orders d and d', respectively.

The two-parameter quantum superalgebra $U_{r,s}(\mathrm{osp}(1,2))$ was introduced in [25]. It is a generalization of the one-parameter quantum superalgebra $U_q(\mathrm{osp}(1,2))$, where the parameter q is replaced by two parameters r and s, with $q = (rs^{-1})^{1/2}$. In [26], various algebraic relations involving the Casimir operator, the Scasimir operator, and powers of generators were studied. However, the finite-dimensional representation theory of $U_{r,s}(\mathrm{osp}(1,2))$, particularly in the case at the root of unity, has not been well developed. This is mainly due to the more complicated algebraic structure caused by the two parameters. This gap led us to investigate the finite-dimensional simple modules of $U_{r,s}(\mathrm{osp}(1,2))$ and its restricted form $\overline{U}_{r,s}(\mathrm{osp}(1,2))$ when $q = (rs^{-1})^{1/2}$ is a root of unity. Our results extend the known classification theorems from the one-parameter case to the two-parameter setting and complete the finite-dimensional representation theory of $U_{r,s}(\mathrm{osp}(1,2))$ at roots of unity.

This paper is organized as follows. In Section 2, we recall the definition and some basic properties of the two-parameter quantum superalgebra $U_{r,s}(\mathrm{osp}(1,2))$. In Section 3, by analyzing the Scasimir operator and the relations among generators, we classify all finite-dimensional simple modules of $U_{r,s}(\mathrm{osp}(1,2))$ up to isomorphism under the assumption that $q = (rs^{-1})^{1/2}$ is a primitive l-th root of unity. In Section 4, we define the restricted quantum superalgebra $\overline{U}_{r,s}(\mathrm{osp}(1,2))$ and provide a simplified classification of its finite-dimensional simple modules.

2. Preliminaries

First, we introduce some notations.

Let m be an integer and let $\upsilon \in \mathbb{K}$, where $\mathbb{K}$ is a field and $\nu \neq 0, \pm 1$. We define the quantum integer $[m]_\upsilon$ as

$$[m]_\upsilon = \frac{\upsilon^m - \upsilon^{-m}}{\upsilon - \upsilon^{-1}} = \upsilon^{m-1} + \upsilon^{m-3} + \cdots + \upsilon^{-m+3} + \upsilon^{-m+1}.$$

Let r, s be generic with $r^2 \neq s^2$ and $q = (rs^{-1})^{\frac{1}{2}}$, $q' = -q$, and $\eta = (r^{\frac{1}{2}} + s^{\frac{1}{2}})(r - s)$.

In the following, we always assume that q is a primitive l-th-root of unity ($l > 2$), and

$$L = \begin{cases} l, & \text{if } l \text{ is even;} \\ 2l, & \text{if } l \text{ is odd;} \end{cases} \qquad l' = \begin{cases} \frac{l}{2}, & \text{if } l \text{ is twice an odd integer;} \\ L, & \text{otherwise.} \end{cases} \tag{2.1}$$

It is easy to see that L is always even, $q^L = 1$, and $[L]_q = 0$. Also, $[l']_q = 0$ and $-q$ is the l'-th root of unity.

The two-parameter quantum superalgebra $U_{r,s}(\mathrm{osp}(1,2))$ is the unital associative algebra generated by $E, F, \omega^{\pm 1}, \omega'^{\pm 1}$ and subject to the following relations [26]:

$$\begin{aligned}
&\omega\omega' = \omega'\omega, \quad \omega\omega^{-1} = \omega^{-1}\omega = 1 = \omega'\omega'^{-1} = \omega'^{-1}\omega', \\
&\omega E\omega^{-1} = qE, \quad \omega' E\omega'^{-1} = q^{-1}E, \\
&\omega F\omega^{-1} = q^{-1}F, \quad \omega' F\omega'^{-1} = qF, \text{ and} \\
&EF + FE = \frac{\omega - \omega'}{r - s}.
\end{aligned} \tag{2.2}$$

Note that the quantum superalgebra $U_q(\mathrm{osp}(1, 2, \mathrm{c}))$ in [12] is a special case of $U_{r,s}(\mathrm{osp}(1, 2))$, where $\omega' = \omega^{-1}\mathbf{c}^r$ and $\mathbf{c}$ is an invertible element that commutes with $\omega, \omega^{-1}, E, F$.

The Scasimir operator $S_{r,s}$ is

$$S_{r,s} = r^{\frac{1}{2}}\omega - s^{\frac{1}{2}}\omega' - \eta FE,$$

where $\eta = (r^{\frac{1}{2}} + s^{\frac{1}{2}})(r - s)$. It is straightforward to check that

$$S_{r,s}F = -FS_{r,s}, \quad S_{r,s}E = -ES_{r,s}, \quad S_{r,s}\omega = \omega S_{r,s}, \text{ and } S_{r,s}\omega' = \omega' S_{r,s}.$$

Set $t = \sqrt{q}\mathbf{i}$ and

$$[\omega; \omega'; m]_t = \frac{t^m\omega - t^{-m}\omega'}{r - s}.$$

For $m, n \in \mathbb{N}$, it is straightforward to see that

$$EF^m = (-1)^m F^m E + (-1)^{m-1} F^{m-1} \left(\frac{(q')^{-m} - 1}{(q')^{-1} - 1}\omega - \frac{(q')^m - 1}{q' - 1}\omega' \right) (r - s)^{-1}$$

and

$$FE^m = (-1)^m E^m F + (-1)^{m-1} E^{m-1} \left(\frac{(q')^{m} - 1}{(q')^{-1} - 1}\omega - \frac{(q')^{-m} - 1}{q' - 1}\omega' \right) (r - s)^{-1},$$

which can be written as

$$EF^m = (-1)^m F^m E + (-1)^{m-1}[m]_t F^{m-1}[\omega; \omega'; 1 - m]_t \tag{2.3}$$

and

$$FE^m = (-1)^m E^m F + (-1)^{m-1}[m]_t E^{m-1}[\omega; \omega'; m - 1]_t. \tag{2.4}$$

Let us review the structure of the center of $U_{r,s}(\mathrm{osp}(1, 2))$.

Theorem 2.1. *[26] When rs^{-1} is a root of unity, the center of $U_{r,s}(\mathrm{osp}(1, 2))$ contains*

$$\mathbb{C}[\omega\omega', S_{r,s}^2, \omega^{\pm 1}, \omega'^{\pm 1}, F^L] + \mathbb{C}[\omega\omega', S_{r,s}^2, \omega^{\pm 1}, \omega'^{\pm 1}, E^L].$$

(1) If l is odd, this is the center of the algebra.
(2) If l is even, the center is the free module on

$$\mathbb{C}[\omega\omega', S_{r,s}^2, \omega^{\pm 1}, \omega'^{\pm 1}, F^L] + \mathbb{C}[\omega\omega', S_{r,s}^2, \omega^{\pm 1}, \omega'^{\pm 1}, E^L]$$

with basis 1, $\omega^{\frac{1}{2}}S_{r,s}$, $\omega'^{\frac{1}{2}}S_{r,s}$.

3. Representations of the quantum superalgebra

Recall that $q = (rs^{-1})^{\frac{1}{2}}$ is an l-primitive root of unity, $q' = -q$, and

$$S_{r,s} = r^{\frac{1}{2}}\omega - s^{\frac{1}{2}}\omega' - \eta FE,$$

where $\eta = (r^{\frac{1}{2}} + s^{\frac{1}{2}})(r - s)$.

Lemma 3.1. *Keeping the notations above, there is no finite-dimensional simple $U_{r,s}(\mathrm{osp}(1,2))$-module of dimension greater than L.*

Proof. Let V be a finite-dimensional $U_{r,s}(\mathrm{osp}(1,2))$-module with $\dim V > L$. We claim that there exists a nonzero submodule of V whose dimension is at most L.

To see the claim, we consider two cases in the following.

If l is even, then the elements $S_{r,s}^2$, $\omega^{\frac{l}{2}}S_{r,s}$, $\omega'^{\frac{l}{2}}S_{r,s}$, E^L, and F^L belong to the center of $U_{r,s}(\mathrm{osp}(1,2))$. Therefore, the operators ω, ω', $\omega^{\frac{l}{2}}S_{r,s}$, $\omega'^{\frac{l}{2}}S_{r,s}$, $S_{r,s}^2$, E^L, and F^L mutually commute.

If l is odd, then the elements $S_{r,s}^2$, E^L, and F^L are central in $U_{r,s}(\mathrm{osp}(1,2))$, and again ω, ω', $S_{r,s}^2$, E^L, and F^L mutually commute.

In both cases, there exists a common nonzero eigenvector $v \in V$ such that

$$\omega v = \lambda v, \quad \omega' v = \lambda' v, \quad E^L v = av, \quad S_{r,s}v = bv, \text{ and } F^L v = \alpha v$$

for some scalars $\lambda, \lambda', a, b, \alpha \in \mathbb{K}$, where b is determined by λ, λ', a, and α.

Case 1. Suppose $F^L v = 0$, then there exists an integer $0 \leq m < L$ such that $F^m v \neq 0$ but $F^{m+1}v = 0$. Replacing v by $F^m v$, we may assume that $Fv = 0$ without loss of generality. Now we consider the subspace

$$V' = \mathrm{span}\{v, Ev, \ldots, E^{L-1}v\}.$$

We claim that V' is a $U_{r,s}(\mathrm{osp}(1,2))$-submodule of V.

To verify this, it suffices to check that V' is invariant under the actions of the generators E, F, ω, ω'. The invariance under ω, ω' and E is clear: Since the operators ω and ω' act as scalars on v, they preserve each $E^k v$, and hence V'; if $m < L-1$, then $E(E^m v) = E^{m+1}v \in V'$ and $E(E^{L-1}v) = E^L v = av \in V'$. Now, it remains to show that V' is invariant under F. Recall the relation (2.4). Using $Fv = 0$, we compute for $m > 0$

$$\begin{aligned} FE^m v &= (-1)^m E^m F v + (-1)^{m-1}[m]_t E^{m-1}[\omega; \omega'; m-1]_t v \\ &= (-1)^{m-1}[m]_t \left(\frac{t^{m-1}\lambda - t^{1-m}\lambda'}{r - s} \right) E^{m-1} v \in V'. \end{aligned}$$

Thus, F preserves V', and V' is indeed a submodule.

Case 2. Suppose $F^L v = \alpha v \neq 0$ for some $\alpha \neq 0$. Define the subspace

$$V' = \mathrm{span}\{v, Fv, \ldots, F^{L-1}v\}.$$

We claim that V' is a submodule. Again, it suffices to check invariance under E, F, ω, ω'. Indeed, it is obvious that V' is stable under the action of ω, ω' and F. To see that V' is stable under the action of E, we recall that

$$S_{r,s} = r^{\frac{1}{2}}\omega - s^{\frac{1}{2}}\omega' - \eta FE,$$

where $\eta = (r^{\frac{1}{2}} + s^{\frac{1}{2}})(r-s)$. Note that $S_{r,s}F^{L-1}v = (-1)^{L-1}F^{L-1}S_{r,s}v = (-1)^{L-1}bF^{L-1}v$, and we get that

$$\begin{aligned} Ev &= \alpha^{-1}EF^{L}v = \alpha^{-1}(EF)F^{L-1}v \\ &= \alpha^{-1}\left(\frac{\omega-\omega'}{r-s} - \frac{1}{\eta}\left(r^{\frac{1}{2}}\omega - s^{\frac{1}{2}}\omega' - S_{r,s}\right)\right)F^{L-1}v \\ &= \left(\alpha^{-1}\gamma\right)F^{L-1}v, \end{aligned}$$

where

$$\begin{aligned} \gamma &= \frac{q^{1-L}\lambda - q^{L-1}\lambda'}{r-s} - \frac{1}{\eta}\left(r^{\frac{1}{2}}q^{1-L}\lambda - s^{\frac{1}{2}}q^{L-1}\lambda' + (-1)^{L}b\right) \\ &= \frac{1}{\eta}\left(s^{\frac{1}{2}}q^{1-L}\lambda - r^{\frac{1}{2}}q^{L-1}\lambda' - b\right) \\ &= \frac{1}{\eta}\left(r^{\frac{1}{2}}\lambda - s^{\frac{1}{2}}\lambda' - b\right). \end{aligned} \tag{3.1}$$

That is,

$$b = r^{\frac{1}{2}}\lambda - s^{\frac{1}{2}}\lambda' - \eta\alpha\beta,$$

where $\beta = \alpha^{-1}\gamma$. For any $0 \le m < L-1$, we have

$$\begin{aligned} EF^{m+1}v &= (-1)^{m}[m+1]_{t}F^{m}[\omega;\omega';-m]_{t}v + (-1)^{m+1}F^{m+1}Ev \\ &= (-1)^{m}\left([m+1]_{t}\frac{t^{-m}\lambda - t^{m}\lambda' m}{r-s} - \gamma\right)F^{m}v. \end{aligned}$$

This implies that V' is stable under the action of E.

This completes the proof. □

Recall that q is a primitive l-th root of unity with $l > 2$.

(1) Let $V_{\lambda,\lambda',a}$ be a vector space with basis $v_0, \ldots, v_{L-1}$, with the convention that $v_L = v_0$. We define the action of $U_{r,s}(\text{osp}(1,2))$ on $V_{\lambda,\lambda',a}$ by

$$\begin{aligned} &\omega v_m = q^{m}\lambda v_m, \quad \omega' v_m = q^{-m}\lambda' v_m, \\ &\omega v_{L-1} = q^{L-1}\lambda v_{L-1}, \quad \omega' v_{L-1} = q^{1-L}\lambda' v_{L-1}, \\ &Ev_m = v_{m+1}, \quad Ev_{L-1} = av_0, \\ &Fv_{m+1} = (-1)^{m}[m+1]_{t}\left(\frac{t^{m}\lambda - t^{-m}\lambda'}{r-s}\right)v_m, \quad \text{and} \quad Fv_0 = 0, \end{aligned}$$

where $0 \le m < L-1$.

We claim that $V_{\lambda,\lambda',a}$ is an L-dimensional $U_{r,s}(\text{osp}(1,2))$-module. Indeed, we observe that the basis of $V_{\lambda,\lambda',a}$ can be written as $v_0, Ev_0, \ldots, E^{L-1}v_0$. Then the module structure follows directly from Lemma 3.1 (*Case*1). If l is odd and $\lambda' = q^{-1}\lambda$, $V_{\lambda,\lambda',a}$ has a proper submodule $\{v_0 + v_l, v_1 + v_{l+1}, \ldots, v_{l-1} + v_{L-1}\}$ of dimension l. Otherwise, $V_{\lambda,\lambda',a}$ of dimension L is simple for $a \neq 0$ if l is even or $\lambda' \neq q^{-1}\lambda$.

(2) Let $V_{\lambda,\lambda',\alpha,\beta}$ be an L-dimensional vector space with the basis $\{v_0, v_1, \ldots, v_{L-1}\}$. The action of $U_{r,s}(\text{osp}(1,2))$ on $V_{\lambda,\lambda',\alpha,\beta}$ is given by

$$\omega v_m = q^{-m}\lambda v_m, \quad \omega' v_m = q^{m}\lambda' v_m,$$

$$\omega v_{L-1} = q^{1-L}\lambda v_{L-1}, \quad \omega' v_{L-1} = q^{L-1}\lambda' v_{L-1},$$
$$Ev_{m+1} = (-1)^m \left([m+1]_t \frac{t^{-m}\lambda - t^m\lambda'}{r-s} - \alpha\beta \right) v_m, \quad Ev_0 = \beta v_{L-1},$$
$$Fv_m = v_{m+1}, \quad \text{and} \quad Fv_{L-1} = \alpha v_0$$

for $0 \le m < L-1$.

We claim that $V_{\lambda,\lambda',\alpha,\beta}$ is an L-dimensional $U_{r,s}(\mathrm{osp}(1,2))$-module. Indeed, we observe that the basis of $V_{\lambda,\lambda',\alpha,\beta}$ can be written as $v_0, Fv_0, \ldots, F^{L-1}v_0$. Then the module structure follows directly from Lemma 3.1 (Case 2). If l is odd and $\alpha\beta = \dfrac{q\lambda - \lambda'}{(q+1)(r-s)}$, $V_{\lambda,\lambda',\alpha,\beta}$ has a proper submodule $\{v_0 + v_l, v_1 + v_{l+1}, \ldots, v_{l-1} + v_{L-1}\}$ of dimension l. Otherwise, $V_{\lambda,\lambda',\alpha,\beta}$ of dimension L is simple for $\alpha \neq 0$ if l is even or $\alpha\beta \neq \dfrac{q\lambda - \lambda'}{(q+1)(r-s)}$.

(3) Let $V_{\lambda,\lambda',n}$ be an n-dimensional vector space with basis $\{v_0, v_1, \ldots, v_{n-1}\}$. The action of the quantum superalgebra $U_{r,s}(\mathrm{osp}(1,2))$ on $V_{\lambda,\lambda',n}$ is defined as follows:

$$\begin{aligned} &\omega v_m = q^{-m}\lambda v_m, \quad \omega' v_m = q^m \lambda' v_m, \\ &\omega v_{n-1} = q^{-n+1}\lambda v_{n-1}, \quad \omega' v_{n-1} = q^{n-1}\lambda' v_{n-1}, \\ &Ev_{m+1} = (-1)^m \left([m+1]_t \frac{t^{-m}\lambda - t^m\lambda'}{r-s} \right) v_m, \quad Ev_0 = 0, \\ &Fv_m = v_{m+1}, \quad \text{and} \quad Fv_{n-1} = 0 \end{aligned} \tag{3.2}$$

for $0 \le m < n-1$, where $n = l'$ and $\lambda' \neq (q')^{i+1}\lambda$ for all $1 \le i \le l'-1$ or $\lambda' = (q')^{-n+1}\lambda$ for $1 \le n \le l'-1$. We claim that $V_{\lambda,\lambda',n}$ is an n-dimensional $U_{r,s}(\mathrm{osp}(1,2))$-module. Indeed, we set

$$\omega v_0 = \lambda v_0, \quad \omega' v_0 = \lambda' v_0, \quad S_{r,s} v_0 = b v_0, \quad Ev_0 = 0, \quad F^n v_0 = 0, \text{ and } F^m v_0 = v_m \quad (0 \le m < n).$$

So, we can establish the module structure as stated in Eq (3.2). Note that if $n - 1 \ge l'$, this module is not simple since $v_{l'}, \ldots, v_{n-1}$ span a proper submodule.

Now we determine when $V_{\lambda,\lambda',n}$ is simple for $n \le l'$. First, since

$$EFv_{n-1} + FEv_{n-1} = \frac{\omega - \omega'}{r-s} v_{n-1},$$

we get that

$$(-1)^{n-1}[n]_t \left(\frac{t^{-n+1}\lambda - t^{n-1}\lambda'}{r-s} \right) = 0, \tag{3.3}$$

which implies either $t^{2n} = 1$ or $\lambda' = t^{-2n+2}\lambda$. Therefore,

$$n = l' \quad \text{or} \quad \lambda' = (q')^{-n+1}\lambda. \tag{3.4}$$

Moreover, the vector v_{n-1} is an eigenvector of ω, ω' and $S_{r,s}$, and $Fv_{n-1} = 0$. According to [26], the algebra $U_{r,s}(\mathrm{osp}(1,2))$ is a T-module with basis $1, F, E, F^2, E^2, \ldots$, where $T = \mathbb{C}[S_{r,s}, \omega^{\pm 1}, \omega'^{\pm 1}]$. This implies that acting with powers of E on v_{n-1} generates a submodule. Hence, $V_{\lambda,\lambda',n}$ is simple if and only if $E^{n-1}v_{n-1} \neq 0$, or equivalently, $F^{n-1}E^{n-1}v_{n-1} \neq 0$.

By [26, Proposition 2.2], we have

$$\prod_{i=0}^{n-2}\left(S_{r,s}-(q')^i r^{\frac{1}{2}}\omega+(q')^{-i}s^{\frac{1}{2}}\omega'\right)=\varepsilon(n)(-\eta)^{n-1}F^{n-1}E^{n-1},$$

where

$$\varepsilon(n)=\begin{cases}1, & n=0 \text{ or } 1 \pmod 4;\\ -1, & n=2 \text{ or } 3 \pmod 4).\end{cases}$$

Thus, the nonvanishing of $F^{n-1}E^{n-1}v_{n-1}$ reduces to checking that

$$\prod_{i=1}^{n-1}((q')^i-1)\left(r^{\frac{1}{2}}q^{-n}\lambda+s^{\frac{1}{2}}q^n(q')^{-i}\lambda'\right)\neq 0. \tag{3.5}$$

Combining this with the condition (3.4), we conclude that $V_{\lambda,\lambda',n}$ is a simple $U_{r,s}(\mathrm{osp}(1,2))$-module if and only if either $\lambda'=(q')^{-n+1}\lambda$ for $1\le n\le l'-1$ or $n=l'$ with $\lambda'\neq(q')^{i+1}\lambda$ for all $1\le i\le l'-1$. We denote the corresponding simple module by $V_{\lambda,\lambda',n}$.

We have the following theorem.

Theorem 3.1. *Finite-dimensional simple modules of* $U_{r,s}(\mathrm{osp}(1,2))$ *up to isomorphism are the following lists:*

(1) $V_{\lambda,\lambda',a}$ *with* $a\neq 0$ *of dimension L, if l is even or* $\lambda'\neq q^{-1}\lambda$*;*
(2) $V_{\lambda,\lambda',a}$ *with* $a\neq 0$ *of dimension l, if l is odd and* $\lambda'=q^{-1}\lambda$*;*
(3) $V_{\lambda,\lambda',\alpha,\beta}$ *with* $\alpha\neq 0$ *of dimension L, if l is even or* $\alpha\beta\neq\dfrac{q\lambda-\lambda'}{(q+1)(r-s)}$*;*
(4) $V_{\lambda,\lambda',\alpha,\beta}$ *with* $\alpha\neq 0$ *of dimension l, if l is odd and* $\alpha\beta=\dfrac{q\lambda-\lambda'}{(q+1)(r-s)}$*;*
(5) $V_{\lambda,\lambda',n}$ *of dimension n, where n is determined by weights* λ,λ' *and*
  (1) $\lambda'=(q')^{-n+1}\lambda$ *for* $1\le n\le l'-1$*;*
  (2) $n=l'$ *and* $\lambda'\neq(q')^{i+1}\lambda$ *for all* $1\le i\le l'-1$.

Proof. Let V be a finite-dimensional simple module over $U_{r,s}(\mathrm{osp}(1,2))$. By Lemma 3.1, we see that the dimension of any simple $U_{r,s}(\mathrm{osp}(1,2))$-module is less than or equal to L.

Now suppose that V is a simple $U_{r,s}(\mathrm{osp}(1,2))$-module with $\dim V=m\le L$.

From the proof of Case 1 in Lemma 3.1, we see that V has a basis $\{v_0,v_1,\dots,v_{m-1}\}$, and the actions of the generators are given as follows:

$$\begin{aligned}&\omega v_i=q^i\lambda v_i,\quad \omega' v_i=q^{-i}\lambda' v_i,\\ &\omega v_{m-1}=q^{m-1}\lambda v_{m-1},\quad \omega' v_{m-1}=q^{-m+1}\lambda' v_{m-1},\\ &Ev_i=v_{i+1},\quad Ev_{m-1}\in V,\\ &Fv_{i+1}=(-1)^i[i+1]_t\left(\frac{t^i\lambda-t^{-i}\lambda'}{r-s}\right)v_i,\quad \text{and}\quad Fv_0=0\end{aligned}$$

for $0\le i<m-1$.

Let $Ev_{m-1}=\sum_{0\le j\le m-1}a_jv_j$.

- Suppose $Ev_{m-1} = 0$ and set

$$y_0 = v_{m-1},\ y_1 = Fy_0,\ \ldots,\ y_{m-1} = F^{m-1}y_0.$$

By the actions of ω, ω', F, E, we have

$$\begin{aligned}
&\omega y_i = q^{-i}\mu y_i, \quad \omega' y_i = q^i \mu' y_i,\\
&\omega y_{m-1} = q^{-m+1}\mu y_{m-1}, \quad \omega' y_{m-1} = q^{m-1}\mu' y_{m-1},\\
&Ey_{i+1} = (-1)^i\left([i+1]_t \frac{t^{-i}\mu - t^i\mu'}{r-s}\right)y_i, \quad \text{and} \quad Ey_0 = 0,\\
&Fy_i = y_{i+1}, \quad \text{and} \quad Fy_{m-1} = 0,
\end{aligned}$$

where $\mu = q^{m-1}\lambda$, $\mu' = q^{1-m}\lambda'$ for $0 \le i < m-1$. By similar arguments from Eq (3.3) to Eq (3.5), we obtain that $m = l'$ and $\lambda' \neq (q')^{j+1}\lambda$ for all $1 \le j \le l'-1$ or $\lambda' = (q')^{-m+1}\lambda$ for $1 \le m \le l'-1$. In these cases, the simple modules V can be identified as $V_{\lambda,\lambda',n}$.

- Suppose $Ev_{m-1} \neq 0$. We now consider two cases:

(1) Assume $m < L$. Since $\omega Ev_{m-1} = q^m\lambda Ev_{m-1}$, we have

$$\lambda \sum_{0\le j\le m-1} q^j a_j v_j = q^m \lambda \sum_{0\le j\le m-1} a_j v_j. \tag{3.6}$$

Hence, there exists some j such that $q^{m-j} = 1$. Moreover, the relation

$$EFv_{m-1} + FEv_{m-1} = \frac{\omega - \omega'}{r-s}v_{m-1}$$

must hold. Thus, we obtain $m = l$ (this occurs only when l is odd and $\lambda' = q^{-1}\lambda$), that is, $Ev_{l-1} = av_0$. In this case, we obtain a simple $U_{r,s}(\mathrm{osp}(1,2))$-module of dimension l, denoted by $V_{\lambda,\lambda',a}$, with the following actions:

$$\begin{aligned}
&\omega v_i = q^i\lambda v_i, \quad \omega' v_i = q^{-i}\lambda' v_i,\\
&\omega v_{l-1} = q^{l-1}\lambda v_{l-1}, \quad \omega' v_{l-1} = q^{-l+1}\lambda' v_{l-1},\\
&Ev_i = v_{i+1}, \quad Ev_{l-1} = av_0 \quad (a \neq 0),\\
&Fv_{i+1} = (-1)^i[i+1]_t\left(\frac{t^i\lambda - t^{-i}\lambda'}{r-s}\right)v_i, \quad \text{and} \quad Fv_0 = 0,
\end{aligned}$$

where $0 \le i < l-1$ and $\lambda' = q^{-1}\lambda$.

(2) If $m = L$, then by Lemma 3.1, V has a basis $\{v_0, v_1, \ldots, v_{L-1}\}$, and the following relations hold:

$$\begin{aligned}
&\omega v_i = q^i\lambda v_i, \quad \omega' v_i = q^{-i}\lambda' v_i,\\
&\omega v_{L-1} = q^{L-1}\lambda v_{L-1}, \quad \omega' v_{L-1} = q^{-L+1}\lambda' v_{L-1},\\
&Ev_i = v_{i+1}, \quad Ev_{L-1} = av_0,\\
&Fv_{i+1} = (-1)^i[i+1]_t\left(\frac{t^i\lambda - t^{-i}\lambda'}{r-s}\right)v_i, \quad \text{and} \quad Fv_0 = 0
\end{aligned}$$

for $0 \le i < L-1$.

Assume that $Fv_i \neq 0$ for all $1 \le i \le L-1$, which implies that $\lambda' \neq (q')^{i-1}\lambda$. In this case, we can choose a new basis

$$\{y_0 = v_{L-1},\ y_1 = Fy_0,\ \ldots,\ y_{L-1} = F^{L-1}y_0\}$$

and

$$\omega y_m = \mu q^{-m} y_m, \quad \omega' y_m = \mu' q^m y_m, \quad E y_m = \kappa_m y_{m-1}, \quad F y_m = y_{m+1}, \text{ and } F y_{L-1} = 0,$$

where $\mu = q^{L-1}\lambda \neq 0, \mu' = q^{1-L}\lambda' \neq 0$ and $\kappa_m \neq 0$ for $0 \le m \le L-1$. Let $v = \sum_{0\le i\le l-1} a_i y_i \neq 0$, where $a_i \neq 0$. Then

$$\omega^j v = \sum_{0\le i\le L-1} a_i \mu^j q^{-ij} y_i.$$

By choosing suitable values of j, we can express each y_i as a linear combination of the vectors $\omega^j \cdot v$ using Cramer's rule and the nonvanishing of the Vandermonde determinant. In particular, y_0 can be generated by v over $\mathbb{K}[\omega^{\pm 1}, \omega'^{\pm 1}]$. Since y_0 generates the entire module $V_{\lambda,\lambda',a}$ under the action of $U_{r,s}(\mathrm{osp}(1,2))$, it follows that v also generates $V_{\lambda,\lambda',a}$. Moreover, this holds for any nonzero vector $v \in V_{\lambda,\lambda',a}$. We conclude that $V_{\lambda,\lambda',a}$ has no nontrivial submodules. Therefore, $V_{\lambda,\lambda',a} = U_{r,s}(\mathrm{osp}(1,2)) \cdot v$ is a simple module.

Next, if $Fv_i = 0$ for some $1 \le i \le L-1$ with $i \neq l$, then $\lambda' = (q')^{i-1}\lambda$. In this case, the module $V_{\lambda,\lambda',a}$ remains simple by arguments analogous to the above. Note that if $L = 2l$ where l is odd and $\lambda' = q^{-1}\lambda$, then V is not simple since $\{v_0 + v_l, v_1 + v_{l+1}, \ldots, v_{l-1} + v_{L-1}\}$ spans a proper simple submodule of dimension l.

Consequently, $V_{\lambda,\lambda',a}$ with $a \neq 0$ is a simple module of dimension l if l is odd and $\lambda' = q^{-1}\lambda$, and of dimension L otherwise.

Recall the proof of Case 2 in Lemma 3.1. If $F^L v = \alpha v$ for some $\alpha \neq 0$, then V is spanned by the linearly independent vectors $v, Fv, \ldots, F^{L-1}v$. Let $v_0 = v$ and $v_m = F^m v_0$ for $0 \le m \le L-1$. Then

$$\omega v_m = q^{-m}\lambda v_m, \quad \omega' v_m = q^m \lambda' v_m, \quad E^L v_0 = a v_0, \text{ and } F v_{L-1} = \alpha v_0$$

for some scalars $\lambda, \lambda', a, \alpha \in \mathbb{K}$.

We see that

$$\begin{aligned}
Ev_0 &= \alpha^{-1} E F^L v_0 = \alpha^{-1}(EF)F^{L-1}v_0 \\
&= \alpha^{-1}\left(\frac{\omega - \omega'}{r-s} - \frac{1}{\eta}\left(r^{\frac{1}{2}}\omega - s^{\frac{1}{2}}\omega' - S_{r,s}\right)\right)F^{L-1}v_0 \\
&= \left(\alpha^{-1}\gamma\right)F^{L-1}v_0 \\
&= \left(\alpha^{-1}\gamma\right)v_{L-1},
\end{aligned}$$

where $\beta = \alpha^{-1}\gamma$, so that $\gamma = \alpha\beta$, and γ is as defined in (3.1). For any $0 \le m < L-1$, we have

$$\begin{aligned}
EF^{m+1}v_0 &= (-1)^m [m+1]_t F^m [\omega; \omega'; -m]_t v_0 + (-1)^{m+1} F^{m+1} E v_0 \\
&= (-1)^m \left([m+1]_t \frac{t^{-m}\lambda - t^m \lambda'}{r-s} - \gamma\right) F^m v_0.
\end{aligned}$$

Thus, the actions of the generators on the basis vectors $v_m = F^m v_0$ are given by

$$\omega v_m = q^{-m}\lambda v_m, \quad \omega' v_m = q^m \lambda' v_m,$$
$$\omega v_{L-1} = q^{1-L}\lambda v_{L-1}, \quad \omega' v_{L-1} = q^{L-1}\lambda' v_{L-1},$$
$$E v_{m+1} = (-1)^m \left([m+1]_t \frac{t^{-m}\lambda - t^m\lambda'}{r-s} - \alpha\beta\right) v_m, \quad E v_0 = \beta v_{L-1},$$
$$F v_m = v_{m+1}, \quad \text{and} \quad F v_{L-1} = \alpha v_0$$

for $0 \le m < L-1$. If $L = 2l$ where l is odd and $\alpha\beta = \dfrac{q\lambda - \lambda'}{(q+1)(r-s)}$, then V is not simple since $\{v_0 + v_l, v_1 + v_{l+1}, \ldots, v_{l-1} + v_{L-1}\}$ spans a proper simple submodule of dimension l. Otherwise, the module $V_{\lambda,\lambda',\alpha,\beta}$ is simple by analogous arguments.

The proof is finished. □

4. Representations of the restricted quantum superalgebra

First, we give the definition of the restricted quantum superalgebra $\overline{U}_{r,s}(\mathrm{osp}(1,2))$.

Definition 4.1. *The restricted quantum superalgebra* $\overline{U}_{r,s}(\mathrm{osp}(1,2))$ *is the associative unital algebra generated by* $\omega^{\pm}, \omega'^{\pm}, E, F$*, subject to the relations in* (2.2) *and the additional conditions*

$$\omega^L = 1, \quad \omega'^L = 1, \quad E^L = 0, \text{ and } F^L = 0,$$

where $q = (rs^{-1})^{1/2}$ *and* L *is defined in* (2.1).

Proposition 4.1. *The set* $\{F^i E^j \omega^k \omega'^l | 0 \le i, j < L, k, l \in \mathbb{Z}_L\}$ *is a basis of* $\overline{U}_{r,s}(\mathrm{osp}(1,2))$.

Proof. Similar to the proof of [26, Proposition 2.3], it can be easily obtained. □

Let V be a nonzero $\overline{U}_{r,s}(\mathrm{osp}(1,2))$-module with dimension greater than L. There exists a nonzero vector $v \in V$ such that

$$\omega v = \lambda v, \quad \omega' v = \lambda' v, \quad E^L v = 0, \text{ and } F^L v = 0$$

for some scalars $\lambda, \lambda' \in \mathbb{K}$.

Since $\omega^L = 1$, we have $\lambda^L = 1$, $\lambda'^L = 1$, so λ, λ' are primitive L-th roots of unity. Since $F^L = 0$, the operator F is nilpotent. Thus, without loss of generality, we can choose $v \neq 0$ such that $Fv = 0$. Now consider the subspace

$$V' = \mathrm{span}\{v, Ev, E^2 v, \ldots, E^{L-1} v\}.$$

Since $E^L = 0$, we get that $E^L v = 0$, and hence $\dim V' \le L$.

We claim that V' is invariant under the actions of all generators. For $0 \le m < L$, we have

$$E(E^m v) = E^{m+1} v \in V', \quad \omega E^m v = q^m \lambda E^m v \in V', \text{ and } \omega' E^m v = q^{-m}\lambda' E^m v \in V'.$$

Moreover, we have

$$F E^m v = \left((-1)^m E^m F + (-1)^{m-1}[m]_t E^{m-1}[\omega; \omega'; m-1]_t\right) v$$

$$= (-1)^{m-1}[m]_t \left(\frac{t^{m-1}\lambda - t^{1-m}\lambda'}{r-s} \right) E^{m-1} v \in V'.$$

Thus, V' is a $\overline{U}_{r,s}(\mathrm{osp}(1,2))$-submodule of V, and its dimension is at most L.

Therefore, there is no finite dimensional simple $\overline{U}_{r,s}(\mathrm{osp}(1,2))$ of dimension greater than L.

Theorem 4.1. *The finite-dimensional simple $\overline{U}_{r,s}(\mathrm{osp}(1,2))$-module up to isomorphism is $V_{\lambda,\lambda',n}$ of dimension $n \le l'$, where n is determined by weights λ, λ' and*

(1) $\lambda' = (q')^{-n+1}\lambda$ for $1 \le n \le l'-1$, where λ is a primitive L-th root of unity;
(2) $n = l'$ and $\lambda' \neq (q')^{i+1}\lambda$ for all $1 \le i \le l'-1$, where λ, λ' are primitive L-th roots of unity.

Proof. Let V be a finite-dimensional simple $\overline{U}_{r,s}(\mathrm{osp}(1,2))$-module of dimension $n \le L$, with a basis

$$\{v_0, v_1, \ldots, v_{n-1}\}.$$

Since $F^L = 0$, we may assume that $Fv_0 = 0$ without loss of generality, and the actions are as follows:

$$\begin{aligned}
&\omega v_m = q^m \lambda v_m, \quad \omega' v_m = q^{-m}\lambda' v_m \quad \text{for } 0 \le m \le n-1,\\
&E v_m = v_{m+1} \quad \text{for } 0 \le m < n-1, \quad E v_{n-1} \in V,\\
&F v_{m+1} = (-1)^m [m+1]_t \left(\frac{t^m \lambda - t^{-m}\lambda'}{r-s} \right) v_m \quad \text{for } 0 \le m < n-1, \quad \text{and} \quad F v_0 = 0,
\end{aligned}$$

where λ, λ' are primitive L-th roots of unity.

Let $Ev_{n-1} = \sum_{0 \le i \le n-1} a_i v_i$.

• Suppose $Ev_{n-1} = 0$. From the relation

$$EFv_{n-1} + FEv_{n-1} = \frac{\omega - \omega'}{r-s} v_{n-1},$$

we get

$$(-1)^{n-1}[n]_t \left(\frac{t^{n-1}\lambda - t^{-n+1}\lambda'}{r-s} \right) = 0.$$

Thus, either $t^{2n} = 1$ or $\lambda' = t^{-2n+2}\lambda$. If $t^{2n} = 1$, then $n = l'$. If $\lambda' = t^{-2n+2}\lambda$ for $n \ge l'+1$, then there exist a submodule generated by $v_{l'}, \ldots, v_{n-1}$, which implies that V is not simple unless $n \le l'$.

Moreover, as the discussion of Eq (3.5) shows, $V_{\lambda,\lambda',n}$ is simple if and only if one of the following holds:

(1) $\lambda' = (q')^{-n+1}\lambda$ for $1 \le n \le l'-1$, where λ is a primitive L-th root of unity;
(2) $n = l'$ and $\lambda' \neq (q')^{i+1}\lambda$ for all $1 \le i \le l'-1$, where λ, λ' are primitive L-th roots of unity.

• Suppose $Ev_{n-1} \neq 0$. From the relation $EFv_{m-1} + FEv_{m-1} = \frac{\omega-\omega'}{r-s} v_{m-1}$ and Eq (3.6), we obtain the following two cases:

(1) If $n < L$, then $n = l$ (this occurs only when l is odd and $\lambda' = q^{-1}\lambda$), that is, $Ev_{l-1} = av_0$. However, $E^L = 0$ implies $Ev_{l-1} = 0$, contradicting $Ev_{n-1} = av_0 \neq 0$.
(2) If $n = L$ and L is even, then $E^L = 0$ implies $Ev_{L-1} = 0$, again contradicting $Ev_{n-1} \neq 0$.

The proof is finished. □

5. Conclusions

In this paper, we completely determine all finite-dimensional simple modules up to isomorphism for the two-parameter quantum superalgebra $U_{r,s}(\mathrm{osp}(1,2))$ and its restricted form at roots of unity. These results may help us to understand the general representation theory of two-parameter quantum superalgebras.

Author contributions

S. Yang contributed the creative ideas and proof techniques for this paper, including the construction of the article structure and the revision of the content; Y. Gao consulted the relevant background of the paper and wrote the article. All authors have read and approved the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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